A variant of the Gleason-Kahane-Żelazko theorem

Tomasz Kochanek

University of Silesia
Institute of Mathematics
Katowice, Poland

50^{th} International Symposium on Functional Equations

Hajdúszoboszló, Hungary

June 17–24, 2012

Gleason-Kahane-Żelazko theorem

Theorem (A.M. Gleason, J.P. Kahane, W. Żelazko, 1967-68)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$ be a linear functional with $f(\mathbf{1}) = 1$.

Gleason-Kahane-Żelazko theorem

Theorem (A.M. Gleason, J.P. Kahane, W. Żelazko, 1967-68)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$ be a linear functional with $f(\mathbf{1}) = 1$. Then f is multiplicative (i.e. $f(xy) = f(x)f(y)$ for any $x, y \in \mathscr{A}$) if and only if

$$f(x) \in \sigma(x) \quad \text{for } x \in \mathscr{A},$$

Gleason-Kahane-Żelazko theorem

Theorem (A.M. Gleason, J.P. Kahane, W. Żelazko, 1967-68)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$ be a linear functional with $f(\mathbf{1}) = 1$. Then f is multiplicative (i.e. $f(xy) = f(x)f(y)$ for any $x, y \in \mathscr{A}$) if and only if

$$f(x) \in \sigma(x) \quad \text{for } x \in \mathscr{A},$$

equivalently: if and only if $f(x) \neq 0$ for every invertible $x \in \mathscr{A}$.

Gleason-Kahane-Żelazko theorem

Theorem (A.M. Gleason, J.P. Kahane, W. Żelazko, 1967-68)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$ be a linear functional with $f(\mathbf{1}) = 1$. Then f is multiplicative (i.e. $f(xy) = f(x)f(y)$ for any $x, y \in \mathscr{A}$) if and only if

$$f(x) \in \sigma(x) \quad \text{for } x \in \mathscr{A},$$

equivalently: if and only if $f(x) \neq 0$ for every invertible $x \in \mathscr{A}$.

Recall that for complex Banach algebras $\mathscr{A}$ having a unit $\mathbf{1}$, and for each $x \in \mathscr{A}$, the **spectrum** of x is defined as

$$\sigma(x) = \{\lambda \in \mathbb{C}\colon \lambda\mathbf{1} - x \text{ is non-invertible}\}.$$

Gleason-Kahane-Żelazko theorem

Observe that the "only if" part of the above theorem is obvious.

Gleason-Kahane-Żelazko theorem

Observe that the "only if" part of the above theorem is obvious. Indeed, if $f\colon \mathscr{A} \to \mathbb{C}$ is linear and multiplicative and $f(\mathbf{1}) = 1$, then for any $x \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, $\lambda \notin \sigma(x)$ the element $\lambda\mathbf{1} - x$ is invertible, hence the multiplicativity of f implies $f(\lambda\mathbf{1} - x) \neq 0$,

Gleason-Kahane-Żelazko theorem

Observe that the "only if" part of the above theorem is obvious. Indeed, if $f\colon \mathscr{A} \to \mathbb{C}$ is linear and multiplicative and $f(\mathbf{1}) = 1$, then for any $x \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, $\lambda \notin \sigma(x)$ the element $\lambda\mathbf{1} - x$ is invertible, hence the multiplicativity of f implies $f(\lambda\mathbf{1} - x) \neq 0$, that is, $\lambda \neq f(x)$. This proves $f(x) \in \sigma(x)$ for each $x \in \mathscr{A}$.

Gleason-Kahane-Żelazko theorem

Observe that the "only if" part of the above theorem is obvious. Indeed, if $f\colon \mathscr{A} \to \mathbb{C}$ is linear and multiplicative and $f(\mathbf{1}) = 1$, then for any $x \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, $\lambda \notin \sigma(x)$ the element $\lambda\mathbf{1} - x$ is invertible, hence the multiplicativity of f implies $f(\lambda\mathbf{1} - x) \neq 0$, that is, $\lambda \neq f(x)$. This proves $f(x) \in \sigma(x)$ for each $x \in \mathscr{A}$.

On the other hand, the "if" part is a deep result.

Gleason-Kahane-Żelazko theorem

Observe that the "only if" part of the above theorem is obvious. Indeed, if $f\colon \mathscr{A} \to \mathbb{C}$ is linear and multiplicative and $f(\mathbf{1}) = 1$, then for any $x \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, $\lambda \notin \sigma(x)$ the element $\lambda\mathbf{1} - x$ is invertible, hence the multiplicativity of f implies $f(\lambda\mathbf{1} - x) \neq 0$, that is, $\lambda \neq f(x)$. This proves $f(x) \in \sigma(x)$ for each $x \in \mathscr{A}$.

On the other hand, the "if" part is a deep result.

Another formulation:

Theorem (A.M. Gleason, J.P. Kahane, W. Żelazko, 1967-68)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element and let $\mathscr{M} \subset \mathscr{A}$ be a one codimensional subspace of $\mathscr{A}$.

Gleason-Kahane-Żelazko theorem

Observe that the "only if" part of the above theorem is obvious. Indeed, if $f\colon \mathscr{A} \to \mathbb{C}$ is linear and multiplicative and $f(\mathbf{1}) = 1$, then for any $x \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, $\lambda \notin \sigma(x)$ the element $\lambda\mathbf{1} - x$ is invertible, hence the multiplicativity of f implies $f(\lambda\mathbf{1} - x) \neq 0$, that is, $\lambda \neq f(x)$. This proves $f(x) \in \sigma(x)$ for each $x \in \mathscr{A}$.

On the other hand, the "if" part is a deep result.

Another formulation:

Theorem (A.M. Gleason, J.P. Kahane, W. Żelazko, 1967-68)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element and let $\mathscr{M} \subset \mathscr{A}$ be a one codimensional subspace of $\mathscr{A}$. If every $x \in \mathscr{M}$ belongs to some proper ideal $\mathcal{I}_x$ in $\mathscr{A}$, then $\mathscr{M}$ is actually an ideal (of course, maximal).

Gleason-Kahane-Żelazko theorem

It is well-known that the Gleason-Kahane-Żelazko theorem **does not hold** for **real** Banach algebras.

Gleason-Kahane-Żelazko theorem

It is well-known that the Gleason-Kahane-Żelazko theorem **does not hold** for **real** Banach algebras. Indeed, consider the Banach algebra $\mathscr{A} = C_{\mathbb{R}}[0,1]$ and the linear functional on $\mathscr{A}$ defined by

$$C_{\mathbb{R}}[0,1] \ni f \mapsto \varphi(f) = \int_0^1 f(t)\,\mathrm{d}t.$$

Gleason-Kahane-Żelazko theorem

It is well-known that the Gleason-Kahane-Żelazko theorem **does not hold** for **real** Banach algebras. Indeed, consider the Banach algebra $\mathscr{A} = C_{\mathbb{R}}[0,1]$ and the linear functional on $\mathscr{A}$ defined by

$$C_{\mathbb{R}}[0,1] \ni f \mapsto \varphi(f) = \int_0^1 f(t)\,\mathrm{d}t.$$

Despite the fact that for each $f \in \mathscr{A}$ we have $\varphi(f) \in \sigma(f) = f([0,1])$ (the spectrum of f is just the range of f), the functional φ fails to be multiplicative.

Gleason-Kahane-Żelazko theorem

It is well-known that the Gleason-Kahane-Żelazko theorem **does not hold** for **real** Banach algebras. Indeed, consider the Banach algebra $\mathscr{A} = C_{\mathbb{R}}[0,1]$ and the linear functional on $\mathscr{A}$ defined by

$$C_{\mathbb{R}}[0,1] \ni f \mapsto \varphi(f) = \int_0^1 f(t)\,\mathrm{d}t.$$

Despite the fact that for each $f \in \mathscr{A}$ we have $\varphi(f) \in \sigma(f) = f([0,1])$ (the spectrum of f is just the range of f), the functional φ fails to be multiplicative.

Theorem (N. Farnum, R. Whitley, 1978)

Let K be a compact Hausdorff space. Then the Banach algebra $C_{\mathbb{R}}(K)$ satisfies the assertion of the Gleason-Kahane-Żelzako theorem if and only if K is totally disconnected.

Gleason-Kahane-Żelazko theorem

A real counterpart of the Gleason-Kahane-Żelazko theorem was given by N.H. Kulkarni:

Gleason-Kahane-Żelazko theorem

A real counterpart of the Gleason-Kahane-Żelazko theorem was given by N.H. Kulkarni:

Theorem (N.H. Kulkarni, 1984)

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$. Suppose that $f\colon \mathscr{A} \to \mathbb{C}$ is a linear map satisfying $f(\mathbf{1}) = 1$ and

$$f(x)^2 + f(y)^2 \neq 0$$

for all $x, y \in \mathscr{A}$ such that $xy = yx$ and $x^2 + y^2$ is invertible.

Gleason-Kahane-Żelazko theorem

A real counterpart of the Gleason-Kahane-Żelazko theorem was given by N.H. Kulkarni:

Theorem (N.H. Kulkarni, 1984)

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$. Suppose that $f\colon \mathscr{A} \to \mathbb{C}$ is a linear map satisfying $f(\mathbf{1}) = 1$ and

$$f(x)^2 + f(y)^2 \neq 0$$

for all $x, y \in \mathscr{A}$ such that $xy = yx$ and $x^2 + y^2$ is invertible. Then f is multiplicative.

Gleason-Kahane-Żelazko theorem

A real counterpart of the Gleason-Kahane-Żelazko theorem was given by N.H. Kulkarni:

Theorem (N.H. Kulkarni, 1984)

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$. Suppose that $f\colon \mathscr{A} \to \mathbb{C}$ is a linear map satisfying $f(\mathbf{1}) = 1$ and

$$f(x)^2 + f(y)^2 \neq 0$$

for all $x, y \in \mathscr{A}$ such that $xy = yx$ and $x^2 + y^2$ is invertible. Then f is multiplicative.

We shall propose another variant of the Gleason-Kahane-Żelazko theorem, also valid for **real** Banach algebras, and inspired by the following result of S. Kowalski and Z. Słodkowski.

Kowalski-Słodkowski theorem

Theorem (S. Kowalski, Z. Słodkowski, 1980)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element $\mathbf{1}$. Suppose that $f\colon \mathscr{A} \to \mathbb{C}$ satisfies: $f(0) = 0$ and

$$f(x) - f(y) \in \sigma(x - y) \quad \text{for } x, y \in \mathscr{A}.$$

Kowalski-Słodkowski theorem

Theorem (S. Kowalski, Z. Słodkowski, 1980)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element $\mathbf{1}$. Suppose that $f\colon \mathscr{A} \to \mathbb{C}$ satisfies: $f(0) = 0$ and

$$f(x) - f(y) \in \sigma(x - y) \quad \text{for } x, y \in \mathscr{A}.$$

Then f is linear and multiplicative.

Kowalski-Słodkowski theorem

Theorem (S. Kowalski, Z. Słodkowski, 1980)

Let $\mathscr{A}$ be a **complex** Banach algebra with a unit element $\mathbf{1}$. Suppose that $f\colon \mathscr{A} \to \mathbb{C}$ satisfies: $f(0) = 0$ and

$$f(x) - f(y) \in \sigma(x - y) \quad \text{for } x, y \in \mathscr{A}.$$

Then f is linear and multiplicative.

Let us stress that here, in contrast to the original version of the Gleason-Kahane-Żelazko theorem, the linearity of f is a part of the assertion.

Main result

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Main result

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Then f is linear and multiplicative.

Main result

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Then f is linear and multiplicative.

Since all linear and multiplicative maps on $C_{\mathbb{R}}[0,1]$ are just evaluation functionals, condition $(*)$ requires that for any two functions $g, h \in C_{\mathbb{R}}[0,1]$ we would have

$$\int_0^1 g(t)\,\mathrm{d}t = g(a) \quad \text{and} \quad \int_0^1 h(t)\,\mathrm{d}t = h(a)$$

Main result

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Then f is linear and multiplicative.

Since all linear and multiplicative maps on $C_{\mathbb{R}}[0,1]$ are just evaluation functionals, condition $(*)$ requires that for any two functions $g, h \in C_{\mathbb{R}}[0,1]$ we would have

$$\int_0^1 g(t)\,\mathrm{d}t = g(a) \quad \text{and} \quad \int_0^1 h(t)\,\mathrm{d}t = h(a)$$

for some point $a \in [0,1]$ (depending on g and h).

Main result

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Then f is linear and multiplicative.

Since all linear and multiplicative maps on $C_{\mathbb{R}}[0,1]$ are just evaluation functionals, condition $(*)$ requires that for any two functions $g, h \in C_{\mathbb{R}}[0,1]$ we would have

$$\int_0^1 g(t)\,\mathrm{d}t = g(a) \quad \text{and} \quad \int_0^1 h(t)\,\mathrm{d}t = h(a)$$

for some point $a \in [0,1]$ (depending on g and h). This is however impossible in general.

Auxiliary tools

Theorem (P. Mankiewicz, 1974)

If $X\colon X \to \mathbb{C}$ is a Lipschitz map defined on a separable Fréchet space, then it has $\mathbb{R}$-linear Gateaux differentials everywhere on X except some **zero set**.

Auxiliary tools

Theorem (P. Mankiewicz, 1974)

If $X\colon X \to \mathbb{C}$ is a Lipschitz map defined on a separable Fréchet space, then it has $\mathbb{R}$-linear Gateaux differentials everywhere on X except some **zero set**.

Let $\mathcal{Q}$ be the **Hilbert cube**, i.e.

$$\mathcal{Q} = \prod_{n\in\mathbb{N}} [-2^{-n}, 2^{-n}],$$

equipped with the natural product measure μ.

Theorem (P. Mankiewicz, 1974)

If $X\colon X\to\mathbb{C}$ is a Lipschitz map defined on a separable Fréchet space, then it has $\mathbb{R}$-linear Gateaux differentials everywhere on X except some **zero set**.

Let $\mathcal{Q}$ be the **Hilbert cube**, i.e.

$$\mathcal{Q}=\prod_{n\in\mathbb{N}}[-2^{-n},2^{-n}],$$

equipped with the natural product measure μ. A subset $Z\subset X$ is called a **zero set** if for every continuous affine map $\psi\colon\mathcal{Q}\to X$ with linearly dense image we have

$$\mu(\psi^{-1}(Z))=0.$$

Auxiliary tools

Theorem (S. Kowalski, Z. Słodkowski, 1980)

Let X be a separable complex Fréchet space and let $f\colon U \to \mathbb{C}$ be a locally Lipschitz map defined on an open set $U \subset X$.

Auxiliary tools

Theorem (S. Kowalski, Z. Słodkowski, 1980)

Let X be a separable complex Fréchet space and let $f\colon U \to \mathbb{C}$ be a locally Lipschitz map defined on an open set $U \subset X$. If f has a $\mathbb{C}$-linear Gateaux differential at every point except some zero set, then f is holomorphic in U.

Auxiliary tools

Theorem (S. Kowalski, Z. Słodkowski, 1980)

Let X be a separable complex Fréchet space and let $f\colon U \to \mathbb{C}$ be a locally Lipschitz map defined on an open set $U \subset X$. If f has a $\mathbb{C}$-linear Gateaux differential at every point except some zero set, then f is holomorphic in U.

The method of the proof of our main result is based on the methods invented by Kowalski and Słodkowski and on the complexification process.

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Then f is linear and multiplicative.

Sketch of the proof

- First, observe that we may assume that $\mathscr{A}$ is a **separable** Banach algebra.

Sketch of the proof

- First, observe that we may assume that $\mathscr{A}$ is a **separable** Banach algebra. In fact, we shall prove that for any two elements $x_1, x_2 \in \mathscr{A}$ the map f is linear and multiplicative on the subalgebra $\langle x_1, x_2 \rangle$ generated by x_1 and x_2, which obviously is separable.

Sketch of the proof

- First, observe that we may assume that $\mathscr{A}$ is a **separable** Banach algebra. In fact, we shall prove that for any two elements $x_1, x_2 \in \mathscr{A}$ the map f is linear and multiplicative on the subalgebra $\langle x_1, x_2 \rangle$ generated by x_1 and x_2, which obviously is separable.
- Let $\mathscr{A}_{\mathbb{C}} = \mathscr{A} \times \mathscr{A}$ be the complexification of the algebra $\mathscr{A}$. Then $\mathscr{A}_{\mathbb{C}}$ is a complex Banach algebra with a certain norm $\|\cdot\|$ such that $x \mapsto (x, 0)$ is an isometry and

$$\|(x,y)\| \leq \|x\| + \|y\| \leq 2\sqrt{2}\|(x,y)\| \text{ for } (x,y) \in \mathscr{A}_{\mathbb{C}}.$$

Sketch of the proof

- First, observe that we may assume that $\mathscr{A}$ is a **separable** Banach algebra. In fact, we shall prove that for any two elements $x_1, x_2 \in \mathscr{A}$ the map f is linear and multiplicative on the subalgebra $\langle x_1, x_2 \rangle$ generated by x_1 and x_2, which obviously is separable.
- Let $\mathscr{A}_{\mathbb{C}} = \mathscr{A} \times \mathscr{A}$ be the complexification of the algebra $\mathscr{A}$. Then $\mathscr{A}_{\mathbb{C}}$ is a complex Banach algebra with a certain norm $\|\cdot\|$ such that $x \mapsto (x, 0)$ is an isometry and

$$\|(x,y)\| \leq \|x\| + \|y\| \leq 2\sqrt{2}\|(x,y)\| \text{ for } (x,y) \in \mathscr{A}_{\mathbb{C}}.$$

- Define $\widetilde{f}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$ by

$$\widetilde{f}(x,y) = f(x) + if(y).$$

Sketch of the proof

- For any $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathscr{A}_{\mathbb{C}}$ we have

$$\begin{aligned} |\widetilde{f}(\mathbf{x}) - \widetilde{f}(\mathbf{y})| &\leq |f(x_1) - f(y_1)| + |f(x_2) - f(y_2)| \\ &= |\varphi_{x_1,y_1}(x_1 - y_1)| + |\varphi_{x_2,y_2}(x_2 - y_2)| \end{aligned}$$

Sketch of the proof

- For any $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathscr{A}_{\mathbb{C}}$ we have

$$\begin{aligned} |\widetilde{f}(\mathbf{x}) - \widetilde{f}(\mathbf{y})| &\leq |f(x_1) - f(y_1)| + |f(x_2) - f(y_2)| \\ &= |\varphi_{x_1,y_1}(x_1 - y_1)| + |\varphi_{x_2,y_2}(x_2 - y_2)| \end{aligned}$$

- For any $\mathbb{R}$-linear and multiplicative $\varphi\colon \mathscr{A} \to \mathbb{C}$ the formula $\widetilde{\varphi}(x, y) = \varphi(x) + i\varphi(y)$ defines a $\mathbb{C}$-linear and multiplicative functional $\widetilde{\varphi}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$.

Sketch of the proof

- For any $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathscr{A}_{\mathbb{C}}$ we have

$$\begin{aligned}|\widetilde{f}(\mathbf{x}) - \widetilde{f}(\mathbf{y})| &\leq |f(x_1) - f(y_1)| + |f(x_2) - f(y_2)| \\ &= |\varphi_{x_1,y_1}(x_1 - y_1)| + |\varphi_{x_2,y_2}(x_2 - y_2)|\end{aligned}$$

- For any $\mathbb{R}$-linear and multiplicative $\varphi\colon \mathscr{A} \to \mathbb{C}$ the formula $\widetilde{\varphi}(x, y) = \varphi(x) + i\varphi(y)$ defines a $\mathbb{C}$-linear and multiplicative functional $\widetilde{\varphi}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$. Since

$$\varphi_{x_i,y_i}(x_i - y_i) = \widetilde{\varphi}_{x_i,y_i}(x_i - y_i, 0) \in \sigma(x_i - y_i, 0) \text{ for } i = 1, 2,$$

we have

$$\begin{aligned}|\widetilde{f}(\mathbf{x}) - \widetilde{f}(\mathbf{y})| &\leq \|(x_1 - y_1, 0)\| + \|(x_2 - y_2, 0)\| \\ &= \|x_1 - y_1\| + \|x_2 - y_2\| \leq 2\sqrt{2}\|\mathbf{x} - \mathbf{y}\|,\end{aligned}$$

Sketch of the proof

- For any $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathscr{A}_{\mathbb{C}}$ we have

$$\begin{aligned} |\widetilde{f}(\mathbf{x}) - \widetilde{f}(\mathbf{y})| &\leq |f(x_1) - f(y_1)| + |f(x_2) - f(y_2)| \\ &= |\varphi_{x_1,y_1}(x_1 - y_1)| + |\varphi_{x_2,y_2}(x_2 - y_2)| \end{aligned}$$

- For any $\mathbb{R}$-linear and multiplicative $\varphi\colon \mathscr{A} \to \mathbb{C}$ the formula $\widetilde{\varphi}(x, y) = \varphi(x) + i\varphi(y)$ defines a $\mathbb{C}$-linear and multiplicative functional $\widetilde{\varphi}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$. Since

$$\varphi_{x_i,y_i}(x_i - y_i) = \widetilde{\varphi}_{x_i,y_i}(x_i - y_i, 0) \in \sigma(x_i - y_i, 0) \text{ for } i = 1, 2,$$

we have

$$\begin{aligned} |\widetilde{f}(\mathbf{x}) - \widetilde{f}(\mathbf{y})| &\leq \|(x_1 - y_1, 0)\| + \|(x_2 - y_2, 0)\| \\ &= \|x_1 - y_1\| + \|x_2 - y_2\| \leq 2\sqrt{2}\|\mathbf{x} - \mathbf{y}\|, \end{aligned}$$

which proves that **$\widetilde{\mathbf{f}}$ is a Lipschitz map.**

Sketch of the proof

- By Mankiewicz's theorem, $\widetilde{f}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$ has $\mathbb{R}$-linear Gateaux differentials "almost everywhere". On the other hand, every such has to be of the form

$$\mathrm{D}\widetilde{f}(a)(x,y) = \mathrm{D}f(a)x + i\mathrm{D}f(a)y$$

(whenever it exists), where $\mathrm{D}f(a)$ is also $\mathbb{R}$-linear "almost everywhere" .

Sketch of the proof

- By Mankiewicz's theorem, $\widetilde{f}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$ has $\mathbb{R}$-linear Gateaux differentials "almost everywhere". On the other hand, every such has to be of the form
$$\mathrm{D}\widetilde{f}(a)(x, y) = \mathrm{D}f(a)x + i\mathrm{D}f(a)y$$
(whenever it exists), where $\mathrm{D}f(a)$ is also $\mathbb{R}$-linear "almost everywhere" .
- Consequently, $\mathrm{D}\widetilde{f}(a)$ exists and is $\mathbb{C}$-linear, thus the result of Kowalski and Słodkowski implies that $\widetilde{f}$ is holomorphic on $\mathscr{A}_{\mathbb{C}}$.

Sketch of the proof

- By Mankiewicz's theorem, $\widetilde{f}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$ has $\mathbb{R}$-linear Gateaux differentials "almost everywhere". On the other hand, every such has to be of the form
$$\mathrm{D}\widetilde{f}(a)(x,y) = \mathrm{D}f(a)x + i\mathrm{D}f(a)y$$
(whenever it exists), where $\mathrm{D}f(a)$ is also $\mathbb{R}$-linear "almost everywhere" .
- Consequently, $\mathrm{D}\widetilde{f}(a)$ exists and is $\mathbb{C}$-linear, thus the result of Kowalski and Słodkowski implies that $\widetilde{f}$ is holomorphic on $\mathscr{A}_{\mathbb{C}}$.
- The function $\widetilde{f}_{a,b}\colon \mathbb{C} \to \mathbb{C}$, defined for every $a, b \in \mathscr{A}_{\mathbb{C}}$ by the formula
$$\widetilde{f}_{a,b}(z) = \widetilde{f}(az+b) \quad (z \in \mathbb{C}),$$
is **Lipschitz** and **entire**, hence – it is **affine**.

Sketch of the proof

- By Mankiewicz's theorem, $\widetilde{f}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$ has $\mathbb{R}$-linear Gateaux differentials "almost everywhere". On the other hand, every such has to be of the form
$$\mathrm{D}\widetilde{f}(a)(x, y) = \mathrm{D}f(a)x + i\mathrm{D}f(a)y$$
(whenever it exists), where $\mathrm{D}f(a)$ is also $\mathbb{R}$-linear "almost everywhere" .
- Consequently, $\mathrm{D}\widetilde{f}(a)$ exists and is $\mathbb{C}$-linear, thus the result of Kowalski and Słodkowski implies that $\widetilde{f}$ is holomorphic on $\mathscr{A}_{\mathbb{C}}$.
- The function $\widetilde{f}_{a,b}\colon \mathbb{C} \to \mathbb{C}$, defined for every $a, b \in \mathscr{A}_{\mathbb{C}}$ by the formula
$$\widetilde{f}_{a,b}(z) = \widetilde{f}(az + b) \quad (z \in \mathbb{C}),$$
is **Lipschitz** and **entire**, hence – it is **affine**.
- From this it is not difficult to derive that $\widetilde{f}$ is $\mathbb{C}$-linear, thus f itself is $\mathbb{R}$-linear.

Sketch of the proof

- By Mankiewicz's theorem, $\widetilde{f}\colon \mathscr{A}_{\mathbb{C}} \to \mathbb{C}$ has $\mathbb{R}$-linear Gateaux differentials "almost everywhere". On the other hand, every such has to be of the form
$$\mathrm{D}\widetilde{f}(a)(x,y) = \mathrm{D}f(a)x + i\mathrm{D}f(a)y$$
(whenever it exists), where $\mathrm{D}f(a)$ is also $\mathbb{R}$-linear "almost everywhere" .
- Consequently, $\mathrm{D}\widetilde{f}(a)$ exists and is $\mathbb{C}$-linear, thus the result of Kowalski and Słodkowski implies that $\widetilde{f}$ is holomorphic on $\mathscr{A}_{\mathbb{C}}$.
- The function $\widetilde{f}_{a,b}\colon \mathbb{C} \to \mathbb{C}$, defined for every $a, b \in \mathscr{A}_{\mathbb{C}}$ by the formula
$$\widetilde{f}_{a,b}(z) = \widetilde{f}(az + b) \quad (z \in \mathbb{C}),$$
is **Lipschitz** and **entire**, hence – it is **affine**.
- From this it is not difficult to derive that $\widetilde{f}$ is $\mathbb{C}$-linear, thus f itself is $\mathbb{R}$-linear.
- By the Gleason-Kahane-Żelazko theorem (we use our assumption once more), $\widetilde{f}$ is also multiplicative, thus f itself is multiplicative as well.

A question on assumptions

The assertion of our main result is purely algebraic, as well as its main assumption (about the graph of the given function f).

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Then f is linear and multiplicative.

A question on assumptions

The assertion of our main result is purely algebraic, as well as its main assumption (about the graph of the given function f).

Theorem

Let $\mathscr{A}$ be a **real** Banach algebra with a unit element $\mathbf{1}$ and let $f\colon \mathscr{A} \to \mathbb{C}$. Suppose that for any $x, y \in \mathscr{A}$ there exists a linear and multiplicative map $\varphi_{x,y}\colon \mathscr{A} \to \mathbb{C}$ such that

$$(*) \qquad f(x) = \varphi_{x,y}(x) \quad \text{and} \quad f(y) = \varphi_{x,y}(y).$$

Then f is linear and multiplicative.

Question

Is the Banach algebra structure essential in our Theorem?