Matematika A1a – Analízis, 11–12. hét

Határozatlan integrál megoldása

I. Számítsuk ki a következő integrálokat.

1. $\displaystyle\int \frac{1+x}{\sqrt{x}}\,\mathrm{d}\,x = \int x^{-\frac{1}{2}} + x^{\frac{1}{2}}\,\mathrm{d}\,x = 2\sqrt{x} + \frac{2}{3}x^{\frac{3}{2}} + C$
2. $\displaystyle\int \frac{1}{x^2}\,\mathrm{d}\,x = -\frac{1}{x} + C$
3. $\displaystyle\int 1 + \mathrm{e}^{x-1}\,\mathrm{d}\,x = x + \mathrm{e}^{x-1} + C$
4. $\displaystyle\int \frac{x^2-1}{x^2+1}\,\mathrm{d}\,x = \int \frac{(x^2+1)-2}{x^2+1}\,\mathrm{d}\,x = x - 2\,\mathrm{arctg}(x) + C$
5. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}\,x = \mathrm{arsh}(x) + C$
6. $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}\,x = \mathrm{arch}(x) + C$
7. $\displaystyle\int \frac{1}{\sin^2(x)}\,\mathrm{d}\,x = -\,\mathrm{ctg}(x) + C$
8. $\displaystyle\int \frac{1}{\cos^2(x)}\,\mathrm{d}\,x = \mathrm{tg}(x) + C$
9. $\displaystyle\int \frac{1}{\mathrm{sh}^2(x)}\,\mathrm{d}\,x = -\,\mathrm{cth}(x) + C$
10. $\displaystyle\int \frac{1}{\mathrm{ch}^2(x)}\,\mathrm{d}\,x = \mathrm{th}(x) + C$
11. $\displaystyle\int \mathrm{sh}(x)\,\mathrm{d}\,x = \mathrm{ch}(x) + C$
12. $\displaystyle\int \mathrm{ch}(x)\,\mathrm{d}\,x = \mathrm{sh}(x) + C$

II. Parciális integrálás segítségével határozzuk meg az alábbi integrálokat, ahol $a, b \in \mathbb{R} \setminus \{0\}$.

1. $\int x e^{ax} \,\mathrm{d}\, x = \frac{1}{a^2} \mathrm{e}^{ax}(ax - 1) + C$
2. $\int x^2 \mathrm{e}^{-ax} \,\mathrm{d}\, x = -\frac{1}{a^3} \mathrm{e}^{-ax}(a^2x^2 + 2ax + 2) + C$
3. $\int x \sin(x) \,\mathrm{d}\, x = \sin(x) - x\cos(x) + C$
4. $\int \mathrm{e}^{ax} \sin(bx) \,\mathrm{d}\, x = \frac{\mathrm{e}^{ax}}{a^2 + b^2}(a \sin(bx) - b\cos(bx)) + C$
5. $\int \mathrm{e}^{x} \cos(x) \,\mathrm{d}\, x = \frac{\mathrm{e}^{x}}{2}(\sin(x) + \cos(x)) + C$
6. $\int \sqrt{1 - x^2} \,\mathrm{d}\, x = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\arcsin(x) + C$
7. $\int \sqrt{1 + x^2} \,\mathrm{d}\, x = \frac{1}{2}x\sqrt{1 + x^2} + \frac{1}{2}\operatorname{arsh}(x) + C$
8. $\int \sqrt{x^2 - 1} \,\mathrm{d}\, x = \frac{1}{2}x\sqrt{x^2 - 1} + \frac{1}{2}\operatorname{arch}(x) + C$
9. $\int \arcsin(x) \,\mathrm{d}\, x = x\arcsin(x) + \sqrt{1 - x^2} + C$
10. $\int \operatorname{arctg}(x) \,\mathrm{d}\, x = x\operatorname{arctg}(x) - \frac{1}{2}\ln(1 + x^2) + C$
11. $\int x \operatorname{arctg}(ax) \,\mathrm{d}\, x = \frac{x^2}{2}\operatorname{arctg}(ax) + \frac{1}{2a^2}\operatorname{arctg}(ax) - \frac{x}{2a} + C$
12. $\int x^3 \ln^2(x) \,\mathrm{d}\, x = \frac{x^4}{4}\ln^2(x) - \frac{x^4}{8}\ln(x) + \frac{x^4}{32} + C$

III. Racionális törtfüggvényekre vonatkozó integrálási szabályt alkalmazva határozzuk meg az alábbi integrálokat.

1. $$\int \frac{1}{1-x^2}\,\mathrm{d}\,x = \int \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{x-1}\,\mathrm{d}\,x = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$$

2. $$\int \frac{1}{x^2-2x-3}\,\mathrm{d}\,x = \int \frac{1}{4}\cdot\frac{1}{x-3} - \frac{1}{4}\cdot\frac{1}{x+1}\,\mathrm{d}\,x = \frac{1}{4}\ln|x-3| - \frac{1}{4}\ln|x+1| + C$$

3. $$\int \frac{1}{x^2+2x+6}\,\mathrm{d}\,x = \frac{1}{\sqrt{5}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{5}}\right) + C$$

4. $$\int \frac{x^2-1}{(x+2)^3}\,\mathrm{d}\,x = \int \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{3}{(x+2)^3}\,\mathrm{d}\,x =$$
$$= \ln|x+2| + \frac{4}{x+2} - \frac{3}{2(x+2)^2} + C$$

5. $$\int \frac{1}{x^3+1}\,\mathrm{d}\,x = \int \frac{1}{3}\cdot\frac{1}{x+1} - \frac{1}{3}\cdot\frac{x-2}{x^2-x+1}\,\mathrm{d}\,x =$$
$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|x^2-x+1\right| + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

6. $$\int \frac{x^4}{(x-2)(x-3)(x-4)}\,\mathrm{d}\,x = \int x + 9 + \frac{8}{x-2} - \frac{81}{x-3} + \frac{128}{x-4}\,\mathrm{d}\,x =$$
$$= \frac{x^2}{2} + 9x + 8\ln|x-2| - 81\ln|x-3| + 128\ln|x-4| + C$$

7. $$\int \frac{16x^2+4x}{x^4+4}\,\mathrm{d}\,x = \int \frac{4x+1}{x^2-2x+2} - \frac{4x+1}{x^2+2x+2}\,\mathrm{d}\,x =$$
$$= 2\ln\left|x^2-2x+2\right| + 5\operatorname{arctg}(x-1) - 2\ln\left|x^2+2x+5\right| + 3\operatorname{arctg}(x+1) + C$$

8. $$\int \frac{x^3}{(x^2+1)^2}\,\mathrm{d}\,x = \int \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\,\mathrm{d}\,x = \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C$$

9. $$\int \frac{x^4+4}{x^3-1}\,\mathrm{d}\,x = \int x + \frac{5}{3}\cdot\frac{1}{x-1} - \frac{5x+7}{3(x^2+x+1)}\,\mathrm{d}\,x =$$
$$= \frac{x^2}{2} + \frac{5}{3}\ln|x-1| - \frac{5}{6}\ln\left|x^2+x+1\right| - \sqrt{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

10. $$\int \frac{1}{(x^2+x+1)^2}\,\mathrm{d}\,x = \frac{2x+1}{3(x^2+x+1)} + \frac{4}{3\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

11. $$\int \frac{x}{(x^2+2x+2)^2}\,\mathrm{d}\,x = -\frac{x+2}{2(x^2+2x+2)} - \frac{1}{2}\operatorname{arctg}(x+1) + C$$

12. $$\int \frac{1}{(x^2+2x+2)^2}\,\mathrm{d}\,x = \frac{x+1}{2(x^2+2x+2)} + \frac{1}{2}\operatorname{arctg}(x+1) + C$$

IV. Alkalmas helyettesítéssel határozzuk meg az alábbi integrálokat, ahol $k \in \mathbb{N}$.

1. $t = x+2$ $\quad \int \frac{x^3}{(x+2)^4}\,\mathrm{d}\,x = \int \frac{(t-2)^3}{t^4}\,\mathrm{d}\,t = \ln|x+2| + \frac{6}{x+2} - \frac{6}{(x+2)^2} + \frac{8}{3(x+2)^3} + C$

2. $t = \sqrt{1+x}$ $\quad \int \frac{1}{\sqrt{1+x}+(\sqrt{1+x})^3}\,\mathrm{d}\,x = \int \frac{2}{1+t^2}\,\mathrm{d}\,t = 2\operatorname{arctg}\left(\sqrt{1+x}\right) + C$

3. $t = \sqrt[4]{x-1}$ $\quad \int x\sqrt[4]{x-1}\,\mathrm{d}\,x = \int 4t^4 + 4t^8\,\mathrm{d}\,t = \frac{4}{5}(x-1)^{\frac{5}{4}} + \frac{4}{9}(x-1)^{\frac{9}{4}} + C$

4. $t = \mathrm{e}^x$ $\quad \int \frac{\mathrm{e}^{4x}}{1+\mathrm{e}^x}\,\mathrm{d}\,x = \int \frac{t^3}{1+t}\,\mathrm{d}\,t = \frac{\mathrm{e}^{3x}}{3} - \frac{\mathrm{e}^{2x}}{2} + \mathrm{e}^x - \ln(1+\mathrm{e}^x) + C$

5. $t = \sqrt{\mathrm{e}^x - 1}$ $\quad \int \sqrt{\mathrm{e}^x-1}\,\mathrm{d}\,x = \int \frac{2t^2}{t^2+1}\,\mathrm{d}\,t = 2\sqrt{\mathrm{e}^x-1} - 2\operatorname{arctg}(\sqrt{\mathrm{e}^x-1}) + C$

6. $t = \sqrt{x}$ $\quad \int \sqrt{x}\,\mathrm{e}^{\sqrt{x}}\,\mathrm{d}\,x = \int 2t^2\,\mathrm{e}^t\,\mathrm{d}\,t = 2\,\mathrm{e}^{\sqrt{x}}(x - 2\sqrt{x} + 2) + C$

7. $t = \arcsin x$ $\quad \int \sqrt{1-x^2}\,\mathrm{d}\,x = \int \cos^2 t\,\mathrm{d}\,t = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$

8. $t = \sin(x)$ $\quad \int \sin^k(x)\cdot\cos(x)\,\mathrm{d}\,x = \int t^k\,\mathrm{d}\,t = \frac{\sin^{k+1}(x)}{k+1} + C$

9. $t = \cos(x)$ $\quad \int \cos^k(x)\cdot\sin(x)\,\mathrm{d}\,x = \int -t^k\,\mathrm{d}\,t = -\frac{\cos^{k+1}(x)}{k+1} + C$

10. $t = \sin(x)$ $\quad \int \operatorname{ctg}(x)\,\mathrm{d}\,x = \int \frac{1}{t}\,\mathrm{d}\,t = \ln|\sin(x)| + C$

11. $t = \cos(x)$ $\quad \int \operatorname{tg}(x)\,\mathrm{d}\,x = \int -\frac{1}{t}\,\mathrm{d}\,t = -\ln|\cos(x)| + C$

12. $t = \operatorname{tg}(x)$ $\quad \int \operatorname{tg}^2(x)\,\mathrm{d}\,x = \int \frac{t^2}{1+t^2}\,\mathrm{d}\,t = \operatorname{tg}(x) - x + C$

13. $t = \operatorname{tg}(x)$ $\quad \int \operatorname{tg}^4(x)\,\mathrm{d}\,x = \int \frac{t^4}{1+t^2}\,\mathrm{d}\,t = \frac{\operatorname{tg}^3(x)}{3} - \operatorname{tg}(x) + x + C$

14. $t = \mathrm{e}^x$ $\quad \int \frac{2}{\mathrm{e}^{3x}-\mathrm{e}^x}\,\mathrm{d}\,x = \int \frac{2}{t^4-t^2}\,\mathrm{d}\,t = \ln|\mathrm{e}^x - 1| - \ln|\mathrm{e}^x + 1| + 2\,\mathrm{e}^{-x} + C$

15. $t = \sqrt{x-1}$ $\quad \int \frac{1}{x+\sqrt{x-1}-1}\,\mathrm{d}\,x = \int \frac{2}{t+1}\,\mathrm{d}\,t = 2\ln(\sqrt{x-1}+1) + C$

V. A $t = \operatorname{tg}(x)$ helyettesítéssel vezessük vissza a következő integrálokat racionális törtfüggvények integráljaira, majd számoljuk ki az integrálokat.

1. $$\int \frac{1}{\operatorname{tg}(x)-1}\,\mathrm{d}\,x = \int \frac{1}{(t-1)(t^2+1)}\,\mathrm{d}\,t = \int \frac{1}{2}\cdot\frac{1}{t-1} - \frac{1}{2\cdot}\frac{t+1}{t^2+1}\,\mathrm{d}\,t =$$
$$= \frac{1}{2}\ln|\operatorname{tg}(x)-1| - \frac{1}{4}\ln(1+\operatorname{tg}^2(x)) - \frac{x}{2} + C$$

2. $$\int \frac{1}{\sin^2(x)+\sin(2x)}\,\mathrm{d}\,x = \int \frac{1}{t(t+2)}\,\mathrm{d}\,t =$$
$$= \int \frac{1}{2t} - \frac{1}{2}\cdot\frac{1}{t+2}\,\mathrm{d}\,t = \frac{1}{2}\ln|\operatorname{tg}(x)| - \frac{1}{2}\ln|2+\operatorname{tg}(x)| + C$$

3. $$\int \frac{1+\operatorname{tg}^2(x)}{1-\operatorname{tg}^2(x)}\,\mathrm{d}\,x = \int \frac{1}{1-t^2}\,\mathrm{d}\,t = \int \frac{1}{2}\cdot\frac{1}{t+1} - \frac{1}{2}\cdot\frac{1}{t-1}\,\mathrm{d}\,t = \frac{1}{2}\ln\left|\frac{\operatorname{tg}(x)+1}{\operatorname{tg}(x)-1}\right| + C$$

4. $$\int \frac{1}{1+3\cos^2(x)}\,\mathrm{d}\,x = \int \frac{1}{t^2+4}\,\mathrm{d}\,t = \frac{1}{2}\operatorname{arctg}\left(\frac{1}{2}\operatorname{tg}(x)\right) + C$$

VI. A következő integrálok kiszámításához gondoljunk az elemi függvények (sin, tg, exp, arctg, arcsin,...) deriváltjára és a deriválásnál megismert láncszabályra.

1. $\displaystyle\int x\,\mathrm{e}^{-x^2}\,\mathrm{d}\,x = -\frac{1}{2}\,\mathrm{e}^{-x^2} + C$

2. $\displaystyle\int \frac{3^x}{1+9^x}\,\mathrm{d}\,x = \frac{1}{\ln 3}\,\mathrm{arctg}\,(3^x) + C$

3. $\displaystyle\int \frac{\mathrm{e}^x}{\sqrt{1-\mathrm{e}^{2x}}}\,\mathrm{d}\,x = \arcsin(\mathrm{e}^x) + C$

4. $\displaystyle\int \frac{\mathrm{e}^x}{\sqrt[3]{1+\mathrm{e}^x}}\,\mathrm{d}\,x = \frac{3}{2}(1+\mathrm{e}^x)^{\frac{2}{3}} + C$

5. $\displaystyle\int \frac{\cos(\ln(x))}{x}\,\mathrm{d}\,x = \sin(\ln|x|) + C$

6. $\displaystyle\int \frac{\cos(x)}{1+\sin^2(x)}\,\mathrm{d}\,x = \mathrm{arctg}(\sin(x)) + C$

VII. Trigonometrikus azonosságok segítségével számoljuk ki az alábbi határozatlan integrálokat.

1. $\displaystyle\int \cos(2x)\cos(5x)\,\mathrm{d}\,x = \int \frac{\cos(3x)}{2} + \frac{\cos(7x)}{2}\,\mathrm{d}\,x = \frac{\sin(3x)}{6} + \frac{\sin(7x)}{14} + C$

2. $\displaystyle\int \cos^5(x)\sin^2(x)\,\mathrm{d}\,x = \int \cos(x)\cdot(\sin^2(x) - 2\sin^4(x) + \sin^6(x))\,\mathrm{d}\,x =$

$\displaystyle= \frac{\sin^3(x)}{3} - \frac{2\sin^5(x)}{5} + \frac{\sin^7(x)}{7} + C$

3. $\displaystyle\int \cos^5(x)\,\mathrm{d}\,x = \int \cos(x)\cdot(1 - 2\sin^2(x) + \sin^4(x))\,\mathrm{d}\,x = \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C$

4. $\displaystyle\int \cos^5(x)\sin^3(x)\,\mathrm{d}\,x = \int \cos(x)\cdot(\sin^3(x) - 2\sin^5(x) + \sin^7(x))\,\mathrm{d}\,x =$

$\displaystyle= \frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{3} + \frac{\sin^8(x)}{8} + C$

5. $\displaystyle\int \sin^4(x)\,\mathrm{d}\,x = \int \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\,\mathrm{d}\,x = \frac{3x}{8} - \frac{\cos(2x)}{4} + \frac{\sin(4x)}{32} + C$

6. $\displaystyle\int \cos^5(x)\sin^5(x)\,\mathrm{d}\,x = \int \cos(x)\cdot(\sin^5(x) - 2\sin^7(x) + \sin^9(x))\,\mathrm{d}\,x =$

$\displaystyle= \frac{\sin^6(x)}{6} - \frac{\sin^8(x)}{4} + \frac{\sin^{10}(x)}{10} + C$