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The true self-repelling motion

Bálint Tóth[1], **Wendelin Werner**[2]

[1] Mathematical Institute of the Hungarian Academy of Sciences, P.O. Box 127, H-1364 Budapest, Hungary
e-mail: balint@math-inst.hu
[2] Département de Mathématiques, Bât. 425, Université Paris-Sud, F-91405 Orsay Cedex, France
e-mail: Wendelin.Werner@math.u-psud.fr

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Abstract. We construct and study a continuous real-valued random process, which is of a new type: It is self-interacting (self-repelling) but only in a local sense: it only feels the self-repellance due to its occupation-time measure density in the 'immediate neighbourhood' of the point it is just visiting. We focus on the most natural process with these properties that we call 'true self-repelling motion'. This is the continuous counterpart to the integer-valued 'true' self-avoiding walk, which had been studied among others by the first author. One of the striking properties of true self-repelling motion is that, although the couple (X_t, occupation-time measure of X at time t) is a continuous Markov process, X is not driven by a stochastic differential equation and is not a semi-martingale. It turns out, for instance, that it has a finite variation of order 3/2, which contrasts with the finite quadratic variation of semi-martingales. One of the key-tools in the construction of X is a continuous system of coalescing Brownian motions similar to those that have been constructed by Arratia [A1, A2]. We derive various properties of X (existence and properties of the occupation time densities $L_t(x)$, local variation, etc.) and an identity that shows that the dynamics of X can be very loosely speaking described as follows: $-dX_t$ is equal to the gradient (in space) of $L_t(x)$, in a generalized sense, even though $x \mapsto L_t(x)$ is not differentiable.

Key words: Self-repelling motion – Self-interacting motion – Polymer measure

Mathematics Subject Classification (1991): 60G18, 60K35, 82C22, 82B41

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1. Introduction

1.1. General introduction

In the present paper, we construct and study a continuous real-valued random process, which is of a new type: It is self-interacting (self-repelling) but only in a local sense (we shall give a more precise meaning to this later). We will focus on the most natural non-trivial process with these properties that we call 'true self-repelling motion'. A more general approach to this class of processes will be developed in a forthcoming paper.

One of the remarkable features of such processes and in particular of "true self-repelling motion" X, is that they are (in general) *not* semi-martingales; for instance, we shall see that X has a finite variation of order $3/2$, which contrasts with the finite quadratic variation of a non-degenerate semi-martingale.

True self-repelling motion is the continuous counterpart of certain self-interacting walks $(S_n, n \geq 0)$ on $\mathbb{Z}$, called 'true self-avoiding walks' (or sometimes also 'myopic' self-avoiding walk; see a more precise definition later in this introduction). It had been conjectured (see

Amit-Parisi-Peliti [APP], Peliti-Pietronero [PP], Madras-Slade [MS]) that the correct asymptotic scaling is $n^{-2/3}S_n$ (i.e. S_n is of order $n^{2/3}$). In Tóth [T1], a limit theorem was proved in this scaling regime. We believe that the true self-repelling motion constructed in the present paper is the scaling limit of several other self-interacting walks or processes (see for instance the example in Section 11, or the polymer measure proposed by Durrett and Rogers in [DR]).

As we shall see, the true self-repelling motion (and our construction will make this apparent) can be interpreted as the first coordinate of a process that explores a certain continuous planar random 'generalized labyrinth'. This 'labyrinth' is in some sense the continuous limit of a certain percolation in $\mathbb{Z} \times \mathbb{N}$; see Section 11, especially Figures 1 and 2, for this discrete analog.

We focus in the present paper on the construction and the basic properties of the continuous process X, and we do not derive invariance principles (stating for instance that X is the weak limit of rescaled true self-avoiding walks) for which delicate tightness arguments are needed, but we plan to do so in future work. However, we remark here that convergence of all finite dimensional distributions of the true self-avoiding walk to those of X. constructed in the present work follows from the arguments of [T1], where convergence of the one-dimensional marginals is stated and proven. In Appendix B, we just give a brief *non-rigourous* phenomenological motivation.

The asymptotic behaviour of various one-dimensional self-interacting walks has been studied extensively. It has been shown that for some 'global' models where the self-interaction is 'very' repulsive (such as the Domb-Joyce model; see for instance [B,GH,K,MS] and the references therein), the measure on self-repelling walks will asymptotically concentrate on ballistic motion (i.e. the continuous limit is a linear function of time). On the other end of the spectrum, if the interaction is 'not too weakly' self-attracting, self-attracting walks will eventually be stuck at some point or at some edge (i.e. the continuous limit is constant) (see e.g. [D1]). Some other 'weakly self-interacting walks' do converge to semimartingales that can be constructed from Brownian motion (for instance to perturbed Brownian motions, [D2, W]). In all these cases, the dynamics of the limiting process after scaling is either trivial or easy to understand; the self-interaction is either so strong or so weak that the limiting process is 'degenerate'. In the present case, something completely different happens. The self-interaction 'passes to the limit' and gives rise to a completely new type of process. The multi-dimensional problem is (as usual for this kind of questions) much more difficult, and not treated here.

1.2. Properties of the true self-repelling motion

We now give a brief summary of the properties of the true self-repelling motion X that we construct in the present paper:

Continuity, recurrence. *Almost surely,* $X_0 = 0$, *the process* $t \mapsto X_t$ *is continuous on* $[0, \infty)$ *and for any* $x \in \mathbb{R}$, $\{t \geq 0 : X_t = x\}$ *is unbounded.*

Scaling. *For all* $a > 0$, $(X_{at}, t \geq 0)$ *and* $(a^{2/3} X_t, t \geq 0)$ *are identical in law.*

This scaling property shows that X is super-diffusive. The local counterpart of this property is the finite variation of order 3/2 that we have already mentioned:

Local variation. *For all* $\varepsilon > 0$, *define by induction* $\theta_0^\varepsilon := 0$ *and for all* $n \geq 1$,

$$\theta_n^\varepsilon := \inf\{t > \theta_{n-1}^\varepsilon : |X_t - X_{\theta_{n-1}^\varepsilon}| = \varepsilon\} \ . \tag{1.1}$$

Then, for all $t \geq 0$,

$$\text{P-}\lim_{\varepsilon \downarrow 0} \varepsilon^{3/2} \sup\{n \geq 0 : \theta_n^\varepsilon \leq t\} = \frac{2}{\sqrt{\pi}} t \ . \tag{1.2}$$

Here and throughout in the paper P-lim stands for limit in probability.

Occupation-time density. *Almost surely, for all* $t \geq 0$, *the occupation-time measure* μ_t *of* X *on the time-interval* $[0, t]$, *defined for all borelian subset* A *of* $\mathbb{R}$ *by*

$$\mu_t(A) := \int_0^t \mathbb{1}_{\{X_s \in A\}} ds \tag{1.3}$$

has a bounded density with respect to the Lebesgue measure and this density has a continuous version that we denote by $L_t(\cdot)$.

L is the analog for X of local times for a semi-martingale. In the sequel by a slight abuse of terminology, we will call $L_t(x)$ the local time of X at time t and level x (even if X is *not* a semi-martingale).

Markov property of (X_t, μ_t). *The process* $(X_t, L_t(\cdot))_{t \geq 0}$ *(or equivalently the process* $(X_t, \mu_t)_{t \geq 0}$*) is a Markov process.*

In other words, the future of X after t depends only on the occupation-time measure at time t (i.e. μ_t), and on the position of X at time t. In this sense, this means that X is a self-interacting motion. So, the process $(X_t, L_t(\cdot))_{t \geq 0}$ can in fact be viewed as a continuous increasing Markov process in the set of pointed continuous positive real-valued functions (that is in the set of pairs (x, f) where $x \in \mathbb{R}$ and f is a continuous positive real-valued function; see Section 10 for more on this approach). The following property is crucial:

Locality. *The self-interaction is local in the following sense: For all* $t \geq 0$, *the law of* X *just after* t *depends only on* L_t *restricted to the immediate neighbourhood of the point* X_t. *More precisely, we define for all open interval* Ω *the process* X^{Ω} (*in short '*X *in* Ω*'*) *as follows: For all* $u \geq 0$,

$$\tau_u^{\Omega} := \inf\left\{ t > 0 : \int_0^t \mathbb{1}_{\{X_s \in \Omega\}} ds > u \right\} \tag{1.4}$$

$$X_u^{\Omega} := X_{\tau_u^{\Omega}} \tag{1.5}$$

(*in plain words:* X^{Ω} *is obtained by glueing together the 'excursions' of* X *in* Ω).

Then, if Ω *and* Ω' *are two disjoint open intervals in* $\mathbb{R}$, X^{Ω} *and* $X^{\Omega'}$ *are independent.*

Moreover, if we introduce the occupation-time density $L_u^{\Omega}(\cdot)$ *of* X^{Ω} (*so that for all* $u \geq 0$ *and* $x \in \Omega$, $L_u^{\Omega}(x) = L_{\tau_u^{\Omega}}(x)$), *then the process* $(X_u^{\Omega}, L_u^{\Omega}(\cdot) - L_u^{\Omega}(X_u^{\Omega}))_{u \geq 0}$ *is a Markov process.*

An alternative – maybe more enlightening – formulation of this property can be found at the end of subsection 5.2.

In other words, the process X is 'feeling' only the self-interaction due to its own past occupation-time measure at the points it is currently visiting. This could at first glance suggest that X is the solution of some sort of 'stochastic differential equation' involving X_t and (formally) $\mathrm{grad}(L_t(\cdot))(X_t)$: Loosely speaking, dX_t does depend on the 'variation' of L_t in the 'immediate neighbourhood' of X_t (roughly: the 'derivative' of L_t in the space-variable, at the point X_t).

The following result gives a more precise meaning to this statement:

Dynamical driving mechanism. *There exists a family of stopping times* $T(x, h)$ *indexed by the points of the halfplane,* $(x, h) \in \mathbb{R} \times \mathbb{R}_+$, *such that almost surely the Lebesgue measure of* $[0, \infty) \setminus \{T(x, h) : (x, h) \in \mathbb{R} \times \mathbb{R}_+\}$ *is* 0, *and such that for all* $(x, h) \in \mathbb{R} \times \mathbb{R}_+$:

$$\begin{aligned} &\text{P-}\lim_{\varepsilon \downarrow 0} \int_0^{T(x,h)} \frac{L_s(X_s + \varepsilon) - L_s(X_s - \varepsilon)}{2\varepsilon} ds \\ &\quad = -X_{T(x,h)} + \frac{1}{4} \left(\sup_{0 \leq s \leq T(x,h)} X_s + \inf_{0 \leq s \leq T(x,h)} X_s \right) . \end{aligned} \tag{1.6}$$

Phenomenologically, this equation states that the motion is driven by the negative gradient (in the space-variable) of the local time at the actual position, as long as the moving point is in the interior of the range swept in the past. This behaviour entitles us to call this process 'truely self-repelling'. In addition, at the edges of this range an instantaneous partial reflection (moving boundary condition) is felt. Indeed: writing (1.6) *formally* in differential form we find:

$$dX_t = -\frac{\partial L_t(X_t)}{\partial x} dt + \left(\text{boundary effects at } \sup_{0 \le s \le t} X_s \text{ and } \inf_{0 \le s \le t} X_s \right) . \tag{1.7}$$

Strictly speaking, (1.7) does not make sense mathematically: the local time process is so singular that a 'differential equation' involving its gradient can not be rigorously defined (we shall see that $L_t(\cdot)$ has the same regularity properties as Brownian motion). Nevertheless, this formal way of writing may help the intuition about the dynamics of the process.

Before giving more details on X and its construction, we just mention now that the flavour of some features of X recalls (even if they are not really related) the results of Rogers-Walsh [RW1, RW2], on certain functionals of linear Brownian motion B, involving the local time of Brownian motion taken at time t and level B_t. Similarly, let us mention that a special degenerate example of locally self-interacting motions (which are of a different nature than true self-repelling motion) is given by the so-called 'perturbed Brownian motions' that only 'feel' the boundary effect (this is Brownian motion perturbed only when it is at its past maximum or minimum; see Carmona-Petit-Yor [CPY], Davis [D2] and Perman-Werner [PW]).

1.3. Idea of the construction and structure of the paper

The law of the occupation-time measures of X at suitable stopping times can be described in a way that can recall the celebrated Ray-Knight Theorems for the local times of Brownian motion. Our construction of the process X is actually based upon this crucial property. The present paper does not technically rely on the results of Tóth [T1], but it is conceptually strongly connected with that paper so that it is relevant, before we explain how we construct X, to recall some results on the so-called 'true self-avoiding random walk' on $\mathbb{Z}$, defined as follows: S_i is a nearest neighbour walk on $\mathbb{Z}$ starting from the origin, $l_i(z) : i \in \mathbb{N},\ z \in \mathbb{Z}$ is its local time on edges, i.e.

$$l_i(z) := \#\{j \in [0, i-1] : \{S_j, S_{j+1}\} = \{z, z+1\}\} \tag{1.8}$$

(note that we use a different notation than that used in Tóth [T1] to have here different symbols for the discrete walk and for the continuous motion). The true self-avoiding walk is governed by the law

$$\begin{aligned}\mathbf{P}(S_{i+1} = S_i + 1 | S_0, \ldots, S_i) &= \frac{\exp(-gl_i(S_i))}{\exp(-gl_i(S_i)) + \exp(-gl_i(S_i - 1))} \\ &= 1 - \mathbf{P}(S_{i+1} = S_i - 1 | S_0, \ldots, S_i)\end{aligned} \tag{1.9}$$

where g is a positive constant. In plain words, S prefers to jump along the neighbouring edge it has visited less often in the past. (For a wider class of related self-interacting random walks and generalized Ray-Knight theorems see also [T2, T3, T4].) The designation 'true' comes from the fact that this is a *true walk* in contrast to the polymer models that are also called self-avoiding walks (e.g. Edwards model, Domb-Joyce model etc.; these do not define a consistent family of probability measures). Define also for all $z \in \mathbb{Z}$ and $m \in \mathbb{N}$, the stopping times (inverse local times)

$$\tau_{z,m} := \min\{i \geq 0 : l_i(z) = m + 1\} \ , \tag{1.10}$$

and the local time process stopped at the inverse local time $\tau_{z,m}$

$$\tilde{l}_{z,m}(\cdot) := l_{\tau_{z,m}}(\cdot) \ . \tag{1.11}$$

The core of the results in Tóth [T1] was a compound Ray-Knight type theorem proved for the properly scaled process $\tilde{l}_{\cdot,\cdot}(\cdot)$. Let $x \in \mathbb{R}$ and $h \in [0, \infty)$ be fixed. Denote by $\Phi_{(x,h)}(y)$ a Brownian motion in $\mathbb{R}_+$ defined for $y \in [x, \infty)$ starting at 'time' x from level h obeying the following boundary conditions at 0: in the time interval $[x, x^+]$ (here and in the sequel, $x^+ = \max\{x, 0\}$), $\Phi_{(x,h)}(\cdot)$ is instantaneously reflected at 0 and in the time interval $[x^+, \infty)$, $\Phi_{(x,h)}(\cdot)$ is absorbed at the first hitting of 0. We shall call such a process *reflected/absorbed Brownian motion*, abbreviated RAB. The compound Ray-Knight type theorem states the following weak convergence: for any $x \in \mathbb{R}$ and $h > 0$ fixed,

$$\frac{\tilde{l}_{[Ax],[\sqrt{A}\sigma h]}([Ay])}{2\sigma\sqrt{A}} \Rightarrow \Phi_{(x,h)}(y), \qquad y \in [x, \infty) \tag{1.12}$$

when $A \to \infty$, as *process* in the time parameter y (the constant σ is an explicit function of g, see (1.23) of [T1]).

It is not explicitly stated, but the methods of the cited paper allow for the proof of a more general, *joint* weak convergence: let finitely many pairs of coordinates $(x_1, h_1), \ldots, (x_p, h_p) \in \mathbb{R} \times (0, \infty)$ be fixed, then

$$\left(\frac{\tilde{l}_{[Ax_1],[\sqrt{A}\sigma h_1]}([Ay_1])}{2\sigma\sqrt{A}},\ldots,\frac{\tilde{l}_{[Ax_p],[\sqrt{A}\sigma h_p]}([Ay_p])}{2\sigma\sqrt{A}}\right)\Rightarrow\left(\Phi_{(x_1,h_1)}(y_1),\ldots,\Phi_{(x_p,h_p)}(y_p)\right)$$
$$y_1 \in [x_1,\infty),\ldots,y_p \in [x_p,\infty) \tag{1.13}$$

when $A \to \infty$, where $(\Phi_{(x_1,h_1)}(\cdot),\ldots,\Phi_{(x_p,h_p)}(\cdot))$ are *independent coalescing Brownian motions* reflected from 0 in the time intervals $[x_k, x_k^+]$, $k = 1,\ldots,p$ and absorbed instanteneously at the first hitting of 0 in the time-intervals $[x_k^+,\infty)$, $k = 1,\ldots,p$, respectively (in other words, independent *coalescing* RABs, or shortly CRABs; see the begining of Section 2 for a precise formal definition of CRABs).

Our construction of the process is based upon the previous observation. In Section 2, we will construct a random map $(x,h) \mapsto \Lambda_{(x,h)}(\cdot)$ on $\mathbb{R} \times (0,\infty)$ that generalizes the finite family of independent CRABs described above. Loosely speaking Λ is an infinite family of independent CRABs *started from each point of* $\mathbb{R} \times (0,\infty)$. Analogous types of 'continuous families' of coalescing Brownian motions or diffusions have been introduced by Arratia [A2] (see also Arratia [A1] and Harris [H]). One of the important features of Λ, that will be of importance in the construction of X is a certain 'self-duality' property, described in Section 2.2. This will allow to construct (deterministically from Λ) a natural 'continuous family' of continuous functions $(\overline{\Lambda}_{(x,h)}(\cdot))_{(x,h)\in\mathbb{R}\times(0,\infty)}$ defined on $\mathbb{R}$ such that

1) For all $(x,h) \in \mathbb{R} \times (0,\infty)$ and $y \geq x$, $\overline{\Lambda}_{(x,h)}(y) = \Lambda_{(x,h)}(y)$
2) For all $(x,h) \neq (x',h')$ in $\mathbb{R} \times (0,\infty)$, the two curves $y \mapsto \overline{\Lambda}_{(x,h)}(y)$ and $y \mapsto \overline{\Lambda}_{(x',h')}(y)$ never cross.
3) The law of $(x,h,y) \mapsto \overline{\Lambda}_{(x,h)}(y)$ and that of $(x,h,y) \mapsto \overline{\Lambda}_{(-x,h)}(-y)$ are identical.

The complete proofs of the construction of $\overline{\Lambda}$ are postponed to Sections 8 and 9, in order to focus on the main point: the construction and analysis of X.

We will see (in Section 3) that the non-crossing property implies that this family of curves induces a (random) total ordering of the halfplane $\mathbb{R} \times (0,\infty)$. For any $(x_1,h_1) \neq (x_2,h_2)$ in $\mathbb{R} \times (0,\infty)$, either $\overline{\Lambda}_{(x_1,h_1)}(x_2) < h_2$ or $\overline{\Lambda}_{(x_2,h_2)}(x_1) < h_1$. In particular, if we define

$$T(x,h) = \int_{-\infty}^{+\infty} \overline{\Lambda}_{(x,h)}(y)\,dy, \tag{1.14}$$

then T is injective from $\mathbb{R} \times (0,\infty)$ into $[0,\infty)$, and its range is almost surely of full Lebesgue measure in $[0,\infty)$. When $t = T(x,h)$ for some (x,h), we say that the position of X at time t is $X_t = x$.

The true self-repelling motion $(X_t, t \geq 0)$ is then extended for all $t \geq 0$ by continuity.

We then check (this is done in Section 4) that the family of curves $\overline{\Lambda}_{(x,h)}$ indeed represents the family of occupation time densities of X. More precisely, we will show that the density of the occupation time measure of X at time t has a continuous version (for all $t \geq 0$) that we denote by $L_t(\cdot)$ and that for all $(x, h) \in \mathbb{R} \times (0, \infty)$, $T(x, h)$ corresponds to the 'inverse local time', i.e. that

$$T(x,h) = \inf\{t \geq 0 : L_t(x) = h\} \tag{1.15}$$

and finally that for any $(x, h) \in \mathbb{R} \times (0, \infty)$, for all $y \in \mathbb{R}$,

$$\overline{\Lambda}_{(x,h)}(y) = L_{T(x,h)}(y) \ . \tag{1.16}$$

In Section 5, we derive the Markov property (and its local version) of the process $(X_t, L_t(\cdot))_{t \geq 0}$.

Section 6 is devoted to the dynamical driving mechanism, i.e. to the proof of (1.6).

In Section 7, we derive the local variation result (1.2) for X as well as an approximation result for the 'local times' via upcrossings by the process X. Also, some other pathwise properties of the process are stated in that section.

Sections 8 and 9 are devoted to the complete proofs of the results presented in Section 2 (respectively, the construction of the continuous family of coalescing reflected/absorbed Brownian motions, and the duality properties).

In Section 10, we make some brief comments on the stationary limit of the process and on other self-avoiding motions.

Finally, in Section 11, we present a discrete counterpart to our construction, together with some pictures that will hopefully help to the understanding of the construction of X.

In Appendix A, we derive estimates of collision times by reflected/absorbed Brownian motions that are used throughout the paper. Appendix B is devoted to a non-rigorous phenomenological derivation of the invariance principle and the dynamical driving mechanism (1.6).

1.4. Notation, preliminaries

$\mathbb{D}$ (resp. $\mathbb{D}_+$) denotes the set of dyadic rational numbers (resp. non-negative dyadic rational numbers). We will use the notation $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$ and $\mathbb{D}_+^* = \mathbb{D}_+ \setminus \{0\}$. We also define

$$\mathbb{E} = \mathbb{R} \times \mathbb{R}_+^* \quad \text{and} \quad \tilde{\mathbb{E}} = \mathbb{D} \times \mathbb{D}_+^* \ . \tag{1.17}$$

We also put

$$\mathbb{F}^+ = \{(x,h,y) \in \mathbb{E} \times \mathbb{R} : y \geq x\} \tag{1.18}$$

and

$$\mathbb{F}^- = \{(x,h,y) \in \mathbb{E} \times \mathbb{R} : y \leq x\} \ . \tag{1.19}$$

B denotes a one-dimensional Brownian motion started from $h \in \mathbb{R}$ under the probability measure P_h. For any $x \in \mathbb{R}$, we define under the probability measure P_h,

$$B^*(y) = |B(y-x)| \quad \text{for all} \quad y \geq x \tag{1.20}$$

and

$$\tau = \inf\{y \geq \max(x,0) : B^*(y) = 0\} \ . \tag{1.21}$$

We then say that the law of $(R(y), y \geq x)$ defined by $R(y) = B^*(\min(y,\tau))$ is that of a reflected/absorbed Brownian motion (in the sequel, we shall use the abreviation RAB) started from h at time x. In plain words, R is a reflecting Brownian motion started from h at time x and killed at its first positive hitting time of 0.

More generally, when $\xi = (x_n, h_n)_{n \in I}$ is a (possibly finite) sequence in $\mathbb{E}$ (where $I = \mathbb{N}$ or $I = \{0, \ldots, p\}$ for some $p \geq 0$), we define a sequence $(R^n_{(\xi)})_{n \in I}$ of independent RABs such that for all $n \in I$, $R^n_{(\xi)}$ is started from h_n at time x_n. We will drop the subscript (ξ) whenever there is no possible confusion. Also, by a slight abuse of notation, and in order to make some statement more explicit, we will sometime denote $R^n_{(\xi)}$ by $R^n_{(x_n,h_n)}$.

We will use several times in this paper estimates of hitting/collision times probabilities for independent RABs. These estimates are stated and derived in Appendix A, at the end of this paper.

For any real number x, $\lfloor x \rfloor$ denotes the integer part of x and $\lceil x \rceil = \lfloor x \rfloor + 1$.

2. The system of forward and backward lines

2.1. Forward lines: systems of coalescing RABs

Let us fix for a while a (possibly finite) sequence $\xi := (x_j, h_j)_{j \in I}$ of points in $\mathbb{E}$. Recall that $(R^j_{(\xi)})_{j \in I}$ (we will drop the subscript (ξ) when it is not necessary) denotes a family of independent RABs such that for each $j \in I$, R^j is started from h_j at time x_j (in other words, $R^j(x)$ is defined for $x \geq x_j$ and $R^j(x_j) = h_j$).

We now define $C^0_{(\xi)} := R^0_{(\xi)}$ and by induction, for all $j \geq 1$ in I:

$$\omega_j := \inf\{x \geq x_j : R^j(x) \in \{C^0(x), \ldots, C^{j-1}(x)\}\} \tag{2.1}$$

$$\nu_j := \min\{k \in \{0, \ldots, j-1\} : \ R^j(\omega_j) = C^k(\omega_j)\} \tag{2.2}$$

$$C^j(x) := \begin{cases} R^j(x) & \text{if } x_j \le x \le \omega_j \\ C^{\nu_j}(x) & \text{if } \omega_j \le x < \infty \end{cases} \tag{2.3}$$

When $I = \{0, \dots, p\}$, we call $(C^0_{(\xi)}, \dots, C^p_{(\xi)})$ a *finite family of independent coalescing RABs started from* $((x_0, h_0), \dots, (x_p, h_p))$.

In the sequel, the abreviation FICRAB will stand for 'family of independent coalescing reflected-absorbed Brownian motions'.

Note that the law of $\{C^0_{(\xi)}, \dots, C^p_{(\xi)}\}$ is in fact independent of the order in which we performed the coalescence. More precisely, if σ denotes a permutation of $\{0, \dots, p\}$, and if $(\xi \circ \sigma)$ denotes the finite sequence $((x_{\sigma(0)}, h_{\sigma(0)}), \dots, (x_{\sigma(p)}, h_{\sigma(p)}))$ then the law of $(C^{\sigma(0)}_{(\xi\circ\sigma)}, \dots, C^{\sigma(p)}_{(\xi\circ\sigma)})$ is identical to that of $(C^0_{(\xi)}, \dots, C^p_{(\xi)})$: this can be viewed as a simple consequence of the strong Markov property of the process $(R^0_{(\xi)}, \dots, R^p_{(\xi)})$.

Note also that, if $I = \mathbb{N}$, the family of processes $(C^n_{(\xi)})_{n\in\mathbb{N}}$ constructed in this way is such that for any finite set $J \subset \mathbb{N}$, $(C^j_{(\xi)})_{j\in J}$ is a finite FICRAB started from $((x_j, h_j))_{j\in J}$. This leads us to the following general definition:

Definition. [Infinite family of independent coalescing reflected/absorbed Brownian motions] *For any non-empty set* $\{(x_a, h_a) : a \in A\} \subset \mathbb{E}$, *we will say that the family* $(C^a)_{a\in A}$ *is a family of independent coalescing reflected-absorbed Brownian motions* (*FICRAB*) *started from* $((x_a, h_a))_{a\in A}$, *if and only if for any finite subset* $\{a_1, \dots, a_p\}$ *of* A, *the law of* $(C^{a_1}, \dots, C^{a_p})$ *is that of a FICRAB started from* $((x_{a_1}, h_{a_1}), \dots, (x_{a_p}, h_{a_p}))$.

The inductive procedure described in (2.1)–(2.3) (when ξ is an infinite sequence) constructs a countable FICRAB $(C^n_{(\xi)})_{n\ge 0}$ started from $((x_n, h_n))_{n\ge 0}$. Suppose now that σ is a bijection between $\mathbb{N}$ and $\mathbb{N}$, and define $\xi \circ \sigma := (x_{\sigma(n)}, h_{\sigma(n)})_{n\in\mathbb{N}}$. Then, the law of $(C^{\sigma(n)}_{(\xi\circ\sigma)})_{n\ge 0}$ is identical to that of $(C^n_{(\xi)})_{n\ge 0}$ (as for all $p \ge 1$, the law of $(C^{\sigma(n)}_{(\xi\circ\sigma)})_{n\in\{0,\dots,p\}}$ is identical to that of $(C^n_{(\xi)})_{n\in\{0,\dots,p\}}$). In other words, the law of a countable FICRAB started from given points is unique, and independent of the chosen order of coalescence (see Arratia [A1, A2] for analogous statements and a more detailed discussion of this).

The following statement shows, loosely speaking, existence and unicity of the law of a certain FICRAB started from $((x, h))_{(x,h)\in\mathbb{E}}$, with certain regularity and monotonicity conditions. As $\mathbb{E}$ is not countable, this is not straightforward. Note also that the order of the quantors becomes important (one cannot interchange 'almost surely' and 'for all $(x, h) \in \mathbb{E}$').

Theorem 2.1. [Existence and unicity of the process of forward lines]

(i) *There exists a random map* $\mathbb{F}^+ \ni (x,h,y) \mapsto \Lambda_{(x,h)}(y) \in \mathbb{R}_+$ *such that*:
(i1) $(\Lambda_{(x,h)})_{(x,h)\in\mathbb{E}}$ *is a FICRAB started from* $((x,h))_{(x,h)\in\mathbb{E}}$.
(i2) *Almost surely, for all* $(x,h)\in\mathbb{E}$, $\Lambda_{(x,h)}(x)=h$.
(i3) *Almost surely, for all* (x_1,h_1), (x_2,h_2) *in* $\mathbb{E}$ *and* $z\geq y\geq \max\{x_1,x_2\}$:

$$\left[\Lambda_{(x_1,h_1)}(y) < \Lambda_{(x_2,h_2)}(y)\right] \quad \Longrightarrow \quad \left[\Lambda_{(x_1,h_1)}(z) \leq \Lambda_{(x_2,h_2)}(z)\right] \ . \tag{2.4}$$

(i4) *Almost surely, for all* $x\leq y$, *the mapping* $h\mapsto \Lambda_{(x,h)}(y)$ *is left-continuous on* $(0,\infty)$.

(ii) *If* $\Lambda' := (\Lambda'_{(x,h)})_{(x,h)\in\mathbb{E}}$ *is another family with these properties, then* Λ *and* Λ' *are identical in law.*

Note that combining (i2) and (i3) shows that almost surely, for all $x\leq y$, the mapping $h\mapsto\Lambda_{(x,h)}(y)$ is non-decreasing on $(0,\infty)$.

A large part of the proof of this proposition is very close to that of similar statements in Arratia [A1, A2]; we give in this subsection only the fundamental ingredients of the construction of the process $\Lambda_{(\cdot,\cdot)}(\cdot)$ – for a complete proof of Theorem 2.1 see Section 8.

Sketch of construction and proof of (i3), (i4). The process $\Lambda_{(\cdot,\cdot)}(\cdot)$ can be explicitly defined as follows: Let $\tilde{\xi} := (\tilde{x}_n,\tilde{h}_n)_{n\in\mathbb{N}}$ denote an arbitrary fixed bijection between $\mathbb{N}$ and $\tilde{\mathbb{E}}$ (this bijection will be fixed and used throughout the paper). We construct the countable FICRAB $(C^n_{(\tilde{\xi})})_{n\geq 0}$ started from $((\tilde{x}_n,\tilde{h}_n))_{n\geq 0}$, by induction just as in (2.1)–(2.3). During the rest of this paper, we will put

$$F_n(x) := C^n_{(\tilde{\xi})}(x) \tag{2.5}$$

for all $n\geq 0$ and $x\geq \tilde{x}_n$. Loosely speaking (F_n) is a 'dense' system of independent CRABs. We then define, for all $(x,h,y)\in\mathbb{F}^+$

$$\Lambda_{(x,h)}(y) := \sup\{F_n(y):\ n\geq 0, \quad \tilde{x}_n < x \quad \text{and} \quad F_n(x) < h\} \tag{2.6}$$

As we shall see in Section 8, it is easy to check that Λ is almost surely well-defined for all $(x,h,y)\in\mathbb{F}^+$ simultaneously, as almost surely, for all $x\in\mathbb{R}$,

$$\inf\{F_n(x):\ n\geq 0 \ \text{ and } \ \tilde{x}_n < x\} = 0 \ . \tag{2.7}$$

By this definition, the mapping $h\mapsto\Lambda_{(x,h)}(y)$ is monotone non-decreasing and left-continuous on $(0,\infty)$ (for all fixed $x\leq y$), so (i4) is indeed satisfied. The definition also implies almost immediately (i3): If $y\geq\max(x_1,x_2)$ and $\Lambda_{(x_1,h_1)}(y) < \Lambda_{(x_2,h_2)}(y)$, then there exists $n\in\mathbb{N}$ such that $\tilde{x}_n < x_2$, $F_n(x_2) < h_2$ and

$$F_n(y) \in (\Lambda_{(x_1,h_1)}(y), \Lambda_{(x_2,h_2)}(y)] \ . \tag{2.8}$$

The definition of Λ also implies that for all $z \geq y$, $F_n(z) \leq \Lambda_{(x_2,h_2)}(z)$. As the F_n's are coalescing continuous processes, this also implies that for any m such that $F_m(y) \leq F_n(y)$ (and therefore also for any m such that $\tilde{x}_m < x_1$ and $F_m(y) \leq \Lambda_{(x_1,h_1)}(y)$), one has

$$F_m(z) \leq F_n(z) \tag{2.9}$$

for all $z \geq y$. Hence, as $\Lambda_{(x_1,h_1)}(z)$ is the supremum of a set bounded above by $F_n(z)$, for all $z \geq y$,

$$\Lambda_{(x_1,h_1)}(z) \leq F_n(z) \leq \Lambda_{(x_2,h_2)}(z) \ . \tag{2.10}$$

This proves (i3). The proofs of (i1), (i2) and (ii) can be found in Section 8.

□Theorem 2.1 (i3), (i4)

From now on we shall refer to Λ as the *system of forward lines*.

Remark 1: Conditions (i1), (i2) and (i3) essentially fix the process Λ, but still leave some freedom of choosing a regularity property. There are three natural choices:

(1) left-continuity in the variable h (our choice so far).
(2) right-continuity in h (see remark 2 below).
(3) instead of *(i4)*, requiring the following 'perfect flow condition':

(i4*) *Almost surely, for any* $(x, h) \in \mathbb{E}$ and $x \leq y \leq z$

$$\Lambda_{(x,h)}(z) = \Lambda_{(y,\Lambda_{(x,h)}(y))}(z) \tag{2.11}$$

and for all fixed $x < y$, $h \mapsto \Lambda_{(x,h)}(y)$ *is almost surely left-continuous* (*or*, *alternatively*, *right-continuous*).

Arratia [A1, A2] is interested in *random flows* and therefore chooses this third option. All three versions could be constructed via F_n's: If, we suppose for a moment that F_n's are independent coalescing Brownian motions (no reflection nor absorbtion) started from a dense subset of $\mathbb{R}^2$, then Arratia's flow would correspond to the definition (2.6) just replacing $F_n(x) < h$ by $F_n(x) \leq h$

Remark 2: We will sometimes use the right-continuous version (i.e. the second option mentioned above). For all $(x, h) \in \overline{\mathbb{E}}$ and $y \geq x$ (note that $h = 0$ is allowed here), we define

$$\Lambda^+_{(x,h)}(y) := \inf\{F_n(y) : \ n \geq 0, \ \tilde{x}_n < x \ \text{ and } \ F_n(x) > h\} \tag{2.12}$$

Using the results that we will derive for Λ, it is then easy to see that Λ^+ satisfies (i1), (i2), (i3) and that almost surely, for all $x \leq y$, the mapping $h \mapsto \Lambda^+_{(x,h)}(y)$ is right-continuous on $\mathbb{R}_+$. Exactly as for Λ (i.e. in Theorem 2.1-(ii)), one can see that the law of Λ^+ is the unique one such that these four properties are almost surely satisfied.

Remark 3: All four conditions (i1)–(i4) are needed to ensure (ii): Consider for instance a family Λ' such that $\Lambda_{(x,h)} = \Lambda'_{(x,h)}$ for all $(x,h) \in \mathbb{E}$ except when x belongs to a random Poissonian set (of zero Lebesgue measure) in which case we put $\Lambda'_{(x,h)}(y) = h$ for all $y \geq x$. Then Λ' is also a FICRAB, it satisfies (i1), (i2), (i4) but not (i3).

Next we are going to summarize the main properties of the process Λ which will be used in the construction and analysis of true self-repelling motion. Similar features also appear in [A1, A2, H] and the proofs will be given in Section 8; the main ingredients in these proofs are meeting time estimates for independent RAB's.

For any $x \leq y$ in $\mathbb{R}$, we define:

$$M(x,y) := \{\Lambda_{(z,h)}(y) : (z,h) \in \mathbb{E} \;\text{ and }\; z < x\} \ . \tag{2.13}$$

In plain words: $M(x,y)$ denotes the trace at y of forward lines which start before x. We also define

$$M(y) := M(y,y) \ . \tag{2.14}$$

Proposition 2.2. [Some properties of the forward lines] *The following properties of the process* Λ *hold almost surely*:

(i) *For all* $x \in \mathbb{R}$, $M(x)$ *is dense in* $\mathbb{R}_+$.

(ii) *For all* $x < y$ *in* $\mathbb{R}$, *the set* $M(x,y)$ *is locally finite and unbounded.*

(iii) *For all fixed* $x_1 < x_2$ *in* $\mathbb{R}$,

$$M(x_1,x_2) = \{\Lambda_{(x_1,h)}(x_2) : h \in \mathbb{D}^*_+\} \ . \tag{2.15}$$

(iv) *For all* $(x,h) \in \mathbb{E}$ *and for all* $\varepsilon > 0$, *there exists* $n \geq 0$ *such that* $\tilde{x}_n < x$, $F_n(x) \in (h-\varepsilon, h)$, *and for all* $y \geq x+\varepsilon$,

$$\Lambda_{(x,h)}(y) = F_n(y) \ . \tag{2.16}$$

In particular, for all $x \leq y$,

$$M(x,y) = \{F_n(y) : n \geq 0 \text{ and } \tilde{x}_n < x\} \ . \tag{2.17}$$

(v) *For all* $(x,h) \in \mathbb{E}$, *the mapping* $y \mapsto \Lambda_{(x,h)}(y)$ *is continuous on* $[x,\infty)$.

The essential feature (ii) shows for instance that all forward lines that are born at time x between level 0 and h coalesce into finitely many CRABS immediately after x; 'there is not enough space' for infinitely many particles not to coalesce.

The results presented in the next subsection (the 'topological structure of the forward lines') can be used to show that (iii) holds in fact almost surely for all $x_1 < x_2$ in $\mathbb{R}$ simultaneously, but we will not need this stronger result.

In other words, (iv) means that almost surely, for all $(x, h) \in \mathbb{E}$, there exists a sequence y_n decreasing to x and a sequence $(m(n))_{n\geq 0}$ in $\mathbb{N}$, such that

$$\Lambda_{(x,h)}(y) = F_{m(n)}(y) \tag{2.18}$$

for all $y \geq y_n$ and all $n \geq 0$. Loosely speaking, every forward line meets one of the countably many lines F_n 'immediately' after its birth (and it then follows F_n after this meeting time).

2.2. Backward lines and topological structure

We define for all $(x, h, y) \in \mathbb{F}^-$

$$\Lambda^*_{(x,h)}(y) := \sup\{h' > 0 : \Lambda_{(y,h')}(x) < h\} \tag{2.19}$$

(with the notation $\sup \emptyset := 0$). Clearly, the process Λ can be recovered from Λ^* by a similar formula: for all $(x, h, y) \in \mathbb{F}^+$

$$\Lambda_{(x,h)}(y) = \sup\{h' > 0 : \Lambda^*_{(y,h')}(x) < h\} \ . \tag{2.20}$$

Given the definition (2.6) of the process Λ, and using Proposition 2.2-(i), for all $(x, h, y) \in \mathbb{F}^-$,

$$\Lambda^*_{(x,h)}(y) = \sup\{F_n(y) : n \geq 0, \quad \tilde{x}_n < y \quad \text{and} \quad F_n(x) < h\} \ . \tag{2.21}$$

We shall refer to the process $\mathbb{F}^- \ni (x, h, y) \mapsto \Lambda^*_{(x,h)}(y) \in \mathbb{R}_+$ as the *system of backward lines.*

We will see in Section 9 that this definition implies readily that forward and backward lines never cross. The family of backward lines is loosely speaking the generalization of a 'dual graph' of the family of forward lines.

Theorem 2.3. [Duality of forward and backward lines.] *The two processes* $\mathbb{F}^+ \ni (x, h, y) \mapsto \Lambda_{(x,h)}(y) \in \mathbb{R}_+$ *and* $\mathbb{F}^+ \ni (x, h, y) \mapsto \Lambda^*_{(-x,h)}(-y) \in \mathbb{R}_+$ *are identical in law.*

For a complete proof of this result, see Section 9. See also Section 11 for an enlightening picture in a discrete case.

Let us stress here that, although these two processes are identical in law, they are by no means independent as Λ^* (resp. Λ) can be constructed deterministically from Λ (resp. Λ^*).

However, some independence results can be stated: Suppose for instance that $z \in \mathbb{R}$ is fixed. Define the sigma algebra $\mathscr{F}_z$ generated by

$$\{\Lambda_{(x,h)}(y) : x \leq y \leq z, \ h > 0\} \tag{2.22}$$

i.e. by

$$\{F_n(y) : n \geq 0, \ \tilde{x}_n < y \leq z\} \ . \tag{2.23}$$

The family $(F_n(\cdot))_{n\geq 0}$ inherits the Markov property from the family of independent RABs. Consequently, we will see (and this is easy using Proposition 2.2) that for any $x \geq z$, for all $h > 0$, $\Lambda_{(x,h)}(\cdot)$ is independent of $\mathscr{F}_z$. An immediate consequence of this is that, keeping z fixed, the family

$$\{\Lambda^*_{(x,h)}(y) : z \leq y \leq x,\ h > 0\} \tag{2.24}$$

is independent from $\mathscr{F}_z$. We will denote $\mathscr{F}^*_z$, the sigma field generated by this family. In plain words: for $z \in \mathbb{R}$ fixed $\mathscr{F}_z$, respectively $\mathscr{F}^*_z$, is the sigma algebra generated by the information to the left, respectively to the right, of the coordinate z.

For all $n \geq 0$, we define for any $y \leq \tilde{x}_n$,

$$F^*_n(y) := \Lambda^*_{(\tilde{x}_n,\tilde{h}_n)}(y) \ . \tag{2.25}$$

Theorem 2.3 shows that the family of functions $(y \mapsto F^*_n(-y))_{n\geq 0}$ is a FICRAB starting from $(-\tilde{x}_n, \tilde{h}_n)_{n\geq 0}$. Theorem 2.1 and Theorem 2.3 show that Λ^* can be recovered from the family (F^*_n) almost (just reflecting the 'time'-direction) as Λ has been defined from the F_n's. Hence, $\mathscr{F}^*_z$ is in fact generated by

$$\{F^*_n(y) : n \geq 0,\quad z \leq y < \tilde{x}_n\} \ . \tag{2.26}$$

The independence between $\mathscr{F}_z$ and $\mathscr{F}^*_z$ then implies immediately that for all n and n' such that $\tilde{x}_n < z < \tilde{x}_{n'}$, the two random variables $F_n(z)$ and $F^*_{n'}(z)$ are independent. As their laws have no atoms, this shows the following useful fact: For any fixed z such that $\tilde{x}_n < z < \tilde{x}_{n'}$,

$$F_n(z) \neq F^*_{n'}(z) \tag{2.27}$$

almost surely.

For all $(x,h) \in \overline{\mathbb{E}}$, we define the number $I(x,h)$ of incoming forward lines at (x,h) as follows:

$$\begin{aligned} I(x,h) := \limsup_{y\uparrow x} \#\Big\{p \in \mathbb{N} :\ \exists (x_1,h_1),\ldots,(x_p,h_p) \in \mathbb{E} \quad \text{such that} \\ \forall i = 1,\ldots,p : x_i \leq y,\quad \Lambda_{(x_i,h_i)}(x) = h \text{ and} \\ \forall z \in [y,x),\ \Lambda_{(x_1,h_1)}(z) < \ldots < \Lambda_{(x_p,h_p)}(z)\Big\} \\ = \limsup_{y\uparrow x} \#\Big\{p \in \mathbb{N} : \exists n_1,\ldots,n_p \in \mathbb{N} \quad \text{such that} \\ \forall i = 1,\ldots,p : \tilde{x}_i \leq y,\ F_{n_i}(x) = h \quad \text{and} \\ \forall z \in [y,x),\ F_{n_1}(z) < \cdots < F_{n_p}(z)\Big\} \ . \end{aligned} \tag{2.28}$$

In plain words, $I(x,h)$ is the number of disjoint forward lines that coalesce *exactly* at time x and level h. Similarly, we define the number

$I^*(x,h)$ of incoming (from the right) backward lines at (x,h). The previous remark (2.27) implies that for all fixed $x \in \mathbb{R}$, almost surely, for all $h > 0$, either $I(x,h) = 0$ or $I^*(x,h) = 0$.

For all $(x,h) \in \overline{\mathbb{E}}$, the pair of integers $[I(x,h), I^*(x,h)]$ will be called the *type* of (x,h) and the integer $I(x,h) + I^*(x,h) + 1$ will be called its multiplicity (for reasons that will become apparent later). The following result gives a more detailed description of the different possible types of points.

Proposition 2.4. (i) *Any fixed* $(x,h) \in \mathbb{E}$ *almost surely has multiplicity* 1, *i.e. it is of type* $[0,0]$.

(ii) *Let x be fixed in* $\mathbb{R}$. *Then almost surely, for any* $h \geq 0$, *the point* (x,h) *has multiplicity at most* 2, *i.e. it is of type* $[0,0]$, $[1,0]$ *or* $[0,1]$.

(iii) *Almost surely, any point* $(x,h) \in \mathbb{E}$ *has multiplicity at most* 3, *i.e. it is of one of the following six types*: $[0,0]$, $[1,0]$, $[0,1]$, $[2,0]$, $[1,1]$, $[0,2]$.

The way we just presented these results seem to indicate that Proposition 2.4 is a consequence of Theorem 2.3. In fact, we will first prove Proposition 2.4–(iii), using collision times estimates for RABs and then use this fact to derive Theorem 2.3. See Section 9, for the complete proofs of these results.

Before proceeding to the actual construction of true self-repelling motion, let us stress some facts: As we have already mentioned, the two systems Λ and Λ^* are not independent; one of the main ingredients in the forthcoming construction is that we will use both systems simultaneously.

In particular, we now define for all $(x,h) \in \mathbb{E}$ and $y \in \mathbb{R}$,

$$\overline{\Lambda}_{(x,h)}(y) := \begin{cases} \Lambda_{(x,h)}(y) & \text{if } (x,h,y) \in \mathbb{F}^+ \\ \Lambda^*_{(x,h)}(y) & \text{if } (x,h,y) \in \mathbb{F}^- \end{cases} \tag{2.29}$$

Theorem 2.3 shows that the two processes $\mathbb{E} \times \mathbb{R} \ni (x,h,y) \mapsto \overline{\Lambda}_{(x,h)}(y) \in \mathbb{R}_+$ and $\mathbb{E} \times \mathbb{R} \ni (x,h,y) \mapsto \overline{\Lambda}_{(-x,h)}(-y) \in \mathbb{R}_+$ are identical in law.

When $(x,h) \in \mathbb{E}$ is fixed, then using the independence between $\mathscr{F}_x$ and $\mathscr{F}^*_x$, the law of $\overline{\Lambda}_{(x,h)}(\cdot)$ can be easily described: $\Lambda_{(x,h)}(\cdot)$ and $\Lambda^*_{(x,h)}(\cdot)$ are independent RABs (more precisely, $\Lambda^*_{(x,h)}$ is a 'backward RAB').

The definition of forward and backward lines and the results presented in this section show that:

(1) Any two forward lines coalesce when they meet
(2) Any two backward lines coalesce when they meet
(3) Any forward line never crosses a backward line

(1), (2) and (3) imply that for all (x,h) and (x',h') in $\mathbb{E}$, the whole curves $\overline{\Lambda}_{(x,h)}$ and $\overline{\Lambda}_{(x',h')}$ never cross (they can collide and/or stick together). This will be a key-remark in the next section.

3. Construction and first properties of true self-repelling motion

For any (x,h) in $\mathbb{E}$, we define the set

$$D(x,h) := \{(x',h') \in \mathbb{E} : h' \le \overline{\Lambda}_{(x,h)}(x')\} \tag{3.1}$$

It is straightforward to check that this is almost surely a bounded set for any $(x,h) \in \mathbb{E}$.

Proposition 3.1. [Ordering of $\mathbb{E}$.] *Almost surely, for all* $(x_1,h_1) \neq (x_2,h_2)$ *in* $\mathbb{E}$, *exactly one of the following two events occurs*:

- *either* $(x_1,h_1) \in D(x_1,h_1) \subset D(x_2,h_2)$, $(x_2,h_2) \notin D(x_1,h_1)$ *and* $D(x_2,h_2) \setminus D(x_1,h_1)$ *contains a non-empty open set.*
- *or* $(x_2,h_2) \in D(x_2,h_2) \subset D(x_1,h_1)$, $(x_1,h_1) \notin D(x_2,h_2)$ *and* $D(x_1,h_1) \setminus D(x_2,h_2)$ *contains a non-empty open set.*

In particular, if we define the relation $\prec$ on $\mathbb{E} \times \mathbb{E}$ as follows:

$$[(x_1,h_1) \prec (x_2,h_2)] \iff [(x_1,h_1) \in D(x_2,h_2)] \ , \tag{3.2}$$

then $\prec$ is a total ordering of $\mathbb{E}$.

Proof. Without loss of generality we may assume $x_1 \le x_2$ and $[x_1 = x_2] \Rightarrow [h_1 < h_2]$. There are two cases to be treated separately:

CASE 1:

$$x_1 \le x_2 \quad \text{and} \quad \Lambda_{(x_1,h_1)}(x_2) < h_2 \ . \tag{3.3}$$

(In particular, this case includes the possibility of $x_1 = x_2$ and $h_1 < h_2$.) By definition of the backward lines we have

$$\Lambda^*_{(x_2,h_2)}(x_1) \ge h_1 \ . \tag{3.4}$$

The non-crossing property of the lines $\overline{\Lambda}_{(x_1,h_1)}(\cdot)$ and $\overline{\Lambda}_{(x_2,h_2)}(\cdot)$, combined with (3.3) shows that $D(x_1,h_1) \subset D(x_2,h_2)$. Moreover, as $\overline{\Lambda}_{(x_1,h_1)}(x_2) < \overline{\Lambda}_{(x_2,h_2)}(x_2)$, the continuity of $\overline{\Lambda}_{(x_1,h_1)}$ and $\overline{\Lambda}_{(x_2,h_2)}$ shows that $D(x_2,h_2) \setminus D(x_1,h_1)$ contains a non-empty open set.

This implies the first alternative of the Proposition.

CASE 2:

$$x_1 < x_2 \quad \text{and} \quad \Lambda_{(x_1,h_1)}(x_2) \ge h_2 \ . \tag{3.5}$$

By left-continuity of $h \mapsto \Lambda_{(x_1,h)}(x_2)$ and local finiteness of $M(x_1,x_2)$ there is an $\varepsilon > 0$ such that for all $h \in (h_1 - \varepsilon, h_1]$ we have $\Lambda_{(x_1,h)}(x_2) = \Lambda_{(x_1,h_1)}(x_2) \ge h_2$ and thus, again by definition of the backward lines we have

$$\Lambda^*_{(x_2,h_2)}(x_1) < h_1 \quad . \tag{3.6}$$

Then, combining (3.5), (3.6) with the non-crossing property and continuity of the lines implies the second alternative.

□Proposition 3.1.

We now define

$$T(x,h) := |D(x,h)| = \int_{-\infty}^{\infty} \overline{\Lambda}_{(x,h)}(y)dy \quad . \tag{3.7}$$

From the previous proposition it follows that $T\colon \mathbb{E} \to \mathbb{R}_+$ is almost surely injective.

Lemma 3.2. *Almost surely, the range of the map* $T\colon \mathbb{E} \to \mathbb{R}_+$ *is dense in* $\mathbb{R}_+$.

Remark: More subtle properties of the map $T\colon \mathbb{E} \to \mathbb{R}_+$ will be summarized in Propositions 4.1 and 4.4: It turns out that T is lower semicontinuous (in particular Borel), it is (Lebesgue) measure preserving and almost surjective, i.e. the complement of its range has zero Lebesgue measure in $\mathbb{R}_+$.

Proof. Assume that for some $\alpha < \alpha'$ in $\mathbb{R}_+$, $T(\mathbb{E}) \cap (\alpha, \alpha') = \emptyset$. Define then $\beta := \sup([0,\alpha] \cap T(\mathbb{E}))$ and $\beta' := \inf([\alpha', \infty) \cap T(\mathbb{E}))$ (β and β' exist because almost surely $\lim_{h\downarrow 0} T(0,h) = 0$ and $\lim_{h\uparrow\infty} T(0,h) = \infty$). Clearly, $0 \le \beta < \beta' < \infty$ and there exist two sequences (x_n, h_n) and (x'_n, h'_n) in $\mathbb{E}$ such that

$$\lim_{n\uparrow\infty} T(x_n,h_n) = \beta \quad \text{and} \quad \lim_{n\uparrow\infty} T(x'_n,h'_n) = \beta' \quad . \tag{3.8}$$

By monotone convergence it is then immediate that the Lebesgue measure of the Borel sets $D := \cup_{n\ge 0} D(x_n,h_n)$ and $D' := \cap_{n\ge 0} D(x'_n,h'_n)$ is β and β', respectively. Proposition 3.1 then also implies immediately that $D \subset D'$ and that the Lebesgue measure of $D' \setminus D$ is $\beta' - \beta > 0$. Hence, $D' \setminus D$ contains at least three different points in $\mathbb{E}$ and for any of these points, $T(x,h) \in [\beta, \beta']$. But as T is injective, this implies that there exists $(x,h) \in D' \setminus D$ such that $T(x,h) \in (\beta, \beta')$, which contradicts the definition of β and β'. Hence, $T(\mathbb{E})$ is dense in $\mathbb{R}_+$.

□Lemma 3.2.

We state and prove now a technical Lemma to be used in the proof of the forthcoming statements.

Lemma 3.3. *Suppose that* $(x,h) \neq (x',h')$ *are two different points in* $\overline{E}$ *such that there exist two sequences* $(x_n,h_n)_{n\geq 0}$ *and* $(x'_n,h'_n)_{n\geq 0}$ *in* $\mathbb{E}$ *and two real numbers* t *and* t', *such that*

$$\lim_{n\uparrow\infty}(x_n,h_n) = (x,h), \quad \lim_{n\uparrow\infty}(x'_n,h'_n) = (x',h') \tag{3.9}$$

and

$$\lim_{n\uparrow\infty} T(x_n,h_n) = t, \quad \lim_{n\uparrow\infty} T(x'_n,h'_n) = t' \ . \tag{3.10}$$

Then: $t \neq t'$.

Proof. Assume (3.9) and (3.10):

CASE 1: If $x = x'$ and $h + \varepsilon < h' - \varepsilon$ then clearly there is an $n_0 < \infty$ such that for any $n \geq n_0$, $(x_n,h_n) \in D(x,h+\varepsilon)$ and $(x'_n,h'_n) \notin D(x,h'-\varepsilon)$, thus by Proposition 3.1 $(x,h'-\varepsilon) \in D(x'_n,h'_n)$. Hence we conclude that for $n \geq n_0$ $T(x_n,h_n) < T(x,h+\varepsilon) < T(x,h'-\varepsilon) < T(x'_n,h'_n)$ and this implies $t \neq t'$
CASE 2: Suppose $x < r < q < q' < r' < x'$ with $r,q,q',r' \in \mathbb{D}^*_+$. We assume (possibly choosing subsequences) that $T(x_n,h_n)$ and $T(x'_n,h'_n)$ are monotone sequences and that $x_n < r$ and $x'_n > r'$ for all $n \in \mathbb{N}$. As $T(x_n,h_n)$ and $T(x'_n,h'_n)$ are monotone, due to Proposition 3.1 so are the sequences $\overline{\Lambda}_{(x_n,h_n)}(y)$ and $\overline{\Lambda}_{(x'_n,h'_n)}(y)$ for all $y \in \mathbb{R}$. In particular, as $M(r,q)$ and $M^*(r',q')$ are a.s. locally finite, using Proposition 2.2 this implies that there exist $n_0 \geq 0$, $k \geq 0$ and $k' \geq 0$ such that $\tilde{x}_k \leq r$ and $\tilde{x}_{k'} \geq r'$ and that for all $n \geq n_0$,

$$\Lambda_{(x_n,h_n)}(y) = F_k(y), \quad \text{for} \quad y \in [q,\infty) \ , \tag{3.11}$$

$$\Lambda^*_{(x'_n,h'_n)}(y) = F^*_{k'}(y), \quad \text{for} \quad y \in (-\infty,q'] \ . \tag{3.12}$$

(2.27) implies that a.s. $F_k(q) \neq F^*_{k'}(q)$ (because this is true simultaneously for all $q \in \mathbb{D}$, k and k' in $\mathbb{N}$ such that $\tilde{x}_k < q < \tilde{x}_{k'}$).

Suppose for instance that $F_k(q) > F^*_{k'}(q)$ (the opposite case is treated similarly). Using Proposition 3.1 once again, this implies that for $n \geq n_0$, $D(x'_n,h'_n) \subset D(x_n,h_n)$, and that

$$\begin{aligned} T(x_n,h_n) - T(x'_n,h'_n) &= \int_{-\infty}^{\infty} \left(\overline{\Lambda}_{(x_n,h_n)}(y) - \overline{\Lambda}_{(x'_n,h'_n)}(y)\right) dy \\ &\geq \int_q^{q'} (F_k(y) - F^*_{k'}(y))\, dy > 0 \ . \end{aligned} \tag{3.13}$$

This implies indeed $t = \lim_{n\uparrow\infty} T(x_n,h_n) \neq \lim_{n\uparrow\infty} T(x'_n,h'_n) = t'$.

□Lemma 3.3.

We are ready now to define the main object of the present paper: the true self-repelling motion. For any $t \geq 0$, we define the set

$$P_t := \bigcap_{\varepsilon > 0} \overline{\{(x,h) \in \mathbb{E} : T(x,h) \in (t-\varepsilon, t+\varepsilon)\}} \ . \tag{3.14}$$

Lemma 3.4. *Almost surely, for all $t \in [0,\infty)$, P_t is a singleton.*

Proof. Since P_t is the intersection of a family of nested compact sets, it contains at least one point. By definition (3.14) of P_t, for any $(x,h) \in P_t$ there exists a sequence (x_n, h_n) converging to (x,h) with $\lim_{n\uparrow\infty} T(x_n, h_n) = t$. Hence, by Lemma 3.3, P_t can not contain more than one point. □Lemma 3.4.

Definition. *For all $t \geq 0$, we denote*

$$P_t =: \{(X_t, H_t)\} \tag{3.15}$$

and call $\mathbb{R}_+ \ni t \mapsto X_t \in \mathbb{R}$ the true self-repelling motion.

Remark: As we shall soon see

$$H_t = L_t(X_t) \tag{3.16}$$

where $L_t(x)$ is the occupation time density of $X_\cdot$.

In the next Proposition we summarize the first important properties of true self-repelling motion:

Proposition 3.5. [First properties of X_t]

(i) *Almost surely, $t \mapsto (X_t, H_t)$ is continuous on $[0,\infty)$ and $(X_0, H_0) = (0,0)$.*

(ii) *Almost surely, the set $\{t \in \mathbb{R}_+ : X_t = x\}$ is unbounded for any $x \in \mathbb{R}$.*

(iii) *The processes $t \mapsto (X_t, H_t)$ and $t \mapsto (-X_t, H_t)$ are identical in law.*

(iv) *For any $a > 0$, the processes $t \mapsto (X_{at}, H_{at})$ and $t \mapsto (a^{2/3} X_t, a^{1/3} H_t)$ are identical in law.*

Proof. (i) Suppose that there exist two sequences $(t_n)_{n\in\mathbb{N}}$ and $(t'_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$, both converging to $t \geq 0$ and that $\lim_{n\uparrow\infty} (X_{t_n}, H_{t_n})$ and $\lim_{n\uparrow\infty} (X_{t'_n}, H_{t'_n})$ exist. Then, the definition of (X_t, H_t) implies that for any n fixed there exist two sequences $(x_{n,k}, h_{n,k})_{k\in\mathbb{N}}$ and $(x'_{n,k}, h'_{n,k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\uparrow\infty} (x_{n,k}, h_{n,k}) = (X_{t_n}, H_{t_n}), \qquad \lim_{k\uparrow\infty} T(x_{n,k}, h_{n,k}) = t_n \tag{3.17}$$

$$\lim_{k\uparrow\infty} (x'_{n,k}, h'_{n,k}) = (X_{t'_n}, H_{t'_n}), \qquad \lim_{k\uparrow\infty} T(x'_{n,k}, h'_{n,k}) = t'_n \tag{3.18}$$

Hence, by a 'diagonal subsequence trick' we find two sequences $(k_n)_{n\in\mathbb{N}}$ and $(k'_n)_{n\in\mathbb{N}}$ so that

$$\lim_{n\uparrow\infty}(x_{n,k_n},h_{n,k_n}) = \lim_{n\uparrow\infty}\left(X_{t_n},H_{t_n}\right), \quad \lim_{n\uparrow\infty}(x'_{n,k_n},h'_{n,k_n}) = \lim_{n\uparrow\infty}\left(X_{t'_n},H_{t'_n}\right), \tag{3.19}$$

and

$$\lim_{n\uparrow\infty} T(x_{n,k_n},h_{n,k_n}) = t = \lim_{n\uparrow\infty} T(x'_{n,k_n},h'_{n,k_n}) \tag{3.20}$$

Using Lemma 3.3, this implies that

$$\lim_{n\uparrow\infty}\left(X_{t_n},H_{t_n}\right) = \lim_{n\uparrow\infty}\left(X_{t'_n},H_{t'_n}\right) \ . \tag{3.21}$$

Hence, $t\mapsto \left(X_t,H_t\right)$ is almost surely continuous on $[0,\infty)$.

(ii) is a simple consequence of the fact that almost surely, for all $x\in\mathbb{R}$, $\lim_{h\uparrow\infty} T(x,h) = +\infty$, which can be easily derived and is safely left to the reader. (iii) is a simple consequence of Proposition 2.2, and (iv) follows immediately from the scaling property of Brownian motion (and hence that of $\overline{\Lambda}$).

□Proposition 3.5.

4. Occupation-time density

For all $t\in[0,\infty)$ we define the set

$$D_t := T^{-1}([0,t]) = \{(x,h)\in\mathbb{E} : T(x,h)\in[0,t]\} \ . \tag{4.1}$$

We clearly have almost surely for all $(x,h)\in\mathbb{E}$

$$D_{T(x,h)} = D(x,h) \ . \tag{4.2}$$

Indeed: $\left[(x',h')\in D_{T(x,h)}\right] \Leftrightarrow [T(x',h')\le T(x,h)] \Leftrightarrow [(x',h')\in D(x,h)]$.

We already know some simple regularity properties of the map $T\colon\mathbb{E}\to\mathbb{R}_+$: it is injective and its range is dense in $\mathbb{R}_+$. Propositions 4.1 and 4.4 summarize the most important properties of T:

Proposition 4.1. [Further properties of the map $T\colon\mathbb{E}\to\mathbb{R}_+$.]

The map $T\colon\mathbb{E}\to\mathbb{R}_+$ almost surely has the following properties:
(i) *T is injective; for all $x\in\mathbb{R}$, $h\mapsto T(x,h)$ is strictly increasing and continuous from the left.*
(ii) *T is lower semicontinuous; in particular, it is Borel.*
(iii) *T preserves Lebesgue measure, that is, for all $t\in\mathbb{R}_+$*

$$|D_t| = t \ . \tag{4.3}$$

Proof. (i) follows immediately from Proposition 3.1 and from left-continuity of $h \mapsto \overline{\Lambda}_{(x,h)}(y)$, for all $x, y \in \mathbb{R}$.

The proof of (ii) and (iii) relies on Lemma 3.2:

(ii) We prove

$$D_t = \bigcap_{T(x,h)>t} D(x,h) =: \widetilde{D}_t \ . \tag{4.4}$$

Since $D(x,h)$ is closed for all $(x,h) \in \mathbb{E}$ (in the euclidean topology *restricted to* $\mathbb{E}$), lower semicontinuity of T follows from (4.4). $D_t \subset \widetilde{D}_t$ is straightforward. $(x',h') \in \widetilde{D}(t)$ if and only if $T(x',h') \leq \inf\{T(x,h) : T(x,h) > t\}$. Due to the density of $T(\mathbb{E})$ in $\mathbb{R}_+$ (Lemma 3.2) this is equivalent to $T(x',h') \leq t$. Hence $\widetilde{D}_t \subset D_t$.

(iii) Applying (4.4), the monotone class theorem and Lemma 3.2, we get:

$$\begin{aligned} |D_t| &= \inf\{|D(x,h)| : T(x,h) > t\} \\ &= \inf\{T(x,h) : T(x,h) > t\} = t \ . \end{aligned} \tag{4.5}$$

□Proposition 4.1.

Before stating the main results of the present section we have to define some relevant quantities related to the 'occupation time measure' of X_t: Beside $T(x,h)$ we shall need later also for all $(x,h) \in \overline{\mathbb{E}}$,

$$T^+(x,h) := \lim_{\varepsilon \downarrow 0} T(x, h+\varepsilon) \ . \tag{4.6}$$

For all $t \geq 0$ and $y \in \mathbb{R}$, we now define

$$L_t(y) := \sup\{h > 0 : T(y,h) < t\} \tag{4.7}$$

with the convention $\sup \emptyset = 0$. Since almost surely, for all $y \in \mathbb{R}$, $h \mapsto T(y,h)$ is *strictly* increasing, $t \mapsto L_t(y)$ is necessarily continuous, and of course, also monotone non-decreasing. In particular, for all $t \geq 0$ and $y \in \mathbb{R}$,

$$L_t(y) = \inf\{h > 0 : T(y,h) \geq t\} \ . \tag{4.8}$$

Inverting (4.7) and (4.8), we get:

$$T(x,h) = \sup\{t \in [0,\infty) : L_t(y) < h\} = \inf\{t \in [0,\infty) : L_t(x) \geq h\} \tag{4.9}$$

and

$$T^+(x,h) = \sup\{t \in [0,\infty) : L_t(y) \leq h\} = \inf\{t \in [0,\infty) : L_t(x) > h\} \tag{4.10}$$

Note that the definition of L_t implies that for all $t \geq 0$,

$$H_t = L_t(X_t) \ . \tag{4.11}$$

We define for all $t \geq 0$ the occupation-time measure μ_t as in the introduction: For all Borel set $A \subset \mathbb{R}$,

$$\mu_t(A) := \int_0^t \mathbb{1}_{\{X_s \in A\}} \, ds \quad . \tag{4.12}$$

The following statement shows that $L_t(\cdot)$ is indeed the Radon-Nikodym derivative of the occupation time measure of μ_t with respect the Lebesgue measure, i.e. it is the 'occupation-time density'.

Theorem 4.2. [Existence and continuity of occupation-time density.]

The following statements hold almost surely:

(i) *For all* $t \geq 0$, μ_t *is absolutely continuous with respect to the Lebesgue measure and its density is* $L_t(\cdot)$. *In other words, for any measurable, bounded, real-valued function* $f : \mathbb{R} \to \mathbb{R}$

$$\int_0^t f(X_s) \, ds = \int_{-\infty}^{\infty} f(y) L_t(y) \, dy \quad . \tag{4.13}$$

(ii) *More generally*: *for any measurable, bounded, real-valued function* $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *and for any time* $t \geq 0$,

$$\int_0^t g(s, X_s) \, ds = \int_{-\infty}^{\infty} \left\{ \int_0^t g(s, y) \, d_s L_s(y) \right\} dy \quad . \tag{4.14}$$

(iii) *The mapping* $t \mapsto L_t(\cdot)$ *is non-decreasing continuous from* $[0, \infty)$ *into the space of continuous real-valued functions with compact support, with topology induced by uniform convergence on compact intervals.*

Proof. (i) We first check (4.13) in the case when $f = \chi_A$ is the indicator function of an open set $A \subset \mathbb{R}$. As X is continuous, $I_t(A) := \{s \in (0, t) : X_s \in A\}$ is an open subset of $(0, \infty)$. Hence, there exists a finite or countable family of disjoint open intervals $\big((\alpha_i, \beta_i)\big)_{i \in J}$ such that $I_t(A) = \cup_{i \in J} (\alpha_i, \beta_i)$. The definition of D_t and $I_t(A)$ implies that

$$\{(x, h) \in \mathbb{E} : x \in A \text{ and } T(x, h) \leq t\} = \bigcup_{i \in J} \big(D_{\beta_i} \setminus D_{\alpha_i}\big) \quad . \tag{4.15}$$

Hence, the Lebesgue area of these two sets are equal, so that almost surely,

$$\begin{aligned} \int \chi_A(y) L_t(y) \, dy &= |\{(x, h) \in \mathbb{E} : x \in A \text{ and } L_t(x) \geq h\}| \\ &= \left| \bigcup_{i \in J} \big(D_{\beta_i} \setminus D_{\alpha_i}\big) \right| \quad . \end{aligned} \tag{4.16}$$

But Proposition 3.1 implies that the sets $D_{\beta_i} \setminus D_{\alpha_i}$ for $i \in J$ are disjoint so that

$$\begin{aligned}\int \chi_A(y) L_t(y)\,dy &= \sum_{i\in J} |D_{\beta_i} \backslash D_{\alpha_i}| \\ &= \sum_{i\in J} (\beta_i - \alpha_i) \\ &= \int_0^t \mathbb{1}_{\{X_s \in A\}}\,ds \ .\end{aligned} \tag{4.17}$$

Using this result, it is then very easy derive (4.13) for arbitrary bounded and measurable f.

(ii) First note that (4.14) for $g(s,y) = h(s)f(y)$, with h and f measurable functions, follows directly from (4.13). For general, measurable $g(s,y)$ apply approximation by linear combination of factorizable ones. This procedure is standard, we leave the details for the reader. (See e.g. Exercise VI.1.15 in [RY])

(iii) This is simply due to the fact that almost surely, L_t is a non-decreasing family of continuous functions with compact support, such that for all $y \in \mathbb{R}$, $t \mapsto L_t(y)$ is continuous.

□Theorem 4.2.

$L_t(\cdot)$ being identified as the occupation-time density, or local time of our process, (4.9) and (4.10) show that $T(x,h)$, respectively, $T^+(x,h)$ are actually the left-continuous, respectively, right-continuous versions of the so-called inverse local time. This implies the general Ray-Knight Theorem for the occupation-time density of X:

Theorem 4.3. [Ray-Knight theorems.]

(i) *Almost surely, for all* $(x,h) \in \mathbb{E}$ and $y \in \mathbb{R}$

$$L_{T(x,h)}(y) = \overline{\Lambda}_{(x,h)}(y) \tag{4.18}$$

(ii) *Almost surely, for all* $(x,h) \in \overline{\mathbb{E}}$ *and* $y \in \mathbb{R}$

$$L_{T^+(x,h)}(y) = \overline{\Lambda}^+_{(x,h)}(y) := \lim_{\varepsilon \downarrow 0} \overline{\Lambda}_{(x,h+\varepsilon)}(y) \ , \tag{4.19}$$

Proof. These are direct consequences of the definition (4.7) of $L_t(\cdot)$.

□Theorem 4.3.

Remark 1: Let us stress here that (i) and (ii) are much ‘stronger’ than the ‘usual’ Ray-Knight Theorems in the sense that we have a description of all $L_{T(x,h)}$ (respectively $L_{T^+(x,h)}$) simultaneously for all $(x,h) \in \mathbb{E}$.

Remark 2: Note that in fact, it is easy to notice that for all fixed $x \in \mathbb{R}$, if

$$\tau_x := \inf\{t \geq 0 : X_t = x\} \tag{4.20}$$

denotes the first hitting time of x by X then almost surely $\tau_x = T^+(x, 0)$, so that (ii) contains also a 'Ray-Knight theorem' at first hitting times. But, of course, there exist infinitely many (random) points such that $\tau_x \neq T^+(x, 0)$ (for instance if $x = \sup\{X_t : t \in [0, T(0, h)]\}$).

Before proceeding to the next section, we complete the list of properties of the function $T : \mathbb{E} \to \mathbb{R}_+$:

Proposition 4.4. *Almost surely the Lebesgue measure of the set* $\mathbb{R}_+ \setminus T(\mathbb{E})$ *is* 0.

Proof. We will use Theorem 4.2: Clearly, if $t \notin T(\mathbb{E})$, then either $H_t = 0$ or $t \neq T(X_t, H_t)$. In the second case, as $t \in [T(X_t, H_t), T^+(X_t, H_t)]$, it implies that $T^+(X_t, H_t) \neq T(X_t, H_t)$. Hence

$$\int_0^\infty \mathbb{1}_{\{t \notin T(\mathbb{E})\}}\, dt \leq \int_0^\infty \mathbb{1}_{\{H_t = 0\}} dt + \int_0^\infty \mathbb{1}_{\{T(X_t, L_t(X_t)) \neq T^+(X_t, L_t(X_t))\}}\, dt \ . \tag{4.21}$$

But Theorem 4.2(ii) readily shows that

$$\int_0^\infty \mathbb{1}_{\{L_t(X_t) = 0\}}\, dt = \int_{\mathbb{R}} 0\, dy = 0 \tag{4.22}$$

and that

$$\int_0^\infty \mathbb{1}_{\{T(X_t, L_t(X_t)) \neq T^+(X_t, L_t(X_t))\}}\, dt = \int_{\mathbb{E}} \mathbb{1}_{\{T(y,h) \neq T^+(y,h)\}}\, dy\, dh. \tag{4.23}$$

But for all $(y, h) \in \mathbb{E}$, almost surely, $T(y, h) = T^+(y, h)$. Hence,

$$\int_{\mathbb{E}} \mathbb{1}_{\{T(y,h) \neq T^+(y,h)\}}\, dy\, dh = 0 \tag{4.24}$$

almost surely. Finally,

$$\int_0^\infty \mathbb{1}_{\{t \notin T(\mathbb{E})\}}\, dt = 0 \tag{4.25}$$

almost surely.

□Proposition 4.4.

5. Markov properties

5.1. Markov property of $(X_t, L_t(\cdot))$

In this subsection, we are going to derive the Markov property of the process $(X_t, L_t(\cdot))$. We start with a Markov Property 'in space':

We first need to put down some notation. Denote by C_0 the space of continuous functions with compact support $\lambda : \mathbb{R} \to \mathbb{R}_+$. For $\lambda \in C_0$ we define $\mathscr{B}_\lambda$ (respectively $\mathscr{A}_\lambda$) the sigma-algebra generated by the data **b**elow (resp. **a**bove) the curve λ. More precisely, for all $n \in \mathbb{N}$ and $\lambda \in C_0$ we define the stopping time

$$\rho_{\lambda,n} := \inf\{x \geq \tilde{x}_n : F_n(x) = \lambda(x)\} \ . \tag{5.1}$$

Then,

$$\mathscr{B}_\lambda := \sigma\big(\{F_n(x) : n \geq 0, \quad \tilde{h}_n < \lambda(\tilde{x}_n) \quad \text{and} \quad x \in [\tilde{x}_n, \rho_{\lambda,n}]\}\big) \tag{5.2}$$

$$\mathscr{A}_\lambda := \sigma\big(\{F_n(x) : n \geq 0, \quad \tilde{h}_n > \lambda(\tilde{x}_n) \quad \text{and} \quad x \in [\tilde{x}_n, \rho_{\lambda,n}]\}\big) \ . \tag{5.3}$$

For all $(x, h) \in \mathbb{E}$, we define the 'past' and 'future' algebras with respect to $T(x, h)$ as follows (we use respectively the notation $\mathscr{B}$ and $\mathscr{A}$ for 'before/below' and 'after/above'):

$$\mathscr{B}_{(x,h)} := \big\{E \in \sigma(\Lambda) : \forall \lambda \in C_0 \ \text{ such that } \ \lambda(x) > h, \\ E \cap \{\overline{\Lambda}_{(x,h)} \leq \lambda\} \in \mathscr{B}_\lambda\big\} \tag{5.4}$$

$$\mathscr{A}_{(x,h)} := \big\{E \in \sigma(\Lambda) : \forall \lambda \in C_0 \ \text{ such that } \ \lambda(x) < h, \\ E \cap \{\overline{\Lambda}_{(x,h)} \geq \lambda\} \in \mathscr{A}_\lambda\big\} \tag{5.5}$$

One can easily check that $\mathscr{B}_{(x,h)}$ and $\mathscr{A}_{(x,h)}$ are indeed sigma algebras. $\mathscr{B}_{(x,h)}$ (respectively $\mathscr{A}_{(x,h)}$) contains the information below (respectively, above) the (random) curve $\overline{\Lambda}_{(x,h)}$. Note that $\overline{\Lambda}_{(x,h)}$ and consequently $T(x, h)$ are $\mathscr{A}_{(x,h)} \cap \mathscr{B}_{(x,h)}$ measurable.

Proposition 5.1. *Let $(x_0, h_0) \in \mathbb{E}$ and $\lambda \in C_0$ be fixed. Then, given $\overline{\Lambda}_{(x_0,h_0)} = \lambda$, the algebras $\mathscr{B}_{(x_0,h_0)}$ and $\mathscr{A}_{(x_0,h_0)}$ are independent.*

Proof. This result is intuitively clear: The law of $(X_t, t \geq T(x_0, h_0))$ depends only on the lines 'above' $\overline{\Lambda}_{(x_0,h_0)}$ and the law of the system of lines above this curve depends only on this curve as the forward lines (resp. backward lines) coalesce with $\Lambda_{(x_0,h_0)}$ (resp. $\Lambda^*_{(x_0,h_0)}$). To make this more rigourous, it suffices ((x_0, h_0) being fixed) to define the process X in the following way: Define first a family of coalescing independent RAB's $(E_n, n \geq 0)$ by induction just as the F_n's except that it is started from $(\hat{x}_n, \hat{h}_n)_{n\geq 0}$ where $(\hat{x}_0, \hat{h}_0) = (x_0, h_0)$ and $(\hat{x}_n, \hat{h}_n)$ is dense in $[x_0, \infty) \times (0, \infty)$.

Similarly, define a family $(E'_n, n \geq 0)$ of backward lines started from $(\hat{x}'_n, \hat{h}'_n)$, where $(\hat{x}'_0, \hat{h}'_0) = (x_0, h_0)$ and $(\hat{x}'_n, \hat{h}'_n)$ is dense in $(-\infty, x_0] \times (0, \infty)$.

Theorem 2.1 and Proposition 2.2 show that if we define for all $y' \leq x' < x_0 < x \leq y$ and $h > 0$,

$$\Lambda^1_{(x,h)}(y) = \sup\{E_n(y) : E_n(x) < h,\ n \geq 0 \text{ and } \hat{x}_n < x\} \tag{5.6}$$

$$\Lambda^2_{(x',h)}(y') = \sup\{E'_n(y') : E'_n(x') < h,\ n \geq 0 \text{ and } \hat{x}'_n > x'\} \tag{5.7}$$

then the law of Λ^1 (resp. Λ^2) defined on $\mathbb{F}^1 = \{(x,h,y) \in \mathbb{F}^+ : x > x_0\}$ (resp. $\mathbb{F}^2 = \{(x,h,y) \in \mathbb{F}^- : x < x_0\}$) is identical to that of the restriction to this set of Λ (resp. Λ^*). In particular, using the duality property (and the independence between $\mathscr{F}_{x_0}$ and $\mathscr{F}^*_{x_0}$), there exists a unique version of Λ (with the law described in Theorem 2.1) such that $\Lambda = \Lambda^1$ and $\Lambda^* = \Lambda^2$ on $\mathbb{F}^1$ and $\mathbb{F}^2$ respectively. Note that with this construction, we can define first E_0 and E'_0 (i.e. $\overline{\Lambda}_{(x_0,h_0)}$) and then $\overline{\Lambda}_{(x,h)}$ for the other values of (x,h): We have explicitly the law of $\overline{\Lambda}$ conditional on $\overline{\Lambda}_{(x_0,h_0)}$.

As (E_n)'s (and (E'_n)) are independent coalescing processes, it is clear that the law of $(E_n : n \geq 0 \text{ and } E_0(\hat{x}_n) \leq \hat{h}_n)$ (and that of $(E'_n : n \geq 0 \text{ and } E_0(\hat{x}'_n) \leq \hat{h}'_n))$ depends only on E_0 (and E'_0). In particular, conditionally on E_0 and E'_0, these processes are independent from $(E_n : n \geq 0 \text{ and } E_0(\hat{x}_n) \geq \hat{h}_n)$ and $(E'_n : n \geq 0 \text{ and } E_0(\hat{x}'_n) \geq \hat{h}'_n)$. But the definitions of Λ^1, Λ^2 and Λ show that

$$\mathscr{B}_{(x_0,h_0)} = \sigma\Big(\{E_n : n \geq 0 \text{ and } E_0(\hat{x}_n) \leq \hat{h}_n\} \cup \{E'_n : n \geq 0 \text{ and } E_0(\hat{x}'_n) \leq \hat{h}'_n\}\Big) \tag{5.8}$$

$$\mathscr{A}_{(x_0,h_0)} = \sigma\Big(\{E_n : n \geq 0 \text{ and } E_0(\hat{x}_n) \geq \hat{h}_n\} \cup \{E'_n : n \geq 0 \text{ and } E_0(\hat{x}'_n) \geq \hat{h}'_n\}\Big) . \tag{5.9}$$

Hence, conditionally on E_0 and E'_0 (i.e. on $\overline{\Lambda}_{(x_0,h_0)}$), the two sigma-fields $\mathscr{A}_{(x_0,h_0)}$ and $\mathscr{B}_{(x_0,h_0)}$ are independent.

$\square$Proposition 5.1

We now introduce the past and future algebras with respect to deterministic time $t \geq 0$:

$$\mathscr{B}_t := \big\{E \in \sigma(\Lambda) \ : \ \forall (x,h) \in \mathbb{E},\ E \cap \{T(x,h) \geq t\} \in \mathscr{B}_{(x,h)}\big\} \tag{5.10}$$

$$\mathscr{A}_t := \big\{E \in \sigma(\Lambda) \ : \ \forall (x,h) \in \mathbb{E},\ E \cap \{T(x,h) \leq t\} \in \mathscr{A}_{(x,h)}\big\} \tag{5.11}$$

Proposition 5.2. *Let $t \geq 0$ be fixed. Then*

$$\sigma\big(\{X_s : 0 \leq s \leq t\}\big) \subset \mathscr{B}_t \tag{5.12}$$

$$\sigma\big(\{X_t, L_t(\cdot)\}\big) \vee \sigma\big(\{X_s : s > t\}\big) \subset \mathscr{A}_t \ . \tag{5.13}$$

Proof. We prove (5.12), the proof of (5.13) is identical. Let $\lambda \in C_0$. Then $T(x,h) \geq t$ and $\Lambda_{(x,h)} \leq \lambda$ imply that

$$T^{-1}[0,t] \subset \{(x,h) \in \mathbb{E} : h \leq \lambda(x)\} \ . \tag{5.14}$$

Hence, given the construction (3.14) it follows that for any $0 \leq s_1 \leq \cdots \leq s_p \leq t$ and any intervals $(a_i, b_i) \subset \mathbb{R}$ ($i \in \{1,\dots,p\}$)

$$\bigcap_{i\in\{1,\dots,p\}} \big\{X_{s_i} \in (a_i,b_i)\big\} \cap \big\{T(x,h) \geq t\big\} \cap \big\{\Lambda_{(x,h)} \leq \lambda\big\} \in \mathscr{B}_\lambda \tag{5.15}$$

and this readily implies (5.12).

□Proposition 5.2

Theorem 5.3. *Let $t \geq 0$, $x \in \mathbb{R}$ and $\lambda \in C_0$ with*

$$\int_{-\infty}^{\infty} \lambda(y)\, dy = t \tag{5.16}$$

be fixed. Then, conditionally on $X_t = x$ and $L_t(\cdot) = \lambda(\cdot)$, the algebras $\mathscr{B}_t$ and $\mathscr{A}_t$ are independent.

Proof. Denote

$$h := \lambda(x) \ . \tag{5.17}$$

For any $(x,h) \in \mathbb{E}$ fixed we define the event

$$B_{(x,h)} := \{T(x,h) = T^+(x,h)\} \ . \tag{5.18}$$

Similarly for $t \geq 0$ fixed, we define the event

$$B_t := \{T(X_t,H_t) = T^+(X_t,H_t)\} \ . \tag{5.19}$$

These events are of full measure:

$$\mathbf{P}(B_{(x,h)}) = 1 = \mathbf{P}(B_t) \tag{5.20}$$

and

$$\{X_t = x \text{ and } L_t = \lambda\} \cap B_t = \{\overline{\Lambda}_{(x,h)} = \lambda\} \cap B_{(x,h)} \ . \tag{5.21}$$

Note also that

$$\{\overline{\Lambda}_{(x,h)} = \lambda\} \subset \{T(x,h) = t\} = \{T(x,h) \geq t\} \cap \{T(x,h) \leq t\} \tag{5.22}$$

Let $E \in \mathscr{B}_t$ and $F \in \mathscr{A}_t$ two arbitrary events. Then the following chain of equalities holds

$$
\begin{aligned}
&\mathbf{P}(E \cap F | X_t = x,\ L_t = \lambda) \\
&\quad = \mathbf{P}(E \cap F | \overline{\Lambda}_{(x,h)} = \lambda) && (5.23)\\
&\quad = \mathbf{P}\big((E \cap \{T(x,h) \geq t\}\big) \cap \big(F \cap \{T(x,h) \leq t\}\big) | \overline{\Lambda}_{(x,h)} = \lambda) && (5.24)\\
&\quad = \mathbf{P}(E \cap \{T(x,h) \geq t\} | \overline{\Lambda}_{(x,h)} = \lambda) \\
&\qquad\qquad \mathbf{P}(F \cap \{T(x,h) \leq t\} | \overline{\Lambda}_{(x,h)} = \lambda) && (5.25)\\
&\quad = \mathbf{P}(E | \overline{\Lambda}_{(x,h)} = \lambda) \mathbf{P}(F | \overline{\Lambda}_{(x,h)} = \lambda) && (5.26)\\
&\quad = \mathbf{P}(E | X_t = x,\ L_t = \lambda) \mathbf{P}(F | X_t = x,\ L_t = \lambda) && (5.27)
\end{aligned}
$$

In (5.23) and (5.27) we use (5.20) and (5.21). In (5.24) and (5.26) we use (5.22). Finally, in (5.25) we use Proposition 5.1 and the fact that by definitions (5.10), respectively (5.11), of $\mathscr{B}_t$, respectively $\mathscr{A}_t$:

$$E \cap \{T(x,h) \geq t\} \in \mathscr{B}_{(x,h)} \ , \tag{5.28}$$

$$F \cap \{T(x,h) \leq t\} \in \mathscr{A}_{(x,h)} \ . \tag{5.29}$$

$\square$Theorem 5.3

5.1. *'Local Markov property'*

In this subsection, we derive the locality property stated in the introduction and some other related results.

We first introduce some new sigma-algebras: For all $x < y$ in $\mathbb{R}$, we define

$$\mathscr{F}_{(x,y)} := \sigma\Big(\{F_n(x') : n \geq 0 \quad \text{and} \quad x < \tilde{x}_n \leq x' < y\}\Big) = \mathscr{F}_{y} \cap \mathscr{F}^*_{x} \ . \tag{5.30}$$

In plain words: $\mathscr{F}_{(x,y)}$ contains the data between the 'times' x and y; the sigma algebras $\mathscr{F}_x$ and $\mathscr{F}^*_x$ defined in section 2 correspond respectively to $\mathscr{F}_{(-\infty,x)}$ and $\mathscr{F}_{(x,\infty)}$. This definition implies immediately that if $\Omega_1, \ldots, \Omega_p$ are p disjoint open intervals in $\mathbb{R}$, then $\mathscr{F}_{\Omega_1}, \ldots, \mathscr{F}_{\Omega_p}$ are p independent sigma-fields.

Suppose now that $\Omega \subset \mathbb{R}$ is an open interval. Define as in the introduction, for all $t \geq 0$ and $u > 0$,

$$A^\Omega_t := \int_0^t \mathbb{1}_{\{X_s \in \Omega\}} ds \tag{5.31}$$

$$\tau^\Omega_u := \inf\{t > 0 : A^\Omega_t > u\} \tag{5.32}$$

$$X^\Omega_u := X_{\tau^\Omega_u} \ . \tag{5.33}$$

We say that X^Ω is *reflecting true self-repelling motion* in the interval Ω started from X^Ω_0. Using Proposition 2.2, It is easy (and left to the reader) to see that X^{Ω} can be described as follows: Introduce for all $x \in \Omega$, $h > 0$,

$$T^{\Omega}(x,h) := \int_{\Omega} \overline{\Lambda}_{(x,h)}(y)\,dy \tag{5.34}$$

and then define $(X_u^{\Omega}, H_u^{\Omega})_{u \geq 0}$ using T^{Ω} exactly as $(X_u, H_u)_{u \geq 0}$ has been derived from T. In particular, this shows that the process X^{Ω} is $\mathscr{F}_{\Omega}$ measurable. Hence, if $\Omega_1, \ldots, \Omega_p$ are disjoint open intervals in $\mathbb{R}$, then the processes $(X^{\Omega_1}, \ldots, X^{\Omega_p})$ are independent.

Suppose now that the open interval Ω is fixed. Almost exactly as for X, one can derive Markov properties for reflecting true self-repelling motion X^{Ω}. Define first the occupation-time densities corresponding to X^{Ω}: For all $u \geq 0$ and $x \in \Omega$,

$$L_u^{\Omega}(x) := L_{\tau_u^{\Omega}}(x) \tag{5.35}$$

(Theorem 4.2 implies that L_u^{Ω} is exactly the occupation-time density of X^{Ω}).

Proposition 5.4. *(X^{Ω}, L^{Ω}) is a Markov process.*

The proof goes along exactly the same lines as that of Theorem 5.3 and we leave it to the reader. In fact, it is also immediate to notice that if we define

$$DL_u^{\Omega}(x) := L_u^{\Omega}(x) - L_u^{\Omega}(X_u^{\Omega}) \tag{5.36}$$

then $(X_u^{\Omega}, DL_u^{\Omega})_{u \geq 0}$ is also a Markov process. Combining this with the independence stated above yields the locality property stated in the introduction.

Remark- As a byproduct of the proof, we also get the following counterpart of Proposition 5.1:

Proposition 5.5. *For all $x \in \mathbb{R}$ and $h > 0$, conditionally on $DL^{\Omega}_{T^{\Omega}(x,h)}$, $(X_u^{\Omega}, u \leq T^{\Omega}(x,h))$ and $(X_u^{\Omega}, u \geq T^{\Omega}(x,h))$ are independent.*

One could then use this and a similar proof as that of Theorem 5.3 to show that for all $t_0 \geq 0$ and $\varepsilon > 0$, if we define

$$t_0^{\varepsilon} := \inf\{t \geq t_0 : |X_t - X_{t_0}| = \varepsilon\} \ , \tag{5.37}$$

then the law of

$$(X_t - X_{t_0}, t \in [t_0, t_0^{\varepsilon}]) \tag{5.38}$$

depends only on $(L_{t_0}(X_{t_0} + x) - L_{t_0}(X_{t_0}), x \in [-\varepsilon, \varepsilon])$.

6. Dynamics

We derive the dynamical driving mechanism of true self repelling motion. A non-rigorous, but still instructive phenomenological derivation is presented in Appendix B. Consulting that argument before reading the forthcoming rather technical proof might be illuminating.

In this section, to avoid heavy notation, as there is no danger of confusion, we will write Λ instead of $\overline{\Lambda}$.

Theorem 6.1. *For any* $(x_0, h_0) \in \mathbb{E}$ *fixed*

$$\text{P-}\lim_{\varepsilon \downarrow 0} \int_0^{T_{(x_0,h_0)}} \frac{L_s(X_s+\varepsilon) - L_s(X_s-\varepsilon)}{2\varepsilon}\, ds = -X_{T_{(x_0,h_0)}} + \frac{1}{4}\left(\sup_{0 \le s \le T_{(x_0,h_0)}} X_s + \inf_{0 \le s \le T_{(x_0,h_0)}} X_s \right) \quad (6.1)$$

Proof of Theorem 6.1. Define

$$\alpha = \alpha_{(x_0,h_0)} := \inf\{y : \Lambda_{(x_0,h_0)}(y) > 0\} \quad (6.2)$$

$$\omega = \omega_{(x_0,h_0)} := \sup\{y : \Lambda_{(x_0,h_0)}(y) > 0\} \ . \quad (6.3)$$

Clearly:

$$\alpha = \inf\{X_s : 0 \le s \le T_{(x_0,h_0)}\} \ , \quad (6.4)$$

$$x_0 = X_{T_{(x_0,h_0)}} \quad (6.5)$$

$$\omega = \sup\{X_s : 0 \le s \le T_{(x_0,h_0)}\} \ . \quad (6.6)$$

From Theorem 4.2(ii) we know that almost surely for any bounded Borel function $\mathbb{R}_+ \times \mathbb{R} \ni (s,x) \mapsto f(s,x)$ the following identity holds:

$$\int_0^t f(s, X_s)\, ds = \int_{-\infty}^{\infty} \left\{ \int_0^t f(s,y)\, d_s L_s(y) \right\} dy \ . \quad (6.7)$$

On the other hand, we know that almost surely and for almost all $(y,h) \in \mathbb{E}$ (with respect to the Lebesgue measure in $\mathbb{E}$) $T(y,h) = T^+(y,h)$, so that a change of variable yields that almost surely, for all bounded Borel function f on $\mathbb{R}_+ \times \mathbb{R}$

$$\int_{-\infty}^{\infty} \left\{ \int_0^t f(s,y)\, d_s L_s(y) \right\} dy = \int_{-\infty}^{\infty} \left\{ \int_0^{L_t(y)} f(T_{(y,h)}, y)\, dh \right\} dy \ . \quad (6.8)$$

Applying (6.7)–(6.8) to $f(s,x) := L_s(x \pm \varepsilon)$, due to Theorem 4.3 we get:

$$\int_0^{T_{(x_0,h_0)}} L_s(X_s \pm \varepsilon)\,ds = \int_{-\infty}^{\infty}\left\{\int_0^{L_{T_{(x_0,h_0)}}(y)} L_{T_{(y,h)}}(y\pm\varepsilon)\,dh\right\}dy$$
$$= \int_{-\infty}^{\infty}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)} \Lambda_{(y,h)}(y\pm\varepsilon)\,dh\right\}dy\ . \qquad (6.9)$$

By use of (6.4)–(6.6) and (6.9), (6.1) transforms to

$$\text{P-}\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_{-\infty}^{\infty}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\big(\Lambda_{(y,h)}(y+\varepsilon)-\Lambda_{(y,h)}(y-\varepsilon)\big)\,dh\right\}dy$$
$$=\frac{1}{4}(\alpha+\omega)-x_0\ . \qquad (6.10)$$

and this is what we are going to prove now.

We write the integral on the left hand side of (6.10) as

$$\int_{-\infty}^{\infty}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\big(\Lambda_{(y,h)}(y+\varepsilon)-\Lambda_{(y,h)}(y-\varepsilon)\big)\,dh\right\}dy = I_1 + I_2 \quad (6.11)$$

where

$$I_1 := \int_\alpha^{x_0}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\big(\Lambda_{(y,h)}(y+\varepsilon)-\Lambda_{(y,h)}(y-\varepsilon)\big)\,dh\right\}dy$$
$$= -\int_\alpha^{x_0}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\Lambda_{(y,h)}(y-\varepsilon)\,dh - \int_0^{\Lambda_{(x_0,h_0)}(y-\varepsilon)}\Lambda_{(y-\varepsilon,h)}(y)\,dh\right\}dy$$
$$+\int_{x_0-\varepsilon}^{x_0}\int_0^{\Lambda_{(x_0,h_0)}(y)}\Lambda_{(y,h)}(y+\varepsilon)\,dh\ \ dy \qquad (6.12)$$

$$I_2 := \int_{x_0}^{\omega}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\big(\Lambda_{(y,h)}(y+\varepsilon)-\Lambda_{(y,h)}(y-\varepsilon)\big)\,dh\right\}dy$$
$$= \int_{x_0}^{\omega}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\Lambda_{(y,h)}(y+\varepsilon)\,dh - \int_0^{\Lambda_{(x_0,h_0)}(y+\varepsilon)}\Lambda_{(y+\varepsilon,h)}(y)\,dh\right\}dy$$
$$-\int_{x_0}^{x_0+\varepsilon}\int_0^{\Lambda_{(x_0,h_0)}(y)}\Lambda_{(y,h)}(y-\varepsilon)\,dh\,dy \qquad (6.13)$$

The statement of the theorem will follow from

$$\text{P-}\lim_{\varepsilon\downarrow 0}\frac{I_1}{2\varepsilon}=\frac{1}{4}(\alpha-x_0)-\frac{1}{2}\max\{0,x_0\} \qquad (6.14)$$

$$\text{P-}\lim_{\varepsilon\downarrow 0}\frac{I_2}{2\varepsilon}=\frac{1}{4}(\omega-x_0)-\frac{1}{2}\min\{0,x_0\} \qquad (6.15)$$

We are going to prove (6.15), the proof of (6.14) is completely identical due to the duality of the forward and backward lines.

The following simple observation is of crucial importance:

Lemma 6.2. *For any* $(x_0, h_0) \in \mathbb{E}$ *and* $z > y > x_0$

$$\begin{aligned}
&\int_0^{\Lambda_{(x_0,h_0)}(y)} \Lambda_{(y,h)}(z)\, dh - \int_0^{\Lambda_{(x_0,h_0)}(z)} \Lambda_{(z,h)}(y)\, dh \\
&\quad = \frac{1}{2}\left(\Lambda_{(x_0,h_0)}(z) - \Lambda_{(x_0,h_0)}(y)\right)^2 + 2\int_0^{\Lambda_{(x_0,h_0)}(y)} \left(\Lambda_{(y,h)}(z) - h\right) dh \\
&\qquad + \frac{1}{2}\Lambda^2_{(x_0,h_0)}(y) - \frac{1}{2}\Lambda^2_{(x_0,h_0)}(z)
\end{aligned} \tag{6.16}$$

Proof. This proof is elementary. As $M(y,z)$ (defined in (2.13)) is locally finite, there almost surely exists an integer $n \geq 0$, and two finite increasing sequences $0 = a_0 < a_1 < \cdots < a_n = \Lambda_{(x_0,h_0)}(y)$ and $0 = b_0 < b_1 < \cdots < b_{n-1} = \Lambda_{(x_0,h_0)}(z)$ such that for all $i \in \{1, \ldots, n\}$ and $h \in (a_{i-1}, a_i]$, we have $\Lambda_{(y,h)}(z) = b_{i-1}$ and, on the other hand for all $i \in \{1, \ldots, n-1\}$ and $h' \in (b_{i-1}, b_i]$ we have $\Lambda_{(z,h')}(y) = a_i$. Elementary computations show that both sides of the equation (6.16) are equal to

$$2\sum_{i=1}^{n}(a_i - a_{i-1})b_{i-1} - a_n b_{n-1} - a_n^2/2 \ . \tag{6.17}$$

$\square$Lemma 6.2

Applying this identity, with $z = y + \varepsilon$, to (6.13) we get:

$$\begin{aligned}
I_2 = \ & \frac{1}{2}\int_{x_0}^{\omega} \left\{\Lambda_{(x_0,h_0)}(y+\varepsilon) - \Lambda_{(x_0,h_0)}(y)\right\}^2 dy \\
& + 2\int_{x_0}^{\omega}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)} \left(\Lambda_{(y,h)}(y+\varepsilon) - h\right) dh\right\} dy \\
& - \int_{x_0}^{x_0+\varepsilon}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)} \Lambda_{(y,h)}(y-\varepsilon)\, dh - \frac{1}{2}\Lambda^2_{(x_0,h_0)}(y)\right\} dy \\
=: \ & I_{2,1} + I_{2,2} - I_{2,3}
\end{aligned} \tag{6.18}$$

We easily get rid of the error term $I_{2,3}$: by almost sure continuity of $y \mapsto \Lambda_{(x.h)}(y)$, for all $(x,h) \in \mathbb{E}$ and a simple dominated convergence argument we have indeed almost surely,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}I_{2,3} = \quad & \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int_{x_0}^{x_0+\varepsilon}\int_0^{\Lambda_{(x_0,h_0)}(y)}\Lambda_{(y,h)}(y-\varepsilon)\,dh\,dy \\ & -\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\int_{x_0}^{x_0+\varepsilon}\frac{1}{2}\Lambda^2_{(x_0,h_0)}(y)\,dy \\ = & \frac{h_0^2}{2}-\frac{h_0^2}{2}=0 \ . \end{aligned} \tag{6.19}$$

Next we look at $I_{2,1}$: since $\Lambda_{(x_0,h_0)}(\cdot)$ is a Brownian motion, from standard arguments (for instance computing the first two moments of $I_{2,1}$) we get:

$$\mathbf{P}\text{-}\lim_{\varepsilon\downarrow 0}\frac{I_{2,1}}{2\varepsilon}=\frac{1}{4}(\omega-x_0) \ . \tag{6.20}$$

It now remains to study $I_{2,2}$. For $(y,h)\in\mathbb{E}$ and $z\geq y$ we denote

$$\mathbf{E}\Big(\Lambda_{(y,h)}(z)\Big)=:m_{(y,h)}(z)=:h+\widetilde{m}_{(y,h)}(z) \ . \tag{6.21}$$

$$\Lambda_{(y,h)}(z)-m_{(y,h)}(z)=:\widetilde{\Lambda}_{(y,h)}(z) \tag{6.22}$$

Elementary computations show that (one could also use Tanaka's formula to estimate these quantities)

$$\widetilde{m}_{(y,h)}(z)=\begin{cases} 2\sqrt{|z-y|}\int_{|z-y|^{-1/2}h}^{\infty}\phi(\xi)\Big(\xi-|z-y|^{-1/2}h\Big)\,d\xi & \text{if } y\leq z\leq 0 \\ 2\sqrt{|y|}\int_{|y|^{-1/2}h}^{\infty}\phi(\xi)\Big(\xi-|y|^{-1/2}h\Big)\,d\xi & \text{if } y\leq 0\leq z \\ 0 & \text{if } 0\leq y\leq z \end{cases} \tag{6.23}$$

Where $\phi(\xi)=(2\pi)^{-1/2}\exp(-\xi^2/2)$. We write

$$\begin{aligned} I_{2,2} = \quad & 2\int_{x_0}^{\omega}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\widetilde{\Lambda}_{(y,h)}(y+\varepsilon)\,dh\right\}dy \\ & +2\int_{x_0}^{\omega}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\widetilde{m}_{(y,h)}(y+\varepsilon)\,dh\right\}dy \\ =: & I_{2,2,1}+I_{2,2,2} \end{aligned} \tag{6.24}$$

First we deal with $I_{2,2,2}$:

$$\begin{aligned} I_{2,2,2} = \quad & \mathbb{1}_{\{x_0<0\}}2\int_{x_0}^{0}\left\{\int_0^{\infty}\widetilde{m}_{(y,h)}(y+\varepsilon)\,dh\right\}dy \\ & -\mathbb{1}_{\{x_0<0\}}2\int_{x_0}^{0}\left\{\int_{\Lambda_{(x_0,h_0)}(y)}^{\infty}\widetilde{m}_{(y,h)}(y+\varepsilon)\,dh\right\}dy \\ =: & I_{2,2,2,1}-I_{2,2,2,2} \end{aligned} \tag{6.25}$$

$I_{2,2,2,1}$ is deterministic and it is easy to compute. After an elementary integration we get:

$$\int_0^\infty \left\{ \sqrt{t} \int_{t^{-1/2}h}^\infty \phi(\xi)(\xi - t^{-1/2}h)\, d\xi \right\} dh = \frac{t}{4} \tag{6.26}$$

and hence

$$\lim_{\varepsilon \downarrow 0} \frac{I_{2,2,2,1}}{2\varepsilon} = \mathbb{1}_{\{x_0<0\}} \lim_{\varepsilon \downarrow 0} \frac{-x_0\varepsilon - \varepsilon^2/2}{2\varepsilon} = -\frac{1}{2}\min\{0, x_0\} \tag{6.27}$$

Next we estimate $I_{2,2,2,2}$. Another elementary computation yields:

$$\int_H^\infty \left\{ \sqrt{t} \int_{t^{-1/2}h}^\infty \phi(\xi)(\xi - t^{-1/2}h)\, d\xi \right\} dh \le \frac{t}{4} \mathrm{e}^{-H^2/(2t)} \tag{6.28}$$

and hence we get for any $x_0 \le y \le 0$:

$$\begin{aligned} \mathbf{E}\Big(\int_{\Lambda_{(x_0,h_0)}(y)}^\infty \widetilde{m}_{(y,h)}(y+\varepsilon)\, dh \Big) &\le \frac{\varepsilon}{2} \mathbf{E}\Big(\exp\{-\Lambda^2_{(x_0,h_0)}(y)/(2\varepsilon)\}\Big) \\ &\le \frac{\varepsilon}{2} \sqrt{\frac{\varepsilon}{|y - x_0| + \varepsilon}} \end{aligned} \tag{6.29}$$

Inserting this in the definition of $I_{2,2,2,2}$, we get

$$\begin{aligned} \mathbf{E}\Big(\big| I_{2,2,2,2} \big| \Big) = \mathbf{E}\Big(I_{2,2,2,2} \Big) &\le \mathbb{1}_{\{x_0<0\}} \varepsilon \int_{x_0}^0 \sqrt{\frac{\varepsilon}{|y - x_0| + \varepsilon}} dy \\ &\le \mathbb{1}_{\{x_0<0\}} \tfrac{1}{2} \sqrt{|x_0|} \varepsilon^{3/2} \end{aligned} \tag{6.30}$$

Hence

$$\text{P-}\lim_{\varepsilon \downarrow 0} \frac{I_{2,2,2,2}}{2\varepsilon} = 0 \tag{6.31}$$

Finally, we turn to $I_{2,2,1}$ defined in (6.24).

Lemma 6.3. *Let* $(x_0, h_0) \in \mathbb{E}$ *and* $x_1 > x_0$ *be fixed. There is a constant* $C = C(x_0, h_0) < \infty$ *such that*:

$$\mathbf{E}\left(\left\{ \int_{x_0}^{x_1} \left\{ \int_0^{\Lambda_{(x_0,h_0)}(y)} \widetilde{\Lambda}_{(y,h)}(y+\varepsilon)\, dh \right\} dy \right\}^2 \right) \le C|x_1 - x_0|\varepsilon^{5/2} \tag{6.32}$$

Remark: The constant $C = C(x_0, h_0)$ can be chosen as $C'\big(h_0 + \mathbb{1}_{\{x_0<0\}} \sqrt{|x_0|}\big)$, where C' is an absolute constant.

Proof. Recall that $\mathscr{F}_z$ the σ-algebra generated by $\big(\Lambda_{(x,h)}(y) : x \le y \le z, h > 0\big)$ and for the extent of this proof we introduce the shorthand notation:

$$\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y) := \int_0^{\Lambda_{(x_0,h_0)}(y)} \widetilde{\Lambda}_{(y,h)}(y+\varepsilon)\, dh \tag{6.33}$$

Clearly $\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)$ is $\mathscr{F}_{y+\varepsilon}$-measurable and

$$\mathbf{E}\left(\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)\middle|\mathscr{F}_y\right) = 0 \tag{6.34}$$

We express the left hand side of (6.32):

$$\begin{aligned}
&\mathbf{E}\left(\left\{\int_{x_0}^{x_1}\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)\,dy\right\}^2\right)\\
&\qquad= 2\int_{x_0}^{x_1}\left\{\int_y^{x_1}\mathbf{E}\left(\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y')\right)dy'\right\}dy\\
&\qquad= 2\int_{x_0}^{x_1}\left\{\int_y^{x_1}\mathbf{E}\left(\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)\mathbf{E}\left(\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y')\middle|\mathscr{F}_{y+\varepsilon}\right)\right)dy'\right\}dy\\
&\qquad= 2\int_{x_0}^{x_1}\left\{\int_y^{(y+\varepsilon)\wedge x_1}\mathbf{E}\left(\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)\mathbf{E}\left(\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y')\middle|\mathscr{F}_{y+\varepsilon}\right)\right)dy'\right\}dy\\
&\qquad= 2\int_{x_0}^{x_1}\left\{\int_y^{(y+\varepsilon)\wedge x_1}\mathbf{E}\left(\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y')\right)dy'\right\}dy\\
&\qquad\le 2\int_{x_0}^{x_1}\left\{\int_y^{(y+\varepsilon)\wedge x_1}\mathbf{E}\left(\Gamma^{(\varepsilon)2}_{(x_0,h_0)}(y')\right)^{1/2}\mathbf{E}\left(\Gamma^{(\varepsilon)2}_{(x_0,h_0)}(y)\right)^{1/2}dy'\right\}dy\\
&\qquad\le 2\varepsilon|x_1-x_0|\sup_{x_0\le y\le x_1}\mathbf{E}\left(\Gamma^{(\varepsilon)2}_{(x_0,h_0)}(y)\right)
\end{aligned}\tag{6.35}$$

In the second line we use the fact that $\Gamma^{(\varepsilon)}_{(x_0,h_0)}(y)$ is $\mathscr{F}_{y+\varepsilon}$ measurable, in the third line we use (6.34).

Next we estimate the expectation on the right-hand side of (6.35).

$$\begin{aligned}
&\mathbf{E}\left(\Gamma^{(\varepsilon)2}_{(x_0,h_0)}(y)\middle|\mathscr{F}_y\right)\\
&\quad= \int_0^{\Lambda_{(x_0,h_0)}(y)}\int_0^{\Lambda_{(x_0,h_0)}(y)}\mathbf{E}\left(\widetilde{\Lambda}_{(y,h)}(y+\varepsilon)\widetilde{\Lambda}_{(y,h')}(y+\varepsilon)\right)dh\,dh'
\end{aligned}\tag{6.36}$$

We compute the right-hand side of (6.36). Let $z\mapsto R^1_{(y,h_1)}(z)$ and $z\mapsto R^2_{(y,h_2)}(z)$ be two independent (*non-coalescing*) RABs starting at 'time' y from level h_1, respectively h_2. Define their first collision time, as in (A.2):

$$\sigma^{(2)}_{y;h_1,h_2} := \inf\{z>y : R^1_{(y,h_1)}(z) = R^2_{(y,h_2)}(z)\} - y \tag{6.37}$$

Then

$$C^1_{(y,h_1)}(z) := R^1_{(y,h_1)}(z), \tag{6.38}$$

$$C^2_{(y,h_2)}(z) := R^2_{(y,h_2)}(z) + \mathbb{1}_{\{\sigma^{(2)}_{y;h_1,h_2}<z-y\}}\left(R^1_{(y,h_1)}(z) - R^2_{(y,h_2)}(z)\right) \tag{6.39}$$

have the joint law of two CRABs. Using the strong Markov property of $\left(R^1_{(y,h_1)}(\cdot), R^2_{(y,h_2)}(\cdot)\right)$ we get, for any $z \geq y$ (in the next displayed equations, we shall simply write σ instead of $\sigma^{(2)}_{y;h_1,h_2}$ for obvious typographical reasons).

$$\begin{aligned}
&\mathbf{E}\big(\big(\Lambda_{(y,h_1)}(z) - m_{(y,h_2)}(z)\big)\big(\Lambda_{(y,h_1)}(z) - m_{(y,h_2)}(z)\big)\big)\\
&\qquad = \mathbf{E}\big(\big(R^1_{(y,h_1)}(z) - m_{(y,h_2)}(z)\big)\big(R^2_{(y,h_2)}(z) - m_{(y,h_2)}(z)\big)\big)\\
&\qquad\quad + \mathbf{E}\big(\big(R^1_{(y,h_1)}(z) - m_{(y,h_1)}(z)\big)\big(R^1_{(y,h_1)}(z) - R^2_{(y,h_2)}(z)\big)\mathbb{1}_{\{\sigma<z-y\}}\big)\\
&\qquad = \mathbf{E}\Big(\big(R^1_{(y,h_1)}(z) - m_{(y+\sigma,R^1_{(y,h_1)}(y+\sigma))}(z)\big)^2\mathbb{1}_{\{\sigma<z-y\}}\Big)\\
&\qquad \leq \mathbf{E}\big((z-y-\sigma)\mathbb{1}_{\{\sigma<z-y\}}\big)\\
&\qquad \leq (z-y)\mathbf{P}\big(\sigma<z-y\big)\\
&\qquad \leq 2|z-y|\exp\left\{\frac{-(h_2-h_1)^2}{4|z-y|}\right\}\ . \qquad (6.40)
\end{aligned}$$

In the last step we used (A.15). From (6.36) and (6.40):

$$\begin{aligned}
&\mathbf{E}\left(\Gamma^{(\varepsilon)2}_{(x_0,h_0)}(y)\middle|\mathscr{F}_y\right)\\
&\leq 2\varepsilon\int_0^{\Lambda_{(x_0,h_0)}(y)}\int_0^{\Lambda_{(x_0,h_0)}(y)}\exp\{-(h_2-h_1)^2/(4\varepsilon)\}\,dh_1\,dh_2\\
&= 2\varepsilon^2\int_0^{\varepsilon^{-1/2}\Lambda_{(x_0,h_0)}(y)}\int_0^{\varepsilon^{-1/2}\Lambda_{(x_0,h_0)}(y)}\exp\{-(h_2-h_1)^2/4\}\,dh_1\,dh_2\\
&\leq 2\varepsilon^2\int_0^{\varepsilon^{-1/2}\Lambda_{(x_0,h_0)}(y)}\int_{-\infty}^{\infty}\exp\{-(h_2-h_1)^2/4\}\,dh_1\,dh_2\\
&= 4\sqrt{\pi}\Lambda_{(x_0,h_0)}(y)\varepsilon^{3/2} \qquad (6.41)
\end{aligned}$$

Inserting (6.41) in (6.35) yields (6.32)

□Lemma 6.3.

We are ready now to estimate $I_{2,2,1}$ defined in (6.24). Let $\eta > 0$ be fixed. For any $x_1 \geq x_0$,

$$\begin{aligned}
&\mathbf{P}(|(2\varepsilon)^{-1}I_{2,2,1}| > \eta)\\
&\quad \leq \mathbf{P}(\omega > x_1) + \mathbf{P}\left(\frac{1}{\varepsilon}\left|\int_{x_0}^{x_1}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\widetilde{\Lambda}_{(y,h)}(y+\varepsilon)\,dh\right\}dy\right| > \eta\right)\\
&\quad \leq \mathbf{P}(\omega > x_1) + \frac{1}{\eta^2\varepsilon^2}\mathbf{E}\left(\left\{\int_{x_0}^{x_1}\left\{\int_0^{\Lambda_{(x_0,h_0)}(y)}\widetilde{\Lambda}_{(y,h)}(y+\varepsilon)\,dh\right\}dy\right\}^2\right)\\
&\quad \leq C\frac{h_0 + \mathbb{1}_{\{x_0<0\}}\sqrt{|x_0|}}{\sqrt{2\pi|x_1|}} + C(x_0,h_0)|x_0-x_1|\eta^{-2}\varepsilon^{1/2} \qquad (6.42)
\end{aligned}$$

In the last step we used (6.32). Letting $x_1 \uparrow \infty$ and $\varepsilon \downarrow 0$ from (6.42) it follows that

$$\text{P-}\lim_{\varepsilon \downarrow 0} \frac{I_{2,2,1}}{2\varepsilon} = 0 \ . \tag{6.43}$$

Putting together (6.18), (6.20), (6.27), (6.31) and (6.43) we get indeed (6.15). As we already mentioned, (6.14) is derived by an identical reasoning applied to the backward lines. (6.14) and (6.15) together imply (6.10), or equivalently, (6.1).

□Theorem 6.1.

Remark: A similar result can be proved for deterministic (fixed) times $t \geq 0$, too:

$$\text{P-}\lim_{\varepsilon \downarrow 0} \int_0^t \frac{L_s(X_s + \varepsilon) - L_s(X_s - \varepsilon)}{2\varepsilon} \, ds = -X_t + \frac{1}{4} \left(\sup_{0 \leq s \leq t} X_s + \inf_{0 \leq s \leq t} X_s \right) \tag{6.44}$$

But the proof of (6.44) would require further extensive estimates which we avoid here.

7. Upcrossings and local variation

We are now first going to state and derive an approximation theorem for the occupation-time densities using upcrossings (in the same spirit as those for the local times of semimartingales, see e.g. Revuz-Yor [RY], Chapter VI).

For $x \in \mathbb{R}$ and $\varepsilon > 0$, define by induction the sampling times

$$\tau_0(x, \varepsilon) := 0, \tag{7.1}$$

$$\sigma_n(x, \varepsilon) := \inf\{t \geq \tau_{n-1}(x, \varepsilon) : X_t = x\}, \qquad n \geq 1, \tag{7.2}$$

$$\tau_n(x, \varepsilon) := \inf\{t \geq \sigma_n(x, \varepsilon) : X_t = x + \varepsilon\}, \qquad n \geq 1 \ . \tag{7.3}$$

Then, the number of upcrossings from x to $x + \varepsilon$ before time t is defined by

$$U_t^{x \uparrow (x+\varepsilon)} := \sup\{n \geq 0 : \tau_n(x, \varepsilon) \leq t\} \ . \tag{7.4}$$

Finally, for all $(x, h) \in \mathbb{E}$, we put

$$U_{(x,h)}^{\varepsilon} := U_{T(x,h)}^{x \uparrow (x+\varepsilon)} \ . \tag{7.5}$$

Clearly, the definition of X shows that almost surely, for all $(x, h) \in \mathbb{E}$

$$U_{(x,h)}^{\varepsilon} = \#\{\Lambda_{(x,h')}(x + \varepsilon) \ : \ h' \in (0, h]\} - 1 \ . \tag{7.6}$$

We will derive the following result:

Lemma 7.1. (i) *For all* $(x, h) \in \mathbb{E}$,

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}(\sqrt{\varepsilon} U^{\varepsilon}_{(x,h)}) = \frac{h}{\sqrt{\pi}} \quad . \tag{7.7}$$

The convergence is uniform in compact subdomains of $\mathbb{E}$.

(ii) *There exists a constant* $C < \infty$ *such that for all* $(x, h) \in \mathbb{E}$ *and* $\varepsilon \in (0, 1)$

$$\mathbf{Var}(\sqrt{\varepsilon} U^{\varepsilon}_{(x,h)}) \leq Ch\sqrt{\varepsilon} \ . \tag{7.8}$$

By applying Chebyshev's inequality, these estimates immediately imply the following approximation result:

Proposition 7.2. *For all* $(x, h) \in \mathbb{E}$,

$$\text{P-}\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} U^{\varepsilon}_{(x,h)} = \frac{h}{\sqrt{\pi}} \quad . \tag{7.9}$$

Proof of Lemma 7.1. By scaling, it suffices to consider $U^{\varepsilon}_x := U^{\varepsilon}_{(x,1)}$ only. Using (7.6) and monotonicity and left-continuity of $h \mapsto \Lambda_{(x,h)}(x + \varepsilon)$, one has

$$U^{\varepsilon}_x = \lim_{p \uparrow \infty} U^{\varepsilon,p}_x \quad , \tag{7.10}$$

where

$$U^{\varepsilon,p}_x := \sum_{j=0}^{2^p - 1} 1\!\!1 \left\{ \Lambda_{(x,j2^{-p})}(x + \varepsilon) \neq \Lambda_{(x,(j+1)2^{-p})}(x + \varepsilon) \right\} \ . \tag{7.11}$$

For the extent of the present proof we introduce the following shorthand notation: for $p \in \mathbb{N}$ and $j < j'$ in $\mathbb{N}$ we denote the events

$$\mathscr{A}_{p,j,j'} = \mathscr{A}_{p,j,j'}(x, \varepsilon) := \left\{ \Lambda_{(x,j2^{-p})}(x + \varepsilon) < \Lambda_{(x,j'2^{-p})}(x + \varepsilon) \right\} \tag{7.12}$$

$$\mathscr{A}^c_{p,j,j'} = \mathscr{A}^c_{p,j,j'}(x, \varepsilon) := \left\{ \Lambda_{(x,j2^{-p})}(x + \varepsilon) = \Lambda_{(x,j'2^{-p})}(x + \varepsilon) \right\} \tag{7.13}$$

Note that for all $x \in \mathbb{R}$ and $\varepsilon > 0$ fixed, the sequence $U^{\varepsilon,p}_x$ is non-decreasing with p, and (as it is integer-valued and bounded) it is stationary for large enough p. Hence,

$$\mathbf{E}(U^{\varepsilon}_x) = \lim_{p \uparrow \infty} \mathbf{E}(U^{\varepsilon,p}_x) \quad \text{and} \quad \mathbf{E}((U^{\varepsilon}_x)^2) = \lim_{p \uparrow \infty} \mathbf{E}((U^{\varepsilon,p}_x)^2) \ . \tag{7.14}$$

Let $\eta \in (0, 1)$ be fixed. Using (7.11) we write

$$\mathbf{E}(U^{\varepsilon,p}_x) = \sum_{j=0}^{\lfloor 2^p \eta \rfloor} \mathbf{P}(\mathscr{A}_{p,j,j+1}) + \sum_{j=\lceil 2^p \eta \rceil}^{2^p - 1} \mathbf{P}(\mathscr{A}_{p,j,j+1}) \tag{7.15}$$

Using (A.6) of Lemma A.1 we easily estimate the first sum on the right hand side of (7.15):

$$\sum_{j=0}^{\lfloor 2^p\eta\rfloor} \mathbf{P}(\mathscr{A}_{p,j,j+1}) \le \eta\sqrt{\frac{2}{\pi\varepsilon}} \tag{7.16}$$

Next, using the uniform convergence of Lemma A.3. we get the limit of the second sum on the right hand side of (7.15):

$$\lim_{p\uparrow\infty} \sum_{j=\lceil 2^p\eta\rceil}^{2^p-1} \mathbf{P}(\mathscr{A}_{p,j,j+1}) = \frac{1-\eta}{\sqrt{\pi\varepsilon}} \tag{7.17}$$

Putting together (7.15), (7.16) and (7.17) and letting $\eta \downarrow 0$ after $p \uparrow \infty$ we get exactly (7.7).

To derive (ii), note that

$$(U_x^{\varepsilon,p})^2 = U_x^{\varepsilon,p} + 2\sum_{j=0}^{2^p-2} \sum_{j'=j+1}^{2^p-1} 1\!\!1\left\{\mathscr{A}_{p,j,j+1} \cap \mathscr{A}_{p,j',j'+1}\right\} \tag{7.18}$$

and

$$(\mathbf{E}(U_x^{\varepsilon,p}))^2 \ge 2\sum_{j=0}^{2^p-2} \sum_{j'=j+1}^{2^p-1} \mathbf{P}(\mathscr{A}_{p,j,j+1})\mathbf{P}(\mathscr{A}_{p,j',j'+1}) \ . \tag{7.19}$$

Hence,

$$\begin{aligned}\mathbf{Var}(U_x^{\varepsilon,P}) \le \mathbf{E}(U_x^{\varepsilon,p}) + 2\sum_{j=0}^{2^p-2} \sum_{j'=j+1}^{2^p-1} \{&\mathbf{P}(\mathscr{A}_{p,j,j+1} \cap \mathscr{A}_{p,j',j'+1}) \\ &- \mathbf{P}(\mathscr{A}_{p,j,j+1})\mathbf{P}(\mathscr{A}_{p,j',j'+1})\}\end{aligned} \tag{7.20}$$

We claim that for all $j < j'$,

$$\begin{aligned}&\mathbf{P}(\mathscr{A}_{p,j,j+1} \cap \mathscr{A}_{p,j',j'+1}) - \mathbf{P}(\mathscr{A}_{p,j,j+1})\mathbf{P}(\mathscr{A}_{p,j',j'+1}) \\ &\quad \le \mathbf{P}(\mathscr{A}_{p,j,j+1} \cap \mathscr{A}^c_{p,j+1,j'} \cap \mathscr{A}_{p,j',j'+1}) \ .\end{aligned} \tag{7.21}$$

Indeed:

$$\begin{aligned}&\mathbf{P}(\mathscr{A}_{p,j,j+1} \cap \mathscr{A}_{p,j',j'+1}) - \mathbf{P}(\mathscr{A}_{p,j,j+1} \cap \mathscr{A}^c_{p,j+1,j'} \cap \mathscr{A}_{p,j',j'+1}) \\ &\quad = \mathbf{P}(\mathscr{A}_{p,j,j+1} \cap \mathscr{A}_{p,j+1,j'} \cap \mathscr{A}_{p,j',j'+1}) \\ &\quad \le \mathbf{P}(\mathscr{A}_{p,j,j+1})\mathbf{P}(\mathscr{A}_{p,j',j'+1})\end{aligned} \tag{7.22}$$

where, in the last step we use the fact that the forward lines involved are independent as long as they do not meet and coalesce.

Note that for all $j \geq 0$ and p fixed, there exists exactly one $j' > j$ such that

$$\Lambda_{(x,(j+1)2^{-p})}(x+\varepsilon) = \Lambda_{(x,j'2^{-p})}(x+\varepsilon) < \Lambda_{(x,(j'+1)2^{-p})}(x+\varepsilon) \ . \tag{7.23}$$

Hence, for all fixed $p \geq 1$ and $j \in \{0, \dots, 2^p - 2\}$,

$$\sum_{j'=j+1}^{2^p-1} \mathbf{P}(\mathscr{A}_{p,j,j+1} \cap \mathscr{A}^c_{p,j+1,j'} \cap \mathscr{A}_{p,j',j'+1}) \leq \mathbf{P}(\mathscr{A}_{p,j,j+1}) \ . \tag{7.24}$$

Finally, combining (7.20), (7.21) and (7.24) we get

$$\mathbf{Var}(U_x^{\varepsilon,p}) \leq \mathbf{E}(U_x^{\varepsilon,p}) + 2\mathbf{E}(U_x^{\varepsilon,p}) = 3(\varepsilon\pi)^{-1/2} + o(\varepsilon^{-1/2}) \ . \tag{7.25}$$

Hence (7.8). □Lemma 7.1

We are now going to deduce from Proposition 7.2 the approximation result for $L_t(x)$ (where t is a fixed time):

Proposition 7.3. *For all fixed* $t \geq 0$ *and* $x \in \mathbb{R}$,

$$\text{P-}\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} U_t^{x\uparrow(x+\varepsilon)} = \frac{L_t(x)}{\sqrt{\pi}} \ . \tag{7.26}$$

Proof. Suppose that $t > 0$ and $x \in \mathbb{R}$ are fixed. Suppose $(h_n)_{n\geq 1}$ is a dense deterministic sequence in $[0, \infty)$ with $h_1 = 0$, and for $p \geq 1$, put $S_p = \{h_1, \dots, h_p\}$. Then for all $\alpha > 0$ and $\eta > 0$, there exists $p < \infty$ such that

$$\mathbf{P}(S_p \cap (L_t(x) - \alpha, L_t(x)] = \emptyset \text{ or } S_p \cap (L_t(x), L_t(x) + \alpha) = \emptyset) < \eta \ . \tag{7.27}$$

Let us now fix p in such a way that (7.27) holds. Proposition 7.2 then implies that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and for all $j \in \{1, \dots, p\}$,

$$\mathbf{P}\left(\left|\sqrt{\pi\varepsilon} U^{\varepsilon}_{(x,h_j)} - h_j\right| > \alpha\right) < \eta/p \ . \tag{7.28}$$

Hence, combining this with (7.27) and using the fact that $t \mapsto U_t^{x\uparrow(x+\varepsilon)}$ is non-decreasing, implies that

$$\mathbf{P}\left(\left|\sqrt{\pi\varepsilon} U_t^{x\uparrow(x+\varepsilon)} - L_t(x)\right| > 2\alpha\right) < 2\eta \ . \tag{7.29}$$

This completes the proof of Proposition 7.3.

□Proposition 7.3.

A similar method is used to derive the following results, that loosely speaking state that X has a finite variation of order $3/2$: Suppose that for $\varepsilon > 0$ fixed we define by induction, the sequence of sampling times

$$\theta_0^\varepsilon := 0, \qquad \theta_n^\varepsilon := \inf\{t > \theta_{n-1}^\varepsilon : |X_t - X_{\theta_{n-1}^\varepsilon}| = \varepsilon\} \ . \tag{7.30}$$

We also define, for all $t > 0$,

$$N_t^\varepsilon := \sup\{n \geq 0 : \theta_n^\varepsilon \leq t\} \ . \tag{7.31}$$

Theorem 7.4.

(i) *For all* $(x, h) \in \mathbb{E}$,

$$\text{P-}\lim_{\varepsilon \downarrow 0} \varepsilon^{3/2} N_{T(x,h)}^\varepsilon = \frac{2T(x,h)}{\sqrt{\pi}} \ . \tag{7.32}$$

(ii) *For all* $t \geq 0$,

$$\text{P-}\lim_{\varepsilon \downarrow 0} \sum_{n \leq N_t^\varepsilon} \left| X_{\theta_n^\varepsilon} - X_{\theta_{n-1}^\varepsilon} \right|^{3/2} \equiv \text{P-}\lim_{\varepsilon \downarrow 0} \varepsilon^{3/2} N_t^\varepsilon = \frac{2t}{\sqrt{\pi}} \ . \tag{7.33}$$

Proof. (i) Define the number of downcrossings $D_t^{y \downarrow y-\varepsilon}$ from level x to $x - \varepsilon$ before time t by X, analogously to the number of upcrossings. Then,

$$N_{T(x,h)}^\varepsilon = \sum_{n \in \mathbb{Z}} \left\{ U_{T(x,h)}^{n\varepsilon \uparrow (n+1)\varepsilon} + D_{T(x,h)}^{(n+1)\varepsilon \downarrow n\varepsilon} \right\} \ , \tag{7.34}$$

and, clearly, for all $\varepsilon > 0$ and $t > 0$,

$$\left| U_{T(x,h)}^{n\varepsilon \uparrow (n+1)\varepsilon} - D_{T(x,h)}^{(n+1)\varepsilon \downarrow n\varepsilon} \right| = \begin{cases} 1 & \text{if } n \in [0, \lfloor x/\varepsilon \rfloor] \cup [\lceil x/\varepsilon \rceil, 0] \\ \\ 0 & \text{if } n \notin [0, \lfloor x/\varepsilon \rfloor] \cup [\lceil x/\varepsilon \rceil, 0] \end{cases} \tag{7.35}$$

Hence

$$\begin{aligned} &\left| N_{T(x,h)}^\varepsilon - 2 \sum_{n \geq \lceil x/\varepsilon \rceil} U_{T(x,h)}^{n\varepsilon \uparrow (n+1)\varepsilon} - 2 \sum_{n \leq \lfloor x/\varepsilon \rfloor} D_{T(x,h)}^{n\varepsilon \downarrow (n-1)\varepsilon} \right| \\ &\qquad \leq 2 U_{T(x,h)}^{\lfloor x/\varepsilon \rfloor \varepsilon \uparrow \lceil x/\varepsilon \rceil \varepsilon} + \varepsilon^{-1} |x| \ . \end{aligned} \tag{7.36}$$

But

$$U_{T(x,h)}^{\lfloor x/\varepsilon \rfloor \varepsilon \uparrow \lceil x/\varepsilon \rceil \varepsilon} \leq \min \left\{ U_{T(x,h)}^{x \uparrow \lceil x/\varepsilon \rceil \varepsilon}, 1 + D_{T(x,h)}^{x \downarrow \lfloor x/\varepsilon \rfloor \varepsilon} \right\} \ . \tag{7.37}$$

Now, clearly

$$\max\{x - \lfloor x/\varepsilon \rfloor \varepsilon, \lceil x/\varepsilon \rceil \varepsilon - x\} \geq \varepsilon/2 \tag{7.38}$$

and, due to Lemma 7.1, (7.37) and (7.38) readily imply

$$\mathbf{P}\text{-}\lim_{\varepsilon\downarrow 0} \varepsilon^{3/2} U_{T(x,h)}^{\lfloor x/\varepsilon\rfloor\varepsilon\uparrow\lceil x/\varepsilon\rceil\varepsilon} = 0 \ . \tag{7.39}$$

Obviously, we also have $\lim_{\varepsilon\downarrow 0} \varepsilon^{3/2}\varepsilon^{-1}x = 0$. So that it suffices to study the asymptotic behaviour of

$$\varepsilon^{3/2} \sum_{n\geq\lceil x/\varepsilon\rceil} U_{T(x,h)}^{n\varepsilon\uparrow(n+1)\varepsilon} \quad \text{and that of} \quad \varepsilon^{3/2} \sum_{n\leq\lfloor x/\varepsilon\rfloor} D_{T(x,h)}^{n\varepsilon\downarrow(n-1)\varepsilon} \ . \tag{7.40}$$

Actually, it is sufficient to show that, for any fixed $K > x$

$$\mathbf{P}\text{-}\lim_{\varepsilon\downarrow 0} \varepsilon^{3/2} \sum_{n=\lceil x/\varepsilon\rceil}^{\lfloor K/\varepsilon\rfloor} U_{T(x,h)}^{n\varepsilon\uparrow(n+1)\varepsilon} = \pi^{-1/2} \int_x^K \Lambda_{(x,h)}(y)\, dy \tag{7.41}$$

Then, letting $K \to \infty$ (similarly to the argument in (6.42)) we get

$$\mathbf{P}\text{-}\lim_{\varepsilon\downarrow 0} \varepsilon^{3/2} \sum_{n=\lceil x/\varepsilon\rceil}^{\infty} U_{T(x,h)}^{n\varepsilon\uparrow(n+1)\varepsilon} = \pi^{-1/2} \int_x^\infty \Lambda_{(x,h)}(y)\, dy \ . \tag{7.42}$$

By symmetry we get a similar result for the other sum of the down-crossings.

We are going to prove now (7.41): Recall that we denote by $\mathscr{F}_z$ the sigma algebra generated by $\{\Lambda_{(x,h)}(y) : x \leq y \leq z,\ h > 0\}$. Let $n \geq \lceil x/\varepsilon\rceil$ be fixed, then *conditionally on* $\mathscr{F}_{n\varepsilon}$ the law of $U_{T(x,h)}^{n\varepsilon\uparrow(n+1)\varepsilon}$ is identical to that of $U^{\varepsilon}_{(n\varepsilon,\Lambda_{(x,h)}(n\varepsilon))}$.

Let us now fix $K > x$. Then, the previous observation combined with the *uniform convergence* proved in Lemma 7.1-(i) shows that

$$\lim_{\varepsilon\downarrow 0} \varepsilon \sum_{n=\lceil x/\varepsilon\rceil}^{\lfloor K/\varepsilon\rfloor} \left| \mathbf{E}\Big(\varepsilon^{1/2} U_{T(x,h)}^{n\varepsilon\uparrow(n+1)\varepsilon} \Big| \mathscr{F}_{n\varepsilon}\Big) - \frac{1}{\sqrt{\pi}} \Lambda_{(x,h)}(n\varepsilon) \right| = 0 \tag{7.43}$$

On the other hand, from Lemma 7.1-(ii) we get

$$\begin{aligned} \sum_{n=\lceil x/\varepsilon\rceil}^{\lfloor K/\varepsilon\rfloor} \mathbf{Var}\Big(\varepsilon^{3/2} U_{T(x,h)}^{n\varepsilon\uparrow(n+1)\varepsilon} \Big| \mathscr{F}_{n\varepsilon}\Big) &\leq C\varepsilon^{5/2} \sum_{n=\lceil x/\varepsilon\rceil}^{\lfloor K/\varepsilon\rfloor} \Lambda_{(x,h)}(n\varepsilon) \\ &\leq C(K-x)\varepsilon^{3/2} \sup_{y\geq x} \Lambda_{(x,h)}(y) \ . \end{aligned} \tag{7.44}$$

(7.43) and (7.44) imply

$$\mathbf{P}\text{-}\lim_{\varepsilon\downarrow 0} \left\{ \varepsilon^{3/2} \sum_{n=\lceil x/\varepsilon\rceil}^{\lfloor K/\varepsilon\rfloor} U_{T(x,h)}^{n\varepsilon\uparrow(n+1)\varepsilon} - \frac{\varepsilon}{\sqrt{\pi}} \sum_{n=\lceil x/\varepsilon\rceil}^{\lfloor K/\varepsilon\rfloor} \Lambda_{(x,h)}(n\varepsilon) \right\} = 0 \ . \tag{7.45}$$

But obviously, almost surely,

$$\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\sqrt{\pi}}\sum_{n=\lceil x/\varepsilon\rceil}^{\lfloor K/\varepsilon\rfloor}\Lambda_{(x,h)}(n\varepsilon)=\pi^{-1/2}\int_x^K\Lambda_{(x,h)}(y)\,dy\ . \tag{7.46}$$

Combining (7.45) and (7.46) yields (7.41) and, eventually, (7.32).

(ii) Suppose now that $t>0$ is fixed (the case $t=0$ is trivial). The set $T(\tilde{\mathbb{E}})$ is a.s. dense in $\mathbb{R}_+$ (this is a straightforward consequence of the fact that for any $(x,h),(x',h')$ in $\mathbb{E}$ such that $T(x,h)>T(x',h')$, $D(x,h)\backslash D(x',h')$ contains an open set and therefore a point in $\tilde{\mathbb{E}}$). For all $p\geq 1$, we now define

$$S_p:=\{T(\tilde{x}_j,\tilde{h}_j):j=1,2,\ldots,p\}\ . \tag{7.47}$$

The density of $T(\tilde{\mathbb{E}})$ implies readily that for any $\alpha>0$ and $\eta>0$, there exists $p\geq 1$, such that

$$\mathbf{P}(S_p\cap(t-\alpha,t)=\emptyset \text{ or } S_p\cap(t,t+\alpha)=\emptyset)<\eta\ . \tag{7.48}$$

But if α, η and p are fixed, Theorem 7.4 (i) shows that there exists $\varepsilon_0>0$ such that for all $\varepsilon<\varepsilon_0$ and for all $j\in\{1,\ldots,p\}$,

$$\mathbf{P}\left(\left|\frac{\varepsilon^{3/2}\sqrt{\pi}N^{\varepsilon}_{T(\tilde{x}_j,\tilde{h}_j)}}{2}-T(\tilde{x}_j,\tilde{h}_j)\right|<\alpha\right)<\eta/p. \tag{7.49}$$

Combining this with (7.48) and using the fact that $t\mapsto N^{\varepsilon}_t$ is a non-decreasing function of t, implies readily (7.33).

□Theorem 7.4.

Remark. [Further pathwise properties of X]

Using the Markov property (for instance at rational times), it is then possible to make the link between exceptional times for X (times of monotonicity for instance) and the points in $\mathbb{E}$ of exceptional topological type (for the system of lines; see section 2.2). More precisely, if we say that $t>0$ is a time of monotonicity for X if there exists $\varepsilon>0$ such that either for all $u\in(0,\varepsilon)$, $X_{t-u}<X_t<X_{t+u}$ or for all $u\in(0,\varepsilon)$, $X_{t-u}>X_t>X_{t+u}$, then the set of points (X_t,H_t) where t is a time of monotonicity for X corresponds to the set of points of topological type $[1,1]$. Similarly, points of topological type $[0,2]$ and $[2,0]$ correspond to times of local extrema for X, and the points of multiplicity 2 correspond to end-times of excursions away from a point. This can be used to show *existence* of points of topological type [2,0], [1,1] and [0,2].

8. Construction and properties of Λ: proofs

In this section, we give detailed proofs of Theorem 2.1 and Proposition 2.2. As already mentioned, this construction of system of forward lines is very similar to that of Arratia [A1, A2]. This section is divided into two subsections: In subsection 8.1 we complete the proof of Theorem 2.1, i.e. we prove in turn (i1), (i2) and (ii) (recall that (i3) and (i4) were proven in Section 2.1). In subsection 8.2, we complete the proof of Proposition 2.2.

8.1. Proof of Theorem 2.1

Proof of Theorem 2.1 (i1). Define the system of forward lines Λ exactly as in Equation (2.6) (in section 2.1).

We first state and derive a useful Lemma:

Lemma 8.1. *For all fixed* $(x, h) \in \mathbb{E}$, $\Lambda_{(x,h)}(\cdot)$ *is almost surely well-defined on* $[x, \infty)$, and

(i) *There exists a deterministic sequence* $(n(k), k \geq 1)$ *such that* $\tilde{x}_{n(k)} < x$, $\lim_{k\uparrow\infty} \tilde{x}_{n(k)} = x$, $\lim_{k\uparrow\infty} \tilde{h}_{n(k)} = h$, *and for all* $\varepsilon > 0$ *there exists a constant* $C = C(\varepsilon) < \infty$ *such that for all* $k \geq 1$

$$\mathbf{P}\Big(\Lambda_{(x,h)}(y) = F_{n(k)}(y) \quad \text{for all} \quad y \geq x + \varepsilon\Big) \geq 1 - C\frac{2^{-k}}{\sqrt{\varepsilon}} \ . \tag{8.1}$$

(ii) *Almost surely for all* $\varepsilon > 0$, *there exists a integer* $k_0 = k_0(\varepsilon)$ *such that for all* $k \geq k_0$,

$$\Lambda_{(x,h)}(y) = F_{n(k)}(y) \quad \text{for all} \quad y \geq x + \varepsilon \ . \tag{8.2}$$

The random integer $k_0(\varepsilon)$ is $\mathscr{F}_{x_0+\varepsilon}$*-measurable.*

(iii) *Almost surely,*

$$\Lambda_{(x,h)}(x) = \lim_{y\downarrow x} \Lambda_{(x,h)}(y) = h \ . \tag{8.3}$$

Remark: We will see in the next subsections that much more is true, namely that these results have uniform generalisations almost surely, to all $(x, h) \in \mathbb{E}$ simutaneously.

Proof. (i) We are going to squeeze (x, h) between two families of lines $F_{n(k)}$ and $F_{m(k)}$ started before x, such that $F_{n(k)}$ and $F_{m(k)}$ have small probability to have coalesced before x or not to coalesce before $x + \varepsilon$. More precisely, we choose two sequences of points $(\tilde{x}_{n(k)}, \tilde{h}_{n(k)})_{k\geq 1}$ and $(\tilde{x}_{m(k)}, \tilde{h}_{m(k)})_{k\geq 1}$ in $\mathbb{E}$ in such a way that:

$$\tilde{x}_{n(k)} = \tilde{x}_{m(k)} \in (x - 5^{-k}, x) \tag{8.4}$$

and

$$h - 2\cdot 2^{-k} < \tilde{h}_{n(k)} < h - 2^{-k} < h < h + 2^{-k} < \tilde{h}_{m(k)} < h + 2\cdot 2^{-k} \tag{8.5}$$

for all $k \geq 1$ and define the following three events:

$$\mathscr{A}_k := \Big\{\{F_{n(k)}(y) : y \in [\tilde{x}_{n(k)}, x + 5^{-k}]\} \not\subset [h - 3 \cdot 2^{-k}, h]\Big\} \tag{8.6}$$

$$\mathscr{B}_k := \Big\{\{F_{m(k)}(y) : y \in [\tilde{x}_{m(k)}, x + 5^{-k}]\} \not\subset [h, h + 3 \cdot 2^{-k}]\Big\} \tag{8.7}$$

$$\mathscr{C}_{k,\varepsilon} := \Big\{F_{n(k)}(x + \varepsilon) \neq F_{m(k)}(x + \varepsilon)\Big\} \ . \tag{8.8}$$

Elementary estimates on Brownian hitting times yield

$$\mathbf{P}(\mathscr{A}_k) \leq 2\exp\left\{-\frac{5^k}{4^{k+1}}\right\}, \qquad \mathbf{P}(\mathscr{B}_k) \leq 2\exp\left\{-\frac{5^k}{4^{k+1}}\right\} \ . \tag{8.9}$$

Using the upper bound (A.6) from Lemma A.1 we also find

$$\mathbf{P}\big(\mathscr{C}_{k,\varepsilon}\big) \leq C_2' \frac{4 \cdot 2^{-k}}{\sqrt{2\varepsilon}} \tag{8.10}$$

Now, clearly

$$\big\{\Lambda_{(x,h)}(y) \neq F_{n(k)}(y) \quad \text{for some} \quad y \geq x + \varepsilon\big\} \subset \mathscr{A}_k \cup \mathscr{B}_k \cup \mathscr{C}_{k.\varepsilon} \tag{8.11}$$

and (8.9)–(8.11) imply (8.1).

(ii) follows from (8.1) by a simple Borel-Cantelli argument. The fact that the random integer $k_0(\varepsilon)$ is $\mathscr{F}_{x_0+\varepsilon}$-measurable also follows from the proof of (8.1).

Remark: Note that this proof also shows the following useful fact: For all $n \geq 0$, almost surely, for all $y \geq \tilde{x}_n$,

$$\Lambda_{(\tilde{x}_n, \tilde{h}_n)}(y) = F_n(y) \ . \tag{8.12}$$

(iii) Note that

$$\{\sup\{\big|\Lambda_{(x,h)}(y) - h\big| : y \in [x, x + 5^{-k}]\} \geq 3 \cdot 2^{-k}\} \subset \mathscr{A}_k \cup \mathscr{B}_k \tag{8.13}$$

Applying again a Borel-Cantelli argument, from (8.13) and (8.9) we conclude that almost surely there exists a (random) $k_1 < \infty$ such that for all $k \geq k_1$

$$\sup\{\big|\Lambda_{(x,h)}(y) - h\big| : y \in [x, x + 5^{-k}]\} < 3 \cdot 2^{-k} \ . \tag{8.14}$$

This implies (8.3).

$\square$Lemma 8.1.

Now we are ready to prove Theorem 2.1 (i1): Let $(x_1, h_1), \ldots, (x_p, h_p)$ be fixed points in $\mathbb{E}$.

From Lemma 8.1-(ii), it follows that for all $\varepsilon > 0$, for all $j \in \{1, \ldots, p\}$, almost surely $[x_j + \varepsilon, \infty) \ni y \mapsto \Lambda_{(x_j, h_j)}(y)$ is continuous. Combining this with Lemma 8.1-(iii) shows that almost surely for all

$j \in \{1, \ldots, p\}$ $[x_j, \infty) \ni y \mapsto \Lambda_{(x_j,h_j)}(y)$ is continuous and that $\Lambda_{(x_j,h_j)}(x_j) = h_j$. So, it only remains to prove that the finite dimensional distributions of $\left(\Lambda_{(x_1,h_1)}(\cdot), \ldots, \Lambda_{(x_p,h_p)}(\cdot)\right)$ are those of a finite FICRAB starting from $\left((x_j, h_j)\right)_{j\in\{1,\ldots,p\}}$.

Let us choose the sequences $(\tilde{x}_{n_1(k)}, \tilde{h}_{n_1(k)})_{k\geq 1}, \ldots, (\tilde{x}_{n_p(k)}, \tilde{h}_{n_p(k)})_{k\geq 1}$, as in Lemma 8.1 (i) (corresponding respectively to the points $(x_1, h_1), \ldots, (x_p, h_p)$).

Fix an integer $r \geq 1$, a map $j : \{1, \ldots, r\} \to \{1, \ldots, p\}$, r 'time'-variables $y_i > x_{j(i)}$ and r intervals (a_i, b_i), $i = 1, \ldots, r$ and define the events

$$\widetilde{\mathscr{D}} := \left\{\Lambda_{(x_{j(i)},h_{j(i)})}(y_i) \in (a_i, b_i) : i = 1, \ldots, r\right\} \tag{8.15}$$

and

$$\mathscr{D}_k := \left\{F_{n_{j(i)}(k)}(y_i) \in (a_i, b_i) : i = 1, \ldots, r\right\} \tag{8.16}$$

Note that due to Lemma 8.1 (iii) we do not need to consider the possibility of $y_i = x_{j(i)}$ for some $i = 1, \ldots, r$. Also, for a finite FICRAB $(C_{(x_1,h_1)}(\cdot), \ldots, C_{(x_p,h_p)}(\cdot))$ starting from $\left((x_1, h_1), \ldots, (x_p, h_p)\right)$, define the event

$$\mathscr{D} := \left\{C_{(x_{j(i)},h_{j(i)})}(y_i) \in (a_i, b_i) : i = 1, \ldots, r\right\} \tag{8.17}$$

Choose $\varepsilon < \min\{y_i - x_{j(i)} : i = 1, \ldots, r\}$ and denote

$$\mathscr{E}_{k,\varepsilon} := \left\{\Lambda_{(x_j,h_j)}(y_j) \neq F_{n_j(k)}(y_j) \quad \text{for some } j = 1, \ldots, p \text{ and } y_j \geq x_j + \varepsilon\right\} \tag{8.18}$$

Then, from Lemma 8.1 (i) it follows that

$$\mathbf{P}(\mathscr{E}_{k,\varepsilon}) \leq Cp\frac{2^{-k}}{\sqrt{\varepsilon}} \quad . \tag{8.19}$$

Now, clearly

$$|\mathbf{P}(\widetilde{\mathscr{D}}) - \mathbf{P}(\mathscr{D})| \leq |\mathbf{P}(\widetilde{\mathscr{D}}) - \mathbf{P}(\mathscr{D}_k)| + |\mathbf{P}(\mathscr{D}) - \mathbf{P}(\mathscr{D}_k)| \quad . \tag{8.20}$$

and

$$|\mathbf{P}(\widetilde{\mathscr{D}}) - \mathbf{P}(\mathscr{D}_k)| \leq \mathbf{P}(\mathscr{E}_{k,\varepsilon}) \quad . \tag{8.21}$$

On the other hand, since $\lim_{k\uparrow\infty}(\tilde{x}_{n_j(k)}, \tilde{h}_{n_j(k)}) = (x_j, h_j)$, $j = 1, \ldots, p$, we also have

$$\lim_{k\uparrow\infty} \mathbf{P}(\mathscr{D}_k) = \mathbf{P}(\mathscr{D}) \quad . \tag{8.22}$$

Letting $k \uparrow \infty$, (8.19)–(8.22) imply

$$\mathbf{P}(\widetilde{\mathscr{D}}) = \mathbf{P}(\mathscr{D}) \quad . \tag{8.23}$$

That is: the finite dimensional distributions of $\big(\Lambda_{(x_1,h_1)}(\cdot),\dots,\Lambda_{(x_p,h_p)}(\cdot)\big)$ are those of a finite FICRAB starting from $\big((x_j, h_j)\big)_{j\in\{1,\dots,p\}}$.

□Theorem 2.1(i1)

For convenience, we define for all $x \geq 0$,

$$M'(x) = \{F_n(x) : n \geq 0 \text{ and } \tilde{x}_n < x\} \ . \tag{8.24}$$

The next Lemma is used in the proof of the remaining parts of Theorem 2.1:

Lemma 8.2. *Almost surely, for all $y \in \mathbb{R}$, $M'(y)$ is dense in $\mathbb{R}_+$.*

Remark: This implies immediately (2.7) i.e. that $\Lambda_{(x,h)}(y)$ is well-defined for all $(x,h,y) \in \mathbb{F}^+$ simultaneously.

Proof of Lemma 8.2. It suffices to show that for any fixed dyadic numbers $K > 0$ and $h > \alpha > 0$, almost surely for all $x \in (-K,K]$,

$$M'(x) \cap (h-\alpha, h+\alpha) \neq \emptyset \ . \tag{8.25}$$

Let us fix $K > 0$ and $h > \alpha > 0$ in $\mathbb{D}_+$. For all $k \geq 1$, $j \in \mathbb{Z}$, $\Lambda_{(jK2^{-k},h)}$ is a RAB, so that elementary estimates on Brownian hitting times yield that for all integers $-2^k \leq j < 2^k$

$$\mathbf{P}\left(\sup_{y\in[jK2^{-k},(j+1)K2^{-k}]} \left|\Lambda_{(jK2^{-k},h)}(y) - h\right| \geq \alpha\right) \leq 2\exp\left\{\frac{-\alpha^2 2^k}{2K}\right\} \tag{8.26}$$

Note that for all $y \in (-K,K]$ and $j \in \mathbb{Z}$ such that $jK2^{-k} < y \leq (j+1)K2^{-k}$,

$$\Lambda_{(jK2^{-k},h)}(y) \in M'(y) \tag{8.27}$$

as $jK2^{-k}$ and h are dyadics. Hence, for any integer $k \geq 1$,

$$\mathbf{P}(\exists y \in (-K,K] : M'(y) \cap (h-\alpha, h+\alpha) = \emptyset) \leq K2^{k+1}\exp\left\{\frac{-\alpha^2 2^k}{2K}\right\} \ . \tag{8.28}$$

Letting $k \uparrow \infty$ shows that almost surely, $M'(y) \cap (h-\alpha, h+\alpha) \neq \emptyset$ for all $y \in (-K,K]$ and the result follows.

□Lemma 8.2

Proof of Theorem 2.1 (i2). Lemma 8.2 also implies immediately that almost surely, for all $(x,h) \in \mathbb{E}$, $\Lambda_{(x,h)}(x) = h$.

□Theorem 2.1 (i2)

Proof of Theorem 2.1 (ii). Assume that $\mathbb{F}^+ \ni (x,h,y) \mapsto \Lambda'_{(x,h)}(y)$ is a random map satisfying (i1)–(i4).

Let us now define, for all $n \geq 0$,

$$F'_n = \Lambda_{(\tilde{x}_n, \tilde{h}_n)} \ , \tag{8.29}$$

where $(\tilde{x}_n, \tilde{h}_n)_{n \geq 0}$ is the ordering of $\tilde{\mathbb{E}}$ introduced in Section 2. (i1) implies in particular that the law of $(F'_n, n \geq 0)$ is identical to that of $(F_n, n \geq 0)$ (because the law of a countable FICRAB is unique). Lemma 8.2 then implies that, almost surely, for all $x \in \mathbb{R}$, $\{F'_n(x) : n \geq 0 \text{ and } \tilde{x}_n < x\}$ is dense in $\mathbb{R}_+$. Hence, almost surely, for all $(x,h) \in \mathbb{E}$, for all $0 < h'' < h' < h$, there exists n_1 and n_2 such that $\tilde{x}_{n_1} < x$, $\tilde{x}_{n_2} < x$ and

$$h'' < F'_{n_1}(x) < h' < F'_{n_2}(x) < h \ . \tag{8.30}$$

Combining this with (i3) implies that for all $y \geq x$,

$$\Lambda'_{(x,h'')}(y) \leq F'_{n_1}(y) \leq \Lambda'_{(x,h')}(y) \leq F'_{n_2}(y) \leq \Lambda'_{(x,h)}(y) \ . \tag{8.31}$$

Consequently, almost surely, for all $(x,h) \in \mathbb{E}$, for all $y \geq x$,

$$\lim_{h' \uparrow h} \Lambda'_{(x,h')}(y) = \sup\{F'_n(y) : n \geq 0, \ \tilde{x}_n < x \text{ and } \ F'_n(x) < h\} \ . \tag{8.32}$$

On the other hand, (i2), (i3) and (i4) imply that almost surely, for any $0 \leq x \leq y$ the mapping $h \mapsto \Lambda'_{(x,h)}(y)$ is non-decreasing and left-continuous. This implies in particular that almost surely, for all $x \leq y$ and $h > 0$,

$$\lim_{h' \uparrow h} \Lambda'_{(x,h')}(y) = \Lambda'_{(x,h)}(y) \ . \tag{8.33}$$

Hence, almost surely, for all $x \leq y$ and $h > 0$

$$\Lambda'_{(x,h)}(y) = \sup\{F'_n(y) : n \geq 0, \ \tilde{x}_n < x \quad \text{and} \quad F'_n(x) < h\} \ . \tag{8.34}$$

This implies that Λ and Λ' are identical in law (as random maps on $\mathbb{F}^+$) and concludes the proof of Theorem 2.1(ii).

□Theorem 2.1 (ii)

8.2. *Proof of Proposition 2.2*

This subsection contains the remaining proofs of the results stated in Section 2.

Proof of Proposition 2.2 (i). This is another immediate consequence of Lemma 8.2.

□Proposition 2.2 (i)

The following two Lemmas are of crucial importance in the proof of the rest of Proposition 2.2:

Lemma 8.3. *For any fixed* $(x,h) \in \mathbb{E}$ *and* $x' > x$, *almost surely for all* $y \ge x'$,

$$\Lambda_{(x,h)}(y) = \Lambda_{(x',\Lambda_{(x,h)}(x'))}(y) \ . \tag{8.35}$$

Remark: This is the 'flow property' (2.11) stated for fixed $(x,h) \in \mathbb{E}$. As emphasized in Section 2, it does *not* hold simultaneously for all $(x,h) \in \mathbb{E}$.

Proof. It suffices to define a sequence (x_n, h_n) as follows: $(x_0, h_0) = (x,h)$, $x_n = x'$ for all $n \ge 1$, and $(h_n)_{n\ge1}$ is dense in $(0,\infty)$. Then Theorem 2.1-(i1) shows that $(\Lambda_{(x_n,h_n)}(\cdot))_{n\ge0}$ is a countable FICRAB. Hence, using the Markov property, $\Lambda_{(x_0,h_0)}(x')$ is independent of $(\Lambda_{(x_n,h_n)}(\cdot))_{n\ge1}$ and it is then easy to see (using a Borel-Cantelli argument) that, there almost surely exists an increasing sequence $h_{n(k)}$ and a decreasing sequence $h_{m(k)}$ in $\{h_n, n \ge 1\}$ such that $\lim_{k\uparrow\infty} h_{n(k)} = \lim_{k\uparrow\infty} h_{m(k)} = \Lambda_{(x,h)}(x')$ and for all $y > x'$,

$$\Lambda_{(x',h_{n(k)})}(y) = \Lambda_{(x',h_{m(k)})}(y) \tag{8.36}$$

for all large enough k. Combining this with (i3) and the monotonicity of $h' \mapsto \Lambda_{(x',h')}(\cdot)$ implies Lemma 8.3.

□Lemma 8.3

Lemma 8.4. *Suppose that* $x < x'$ *are fixed in* $\mathbb{R}$, *and that* A *and* A' *denote two* $\mathscr{F}_x$ *measurable countable dense subsets of* $\mathbb{R}_+^*$. *Then, almost surely*,

$$\overline{\{\Lambda_{(x,h)}(x') : h \in A\}} = \overline{\{\Lambda_{(x,h)}(x') : h \in A'\}} \ . \tag{8.37}$$

Proof. Lemma 8.3, the Markov property and Theorem 2.1-(i) show immediately that conditionally on $\mathscr{F}_x$, the family $\left(\Lambda_{(x,h)}(\cdot)\right)_{h\in A\cup A'}$ is a countable FICRAB (defined for $y \ge x$). It is then straightforward to see that almost surely, for all $h_0 \in A$,

$$\lim_{\substack{h\uparrow h_0 \\ h\in A'}} \Lambda_{(x,h)}(x') = \Lambda_{(x,h_0)}(x') \tag{8.38}$$

and this implies readily the Lemma.

□Lemma8.4

Proof of Proposition 2.2-(ii). Recall the definition: For all $x < y$

$$M(x,y) = \{\Lambda_{(z,h)}(y) : h > 0 \text{ and } z < x\} \ . \tag{8.39}$$

We break up the proof into two steps:

STEP 1: Let us fix $q < q'$ in $\mathbb{D}$ for the moment. We define

$$M'(q, q') := \{F_n(q') : n \geq 0 \text{ and } \tilde{x}_n = q\} \\ = \{\Lambda_{(q,h)}(q') : h \in D_+^*\} \ . \tag{8.40}$$

Lemma 8.4 and Lemma 8.2 immediately show that for all $K > 0$, almost surely,

$$\overline{M(q, q')} = \overline{M'(q, q')} \ . \tag{8.41}$$

We are now going to show that almost surely,

$$\#\{\Lambda_{(q,h)}(q') : h \in \mathbb{D} \cap (0, K]\} < \infty \ . \tag{8.42}$$

As this is true for all $K \in \mathbb{N} \backslash \{0\}$, this immediately implies that $M'(q, q')$ is locally finite. We define for all $p \geq 1$

$$I_p^K := \{j/2^p : j \in \{1, \ldots, K2^p\}\}, \tag{8.43}$$

$$A_p^K := \{\Lambda_{(q,h)}(q') : h \in I_p^K\}, \tag{8.44}$$

$$N_p^K := \#A_p^K \ . \tag{8.45}$$

As $A_p^K \subset A_{p'}^K$ when $p \leq p'$, N_p^K is a non-decreasing function of p. For any $w \in A_p^K$, we define

$$H_p^K(w) := \sup\{h \in I_p^K : \Lambda_{(q,h)}(q') = w\} \ . \tag{8.46}$$

Note that

$$N_p^K = \#\{h \in I_p^K : H_p^K(\Lambda_{(q,h)}(q')) = h\} \ . \tag{8.47}$$

Clearly, if $h \in I_p^K$, then $H(\Lambda_{(q,h)}(q')) = h$ if and only if one of the following two events occur:

- $h = K$
- The two processes $\Lambda_{(q,h)}$ and $\Lambda_{(q,h+2^{-p})}$ do not meet before q'.

As these two processes are independent RABs before their meeting (i.e. coalescing) time, Lemma A.1 shows that for all $h \in I_p^K$,

$$\mathbf{P}\Big(\Lambda_{(q,h)}(q') < \Lambda_{(q,h+2^{-p})}(q')\Big) \leq C_2' \frac{2^{-p}}{\sqrt{q' - q}} \ . \tag{8.48}$$

Hence,

$$\mathbf{E}(N_p^K) = \sum_{j=1}^{K2^p} \mathbf{P}(H(\Lambda_{(q,j2^{-p})}(q')) = j2^{-p}) \\ = 1 + \sum_{j=1}^{K2^p - 1} \mathbf{P}(\Lambda_{(q,j2^{-p})}(q') < \Lambda_{(q,(j+1)2^{-p})}(q'))$$

$$\leq 1 + K2^p C_2' \frac{2^{-p}}{\sqrt{q' - q}}$$
$$\leq 1 + \frac{KC_2'}{\sqrt{q' - q}} \quad . \tag{8.49}$$

In other words, $\mathbf{E}(N_p^K)$ is bounded, uniformly in $p \in \mathbb{N}$. But $(N_p^K)_{p \geq 0}$ is a non-decreasing sequence: Hence, there almost surely exists $p_0 = p_0(K)$ such that for all $p \geq p_0$, $N_p^K = N_{p_0}^K$. This means in particular that the set

$$\{\Lambda_{(q,h)}(q') : h \in \mathbb{D} \cap (0, K]\} \tag{8.50}$$

is almost surely finite.

Let us now fix a positive number K'. Clearly, there almost surely exists an integer $K > 0$ such that $\Lambda_{(q,K)}(q') > K'$, so that

$$M(q, q') \cap [0, K'] \subset \{\Lambda_{(q,h)}(q') : h \in (0, K]\} \quad . \tag{8.51}$$

But we have seen that this set is in the closure of the set

$$\{\Lambda_{(q,h)}(q') : h \in \mathbb{D} \cap (0, K]\} \quad , \tag{8.52}$$

which is almost surely finite; hence $M(q, q') \cap [0, K']$ is also almost surely finite and

$$M(q, q') = \{\Lambda_{(q,h)}(q') : h \in \mathbb{D}_+^*\} \quad . \tag{8.53}$$

STEP 2: For any integer $n \geq 1$, we define, for all $\tilde{x}_n < x < y$,

$$M^n(x, y) := \{h \in M(x, y) : h < F_n(y)\} \quad . \tag{8.54}$$

It is straightforward to see that almost surely,

$$\overline{\lim_{n \to \infty}} F_n(y) = \infty \tag{8.55}$$

for all $y \in \mathbb{R}$, so that it suffices to check that $M^n(x, y)$ is almost surely finite for all $n \geq 0$ and $x < y$ in $\mathbb{R} \times \mathbb{R}$ in order to show that $M(x, y)$ is a.s. locally finite for all $x < y$.

Clearly, the definition of forward lines (and the fact that two F_n's can not cross) implies that for all $x \leq q \leq q' \leq y$, for all $n \geq 0$, such that $\tilde{x}_n < x$,

$$\# M^n(x, y) \leq \# M^n(q, q') \quad . \tag{8.56}$$

Hence, it is easy, using step 1, to conclude that almost surely, for all $x < y$, $M(x, y)$ is locally finite and that

$$M(x, y) = \{F_n(y) : n \geq 0 \text{ and } \tilde{x}_n < x\} \quad . \tag{8.57}$$

This concludes the proof of Proposition 2.2(ii).

$\square$Proposition 2.2 (ii)

Proof of Proposition 2.2 (iii). This is a simple combination of Proposition 2.2(ii) and Lemma 8.4.

□Proposition 2.2 (iii)

Proof of Proposition 2.2-(iv). Proposition 2.2-(ii) shows in particular that almost surely, for all $(x,h) \in \mathbb{E}$ and $\varepsilon > 0$, there exists $n_0 \geq 0$ such that $\tilde{x}_{n_0} < x$, $F_{n_0}(x) < h$ and

$$\Lambda_{(x,h)}(y) = F_{n_0}(y), \qquad \forall y \geq x + \varepsilon \ . \tag{8.58}$$

Combining this with Lemma 8.2 shows that it is always possible to choose n_0 in such a way that $F_{n_0}(x) \in (h-\varepsilon, h)$. Theorem 2.1-(i3) then implies that almost surely, for all $(x,h) \in \mathbb{E}$ and $\varepsilon > 0$, there exists n_0 such that

$$\tilde{x}_{n_0} < x, \ F_{n_0}(x) \in (h-\varepsilon, h) \text{ and } F_{n_0}(y) = \Lambda_{(x,h)}(y) \tag{8.59}$$

for all $y \geq x + \varepsilon$.This is exactly Proposition 2.2 (ii).

□Proposition 2.2 (iv)

It now remains to prove Proposition 2.2 (v) i.e. that almost surely, $\Lambda_{(x,h)}$ is continuous on $[x, +\infty)$ for all $(x,h) \in \mathbb{E}$. We first focus on the continuity at x:

Lemma 8.5. *Almost surely, for all* $(x,h) \in \mathbb{E}$, $\lim_{y \downarrow x} \Lambda_{(x,h)}(y) = h$.

Proof. It suffices in fact to prove that almost surely, for any $0 < \alpha < h$ and K in $\mathbb{D}_+$, and for all $x \in (-K, K)$, $h'' > h + \alpha$ and $h' < h - \alpha$,

$$h - \alpha \leq \underline{\lim}_{y \downarrow x} \Lambda_{(x,h'')}(y) \leq \overline{\lim}_{y \downarrow x} \Lambda_{(x,h')}(y) \leq h + \alpha \ . \tag{8.60}$$

Exactly as in the proof of Lemma 8.2, one can show that, almost surely, for all large enough k, for all $j \in \{-k, \ldots, k-1\}$,

$$\{\Lambda_{(jK/k,h)}(y) : y \in [jK/k, (j+2)K/k]\} \subset (h-\alpha, h+\alpha) \ . \tag{8.61}$$

In particular (combining this with Theorem 2.1 (i3)), this implies that almost surely, for all large enough k: For all $j \in \{-k, \ldots, k-1\}$, for all $x \in [jK/k, (j+1)K/k]$ and for all $h' < h - \alpha < h + \alpha < h''$, for all $y \in [x, x + K/k]$,

$$h - \alpha \leq \Lambda_{(x,h'')}(y) \leq \Lambda_{(x,h')}(y) \leq h + \alpha \ . \tag{8.62}$$

This implies (8.60) and therefore completes the proof of the Lemma. □Lemma8.5

Proof of Proposition 2.2-(v). Theorem 2.1-(ii) and Lemma 8.5 imply that almost surely, the mapping $y \mapsto \Lambda_{(x,h)}(y)$ is continuous at $y = x$, for all $(x,h) \in \mathbb{E}$. Proposition 2.2 (iv) implies that almost surely, for all $x < x'$ in $\mathbb{R}$ and $h > 0$, $y \mapsto \Lambda_{(x,h)}(y)$ is continuous on $[x', \infty)$ (as all F_n's are continuous). Combining these two facts yields Proposition 2.2(v). $\square$Proposition 2.2-(v)

9. Duality of the systems of lines: Proofs

This section is essentially devoted to the proof of Theorem 2.3 and Proposition 2.4. We proceed in the following way:

(1) First we shall prove that almost surely, for all $(x,h) \in \mathbb{E}$ the backward line $(-\infty, x] \ni y \mapsto \Lambda^*_{(x,h)}(y)$ is *continuous* and that, almost surely, the *backward lines do not cross* (Lemma 9.1).
(2) Next we prove that for any $(x,h) \in \mathbb{E}$ the process$[-x, \infty) \ni y \mapsto \Lambda^*_{(-x,h)}(-y)$ *is a RAB* and that for any finite collection $(x_1, h_1), \ldots, (x_p, h_p)$ the processes $\Lambda^*_{(x_1,h_1)}(\cdot), \ldots, \Lambda^*_{(x_p,h_p)}(\cdot)$ *are independent as long as they stay apart* (Lemma 9.3).
(3) Finally, we prove Proposition 2.4 (iii). From this it follows that any two *backward lines coalesce when they meet* and this consequence completes the proof of Theorem 2.3. Proposition 2.4 (i) and (ii) follow easily from Theorem 2.3. Altogether, Proposition 2.4 sheds light on the *fine topological structure of the systems of forward and backward lines*. This is the most interesting part of the present section.

(1) Note first of all that the definition (2.16) of the backward lines and left-continuity of Λ in the h-variable implies that almost surely for any $(x,h) \in \mathbb{E}$ and $z \le x$ and $\varepsilon > 0$,

$$\Lambda_{(z,\Lambda^*_{(x,h)}(z))}(x) < h \le \Lambda_{(z,\Lambda^*_{(x,h)}(z)+\varepsilon)}(x) \tag{9.1}$$

(with the notation $\Lambda_{(\cdot,0)}(\cdot) = 0$). Hence, almost surely, for all $(x,h) \in \mathbb{E}$, for all $z \le y \le x$ and $\varepsilon > 0$ (using Theorem 2.1-(i3) and the definition of Λ^*),

$$\Lambda_{(z,\Lambda^*_{(x,h)}(z))}(y) \le \Lambda^*_{(x,h)}(y) \tag{9.2}$$

On the other hand, if $y \in [z, x)$ and if $h' := \Lambda_{(z,\Lambda^*_{(x,h)}(z)+\varepsilon)}(y)$, then Proposition 2.2 shows that there exists $n \ge 0$ such that $\tilde{x}_n < y$, $F_n(y) < h'$ and $F_n(x) = \Lambda_{(z,\Lambda^*_{(x,h)}(z)+\varepsilon)}(x) \ge h$. In particular, $\Lambda^*_{(x,h)}(y) \le F_n(y) < \Lambda_{(z,\Lambda^*_{(x,h)}(z)+\varepsilon)}(y)$. Hence (combining this with (9.2)), almost surely, for all $(x,h) \in \mathbb{E}$, for all $z \le y \le x$ and $\varepsilon > 0$,

$$\Lambda_{(z,\Lambda^*_{(x,h)}(z))}(y) \le \Lambda^*_{(x,h)}(y) \le \Lambda_{(z,\Lambda^*_{(x,h)}(z)+\varepsilon)}(y) \ . \tag{9.3}$$

In plain words this means that forward lines and backward lines *never cross*. This will be of crucial importance in the forthcoming proof.

Lemma 9.1. [Continuity and non-crossing of the backward lines] *The dual process* $\mathbb{F}^- \ni (x,h,y) \mapsto \Lambda^*_{(x,h)}(y) \in \mathbb{R}_+$ *defined in* (2.16) *almost surely has the following properties*:

(i) *For all* $(x,h) \in \mathbb{E}$, $\Lambda^*_{(x,h)}(x) = h$.

(ii) *For any* $x \geq y$ *fixed the mapping* $\mathbb{R}^*_+ \ni h \mapsto \Lambda^*_{(x,h)}(y) \in \mathbb{R}_+$ *is left-continuous and non-decreasing.*

(iii) Λ^* *has the non-crossing property* (*analogous to* (i3) *of Theorem* 2.1, *stated for the forward lines*): *for all* (x_1,h_1), (x_2,h_2) *in* $\mathbb{E}$ *and* $z \leq y \leq \min\{x_1,x_2\}$:

$$\left[\Lambda^*_{(x_1,h_1)}(y) < \Lambda^*_{(x_2,h_2)}(y)\right] \quad \Longrightarrow \quad \left[\Lambda^*_{(x_1,h_1)}(z) \leq \Lambda^*_{(x_2,h_2)}(z)\right] \tag{9.4}$$

(iv) *For all* $(x,h) \in \mathbb{E}$, $(-\infty,x] \ni y \mapsto \Lambda^*_{(x,h)}(y)$ *is continuous.*

Proof. (i) and (ii) follow directly from the definition (2.16) of the backward lines Λ^* and the properties of Λ.

(iii) is also a rather simple consequence of the definition of Λ^*: Assume the contrary, namely that there are $(x_1,h_1),(x_2,h_2) \in \mathbb{E}$ and $z < y \leq \min\{x_1,x_2\}$ such that

$$\Lambda^*_{(x_1,h_1)}(y) < \Lambda^*_{(x_2,h_2)}(y) \quad \text{and} \quad \Lambda^*_{(x_1,h_1)}(z) > \Lambda^*_{(x_2,h_2)}(z) \tag{9.5}$$

Then, choosing $\varepsilon > 0$ small enough:

$$\Lambda_{(z,\Lambda^*_{(x_2,h_2)}(z)+\varepsilon)}(z) = \Lambda^*_{(x_2,h_2)}(z) + \varepsilon < \Lambda^*_{(x_1,h_1)}(z) = \Lambda_{(z,\Lambda^*_{(x_1,h_1)}(z))}(z) \ . \tag{9.6}$$

Applying (9.3), on the other hand we get:

$$\Lambda_{(z,\Lambda^*_{(x_2,h_2)}(z)+\varepsilon)}(y) \geq \Lambda^*_{(x_2,h_2)}(y) > \Lambda^*_{(x_1,h_1)}(y) \geq \Lambda_{(z,\Lambda^*_{(x_1,h_1)}(z))}(y) \ . \tag{9.7}$$

(9.6) and (9.7) contradicts (i3) of Theorem 2.1.

(iv) By (9.3) and continuity of the forward lines, almost surely for any $(x,h) \in \mathbb{E}$, $z \leq x$ and $\varepsilon > 0$:

$$\Lambda^*_{(x,h)}(z) \leq \varliminf_{y \downarrow z} \Lambda^*_{(x,h)}(y) \leq \varlimsup_{y \downarrow z} \Lambda^*_{(x,h)}(y) \leq \Lambda^*_{(x,h)}(z) + \varepsilon \tag{9.8}$$

which proves continuity from right. Next, assume that there are $(x,h) \in \mathbb{E}$, $y \leq x$, $\varepsilon > 0$ and an increasing sequence $y_n \uparrow y$, so that

$$\inf_{n \geq 0} \Lambda^*_{(x,h)}(y_n) > \Lambda^*_{(x,h)}(y) + \varepsilon \ . \tag{9.9}$$

By (9.3), for all $n \geq 0$,

$$\Lambda_{(y_n,\Lambda^*_{(x,h)}(y_n))}(y) \leq \Lambda^*_{(x,h)}(y) \tag{9.10}$$

Due to continuity and the non-crossing property ((i3) of Theorem 2.1) of the forward lines:

$$M(y) \cap (\Lambda^*_{(x,h)}(y), \Lambda^*_{(x,h)}(y) + \varepsilon) = \emptyset \tag{9.11}$$

Indeed: assume that for some $z < y$ and $h' > 0$ $\Lambda_{(z,h')}(y) \in (\Lambda^*_{(x,h)}(y), \Lambda^*_{(x,h)}(y) + \varepsilon)$, then for simple topological reasons (9.3), (9.9) and (9.10) imply that for sufficiently large n, $\Lambda_{(y_n, \Lambda^*_{(x,h)}(y_n))}(\cdot)$ would have to cross $\Lambda_{(z,h')}(\cdot)$ and this contradicts (i2) of Theorem 2.1. Hence we conclude that almost surely, for any $(x, h) \in \mathbb{E}$ and $y \leq x$

$$\overline{\lim_{z \uparrow y}} \Lambda^*_{(x,h)}(z) \leq \Lambda^*_{(x,h)}(y) \tag{9.12}$$

By an identical argument we find also

$$\underline{\lim_{z \uparrow y}} \Lambda^*_{(x,h)}(z) \geq \Lambda^*_{(x,h)}(y) \tag{9.13}$$

(9.12) and (9.13) imply continuity from left.

$\square$Lemma 9.1.

(2) Let $A \subset \mathbb{E}$ be an open box of the form $I \times J$ where I and J are two intervals. For all $(x, h) \in \overline{I} \times J$ we define

$$\omega^A_{(x,h)} := \inf\{y > x : \Lambda_{(x,h)}(y) \notin A\} \tag{9.14}$$

$$\omega^{*A}_{(x,h)} := \sup\{y < x : \Lambda^*_{(x,h)}(y) \notin A\} \tag{9.15}$$

In plain words: $\omega^A_{(x,h)}$ (respectively $\omega^{*A}_{(x,h)}$) is the first exit 'time' of the forward line $\Lambda_{(x,h)}(\cdot)$ (respectively, of the backward line $\Lambda^*_{(x,h)}(\cdot)$) from the domain A. We also define the sigma algebra generated by the 'data inside the domain A':

$$\mathscr{R}(A) := \sigma\Big(\Big\{F_n(y) : n \geq 0,\ (\tilde{x}_n, \tilde{h}_n) \in A,\ y \in [\tilde{x}_n, \omega^A_{(\tilde{x}_n, \tilde{h}_n)})\Big\}\Big) \ . \tag{9.16}$$

It is clear from the definition of the F_n's that for any finite collection of pairwise disjoint boxes $A_1, \ldots, A_p \subset \mathbb{E}$ the sigma algebras $\mathscr{R}(A_1), \ldots, \mathscr{R}(A_p)$ are independent.

In particular, for any fixed x_0, if $A^+_{x_0} := (x_0, \infty) \times (0, +\infty)$ and $A^-_{x_0} := (-\infty, x_0) \times (0, +\infty)$, $\mathscr{R}(A^+_{x_0})$ and $\mathscr{R}(A^-_{x_0})$ are independent (note that these two σ-fields are exactly $\mathscr{F}_{x_0}$ and $\mathscr{F}^*_{x_0}$ of Section 2.2; we will keep here the notation $\mathscr{R}(A^+_{x_0})$ to clearly distinguish between what is already proved from what isn't).

Note that for all $x \leq x_0$ and $h > 0$, $(\Lambda_{(x,h)}(y), y \in [x, x_0])$is $\mathscr{R}(A^-_{x_0})$-measurable, whereas by symmetry (this can be for instance deduced from Lemma 8.4 and the definition of Λ^*) for all $x \geq x_0$ and $h > 0$, $(\Lambda^*_{(x,h)}(y), y \in [x_0, x])$is $\mathscr{R}(A^+_{x_0})$-measurable.

Hence, it follows readily (using the definition of Λ^* yet again) that for all fixed x_0, the two families $(\Lambda^*_{(x,h)}(y), y \leq x \leq x_0, h > 0)$ and $(\Lambda^*_{(x,h)}(y),\, x_0 \leq y \leq x, h > 0)$ are independent.

We now state the counterpart for Λ^* of Lemma 8.1:

Lemma 9.2. *For all fixed $x \geq x'$ and $h > 0$, almost surely, for all $y \leq x'$,*

$$\Lambda^*_{(x',\Lambda^*_{(x,h)}(x'))}(y) = \Lambda^*_{(x,h)}(y) \ . \tag{9.17}$$

Proof. For all fixed $h' > 0$, for all $n \geq 0$ such that $\tilde{x}_n < x'$, almost surely, $F_n(x') \neq h'$ (as the law of $F_n(x')$ has no atoms). Hence, it is easy to see (using the fact that all $M(y, x')$ are discrete) that almost surely, for all $y < x'$,

$$\Lambda^*_{(x',h')}(y) = \lim_{\varepsilon \downarrow 0} \Lambda^*_{(x',h'+\varepsilon)}(y) \tag{9.18}$$

Let us now fix $(x,h) \in \mathbb{E}$ with $x > x'$. Then $\Lambda^*_{(x,h)}(x')$ is $\mathscr{R}(\mathscr{A}^+_{x'})$-measurable, and therefore independent from $(\Lambda_{(x',h'')}(y), y \leq x', h'' > 0)$. Combining this with (9.18) (for $h' := \Lambda^*_{(x,h)}(x')$) implies immediately the Lemma.

$\square$Lemma 9.2.

Let us now fix $(x_1,h_1), \ldots, (x_p,h_p)$ in $\mathbb{E}$ such that $x_1 \leq \cdots \leq x_p$. We also fix $x_0 \in \mathbb{R}$ and define $j := \sup\{i \leq p : x_i \leq x_0\}$. The previous Lemma combined with the independence stated above shows in particular that conditionally on

$$(\Lambda^*_{(x_{j+1},h_{j+1})}(x_0), \ldots, \Lambda^*_{(x_p,h_p)}(x_0)) \ , \tag{9.19}$$

the two families of functions

$$[x_0, +\infty) \ni y \mapsto (\Lambda^*_{(x_1,h_1)}(y), \ldots, \Lambda^*_{(x_p,h_p)}(y)) \tag{9.20}$$

and

$$(-\infty, x_0] \ni y \mapsto (\Lambda^*_{(x_1,h_1)}(y), \ldots, \Lambda^*_{(x_p,h_p)}(y)) \tag{9.21}$$

are independent (with for instance the notation $\Lambda^*_{(x,h)} = -\infty$ if $y > x$). This shows that $y \mapsto (\Lambda^*_{(x_1,h_1)}(-y), \ldots, \Lambda^*_{(x_p,h_p)}(-y))$ is a (inhomogeneous) Markov process.

It is easy to identify the transition probabilities of the Markov process $[-x, \infty) \ni y \mapsto \Lambda^*(-x,h)(-y) \in \mathbb{R}_+$: let $(x,h), (x',h'), (x'',h'') \in \mathbb{E}$ be given, with $x \geq x' > x''$, then by construction (2.16) of thebackward lines, left-continuity in the h variable of the forward lines and independence of the sigma algebras $\mathscr{R}(A^+_{x'})$ and $\mathscr{R}(A^-_{x'})$ we have

$$\mathbf{P}(\Lambda^*_{(x,h)}(x'') < h'' | \Lambda^*_{(x,h)}(x') = h') \geq \mathbf{P}(\Lambda_{(x'',h'')}(x') > h') \tag{9.22}$$

$$\mathbf{P}(\Lambda^*_{(x,h)}(x'') \geq h'' | \Lambda^*_{(x,h)}(x') = h') \geq \mathbf{P}(\Lambda_{(x'',h'')}(x') < h') \tag{9.23}$$

Since $\mathbf{P}\big(\Lambda_{(x'',h'')}(x') = h'\big) = 0$, these inequalities imply

$$\mathbf{P}(\Lambda^*_{(x,h)}(x'') < h'' | \Lambda^*_{(x,h)}(x') = h') = \mathbf{P}(\Lambda_{(x'',h'')}(x') > h') \tag{9.24}$$

Thus, $[-x,\infty) \ni y \mapsto \Lambda^*(-x,h)(-y) \in \mathbb{R}_+$ is a Markov process with transition probabilities given in (9.24) and a.s. continuous sample path. These properties imply easily that given $(x,h) \in \mathbb{E}$ fixed $[-x,\infty) \ni y \mapsto \Lambda^*(-x,h)(-y) \in \mathbb{R}_+$ is a RAB: Indeed, if we define a linear Brownian motion B started from h at time 0 under the probability measure $\mathbf{P}_h$, and if A denotes the absorbed Brownian motion obtained by killing B at its first hitting time of 0, then for all positive real numbers x, h_1 and h_2,

$$\begin{aligned}\mathbf{P}_{h_1}(A(x) > h_2) &= \mathbf{P}_{h_1}(B(x) > h_2) - \mathbf{P}_{h_1}(B(x) < -h_2)\\ &= \mathbf{P}_{h_1}(B(x) > h_2) - \mathbf{P}_{h_1}(B(x) > 2h_1 + h_2)\\ &= \mathbf{P}_0(B(x) \in [h_2 - h_1, h_2 + h_1])\\ &= \mathbf{P}_{h_2}(|B(x)| \leq h_1)\ .\end{aligned} \tag{9.25}$$

Next we prove that, given finitely many $(x_1,h_1),\ldots,(x_p,h_p) \in \mathbb{E}$, the processes $\Lambda^*_{(x_1,h_1)}(\cdot),\ldots,\Lambda^*_{(x_p,h_p)}(\cdot)$ are independent as long as they stay apart. We prove this for $p = 2$. The general case is treated identically, only the notation becomes more complicated. Because of Lemma 9.2, and the independence between $\mathscr{R}(A^+_{x_1})$ and $\mathscr{R}(A^-_{x_1})$, it suffices to consider the case where $x_1 = x_2 = x$. We build up the two processes $\Lambda^*_{(x_1,h_1)}(\cdot)$ and $\Lambda^*_{(x_2,h_2)}(\cdot)$ in small steps of 'time span' $x - 5^{-n}k \leq y \leq x - 5^{-n}(k-1)$, in the following way. Let us use the shorthand notation:

$$x_{n,k} := x - 5^{-n}k, \qquad n = 1,2\ldots;\ k = 0,1,2,\ldots \tag{9.26}$$

Define

$$A^{(i)}_{n,k} := (x_{n,k+1}, x_{n,k}] \times (\Lambda^*_{(x,h_i)}(x - 5^{-n}k) - 2^{-n}, \Lambda^*_{(x,h_i)}(x - 5^{-n}k) + 2^{-n}),$$
$$i = 1,2;\ n = 1,2,\ldots;\ k = 0,1,\ldots \tag{9.27}$$

and the stopping times

$$u^{(i)}_n := \inf\{y \geq -x : -y \in [x_{n,k+1}, x_{n,k}] \text{ and } \Lambda^*_{(x,h_i)}(-y) \notin A^{(i)}_{n,k}\} \tag{9.28}$$

$$v_n \ := x - \min\{5^{-n}k : A^{(1)}_{n,k} \cap A^{(2)}_{n,k} \neq \emptyset\} \tag{9.29}$$

By independence of the sigma algebras $\mathscr{R}(A_i)$ for *pairwise disjoint domains* A_i, and using Lemma 9.2, it is clear that the two processes $[-x,\infty) \ni y \mapsto \Lambda^*_{(x,h_1)}(-y)$ and $y \mapsto \Lambda^*_{(x,h_2)}(-y)$ are independent till

$$-y = \min\{u_n^{(1)}, u_n^{(2)}, v_n\} \tag{9.30}$$

Since $\Lambda^*_{(x,h_1)}(\cdot)$ and $\Lambda^*_{(x,h_2)}(\cdot)$ are RABs (this was already proved in the previous paragraphs), by a Borel-Cantelli argument it is easy to see that

$$\lim_{n\uparrow\infty} u_n^{(i)} = \infty, \qquad \text{almost surely, } i = 1, 2. \tag{9.31}$$

On the other hand, $v_{n+1} \geq v_n$ and

$$\lim_{n\uparrow\infty} v_n = \inf\{-y : \Lambda^*_{(x,h_1)}(y) = \Lambda^*_{(x,h_2)}(y)\} \tag{9.32}$$

From these arguments indeed it follows that the law $\Lambda^*_{(x,h_1)}(\cdot)$ and $\Lambda^*_{(x,h_2)}(\cdot)$ are independent as long as they stay apart (i.e. the law of these two processes up to their first meeting time is that of two independent 'backwards' RAB's up to their first meeting time). Summarizing, we get the following

Lemma 9.3. *Given finitely many $(x_1, h_1), \dots, (x_p, h_p) \in \mathbb{E}$ the processes $[-x_j, \infty) \ni y \mapsto \Lambda^*_{(x_j,h_j)}(-y)$, $j = 1, \dots, p$, are RABs and they are independent as long as they stay apart.*

(3) It remains to be proven that the backward lines coalesce when they meet. In order to do this we need the refined topological picture of the system of lines stated in Proposition 2.4. For $(x, h) \in \mathbb{E}$, let $I(x, h)$ be the number of disjoint forward lines coalescing at (x, h) and $O(x, h)$ the number of disjoint forward lines going out from the immediate vecinity of (x, h). The precise definition of $I(x, h)$ was given in (2.28) and

$$\begin{aligned} O(x,h) :=& \lim_{y\downarrow x}\lim_{\varepsilon\downarrow 0} \#\big\{\Lambda_{(x',h')}(y) : (x',h') \in (x, x+\varepsilon) \\ & \qquad \times (h-\varepsilon, h+\varepsilon))\big\} \\ =& \lim_{y\downarrow x}\lim_{\varepsilon\downarrow 0} \#\big\{\Lambda_{(\tilde{x}',\tilde{h}')}(y) : (\tilde{x}',\tilde{h}') \in (x, x+\varepsilon) \\ & \qquad \times (h-\varepsilon, h+\varepsilon) \cap \tilde{\mathbb{E}})\big\}\ . \end{aligned} \tag{9.33}$$

Note that $\#\{\dots\}$ on the right hand side of (9.33) is monotone nonincreasing with decreasing ε and monotone nondecreasing with decreasing y. In the second equality of (9.33) we use (ii) of Proposition 2.2.

Similarly, we define $I^*(x, h)$ and $O^*(x, h)$ as the number of disjoint backward lines coming from the right of x and meeting at (x, h), respctively as the number of disjoint backward lines going out from the immediate vecinity of (x, h):

$$I^*(x,h) := \limsup_{y \downarrow x} \#\{p \in \mathbb{N} : \exists (x_1,h_1),\dots,(x_p,h_p) \in \mathbb{E} \text{ such that } \forall i = 1,\dots,p: \ x_i \geq y,\ \Lambda^*_{(x_i,h_i)}(x) = h \text{ and } \forall z \in (x,y],\ \Lambda^*_{(y,h_1)}(z) < \cdots < \Lambda^*_{(y,h_p)}(z)\} \ . \tag{9.34}$$

$$O^*(x,h) := \lim_{y \uparrow x} \lim_{\varepsilon \downarrow 0} \#\left\{\Lambda^*_{(x',h')}(y) : (x',h') \in (x-\varepsilon,x) \times (h-\varepsilon,h+\varepsilon))\right\} \ . \tag{9.35}$$

It is clear that for all $(x,h) \in \mathbb{E}$: $I(x,h)$, $I^*(x,h) \in \{0,1,2,\dots,\infty\}$ and $O(x,h)$, $O^*(x,h) \in \{1,2,\dots,\infty\}$. Also, from the definition (2.16) of backward lines and (9.3), it follows immediately that for all $(x,h) \in \mathbb{E}$

$$I^*(x,h) = O(x,h) - 1, \qquad O^*(x,h) = I(x,h) + 1 \tag{9.36}$$

We will first prove Proposition 2.4 (iii) and Theorem 2.3 simultaneously. We then derive Proposition 2.4 (i)–(ii) as immediate consequences of Theorem 2.3.

Proof of Proposition 2.4 (iii) and Theorem 2.3. First note that, with probability one, no three (or more) forward lines starting from distinct points $(\tilde{x}_i,\tilde{h}_i)$,$(\tilde{x}_j,\tilde{h}_j)$, $(\tilde{x}_k,\tilde{h}_k)$ in $\tilde{\mathbb{E}}$ will coalesce at the same point $(x,h) \in \mathbb{E}$. Hence, by *(ii)* of Proposition 2.2, we conclude that almost surely, for all $(x,h) \in \mathbb{E}$

$$0 \leq I(x,h) \leq 2 \ . \tag{9.37}$$

Further we proceed in the following way: First we prove that almost surely for all $(x,h) \in \mathbb{E}$

$$O(x,h) \geq 3 \Rightarrow I(x,h) = 0 \tag{9.38}$$

By (9.36) this implies that the backward lines are coalescing:

$$I^*(x,h) \geq 2 \Rightarrow O^*(x,h) = 1 \tag{9.39}$$

and this is what we need to complete the proof of Theorem 2.3. Further on: Theorem 2.3 and (9.36)–(9.39) imply that almost surely for all $(x,h) \in \mathbb{E}$, $I^*(x,h) \leq 2$ so that

$$1 \leq O(x,h) \leq 3 \tag{9.40}$$

and

$$I(x,h) = 2 \Rightarrow O(x,h) \leq 1 \tag{9.41}$$

Finally, (9.37), (9.38), (9.40) and (9.41) imply (iii) of Proposition 2.4.

The proof of (9.38) relies on the following:

Lemma 9.4. *There exists a constant* $C'' < \infty$ *such that for all* $(x,h) \in \mathbb{E}$, $v > 0$ *and* $\delta > 0$:

$$\mathbf{P}\Big(\exists (h_1,h_2,h_3) \in [h, h+\delta]^3 : \Lambda_{(x,h_1)}(x+v) < \Lambda_{(x,h_2)}(x+v) < \Lambda_{(x,h_3)}(x+v)\Big) < C''\left(\frac{\delta}{\sqrt{v}}\right)^3 . \tag{9.42}$$

Proof. From Lemma A.1. we know that for any $v > 0$, $\delta > 0$ and $(x,h) \in \mathbb{E}$

$$\mathbf{P}\Big(\Lambda_{(x,h)}(x+v) < \Lambda_{(x,h+\delta)}(x+v) < \Lambda_{(x,h+2\delta)}(x+v)\Big) < C_3'\left(\frac{2\delta}{\sqrt{v}}\right)^3 . \tag{9.43}$$

Left continuity of $h \mapsto \Lambda_{(x,h)}(x+v)$ implies that

$$\begin{aligned}
&\Big\{\exists (h_1,h_2,h_3) \in [h, h+\delta]^3 : \Lambda_{(x,h_1)}(x+v) < \Lambda_{(x,h_2)}(x+v) < \Lambda_{(x,h_3)}(x+v)\Big\} \\
&\quad \subset \Big\{\exists p \geq 1,\ \exists j \in \{0,\ldots,2^p-2\}: \\
&\qquad \Lambda_{(x,h+\delta j 2^{-p})}(x+v) < \Lambda_{(x,h+\delta(j+1)2^{-p})}(x+v) < \Lambda_{(x,h+\delta(j+2)2^{-p})}(x+v)\Big\}
\end{aligned} \tag{9.44}$$

From the last two relations it follows that

$$\begin{aligned}
&\mathbf{P}\Big(\exists (h_1,h_2,h_3) \in [h, h+\delta]^3 : \Lambda_{(x,h_1)}(x+v) < \Lambda_{(x,h_2)}(x+v) \\
&\quad < \Lambda_{(x,h_3)}(x+v)\Big) \\
&\quad \leq \sum_{p \geq 1} C_2'\left(\frac{2\delta 2^{-p}}{\sqrt{v}}\right)^3 (2^p - 1) \\
&\quad \leq C''\left(\frac{\delta}{\sqrt{v}}\right)^3
\end{aligned} \tag{9.45}$$

with $C'' := 8C_3'/3$ □Lemma 9.4.

We are ready now to prove (9.38). We want to prove that a.s, for all $(x,h) \in \mathbb{E}$, $I(x,h) \geq 1$ implies that $O(x,h) \leq 2$. It is sufficient to prove that for all fixed $n \geq 0$, $\eta > 0$ and $M < \infty$, almost surely, for all $x \in [\tilde{x}_n, \tilde{x}_n + M]$,

$$O^\eta(x, F_n(x)) \leq 2 , \tag{9.46}$$

where

$$O^\eta(x,h) := \inf_{\varepsilon>0} \#\{\Lambda_{(x',h')}(x+\eta) : (x',h') \in (x, x+\varepsilon) \times (h-\varepsilon, h+\varepsilon)\} \ . \tag{9.47}$$

For $m \geq 1$, we divide the interval $[\tilde{x}_n, \tilde{x}_n + M]$ into 5^m equal parts: We define for all $j \geq 0$,

$$x_{j,m} := \tilde{x}_n + jM5^{-m} \quad \text{and} \quad h_{j,m} := F_n(x_{j,m}) \ . \tag{9.48}$$

For $m > 0$ and $j = 1, 2, \ldots, 5^m$ let us define the events

$$\begin{aligned} \mathscr{A}_{m,j} := \Big\{ &\big[\Lambda_{(x_{j-1,m}, h_{j-1,m}+2^{-m})}(x_{j,m}) > h_{j-1,m} + 2 \cdot 2^{-m}\big] \text{ or} \\ &\big[\Lambda_{(x_{j-1,m}, h_{j-1,m}-2^{-m})}(x_{j,m}) < h_{j-1,m} - 2 \cdot 2^{-m}\big] \Big\} \end{aligned} \tag{9.49}$$

A simple Brownian estimate shows that

$$\mathbf{P}(\mathscr{A}_{m,j}) \leq \exp\left\{ -\frac{5^m}{2M4^m} \right\} \tag{9.50}$$

But, if $5^{-m} < \eta/2$, then for all $j \geq 1$,

$$\begin{aligned} &\big\{\exists x \in [x_{j-1,m}, x_{j,m}] : O^\eta(x, F_{n_0}(x)) \geq 3\big\} \cap \mathscr{A}^c_{m,j} \qquad (9.51) \\ &\quad \subset \big\{\exists (h_1, h_2, h_3) \in ([h_{j-1,m} - 2 \cdot 2^{-m}, h_{j-1,m} + 2 \cdot 2^{-m}] \cap \mathbb{R}^*_+)^3 : \\ &\qquad \Lambda_{(x_{j,m}, h_1)}(x_{j,m} + \eta/2) < \Lambda_{(x_{j,m}, h_2)}(x_{j,m} + \eta/2) \\ &\qquad < \Lambda_{(x_{j,m}, h_3)}(x_{j,m} + \eta/2)\big\} \qquad (9.49) \end{aligned}$$

Thus, by (9.42)

$$\mathbf{P}\Big(\big\{\exists x \in [x_{j-1,m}, x_{j,m}] : O^\eta(x, F_{n_0}(x)) \geq 3\big\} \cap \mathscr{A}^c_{m,j}\Big) \leq C(\eta) 2^{-3m} \tag{9.52}$$

where $C(\eta) = C'' 4^3 \sqrt{2} \eta^{-1/2}$. Now, combining (9.50) and (9.52) we conclude

$$\begin{aligned} &\mathbf{P}\Big(\exists x \in [\tilde{x}_{n_0}, \tilde{x}_{n_0} + M] : O^\eta(x, F_{n_0}(x)) \geq 3\Big) \\ &\quad \leq 5^m \left(\exp\left\{ -\frac{5^m}{2M4^m} \right\} + C(\eta) 2^{-3m} \right) \ . \end{aligned} \tag{9.53}$$

Letting $m \uparrow \infty$ this implies (9.46).

□Proposition 2.4(iii) and Theorem 2.3

Proofs of Proposition 2.4 (i)–(ii). These results are almost immediate, using Theorem 2.3: When $(x,h) \in \mathbb{E}$ is fixed, then for all $n \geq 0$ such that $\tilde{x}_n < x$, $F_n(x) \neq h$ almost surely. Similarly, for all $n \geq 0$ such that $\tilde{x}_n > x$, $F^*_n(x) \neq h$ almost surely. This shows that almost surely, $I(x,h) = I^*(x,h) = 0$, and this implies Proposition 2.4 (i).

Suppose now that $x \in \mathbb{R}$ is fixed. For all $n \geq 0$ and $n' \geq 0$ such that $\tilde{x}_n < x$ and $\tilde{x}_{n'} < x$, the meeting 'time' of F_n and $F_{n'}$ is almost surely not equal to x: Hence, almost surely, for all $h > 0$, $I(x,h) \leq 1$. Similarly (by symmetry), almost surely, for all $h > 0$, $I^*(x,h) \leq 1$. This implies Proposition 2.4 (ii).

10. Remarks

The stationary limit- Suppose now that we define a process $\check{X}$ using exactly the same procedure as for X except that the forward lines (F_n) are not RAB's, but Brownian motions that are reflected/absorbed on another independent Brownian motion passing through $(0,0)$ that replaces the line $\{h = 0\}$. Then, the process $\check{X}$ has very similar properties to those of X and some of them are even simpler.

A more precise definition of $\check{X}$ can be the following: Define first a countable family of independent coalescing Brownian motions (without any reflection nor absorbtion) $(U_n)_{n\geq 0}$ started from a dense sequence $(x_n, h_n)_{n\geq 0}$ in $\mathbb{R} \times \mathbb{R}$, with $(x_0, h_0) = 0$.Then define for all $(x,h) \in \mathbb{R} \times \mathbb{R}$ and $y \geq x$,

$$V_{(x,h)}(y) := \sup\{U_n(y) : n \geq 0,\ x_n < x \text{ and } U_n(x) < h\} \ . \tag{10.1}$$

Then, almost exactly as in Sections 8 and 9, one can notice that the law of V defined in this way is unique in a certain sense, and that V has the same self-duality property as Λ: If we define for all $(x,h) \in \mathbb{R} \times \mathbb{R}$ and $y \leq x$,

$$V^*_{(x,h)}(y) := \sup\{U_n(y) : n \geq 0,\ x_n < y \text{ and } U_n(x) < h\} \tag{10.2}$$

then $(x,h,y) \mapsto V_{(x,h)}(y)$ and $(x,h,y) \mapsto V^*_{(-x,h)}(-y)$ are identical in law. In particular, if we define $\overline{V}_{(x,h)}(y)$ for all $y \in \mathbb{R}$, by

$$\overline{V}_{(x,h)}(y) = 1\!\!1_{\{y \geq x\}} V_{(x,h)}(y) + 1\!\!1_{\{y<x\}} V^*_{(x,h)}(y) \ , \tag{10.3}$$

then the law of $\overline{V}_{(0,0)}$ is that of a linear Brownian motion defined on $\mathbb{R}$ with $\overline{V}_{(0,0)}(0) = 0$. We then define the set

$$\mathbb{E}_0 := \{(x,h) \in \mathbb{R} \times \mathbb{R} : \overline{V}_{(0,0)}(x) < h\} \ . \tag{10.4}$$

Then, for all $(x,h) \in \mathbb{E}_0$, we put

$$\check{T}(x,h) = \int_{\mathbb{R}} dy \ \left(\overline{V}_{(x,h)}(y) - \overline{V}_{(0,0)}(y)\right). \tag{10.5}$$

The two-dimensional process $(\check{X}, \check{H})$ is then defined exactly like (X, H), just replacing T by $\check{T}$. In particular, for almost all $t \geq 0$,

$$t = \check{T}(\check{X}_t, \check{H}_t) \ . \tag{10.6}$$

The process $\check{X}$ defined in this way, satisfies all the properties of X enumerated in the introduction; the only difference is that the right-hand side of (1.6) has to be replaced by $-\check{X}_T$ (the perturbation at the boundary of the range disappears). The process $(\check{X}_t,\ t \geq 0)$ will have stationary increments and can be viewed as the weak limit of $(X_{t_0+t} - X_{t_0},\ t \geq 0)$, as $t_0 \uparrow \infty$.

Universality of the 3/2 *variation*- More general self-interacting processes can be defined using other independent coalescing time-homogeneous diffusions than Brownian motion. For example, if we start with a system of independent coalescing powers of Brownian motions (or of Bessel processes), then the same procedure as that described in this paper, would define a self-repelling motion Y with a different scaling behaviour than that of X. More precisely, if we start with independent coalescing processes which each have the law of reflected/absorbed Bessel processes (of dimension $v \in (0, 2)$) at some power $\alpha > 0$, then the corresponding self-repelling motion Y satisfies the following scaling property: $(Y_{at}, t \geq 0)$ and $(a^{2/(2+\alpha)} Y_t, t \geq 0)$ are identical in law. But (as for powers of Bessel processes themselves that are diffusions, and therefore have finite quadratic variation), the local behaviour of Y is similar to that of X i.e. Y satisfies the same type of local variation property as X (with finite variation of order 3/2). This 3/2 variation is in some sense universal for this type of self-interacting processes.The limit theorems proved in [T2] suggest that these processes arise as scaling limits for '*generalized* true self avoiding random walks', with subexponentially self-repelling weights.

11. A discrete construction

We now briefly describe a discrete model that is very similar to true self-repelling motion. Its definition is a little bit more complicated than that of true self-avoiding walk mentioned in the introduction, but it has the advantage that many features (the lines of local times as coalescing random walks, duality of forward and backward lines, definition of the process from the family of coalescing random walks etc) become apparent on a picture. In particular, it can be defined in several different ways: It can be viewed (this will be our third definition) as a self-interacting nearest-neighbour walk in $\mathbb{Z}$; it can also be defined using a family of coalescing reflected/absorbed simple random walks in $\mathbb{N}$.

Before giving a formal definition of this discrete random walk, let us give a first appealing intuitive description that vaguely recalls the totally asymmetric exclusion process in $\mathbb{Z}$: Suppose that at time zero, each site $z \in \mathbb{Z}$ is occupied by a particle if z is an even positive or an odd negative integer; suppose that each odd positive and each even negative site is occupied at time zero by an anti-particle, and that site 0 is (doubly) occupied by both a particle and an anti-particle. The dynamics is the following: particles can move only to the right and anti-particles can move only to the left. At any time n, there is a unique site which is doubly occupied (by two particles, or by a particle and an anti-particle, or by two anti-particles). Toss a fair coin to decide which one of the two will move, and then move it to the right (if a particle is chosen) or to the left (if an anti-particle is chosen). Start the same procedure again at time $n+1$ with this new configuration. It is an easy exercise (that we safely leave to the reader) to check that if S_n denotes the position of the unique doubly occupied site at time n, then $(S_n, n \geq 0)$ is precisely the random walk that we are going to describe in this section.

A second (equivalent) definition of this random walk can be obtained using a family of coalescing random walks (see figure 2 later in this section): This equivalence between the third definition of $(S_n, n \geq 0)$ (the one we are going to describe in the next paragraphs) and the definition via a family of coalescing random walks is very instructive. The construction of true self-repelling motion developed in the present paper can be interpreted as follows: We construct the continuous analog of $(S_n, n \geq 0)$ using a natural generalization/scaling limit of this second definition of $(S_n, n \geq 0)$ (replacing the family of coalescing random walks by a continuous family of coalescing Brownian motions).

We now finally give the third definition of this nearest-neighbour (i.e. $|S_{n+1} - S_n| = 1$) walk $(S_n, n \geq 0)$ defined on $\mathbb{Z}$ and started from $S_0 = 0$. Before describing its transition probabilities, we introduce some notation: define its local time on edges just as for the true self-avoiding walk in the introduction: For all $i \in \mathbb{N}$ and $z \in \mathbb{Z}$,

$$l(i,z) := \#\{j \in [0, i-1] : \{S_j, S_{j+1}\} = \{z, z+1\}\} \tag{11.1}$$

(in plain words: $l(i,z)$ denotes the number of jumps along the edge located to the right of z before step i) and we will also sometimes use the local time on sites:

$$\lambda(i,z) := \#\{j \in [0, i] : S_j = z\} \ . \tag{11.2}$$

We will also use the following function $a(z)$ defined for all $z \in \mathbb{Z}$ by:

$$a(z) := \frac{1 + \mathrm{sgn}(z + 1/2)(-1)^z}{2} \tag{11.3}$$

(i.e. $a(0) = 1$, $a(1) = 0$, $a(2) = 1$ etc. and $a(-1) = 1$, $a(-2) = 0$, $a(-3) = 1$ etc). To define the transition probabilities, we will use $\ell(i, z)$ rather than $l(i, z)$ where we define ℓ for all $i \in \mathbb{N}$ and $z \in \mathbb{Z}$ as follows:

$$\ell(i, z) := l(i, z) + a(z). \tag{11.4}$$

$a(z)$ can be interpretated as the initial data of the local times on edges. It is straightforward to see by induction that for all $i \geq 0$ and $z \in \mathbb{Z}$,

$$\ell(i, z) + \ell(i, z-1) = \begin{cases} 2\lambda(i, S_i) & \text{if} \quad z = S_i \\ 2\lambda(i, z) + 1 & \text{if} \quad z \neq S_i \end{cases} \tag{11.5}$$

In particular, for all $i \geq 0$, $\ell(i, S_i) - \ell(i, S_i - 1)$ is even.

We now define the law of S as follows: For all $i \geq 0$,

$$\mathbf{P}\Big(S_{i+1} = S_i + 1 \Big| S_0, \ldots, S_i\Big) = 1 - \mathbf{P}\Big(S_{i+1} = S_i - 1 \Big| S_0, \ldots, S_i\Big)$$
$$= \begin{cases} 1 & \text{if } \ell(i, S_i) < \ell(i, S_i - 1) \\ 1/2 & \text{if } \ell(i, S_i) = \ell(i, S_i - 1) \\ 0 & \text{if } \ell(i, S_i) > \ell(i, S_i - 1) \end{cases}$$

This definition (combined with (11.5)) implies immediately that for all $i \geq 0$, for all $z \in \mathbb{Z} \backslash \{S_i\}$,

$$\ell(i, z) - \ell(i, z-1) \in \{-1, 1\} \tag{11.7}$$

and that

$$\ell(i, S_i) - \ell(i, S_i - 1) \in \{-2, 0, 2\} \ . \tag{11.8}$$

A simple parity argument shows that for all $i \geq 0$, $\ell(i, S_i) + S_i$ is odd. In particular, if $\ell(i, S_i) = \ell(i, S_i - 1)$ for some $i \geq 0$ then $S_i + \ell(i, S_i)$ is odd and $\ell(i, S_i) + 1 \geq 2$. This implies that for all $i \geq 0$,

$$[\lambda(i, S_i) + S_i \ \text{ is odd}] \Rightarrow [\ell(i, S_i) = \ell(i, S_i - 1)] \ . \tag{11.9}$$

Suppose for a moment that for some $i \geq 0$, $S_i + \lambda(i, S_i)$ is odd, and that $|\ell(i, S_i) - \ell(i, S_i - 1)| = 2$. Combining this with (11.5) implies that $\ell(i, S_i) + S_i$ is even, which contradicts the previous statement.

We are therefore led to define the sets

$$\mathbb{G}^+ := \{(z, h) \in \mathbb{Z} \times \mathbb{N} : h + z \text{ is odd and } h \geq 2\} \tag{11.10}$$

$$\mathbb{G}^- := \{(z, h) \in \mathbb{Z} \times \mathbb{N} : h + z \text{ is even and } h \geq 2\} \ . \tag{11.11}$$

We will also need the corresponding bottom 'lines' $\mathbb{G}_0^+$ and $\mathbb{G}_0^-$ defined as follows

$$\mathbb{G}_0^+ := \{(z, h) \in \mathbb{Z} \times \{0, 1\} : h + z \text{ is odd}\} \setminus \{(-1, 0)\} \tag{11.12}$$

$$\mathbb{G}_0^- := \{(z, h) \in \mathbb{Z} \times \{0, 1\} : h + z \text{ is even}\} \setminus \{(0, 0)\} \ . \tag{11.13}$$

We now define a family $(\xi(z,h),(z,h)\in\mathbb{G}^+)$ of independent random variables with

$$\mathbf{P}\Big(\xi(z,h)=1\Big)=\mathbf{P}\Big(\xi(z,h)=-1\Big)=1/2\ . \tag{11.14}$$

The definition of the law of S and (11.8) shows that S can be defined as follows: For all $i\geq 0$,

$$S_{i+1}-S_i=\begin{cases}1 & \text{if}\quad \ell(i,S_i)=\ell(i,S_i-1)-2\\ \xi(S_i-1,\ell(i,S_i)+1) & \text{if}\quad \ell(i,S_i)=\ell(i,S_i-1)\\ -1 & \text{if}\quad \ell(i,S_i)=\ell(i,S_i-1)+2\end{cases} \tag{11.15}$$

This construction of S turns out to be very convenient: We are now going to see that the local times of S can be described in terms of the random variables ξ and more precisely, in terms of a family of coalescing random walks derived from the family ξ.

We first define deterministically $\xi(z,h)$ when $(z,h)\in\mathbb{G}_0^+$ as follows:

$$\xi(z,1)=\begin{cases}-1 & \text{if } z\geq 0\\ 1 & \text{if } z\leq -2\end{cases} \tag{11.16}$$

and

$$\xi(z,0)=\begin{cases}1 & \text{if } z\geq 1\\ -1 & \text{if } z\leq -3\end{cases} \tag{11.17}$$

(recall that $(-1,0)\notin\mathbb{G}_0^+$).

We define the family of independent coalescing random walks L^+ as follows: For all $(z,h)\in\mathbb{G}^+$, we define a random walk $y\mapsto L^+_{(z,h)}(y)$ defined for $y\geq z$ by induction: $L^+_{(z,h)}(z)=h$ and for all $y\geq z$,

$$L^+_{(z,h)}(y+1)=L^+_{(z,h)}(y)+\xi(y,L^+_{(z,h)}(y))\ . \tag{11.18}$$

Then, as the random variables $(\xi_{(z,h)})_{(z,h)\in\mathbb{G}^+}$ are independent identically distributed, L^+ is a family of coalescing independent simple random walks in $\mathbb{N}\setminus\{1,0\}$. The 'boundary conditions' on the bottom line $\mathbb{G}_0^+$ can be loosely speaking described as follows: L^+ is reflected from the bottom line when $z<0$ and absorbed by the bottom line as soon as $z\geq 0$. This is exactly the discrete counterpart of the system Λ constructed in Sections 2 and 8.

The corresponding dual system is easy to define: For all $(z,h)\in\mathbb{G}^-$, we define a 'backwards' running random walk $L^-_{(z,h)}(\cdot)$ defined for all $y\leq z$ as follows: $L^-_{(z,h)}(z)=h$ and for all $y\leq z$,

$$L^-_{(z,h)}(y-1)=L^-_{(z,h)}(y)-\xi\Big(y-1,L^-_{(z,h)}(y)\Big)\ . \tag{11.19}$$

Then L^- forms a family of independent coalescing 'backwards' random walks reflected from the bottom line $\mathbb{G}_0^-$ 'before' 0 and absorbed 'after' 0. Note that the definitions of L^+ and L^- show that for all $(z,h) \in \mathbb{G}^+$,

$$L^-_{(z+1,h)}(z) - L^-_{(z+1,h)}(z+1) = -\left(L^+_{(z,h)}(z+1) - L^+_{(z,h)}(z)\right) \ . \qquad (11.20)$$

In particular, this shows that the forward walks and the backward walks never cross. Another formulation of this is to say that the trace of the family of backward lines is exactly the (Whitney, topological) dual graph of the trace of the family of forward lines (this can be easily observed on the picture); For instance, for all $(z,h) \in \mathbb{G}^+$, the bond $(z+1,h) \leftrightarrow (z,h+1)$ is occupied (in the graph corresponding to $\mathbb{G}^-$) if and only if $(z,h) \leftrightarrow (z+1,h+1)$ is not occupied (in the graph corresponding to $\mathbb{G}^+$).

It is easy to notice that in fact the law of L^- and of L^+ are very similar: Define for all $(z,h) \in \mathbb{G}^- \cup \mathbb{G}_0^-$,

$$\xi'(z,h) = -\xi(z-1,h) \ . \qquad (11.21)$$

Then, the two families $(\xi(z,h))_{(z,h)\in\mathbb{G}^+\cup\mathbb{G}_0^+}$ and $(\xi'(-1-z,h))_{(z,h)\in\mathbb{G}^+\cup\mathbb{G}_0^+}$ are identical in law. This implies that the mapping $(z,h,y) \mapsto L^-_{(-1-z,h)}(-1-y)$ is identical in law to L^+ as for all $(z,h) \in \mathbb{G}^-$,

$$\begin{aligned} L^-_{(-1-z,h)}(-1-y-1) &= L^-_{(-1-z,h)}(-1-y) \\ &\quad - \xi(-1-y-1, L_{(-1-z,h)}(-1-y)) \\ &= L^-_{(-1-z,h)}(-1-y) \\ &\quad + \xi'(-1-y, L^-_{(-1-z,h)}(-1-y)) \end{aligned} \qquad (11.22)$$

This type of 'self-duality' for discrete families of coalescing simple random walks had been discovered and exploited by Arratia [A1, A2]. The following figure shows the two families of coalescing random walks.

Note that the two families L^+ and L^- create a random maze, with one single connected component (this is a simple consequence of the coalescing property). One single possible path starting at the middle of the 'entrance gate' $[(-1,1),(0,1)]$, i.e. from the point of coordinates $(-1/2,1)$, explores this maze. (See Fig. 2.) The random walk $(S_i)_{i\geq 0}$ can then easily be deterministically constructed from this random maze. More precisely, we define a continuous function $(\tilde{S}_t, H_t)$ on $\mathbb{R}_+$ with $\tilde{S}_0 = -1/2$ and $H_0 = 1$, that explores the maze, at constant speed (the speed is $\sqrt{2}$; in other words, for almost all $t \geq 0$, $|\tilde{S}_t'| = |H_t'| = 1$), as shown on the picture. Note that at all integer times i, $\tilde{S}_i + 1/2 \in \mathbb{Z}$ and $H_i \in \mathbb{N}\backslash\{0\}$ (these points are dotted on the picture).

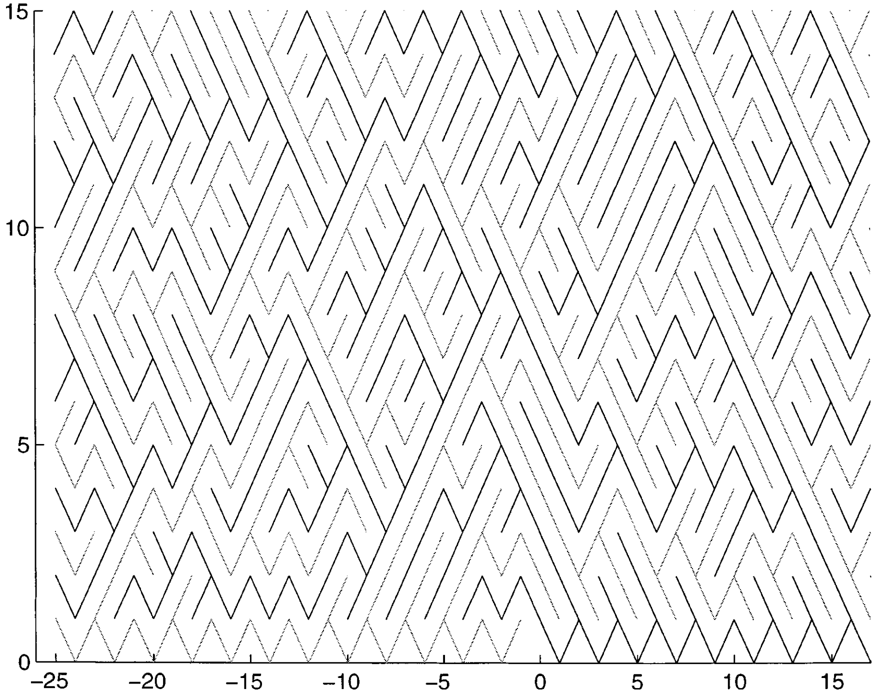

Fig. 1. The 'forward' and the 'backward' systems (resp. in black and grey)

It is then easy to see that the definitions of L^+ and of $\tilde{S}$ in terms of the family $(\xi(z,h))_{(z,h)\in\mathbb{G}^+}$ imply that for all $i \geq 0$,

$$S_i = \tilde{S}_i + 1/2 \tag{11.23}$$

$$H_i = \lambda(i, S_i) \ . \tag{11.24}$$

In order to derive (11.24) and (11.23), it suffices for example to consider the different possible patterns of the maze near the point (S_i, H_i) and check that in all possible cases, $S_{i+1} - S_i$ can indeed be expressed as in (11.15); we safely leave this to the reader.

Note also that at any integer time i, the local times $(\ell(i,y))_{y\in\mathbb{Z}}$ (and therefore also $(\lambda(i,y))_{y\in\mathbb{Z}}$ using (11.5)) can be described in terms of the forward and backward lines L^+ and L^-.

More precisely: Suppose that $i \in \mathbb{N}$ is fixed. Then, define S_i^+ and S_i^- in $\{S_i - 1, S_i\}$ such that $S_i^+ + H_i$ is odd and $S_i^- + H_i$ is even. This definition implies that

$$\ell(i, S_i) = L^+_{(S_i^+, H_i)}(S_i) \tag{11.25}$$

$$\ell(i, S_i - 1) = L^+_{(S_i^-, H_i)}(S_i - 1) \tag{11.26}$$

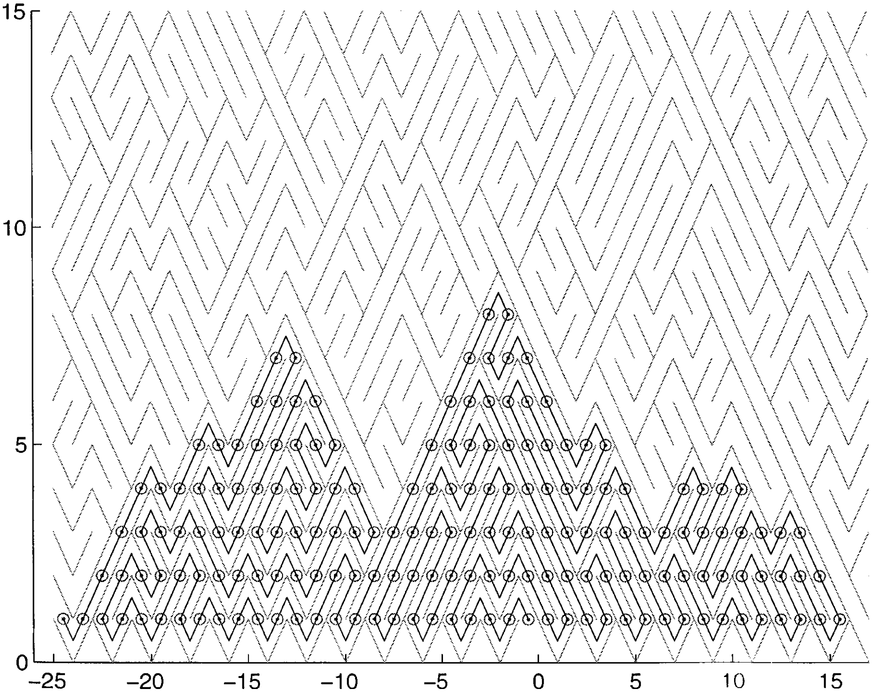

Fig. 2. The maze and the path of $(\tilde{S}_t, H_t)$. The points $(\tilde{S}_i, H_i)$ are circled

(it suffices again to consider the different possible patterns of the maze near (S_i, H_i) and check that this holds in all cases). Then, for all $y \geq S_i$,

$$\ell(i,y) = L^{+}_{(S_i^+, H_i)}(y) \tag{11.27}$$

and for all $y \leq S_i - 1$,

$$\ell(i,y) = L^{-}_{(S_i^-, H_i)}(y) \ . \tag{11.28}$$

Loosely speaking, the forward and backward lines started near the point (S_i, H_i) are exactly the local times of S at time i. Let us now briefly derive (11.27) ((11.28) then follows by symmetry); (11.25) shows that (11.27) holds when $y = S_i$. Suppose now that $y \geq S_i$ is fixed, that (11.27) holds and that $\ell(i,y) \geq 2$ (the case where $(y, \ell(i,y)) \in \mathfrak{G}_0^+$ is obvious). (11.7) shows that $\ell(i, y+1) \in \{\ell(i,y) - 1, \ell(i,y) + 1\}$. In particular (using (11.5)), $\lambda(i, y+1) \geq \ell(i,y) - 1$. Hence, there exists $j \leq i$ such that

$$S_j = y + 1 \text{ and } \lambda(j, y+1) = \ell(i,y) - 1 \ . \tag{11.29}$$

Using (11.10) and (11.9) (note that $\lambda(j, S_j) + S_j = \ell(i,y) + y$ is odd), this implies that

$$S_j = y + 1 \text{ and } \ell(j, S_j) = \ell(j, S_j - 1) = \ell(i,y) - 1 \tag{11.30}$$

It is then easy to notice that if $\xi(y, \ell(i,y)) = +1$ (i.e. if $S_{j+1} = S_j + 1$) then necessarily $\ell(i, y+1) = \ell(i,y) + 1$ (S has to jump a second time on the edge $y+1 \leftrightarrow y+2$ to go to the left of $y+1$ before step i), and that if $\xi(y, \ell(i,y)) = -1$ (i.e. if $S_{j+1} = y$), then $\ell(i, y+1) = \ell(i,y) - 1$ (as S can not jump again on the edge $y \leftrightarrow y+1$ before i). Finally, this shows that indeed

$$\ell(i, y+1) - \ell(i,y) = \xi_{(y, \ell(i,y))} \tag{11.31}$$

and combining this with (11.18) implies (11.27).

Remark: Note that it is also possible, using a similar method, to construct the discrete counterpart of the process $\check{X}$ presented in Section 10.

Appendix A: Preliminary estimates for RAB-s

In the present appendix we collect some a priori estimates on hitting probabilities of Brownian motions and reflected/absorbed Brownian motions. These estimates are used throughout the paper.

Recall the definition of RAB's from the notation section. In this subsection only estimates on independent (non-coalescing) Brownian motions and RABs will be presented.

For any fixed $k \geq 2$ (later on we will only use $k=2$ and $k=3$), $x \in \mathbb{R}$ and $0 \leq h_1 \leq \cdots \leq h_k$, we define k independent RAB's $R^1_{(x,h_1)}, \ldots, R^k_{(x,h_k)}$ (respectively k independent Brownian motions $B^1_{h_1}, \ldots, B^k_{h_k}$) started from $(x, h_1), \ldots, (x, h_k)$ (respectively from levels $h_1, \ldots, h_k$). We then define the following random variables (stopping times for Brownian motions, respectively RABs):

$$\tau^{(k)}_{h_1,\ldots,h_k} := \inf\{y > 0 : B^i_{h_i}(y) = B^j_{h_j}(y) \text{ for some } 1 \leq i < j \leq k\} \tag{A.1}$$

$$\begin{aligned} \sigma^{(k)}_{x;h_1,\ldots,h_k} := \inf\{y > 0 : R^i_{(x,h_i)}(y+x) = R^j_{(x,h_j)}(y+x) \\ \text{for some } 1 \leq i < j \leq k\} \end{aligned} \tag{A.2}$$

(Note the slight abuse of notation: actually $\tau^{(k)}_{h_1,\ldots,h_k} = \tau^{(k)}(B^1_{h_1}, \ldots, B^k_{h_k})$ and $\sigma^{(k)}_{x;h_1,\ldots,h_k} = \sigma^{(k)}(R^1_{(x,h_1)}, \ldots, R^k_{(x,h_k)})$. It is known that for any $v > 0$, and $h_1 \leq h_2 \leq h_3$

$$\mathbf{P}\left(\tau^{(2)}_{h_1,h_2} > v\right) \leq \frac{1}{\sqrt{\pi}} \frac{h_2 - h_1}{\sqrt{v}} =: C_2 \frac{h_2 - h_1}{\sqrt{v}} \tag{A.3}$$

$$\mathbf{P}\left(\tau^{(3)}_{h_1,h_2,h_3} > v\right) \leq C_3 \left(\frac{h_3 - h_1}{\sqrt{v}}\right)^3 . \tag{A.4}$$

Indeed, (A.3) is a straightforward classroom exercise, and (A.4) can be viewed as the hitting time of a wedge of angle $\pi/6$ by the two-dimensional Brownian motion $((B^2 - B^1)/\sqrt{2}, (2B^3 - B^2 - B^1)/\sqrt{6})$ for which such estimates are well-known (using the skew-product decomposition of planar Brownian motion; see for instance Lemma 2.2 in Mountford [M]).

Remark 1: A slightly weaker version of (A.4) with $3 - \varepsilon$ in the exponent, rather than 3 follows from Theorem 2 of Spitzer [S].

Remark 2: Actually there is a stronger recent result of Grabiner [G], that in particular implies that for any $k \geq 2$ there is a finite constant C_k such that for any $v > 0$ and $h_1 \leq \cdots \leq h_k$

$$\mathbf{P}\left(\tau^{(k)}_{h_1,\dots,h_k} > v\right) \leq C_k \left(\frac{h_k - h_1}{\sqrt{v}}\right)^{k(k-1)/2} \tag{A.5}$$

but in the present paper we will not use more than (A.3) and (A.4).

We need estimates similar to (A.3) and (A.4) for the stopping times $\sigma^{(2)}$ and $\sigma^{(3)}$rather than $\tau^{(2)}$ and $\tau^{(3)}$.

Lemma A.1. *For all* $v > 0$, $x \in \mathbb{R}$ *and* $0 \leq h_1 \leq h_2 \leq h_3$

$$\mathbf{P}\left(\sigma^{(2)}_{x;h_1,h_2} > v\right) \leq C'_2 \frac{h_2 - h_1}{\sqrt{v}} \tag{A.6}$$

$$\mathbf{P}\left(\sigma^{(3)}_{x;h_1,h_2,h_3} > v\right) \leq C'_3 \left(\frac{h_3 - h_1}{\sqrt{v}}\right)^3 \ . \tag{A.7}$$

with $C'_2 := \sqrt{2} C_2$ *and* $C'_3 := 3^{3/2} C_3$

Proof. Note first that absorbtion at 0 delays the first collision times $\sigma^{(k)}$ (compared to reflection) so that for any $k \geq 2$, $x \in \mathbb{R}$ and $0 \leq h_1 \leq \cdots \leq h_k$

$$\mathbf{P}\left(\sigma^{(k)}_{-v;h_1,\dots,h_k} > v\right) \leq \mathbf{P}\left(\sigma^{(k)}_{x;h_1,\dots,h_k} > v\right) \leq \mathbf{P}\left(\sigma^{(k)}_{0;h_1,\dots,h_k} > v\right) \tag{A.8}$$

We prove (A.6) by a simple coupling argument. Let $B^1_{h_1}(\cdot)$ and $B^2_{h_2}(\cdot)$ be two independent Brownian motions started from h_1 and h_2 and assume that $R^1_{(0,h_1)}$ and $R^2_{(0,h_2)}$ are exactly $B^1_{h_1}$ and $B^2_{h_2}$ stopped at their first hitting time of zero. Beside the stopping time $\sigma^{(2)}$ we define:

$$\theta^{(2)}_{h_1,h_2} := \inf\{y > 0 : B^1_{h_1}(y) = 0 \text{ or } B^1_{h_1}(y) = B^2_{h_2}(y)\} \ . \tag{A.9}$$

Note that $\theta^{(2)}_{h_1,h_2} \leq \sigma^{(2)}_{0;h_1,h_2}$. We also define

$$\widetilde{\sigma}^{(2)}_{h_1,h_2} := \theta^{(2)}_{h_1,h_2} + 2\left(\sigma^{(2)}_{0;h_1,h_2} - \theta^{(2)}_{h_1,h_2}\right) \ . \tag{A.10}$$

Note that this definition implies that

$$\widetilde{\sigma}^{(2)}_{h_1,h_2} \geq \sigma^{(2)}_{0;h_1,h_2} \ . \tag{A.11}$$

We then define, for all $y \geq 0$,

$$u(y) := \begin{cases} y & \text{if} \quad y < \theta^{(2)}_{h_1,h_2} \\ 2(y - \theta^{(2)}_{h_1,h_2}) + \theta^{(2)}_{h_1,h_2} & \text{if} \quad y \geq \theta^{(2)}_{h_1,h_2} \end{cases} \tag{A.12}$$

and

$$\beta(y) = B^2_{h_2}(u(y)) - R^1_{(0,h_1)}(u(y)) \ . \tag{A.13}$$

Using the strong Markov property it is straightforward to check that $\beta/\sqrt{2}$ is a linear Brownian motion started from $(h_2 - h_1)/\sqrt{2}$ and that $\widetilde{\sigma}^{(2)}_{h_1,h_2}$ is exactly the hitting time of 0 by β. Hence, combining this with (A.11) yields (A.6).

The proof of (A.7) is based on exactly the same idea: This time, one can use a coupling with three independent absorbed Brownian motions. This is safely left to the reader.

□Lemma A.1.

Another straightforward estimate, opposite to (A.3) is the following:

$$\mathbf{P}\left(\tau^{(2)}_{h_1,h_2} < v\right) \leq \mathrm{e}^{-(h_2-h_1)^2/(4v)} \ . \tag{A.14}$$

Again, we shall need a similar estimate for the first collision time of two RABs rather than two BMs.

Lemma A.2. *For all* $v > 0$, $x \in \mathbb{R}$ *and* $0 \leq h_1 \leq h_2$

$$\mathbf{P}\left(\sigma^{(2)}_{x;h_1,h_2} < v\right) \leq 2\mathrm{e}^{-(h_2-h_1)^2/(4v)} \ . \tag{A.15}$$

Proof. Recall from (A.8) that

$$\mathbf{P}\left(\sigma^{(2)}_{x;h_1,h_2} < v\right) \leq \mathbf{P}\left(\sigma^{(2)}_{-v;h_1,h_2} < v\right) \tag{A.16}$$

But a simple reflection argument shows that

$$\mathbf{P}\left(\sigma^{(2)}_{-v;h_1,h_2} < v\right) \leq 2\mathbf{P}\left(\tau^{(2)}_{h_1,h_2} < v\right) \tag{A.17}$$

and, using (A.15), the Lemma follows.

□ LemmaA.2.

Beside (A.3) and (A.14), the reflection principle yields also the precise asymptotics

$$\lim_{\delta \downarrow 0} \delta^{-1} \mathbf{P}\left(\tau^{(2)}_{h,h+\delta} > v\right) = \frac{1}{\sqrt{\pi v}} \ . \tag{A.18}$$

Again, we formulate the analogous statement for RABs:

Lemma A.3. *For all* $x \in \mathbb{R}$, $h > 0$ *and* $v > 0$

$$\lim_{\delta \downarrow 0} \delta^{-1} \mathbf{P}\Big(\sigma^{(2)}_{x;h,h+\delta} > v\Big) = \frac{1}{\sqrt{\pi v}} \quad . \tag{A.19}$$

The convergence is uniform in $(x, h, v) \in (-\infty, \infty) \times [h_0, \infty) \times [v_0, \infty)$ *for any* $h_0 > 0$ *and* $v_0 > 0$ *fixed.*

The proof of this statement is left for the reader.

Appendix B: Phenomenological derivation of the driving mechanism

As mentioned in the Introduction, in Tóth [T1] a limit theorem was proved, essentially for the distribution of $n^{-2/3} S_n$ as $n \uparrow \infty$, but the natural question of the asymptotics of the *process*

$$X^{(A)}_t := A^{-2/3} S_{[At]}, \quad t \in \mathbb{R}_+ \tag{B.1}$$

in the limit $A \uparrow \infty$ remained open. In the following paragraphs we argue that, if the sequence of processes $t \mapsto X^{(A)}_t$ converges in distribution to a process $t \mapsto X^{(\infty)}_t$, as $A \uparrow \infty$, then the limit process is driven by the gradient of its local time, as claimed in (1.6). We warn the reader that the forthcoming argument is based on a somewhat *formal computation* and it is by no means mathematically rigorous, but we hope, it provides a convincing motivation for the construction of a process with the prescribed properties.

Beside the scaled position process $t \mapsto X^{(A)}_t$ defined in (B.1) we define the properly scaled local time process of the true self-avoiding random walk

$$L^{(A)}_t(x) := A^{-1/3} l_{[At]}([A^{2/3} x]), \quad t \in \mathbb{R}_+, \ x \in \mathbb{R} \tag{B.2}$$

and we assume that the sequences of processes $X^{(A)}_\cdot$ and $L^{(A)}(\cdot, \cdot)$ converge jointly weakly:

$$\Big(X^{(A)}_\cdot, L^{(A)}_\cdot(\cdot)\Big) \Rightarrow \Big(X^{(\infty)}_\cdot, L^{(\infty)}_\cdot(\cdot)\Big) \tag{B.3}$$

where $(t, x) \mapsto L^{(\infty)}_t(x)$ is assumed to be the local time of the process $t \mapsto X^{(\infty)}_t$. Let $\mathscr{F}_n$ be the σ-algebra generated by $(S_0, \ldots, S_n)$, then

$$\mathbf{E}\Big(S_{n+1} - S_n \Big| \mathscr{F}_n\Big) = -\tanh(g(l_n(S_n) - l_n(S_n - 1))) \tag{B.4}$$

$$\mathbf{Var}\Big(S_{n+1} - S_n \Big| \mathscr{F}_n\Big) = \cosh^{-2}(g(l_n(S_n) - l_n(S_n - 1))) \tag{B.5}$$

So:

$$S_n + \sum_{k=0}^{n-1} \tanh(g(l_k(S_k) - l_k(S_k - 1))) =: M_n \tag{B.6}$$

where M_n is a martingale with quadratic variation process

$$\langle M \rangle_n = \sum_{k=0}^{n-1} \cosh^{-2}(g(l_k(S_k) - l_k(S_k - 1))) < n \tag{B.7}$$

Our object of study is the scaled form of (B.6):

$$A^{-2/3} S_{[At]} + A^{-2/3} \sum_{k=0}^{[At]-1} \tanh(g(l_k(S_k) - l_k(S_k - 1))) = A^{-2/3} M_{[At]} \tag{B.8}$$

The first term on the left-hand side of (B.8) is just $X_t^{(A)}$. From (B.7) in particular it follows that for any $T < \infty$

$$P\text{-}\lim_{A \uparrow \infty} \left(\sup_{0 \le t \le T} \left| A^{-2/3} M_{[At]} \right| \right) = 0 \tag{B.9}$$

so that the right hand-side of (B.8) is asymptotically negligible. A formal computation of the second term on the left-hand side of (B.8) follows: the first two steps are straightforward transformations using the definitions (B.1) and (B.2) of the scaled process and scaled local time:

$$\begin{aligned} & A^{-2/3} \sum_{k=0}^{[At]-1} \tanh(g(l_k(S_k) - l_k(S_k - 1))) \\ &= A^{-1} \sum_{k=0}^{[At]-1} A^{1/3} \tanh\Big(g\Big(l_{Ak/A}(A^{2/3} X_{k/A}^{(A)}) - l_{Ak/A}\Big(A^{2/3} X_{k/A}^{(A)} - 1\Big)\Big)\Big) \\ &= A^{-1} \sum_{k=0}^{[At]-1} A^{1/3} \tanh\Big(g A^{1/3}\Big(L_{k/A}^{(A)}\Big(X_{k/A}^{(A)}\Big) - L_{k/A}^{(A)}\Big(X_{k/A}^{(A)} - A^{-2/3}\Big)\Big)\Big) \end{aligned} \tag{B.10}$$

The next step is the formal, non-rigorous one: we treat formally $L_t^{(A)}(x)$ as a smooth function and replace $L_t^{(A)}(x) - L_t^{(A)}(x - \delta x)$ by $\frac{\partial L_t^{(A)}(x)}{\partial x} \delta x$ and get

$$\begin{aligned} & A^{-2/3} \sum_{k=0}^{[At]-1} \tanh\big(g(l_k(S_k) - l_k(S_k - 1))\big) \\ \text{“}=\text{”} & A^{-1} \sum_{k=0}^{[At]-1} A^{1/3} \tanh\left(g A^{1/3} A^{-2/3} \frac{\partial L_{k/A}^{(A)}\Big(X_{k/A}^{(A)}\Big)}{\partial x} \right) \\ \text{“}=\text{”} & g A^{-1} \sum_{k=0}^{[At]-1} \frac{\partial L_{k/A}^{(A)}\Big(X_{k/A}^{(A)}\Big)}{\partial x} + \mathcal{O}\Big(A^{-1/3}\Big) \\ \text{“}\Rightarrow\text{”} & g \int_0^t \frac{\partial L_s^{(\infty)}\big(X_s^{(\infty)}\big)}{\partial x} \, ds \end{aligned} \tag{B.11}$$

With the quotation marks "$\cdots$" we intend to emphasize that these last equalities and convergence should not be taken too seriously. Inserting (B.1), (B.9) and (B.11) into (B.8) we get

$$X_t^{(\infty)} + \text{const.} \int_0^t \frac{\partial L_s^{(\infty)}\left(X_s^{(\infty)}\right)}{\partial x} ds = 0 \tag{B.12}$$

which is indeed somewhat reminiscent of (1.6). The effect of 'pushing the boundaries of the range' and the right constant in front of the gradient term can not be recovered on this level of formal computations. We repeat again: this computation is nothing like rigorous, but on the phenomenological level it is convincing.

The same reasoning (on the same level of 'rigour') can be applied to the 'polymer model' proposed by Durrett and Rogers in [DR]:

$$X_t = B_t + \int_0^t ds \int_0^s du f(X_s - X_u) \tag{B.13}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function of compact support and satisfies $f(-x) = -f(x)$ and $\operatorname{sgn} f(x) = \operatorname{sgn} x$. Defining $X_t^{(A)} = A^{-2/3} X_{At}$, in the limit $A \to \infty$ f transforms into δ' and the same dynamical driving mechanism is found.

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