Matematika A1a – Analízis, 14. hét

Határozott integrál alkalmazásainak megoldása

I. Az integrálformulákból közvetlenül adódnak a formulák.

II. 1. Ívhosszszámítás. $L_f = \int_I \sqrt{1+f'(x)^2}\,\mathrm{d}\,x$

a. $L_f = \int_0^b \sqrt{1+a^2}\,\mathrm{d}\,x = \left[x\sqrt{1+a^2}\right]_0^b = b\sqrt{1+a^2}$

b. $L_f = \int_0^a \frac{a}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x = \left[a\arcsin\left(\frac{x}{a}\right)\right]_0^a = \frac{a\pi}{2}$

c. $L_f = \int_{-a}^a \frac{a}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x = \left[a\arcsin\left(\frac{x}{a}\right)\right]_{-a}^a = \pi a$

d. $L_f = \int_0^b \operatorname{ch} x\,\mathrm{d}\,x = [\operatorname{sh} x]_0^b = \operatorname{sh} b$

e. $L_f = \int_0^b \sqrt{1+4a^2x^2}\,\mathrm{d}\,x = \left[\frac{x\sqrt{1+4a^2x^2}}{2} + \frac{\operatorname{arsh}(2ax)}{4a}\right]_0^b = \frac{b\sqrt{1+4a^2b^2}}{2} + \frac{\operatorname{arsh}(2ab)}{4a}$

f. $L_f = \int_0^b \sqrt{1+a^2\,\mathrm{e}^{2ax}}\,\mathrm{d}\,x = \left[\frac{\sqrt{1+a^2\,\mathrm{e}^{2ax}}}{a} - \frac{1}{a}\cdot\operatorname{arth}\left(\frac{1}{\sqrt{1+a^2\,\mathrm{e}^{2ax}}}\right)\right]_0^b =$

$= \frac{\sqrt{1+a^2\,\mathrm{e}^{2ab}}-\sqrt{1+a^2}}{a} - \frac{1}{a}\cdot\operatorname{arth}\left(\frac{1}{\sqrt{1+a^2\,\mathrm{e}^{2ab}}}\right) + \frac{1}{a}\cdot\operatorname{arth}\left(\frac{1}{\sqrt{1+a^2}}\right)$

2. Felszínszámítás. $F_f = 2\pi\int_I f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}\,x$

a. $F_f = 2\pi\int_0^b ax\sqrt{1+a^2}\,\mathrm{d}\,x = \pi\left[ax^2\sqrt{1+a^2}\right]_0^b = \pi ab^2\sqrt{1+a^2}$

b. $F_f = 2\pi\int_0^a a\,\mathrm{d}\,x = 2\pi\,[ax]_0^a = 2\pi a^2$

c. $F_f = 2\pi\int_{-a}^a a\,\mathrm{d}\,x = 2\pi\,[ax]_{-a}^a = 4\pi a^2$

d. $F_f = 2\pi\int_0^b \operatorname{ch}^2 x\,\mathrm{d}\,x = [\pi(x+\operatorname{sh}x\operatorname{ch}x)]_0^b = \pi b + \pi\operatorname{sh}b\operatorname{ch}b$

e. $F_f = 2\pi\int_0^b ax^2\sqrt{1+4a^2x^2}\,\mathrm{d}\,x \overset{t=\operatorname{arsh}(2ax)}{\downarrow \atop =} \frac{\pi}{4a^2}\int\limits_0^{\operatorname{arsh}(2ab)} \operatorname{sh}^2 t\operatorname{ch}^2 t\,\mathrm{d}\,t = \frac{\pi}{16a^2}\int\limits_0^{\operatorname{arsh}(2ab)} \operatorname{sh}^2(2t)\,\mathrm{d}\,t =$

$= \left[\frac{\pi}{32a^2}\left(\frac{\operatorname{sh}(4t)}{4} - t\right)\right]_0^{\operatorname{arsh}(2ab)} = \left[\frac{\pi}{32a^2}\left(\operatorname{sh}t\operatorname{ch}t(1+2\operatorname{sh}^2 t) - t\right)\right]_0^{\operatorname{arsh}(2ab)} =$

$= \frac{\pi b(1+8a^2b^2)\sqrt{1+4a^2b^2}}{16a} - \frac{\pi\operatorname{arsh}(2ab)}{32a^2}$

f. $F_f = 2\pi\int_0^b \mathrm{e}^{ax}\sqrt{1+a^2\,\mathrm{e}^{2ax}}\,\mathrm{d}\,x \overset{t=a\,\mathrm{e}^{ax}}{\downarrow \atop =} \frac{2\pi}{a^2}\int_1^{a\,\mathrm{e}^{ab}} \sqrt{1+t^2}\,\mathrm{d}\,t =$

$= \frac{\pi}{a^2}\left[\operatorname{arsh}t + t\sqrt{1+t^2}\right]_1^{a\,\mathrm{e}^{ab}} = \frac{\pi}{a^2}\left(\operatorname{arsh}(a\,\mathrm{e}^{ab}) - \operatorname{arsh}1 + a\,\mathrm{e}^{ab}\sqrt{1+a^2\,\mathrm{e}^{2ab}} - \sqrt{2}\right)$

3. Térfogatszámítás. $V_f = \pi \int_I f^2(x)\,\mathrm{d}\,x$

a. $V_f = \pi \int_0^b a^2x^2\,\mathrm{d}\,x = \left[\frac{\pi a^2 x^3}{3}\right]_0^b = \frac{\pi a^2 b^3}{3}$

b. $V_f = \pi \int_0^a a^2 - x^2\,\mathrm{d}\,x = \pi \left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{2\pi a^3}{3}$

c. $V_f = \pi \int_{-a}^a a^2 - x^2\,\mathrm{d}\,x = \pi \left[a^2 x - \frac{x^3}{3}\right]_{-a}^a = \frac{4\pi a^3}{3}$

d. $V_f = \pi \int_0^b \operatorname{ch}^2(x)\,\mathrm{d}\,x = \left[\frac{\pi}{2}\operatorname{sh} x \operatorname{ch} x + \frac{\pi x}{2}\right]_0^b = \frac{\pi}{4}\operatorname{sh}(2b) + \frac{\pi b}{2}$

e. $V_f = \pi \int_0^b a^2x^4\,\mathrm{d}\,x = \left[\frac{\pi a^2 x^5}{5}\right]_0^b = \frac{\pi a^2 b^5}{5}$

f. $V_f = \pi \int_0^b \mathrm{e}^{2ax}\,\mathrm{d}\,x = \left[\frac{\pi\,\mathrm{e}^{2ax}}{2a}\right]_0^b = \frac{\pi}{2a}\left(\mathrm{e}^{2ab} - 1\right)$

4. Görbe súlypontja. $x_s(L_f) = \frac{1}{L_f}\int_I x\sqrt{1+f'(x)^2}\,\mathrm{d}\,x$, $y_s(L_f) = \frac{1}{L_f}\int_I f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}\,x$ (Vegyük észre, hogy $y_s = \frac{F_f}{2\pi L_f}$ teljesül.)

a. $x_s = \frac{1}{L_f}\int_0^b x\sqrt{1+a^2}\,\mathrm{d}\,x = \frac{1}{L_f}\left[\frac{x^2\sqrt{1+a^2}}{2}\right]_0^b = \frac{b}{2}$

$y_s = \frac{a}{2}$

b. $x_s = \frac{1}{L_f}\int_0^a \frac{ax}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x = \frac{1}{L_f}\left[-a\sqrt{a^2-x^2}\right]_0^a = \frac{2a}{\pi}$

$y_s = \frac{2a}{\pi}$

c. $x_s = \frac{1}{L_f}\int_{-a}^a \frac{ax}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x = \frac{1}{L_f}\left[-a\sqrt{a^2-x^2}\right]_{-a}^a = 0$

$y_s = \frac{2a}{\pi}$

d. $x_s = \frac{1}{L_f}\int_0^b x\operatorname{ch} x\,\mathrm{d}\,x = \frac{1}{L_f}\left[x\operatorname{sh} x - \operatorname{ch} x\right]_0^b = b - \frac{\operatorname{ch} b - 1}{\operatorname{sh} b}$

$y_s = \frac{b + \operatorname{sh} b \operatorname{ch} b}{2\operatorname{sh} b}$

e. $x_s = \frac{1}{L_f}\int_0^b x\sqrt{1+4a^2x^2}\,\mathrm{d}\,x = \frac{1}{L_f}\left[\frac{\left(1+4a^2x^2\right)^{3/2}}{12a^2}\right]_0^b = \frac{\left(1+4a^2b^2\right)^{3/2} - 1}{12a^2 L_f}$

$y_s = \frac{2ab(1+8a^2b^2)\sqrt{1+4a^2b^2} - \operatorname{arsh}(2ab)}{32a^2b\sqrt{1+4a^2b^2} + 16a\operatorname{arsh}(2ab)}$

5. Síkbeli alakzat súlypontja. $x_s(T_f) = \dfrac{\int_I x f(x)\,\mathrm{d}\,x}{\int_I f(x)\,\mathrm{d}\,x}$, $y_s(T_f) = \dfrac{\int_I f(x)^2\,\mathrm{d}\,x}{2\int_I f(x)\,\mathrm{d}\,x}$ (Vegyük észre, hogy $y_s = \dfrac{V_f}{2\pi\int_I f(x)\,\mathrm{d}\,x}$ teljesül.)

a. $x_s = \dfrac{\int_0^b ax^2\,\mathrm{d}\,x}{\int_0^b ax\,\mathrm{d}\,x} = \dfrac{2b}{3}$ $\qquad y_s = \dfrac{ab}{3}$

b. $x_s = \dfrac{\int_0^a x\sqrt{a^2-x^2}\,\mathrm{d}\,x}{\int_0^a \sqrt{a^2-x^2}\,\mathrm{d}\,x} = \dfrac{\frac{a^3}{3}}{\frac{a^2\pi}{2}} = \dfrac{2a}{3\pi}$ $\qquad y_s = \dfrac{2a}{3\pi}$

c. $x_s = \dfrac{\int_{-a}^a x\sqrt{a^2-x^2}\,\mathrm{d}\,x}{\int_{-a}^a \sqrt{a^2-x^2}\,\mathrm{d}\,x} = \dfrac{0}{a^2\pi} = 0$ $\qquad y_s = \dfrac{2a}{3\pi}$

d. $x_s = \dfrac{\int_0^b x\operatorname{ch} x\,\mathrm{d}\,x}{\int_0^b \operatorname{ch} x\,\mathrm{d}\,x} = \dfrac{b\operatorname{sh} b - \operatorname{ch} b + 1}{\operatorname{sh} b}$ $\qquad y_s = \dfrac{\operatorname{sh} b\operatorname{ch} b + b}{4\operatorname{sh} b}$

e. $x_s = \dfrac{\int_0^b ax^3\,\mathrm{d}\,x}{\int_0^b ax^2\,\mathrm{d}\,x} = \dfrac{3b}{4}$ $\qquad y_s = \dfrac{3ab^2}{10}$

f. $x_s = \dfrac{\int_0^b x\,\mathrm{e}^{ax}\,\mathrm{d}\,x}{\int_0^b \mathrm{e}^{ax}\,\mathrm{d}\,x} = \dfrac{\mathrm{e}^{ab}(ab-1)+1}{a(\mathrm{e}^{ab}-1)}$ $\qquad y_s = \dfrac{\mathrm{e}^{ab}+1}{4}$

6. Forgástest súlypontja. $x_s(V_f) = \dfrac{1}{V_f}\pi\displaystyle\int_I x f(x)^2\,\mathrm{d}\,x$

a. $x_s = \dfrac{\pi}{V_f}\displaystyle\int_0^b a^2x^3\,\mathrm{d}\,x = \dfrac{3b}{4}$

b. $x_s = \dfrac{\pi}{V_f}\displaystyle\int_0^a a^2x - x^3\,\mathrm{d}\,x = \dfrac{3a}{8}$

c. $x_s = \dfrac{\pi}{V_f}\displaystyle\int_{-a}^a a^2x - x^3\,\mathrm{d}\,x = 0$

d. $x_s = \dfrac{\pi}{V_f}\displaystyle\int_0^b x\operatorname{ch}^2 x\,\mathrm{d}\,x = \dfrac{b\operatorname{sh}(2b) + b^2 + 1 - \operatorname{ch}^2 b}{\operatorname{sh}(2b) + 2b}$

e. $x_s = \dfrac{\pi}{V_f}\displaystyle\int_0^b a^2x^5\,\mathrm{d}\,x = \dfrac{5b}{6}$

f. $x_s = \dfrac{\pi}{V_f}\displaystyle\int_0^b x\,\mathrm{e}^{2ax}\,\mathrm{d}\,x = \dfrac{b}{1-\mathrm{e}^{-2ab}} - \dfrac{1}{2a}$