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Discontinuity of the Percolation Density in One Dimensional $1/|x-y|^2$ Percolation Models

M. Aizenman[1,*,**] and C. M. Newman[2,*,***]

[1] Departments of Mathematics and Physics, Rutgers University, New Brunswick, NJ 08903, USA

[2] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

Abstract. We consider one dimensional percolation models for which the occupation probability of a bond – $K_{x,y}$, has a slow power decay as a function of the bond's length. For independent models – and with suitable reformulations also for more general classes of models, it is shown that: i) no percolation is possible if for short bonds $K_{x,y} \leqq p < 1$ and if for long bonds $K_{x,y} \leqq \beta/|x-y|^2$ with $\beta \leqq 1$, regardless of how close p is to 1, ii) in models for which the above asymptotic bound holds with some $\beta < \infty$, there is a discontinuity in the percolation density M ($\equiv P_\infty$) at the percolation threshold, iii) assuming also translation invariance, in the nonpercolative regime, the mean cluster size is finite and the two-point connectivity function decays there as fast as $C(\beta, p)/|x-y|^2$. The first two statements are consequences of a criterion which states that if the percolation density M does not vanish then $\beta M^2 \geqq 1$. This dichotomy resembles one for the magnetization in $1/|x-y|^2$ Ising models which was first proposed by Thouless and further supported by the renormalization group flow equations of Anderson, Yuval, and Hamann. The proofs of the above percolation phenomena involve (rigorous) renormalization type arguments of a different sort.

1. Introduction

The percolation phenomenon is of considerable interest both because it offers an instructive example of a phase transition, and because of the key role which it plays in various situations. The critical behavior which is associated with percolation is nontrivial even in models with noninteracting (i.e. independent) variables. One

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often finds that the analysis of the phase transition in such systems is further rewarded by the insights which it lends to the understanding of the physics and the mathematics of phase transitions in various other models of statistical mechanics.

In this work we shall focus on a new percolation effect, which can be viewed as a natural extension of an Ising spin system phenomenon which was suggested by Thouless [1] and further discussed in the work of Anderson, Yuval and Hamann [2]. The paper has a dual purpose – in line with the above comments on the general subject. One of its goals is to present a self contained contribution to the theory of the percolation transition. At the same time, the general results derived here will form a basis for the rigorous proof of the existence of the Thouless-Anderson-Yuval-Hamann effect in the analogous Ising-spin models. That application of this work is not discussed here, and can be found in the companion paper [3] (co-authored jointly with J.T. Chayes and L. Chayes).

Let us now introduce the models we consider and briefly describe some of our main results.

i) The Setup

We consider here bond percolation models on the one dimensional lattice $\mathbb{Z}$. With *any* pair of sites $(x, y \in \mathbb{Z})$ there is associated a bond $(b = \{x, y\})$, and a random variable n_b (or $n_{x,y}$) whose value indicates whether the bond is "occupied" $(n_b = 1)$ or not $(n_b = 0)$. The system's configuration is given by a set of values of these variables, or equivalently, by the set of the occupied bonds. A model is characterized by the joint distribution of the occupation variables $\{n_b\}$.

For each configuration we regard the occupied bonds as connecting, and decompose the lattice $\mathbb{Z}$ into its connected clusters. In particular, one may ask whether a given site – say the origin $0 \in \mathbb{Z}$, belongs to an infinite connected cluster. The probability of this event is the percolation density (often referred to as P_∞). We shall denote it here by the symbol M.

The main object of our discussion is the dependence of the percolation density M on the model's parameters, and in particular the nature of the phase transition at the percolation threshold. The most striking effects are found within the class of models with a "const/$|x-y|^2$ law" for the asymptotic behavior of the bond occupation probability.

In our guiding examples the bonds are occupied independently of each other, with translation invariant probabilities:

$$\mathrm{Prob}(n_{x,y} = 1) = K_{x-y} . \tag{1.1}$$

We shall usually assume that $K_x < 1$ for all x. In the discussion of general models, where we include systems which are neither independent, nor translation invariant, we shall assume that $K_x^+ < 1$ for all x, where K^+ is the function defined by (3.1), below. Such models will be refered to as *regular*.

There is in fact considerable interest in the so-called correlated percolation models, for which the occupation variables are dependent. Examples of that sort appear in the percolation representations of spin systems (see [3] and the references discussed there). Each of our results is in fact formulated for a larger class than that of the independent models. For some, the necessary restriction is just a bound on the strength of the dependence, whereas for the strongest result we also need a correlation inequality restriction on the nature of the dependence. To keep this introduction simple, we shall discuss here only the independent case. Further details on dependent models can be found in the relevant sections.

ii) Some Background

It is well known that in one dimension there is no percolation in finite range, (independent) translation invariant regular models. It is not difficult to extend this statement to models of unbounded range [4] for which

$$\sum_{x<0}\sum_{y\geqq 0} K_{y-x} \equiv \sum_{z>0} zK_z < \infty . \tag{1.2}$$

For long range models with an asymptotic power law falloff, $K_x \approx \mu/|x|^{-s}$, the above criterion shows the absence of percolation if $s>2$ (regardless of μ and the short range behavior of K_x). In a recent paper, Newman and Schulman [5] showed that for all the other powers, i.e. $s\leqq 2$, percolation may in fact occur. This, for $s>1$, is a non-trivial result – whose borderline case $s=2$ requires particular attention. (In [3] its relation to the result of Fröhlich and Spencer [6], concerning the analogous Ising model problem, is analysed.) Before focusing on this case, let us make two other general remarks.

When considering long range percolation, one should be also aware of the possibility that $M=1$ without any of the bond densities being 1. In fact, for independent models that occurs if and only if

$$\sum_x K_x = \infty . \tag{1.3}$$

This is so because (1.3) implies that with probability one, for any site x there are infinitely many sites y with $n_{x,y}=1$. The above considerations show that the interesting regime for the power in the asymptotic falloff is $2\geqq s>1$ ($M\equiv 0$ for $s>2$, and $M\equiv 1$ for $s\leqq 1$). Furthermore, the general "mean-field" (or Bethe lattice-) bound for independent models [4, 7] shows that if

$$\sum_x K_x < 1 , \tag{1.4}$$

then $M=0$. It follows that if one systematically (and thoroughly) "weakens" any of the models in the above class, for which there is percolation but (1.3) is not satisfied, one would definitely reach a transition from the percolating phase to the non-percolative regime.

One of our main results is that within the "borderline" class of models, with $s=2$, the above phase transition always exhibits a discontinuity in M (i.e. P_∞)!

iii) Main Results for Independent Models

Our key result for independent models is the following proposition:

Proposition 1.1. *If in an independent, translation invariant and regular, one dimensional bond percolation model the percolation density M does not vanish, then*

$$\beta M^2 \geqq 1 , \tag{1.5}$$

where β is defined by

$$\beta = \limsup_{x \to \infty} (K_x |x|^2). \tag{1.6}$$

The above lower bound reminds one of the inequality proposed by Thouless for the spontaneous magnetization in $1/|x-y|^2$ Ising spin systems. The original argument of Thouless is based on rather beautiful "energy versus entropy" considerations (see also [8]) which, however, are not fully conclusive even at the physical level, since they leave an "escape clause" (concerning the possible existence of an intermediate phase just below the critical temperature) which appears now to be operative [9, 10]. The validity of such a bound for Ising systems received further support in the work of Anderson et al. [2], which is remarkable in its early use of "renormalization group" arguments. (Other work on this and related problems will be discussed in [3].)

Our derivation of the dichotomy: "$M=0$ or $\beta M^2 \geqq 1$" is done in two steps, the second of which is based on rigorous renormalization type arguments. In that sense, and in the picture which emerges, our work is close in its spirit to that of Anderson et al. [2]. As has already been indicated, our results are valid for more general (i.e. not only independent) percolation models. In fact, they provide a key ingredient for the rigorous proof of an improved Thouless inequality and the resulting discontinuity in the spontaneous magnetization for Ising systems [3].

Proposition 1.1 has two important implications. The first is essentially the statement which is derived as the first step in the proof (see Propositions 2.1 and 3.1 below), that $M=0$ in any regular model for which $\beta \leqq 1$ (regardless of the short range values of K_x). For example, in a model where $K_x = 1-\exp(-J_x/T)$ with $x^2 J_x$ tending to zero as $x \to \infty$ (no matter how slowly), M remains zero no matter how small the temperature parameter T.

It is interesting to note that this implication complements the result of Newman and Schulman [5] which we shall state below. Together, the two establish the asymptotic falloff $K_x \approx 1/|x|^2$ as the critical rate for the possibility of percolation.

The other implication of Proposition 1.1 is, of course, the discontinuity in the percolation density at that part of the critical manifold (in the parameter space) for which $\beta < \infty$.

To make these points clear, and to demonstrate another application, let us consider here the following family of models with just two continuous parameters ($0 \leqq \beta < \infty$, $0 \leqq p < 1$):

$$K_x = \begin{cases} 1-\exp(-\beta/|x|^2) & \text{for} \quad |x| > L, \\ p & \text{for} \quad |x| \leqq L, \end{cases} \tag{1.7}$$

where L is some fixed integer. (Note that $K_x \approx \beta/|x|^2$.)

The result of Newman and Schulman [5] about the above system is that as long as $\beta > 1$ there is a critical value, $p_c(\beta) < 1$, such that the above models exhibit percolation for $p > p_c$. [That critical value $p_c(\beta)$ is strictly positive only as long as β is less than some β_L. However, by (1.4), β_L can be made arbitrarily large by a choice of sufficiently large L.]

To this rather strong result we may now add the following information on the phase structure (see Fig. 1), and the nature of the phase transition.

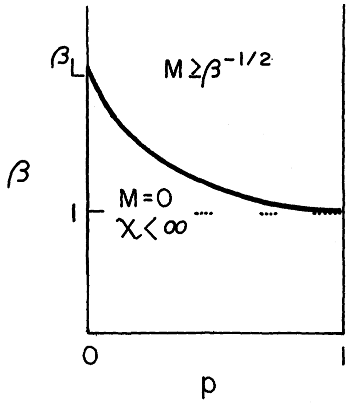

Fig. 1. The phase diagram of the independent percolation model described by (1.7)

a) The value $\beta=1$ is critical in the sense that the system exhibits a percolation transition with $p_c(\beta)<1$ only when $\beta>1$.

b) The percolation density changes discontinuously across the critical line $\{(\beta, p_c(\beta))\,|\,1<\beta<\beta_L\}$. The magnitude of the jump is at least $\beta^{-1/2}$.

c) $p_c(\beta)\to 1$, as $\beta\to 1$.

Remark. While the above statements are direct implications of the main proposition, for the proof of c) one also uses the simple observation that if $M\to 1$ while $\beta \leqq \text{const}$, then necessarily $p\to 1$. A similar argument is employed in proving the absence of percolation for $\beta=1$.

We shall present here also the following bounds on the asymptotic behaviour of the connectivity function $\tau(x,y)\equiv \text{Prob}(x \text{ and } y \text{ are connected})$, one of which applies to all of the nonpercolative regime in one dimensional $\mu/|x-y|^s$ models, and the other to such models (with $s\leqq 2$) at their critical points β_c (defined as the threshold for M to be positive).

Proposition 1.2. *In any translation invariant, independent, percolation model for which $K_x \leqq \text{const}/|x|^s$ with some $s>1$:*

i) *for every $\beta<\beta_c$, the connectivity function obeys the bound*

$$\tau(x,y)\leqq C/|x-y|^s \tag{1.8}$$

with some $C<\infty$ depending on β,

ii) *at $\beta=\beta_c$, for $s\leqq 2$,*

$$\limsup_{|x|\to\infty} \tau(0,x)\,|x|^{2-s}>0\,. \tag{1.9}$$

Remarks. i) An opposite bound to (1.8) is provided by the obvious inequality $\tau(x,y)\geqq K_{x-y}$. Thus, if K_x is given by (1.7) with the power 2 replaced by s (>1), then for all $\beta<\beta_c$: $\lim_{x\to\infty} -\ln\tau(0,x)/\ln|x|=s$. Note that $\beta_c=\infty$ for $s>2$.

ii) For $s=2$ the lower bound (1.9) can be strengthened to: $\tau(0,x)\geqq M^2$ for all x, provided the model is irreducible – in the sense that the lattice cannot be decomposed into two, or more, uncoupled sublattices. This statement uses the recent result of [11].

iii) The proof of (1.8) has two ingredients. One is the Simon inequality technique [12] (adapted to long range models [13]), and the other is the recent result on the finiteness of the expected cluster size below β_c [14]. The latter statement is given here an alternative proof for $s \geqq 2$ which is based on the renormalization type arguments developed below. This alternative proof applies also to certain models with dependent bond variables.

iv) Our result of [7] is also applicable here, showing that χ diverges (with a critical exponent $\gamma \geqq 1$), and hence also $C \to \infty$, as the percolation threshold is approached, e.g. when $p \to p_c(\beta) - 0$ in the model described by (1.7). Thus, when $s=2$ the discontinuity of the order parameter M does not correspond to an ordinary first order phase transition.

As a conclusion of the introduction let us summarize the structure of the remainder of this paper. In Sect. 2 an independent continuum model is presented which clarifies the significance of the values $s=2$ and $\mu=1$ in $\mu/|x-y|^s$ models (when $s=2$, the coefficient μ is referred to as β). The analysis of the continuum model introduces many of the basic techniques used later in the paper including the notions of dissociation and maximal nested sequences (of occupied bonds). In Sect. 3 we prove in Proposition 3.1, which is the main result there, a generalized version of what is mentioned above as the first "implication" of Proposition 1.1: that $\beta < 1$ implies $M=0$. The proof uses a continuum model result of Sect. 2, by means of a comparison argument (Lemma 3.3). The Thouless type inequality of Proposition 1.1, that $\beta M^2 < 1$ implies $M=0$, is proven in Sect. 4 in a generalized form which is applicable to a class of models which need be neither independent nor translation invariant, but have to satisfy a certain "strong FKG" condition. Both this result and its proof are "renormalized" versions of the analysis of Sect. 3, obtained by studying the system's "anchored bonds," for which the quantity $\beta_R = \beta M^2$ plays a role which is somewhat similar to that of β in the original system. In Sect. 5 we derive the results discussed above for the 2-point connectivity function.

2. A Continuum Model with a Phase Transition

As should be clear from the introduction, a special role is played for one dimensional percolation by the "$1/|x|^2$ law" for K_x. It turns out that the significance of both the power $s=2$ and the coefficient $\beta=1$ (the meaning of these parameters is explained in the introduction) can be demonstrated in the properties of a simple continuum model, to which the lattice model on $\mathbb{Z}$ is related via a discretization procedure. The continuum model may also be viewed as a limit of the lattice system under a natural rescaling procedure (in which a certain amount of information is lost). We start this section by presenting this model, which was introduced in the work of Newman and Schulman [5] (see also [2], where a continuum Ising model is discussed).

i) The Continuum Bond Model

In the continuum bond model, occupied bonds between points $x<y$ on the real line form a "gas of particles" with the "fugacity" $\varrho(x,y)=\mu/|x-y|^s$. In probabilistic language, the occupied bonds are described by an inhomogeneous Poisson process on $\mathbb{R}_0^2 \equiv \{(x,y) \mid -\infty<x<y<\infty\}$, with the density $\varrho(x,y)$.

In other words: the numbers of occupied bonds in disjoint regions $V_1, V_2 \ldots$ of $\mathbb{R}_0^2$ are independent random variables. Each such variable has a Poisson distribution with the mean $\int_V \varrho(x,y)dxdy$.

A special feature of the power $s=2$ is that the continuum model is invariant under dilations, as well as translations, of $\mathbb{R}$. In particular, in that case μ is a dimensionless quantity, and hence its value may indeed be of independent significance. (For powers other than 2, μ would change under the scaling $x\to Lx$ to $L^{2-s}\mu$.) The latter point is expressed explicitly by (2.2) below.

The continuum model may be discretized by partitioning $\mathbb{R}$ into unit intervals $\{[x,x+1)\}_{x\in\mathbb{Z}}$, which are naturally associated with the sites of $\mathbb{Z}$. The discrete model bond between $x,y\in\mathbb{Z}$ will be regarded as occupied if there is an occupied continuum bond linking the two corresponding intervals. The probability for such a bond to occur is then

$$K_{x-y}=1-\exp(-\mu J_{x-y})$$

with (2.1)

$$J_{x-y}=\int_{[x,x+1)} du \int_{[y,y+1)} dv|u-v|^{-s}.$$

For $s\geqq 2$, this definition must be modified when $|x-y|=1$ by only considering continuum bonds with $|u-v|>\varepsilon$ [and integrating over only such bonds in (2.1)], since otherwise the divergence in the integral would yield $K_1=1$. Note that (2.1) implies

$$K_z\approx\mu J_z\approx\mu/|z|^s \quad \text{as} \quad |z|\to\infty.$$

When $s=2$ this model is qualitatively the same as the model described in the introduction [see (1.7)]. The parameter β defined by (1.6) takes here the value:

$$\beta=\begin{cases}\mu & \text{for} \quad s=2\\ \infty & \text{for} \quad s<2, \quad \text{and any} \quad \mu>0\\ 0 & \text{for} \quad s>2, \quad \text{and any} \quad \mu<\infty.\end{cases} \tag{2.2}$$

The special role of the value $\beta=1$ (for the independent bond models), and hence also of the power $s=2$, is related to the following somewhat surprising fact about the above continuum system (related to results of Mandelbrot [15]). Although the probability that a *given* finite interval $[a,b]$ has no occupied bond connecting it with its complement vanishes for $s=2$ and any value of $\beta>0$, it turns out that for $\beta<1$ – with probability one, the entire line $\mathbb{R}$ is covered by such intervals. This statement (which does not hold for $\beta\geqq 1$) can be proven in a number of ways. We shall derive it here by the method which we later apply directly to the discrete systems. First however let us present a simple calculation which may offer considerable insight.

ii) The Significance of the Values $s=2$ and $\beta=1$

Consider the event, in the discretized model defined by (2.1), that a site z in the block $\{1,2,\ldots,L\}$ has the property that no occupied bond occurs between any $x<z$ and any y in $\{z+1,\ldots,L\}$ nor between any x in $\{1,\ldots,z-1\}$ and any $y>z$. The probability of this event is:

$$\left(\prod_{x=-\infty}^{z-1}\prod_{y=z+1}^{L}(1-K_{x-y})\right)\cdot\left(\prod_{x=1}^{z-1}\prod_{y=L+1}^{\infty}(1-K_{x-y})\right), \tag{2.3}$$

which can be seen to be independent of z and equal to

$$\exp\left(-\mu\int_0^{L-1}du\int_L^{\infty}dv|u-v|^{-s}\right) = \begin{cases} L^{-\mu} & \text{for } s=2 \\ \exp\{-\mu[(s-1)|2-s|]^{-1}\cdot|L^{2-s}-1|\} & \text{for } 1<s\neq 2.\end{cases} \tag{2.4}$$

The expected number of such z's in $\{1,\ldots,L\}$, which is L times this probability, tends therefore to zero for $s<2$ and to infinity for $s>2$, and in the borderline case of $s=2$ its behavior depends on whether $\mu>1$, in which case the expected number still vanishes, or $\mu<1$, for which it diverges as $L\to\infty$. More succinctly – the behavior depends on whether the value of the parameter β given by (2.2) is above, or below, 1.

The above result for $\beta>1$ is related to the proof in [5] of the existence of percolation in such models. Conversely, for $\beta<1$ the conclusion drawn above suggests that in certain respects the model is like the $s>2$ models satisfying (1.2). In the following discussion we shall see that this is the case even for the continuum model with no cutoff for small $|u-v|$. Henceforth we shall set $s=2$, and thus $\mu=\beta$.

iii) The Dissociation of the Line for $\beta<1$

We turn now to the proof of the claim made at the end of the first part of this section. Although one could pursue an argument based on the calculation (2.4) for the case $\beta<1$, we shall proceed by another route, presenting an argument which forms the basis for the rest of the analysis of this paper.

Before stating the main result, let us introduce the relevant concepts. For a given configuration of occupied bonds, we say that an interval $[a,b]$ *dissociates*, or, is dissociated, if there is no occupied bond linking it with its complement. It is easily seen that the set of dissociated intervals is closed under the operation of union. We shall also refer to the following random events (which we identify with the sets of the continuum bond configurations for which the corresponding conditions are satisfied):

$A_{L,k}$ – there is a dissociated finite interval $[\xi_-,\xi_+]$ with $\xi_+\in[L,kL)$ and $\xi_-\in(-kL,-L]$. (The range of values we consider is $1<k\leqq\infty$.)

$F_{L,k}$ – there is a point $\xi\in(L,kL)$, such that the interval $[0,\xi]$ is not linked by an occupied bond with (ξ,∞).

One may worry here about the question of the measurability of such events. Let us remark therefore that the above two conditions can be generated by countable operations, from the occupation numbers of the rational intervals (to have a countable example). It follows that $A_{L,k}$ and $F_{L,k}$ are measurable in the natural sense. At the same time, it should be noted that for any given (deterministic) interval the dissociation probability is zero – due to the ultraviolet singularity in (2.1).

It is easy to see that the sequence $A_{L,k}$ is increasing in k, and that its limit $A_L \equiv A_{L,\infty}$ is decreasing in L. In fact, $\bigcap A_L$ is the event that $\mathbb{R}$ is entirely covered by finite dissociated intervals. Despite the observation made above, the following is also true.

Proposition 2.1. *In the $s=2$ continuum model (with no cutoff), for each $\beta<1$:*

$$\mathrm{Prob}(\bigcap A_L)=1\,. \tag{2.5}$$

We shall derive this result here in a way which offers a blueprint for the proofs of other assertions made in this paper. Consequently, we shall not mention the fact that for the triple reason of translation invariance, dilation invariance, and measurability at infinity, $\mathrm{Prob}(\bigcap A_L)$ is either zero or one. With or without this observation, the main step in the proof is the derivation of a uniform lower bound, like the one required in the next lemma.

Lemma 2.1. *A sufficient condition for* (2.5) *is the existence of some $1<r<\infty$ and $g>0$ such that*

$$\mathrm{Prob}(F_{L,r})\geqq g\,, \quad \textit{for all} \quad L>0\,. \tag{2.6}$$

The argument we choose for the proof will be used also for the dependent models which will be considered in Sects. 3 and 4. For this reason, at some places it is not the most "economical" argument for the special model considered here. To avoid burdensome complications at this point of our discussion, we postpone the presentation of the proof of Lemma 2.1 until the end of this subsection.

Let us note that in the event $F_{L,\infty}$ there is a well defined (and measurable) *lowest* value of $\xi\in(L,\infty)$ for which the interval $[0,\xi]$ is not linked with (ξ,∞). Referring to *that* value as $\xi_{\min}$, let us define the variables

$$W_L=\begin{cases}\xi_{\min}/L(>1) & \text{in the event } F_{L,\infty}\\ \infty & \text{in the complement of } F_{L,\infty}\,,\end{cases} \tag{2.7}$$

and let $W=W_1$. By the scale invariance of our measures, the variables W_L have identical distributions – which may be deficient (they may even be totally concentrated at $+\infty$). These are clearly related to the probabilities of $F_{L,k}$. One has

$$\mathrm{Prob}(F_{L,k})=\mathrm{Prob}(W_L<k)=\mathrm{Prob}(W<k)\,. \tag{2.8}$$

Comparing (2.8) with (2.6), we see that the criterion formulated in Lemma 2.1 reduces to the proof that there is a positive probability for W to be finite. We shall prove the stronger statement that for any $\beta<1$, W is "almost surely" finite.

Lemma 2.2. *For any $\beta<1$, and $L>0$, W_L is finite with probability* 1. *Alternatively stated:*

$$\operatorname{Prob}(F_{L,k})=\operatorname{Prob}(W_L<k)\to 1\,, \quad as \quad k\to\infty\,, \tag{2.9}$$

and, in particular – the criterion (2.6) *in Lemma 2.1 is satisfied.*

Our analysis of $\xi_{\min}$ and hence of W and of the phenomenon of dissociation is based on notions which will be used again in Sects. 3 and 4. The proof of Lemma 2.2 is given below following the introduction of these concepts and the statement and proof of Lemma 2.3. The definitions given next are intended to apply also in discrete situations.

For a convenient representation of W_L it is useful to consider the process $\{L_n\}$ with $L_0=0$, $L_1=L$, and for $n>1$,

$$L_{n+1}=\begin{cases}\begin{array}{l}\text{supremum (actually – a.s. a maximum)}\\ \quad\text{of } y \text{ in } (L_n,\infty) \text{ such that there is an}\\ \quad\text{an occupied bond between}\\ \quad y \text{ and some } x \text{ in } (L_{n-1},L_n]\end{array} & \text{if there is such a point}\\ \\ \quad L_n & \text{otherwise}.\end{cases} \tag{2.10}$$

Thus, L_n is the point of the "furthest direct reach" from $(L_{n-1},L_n]$. By the construction, for each n there is no occupied bond between $[0,L_n]$ and (L_{n+1},∞), and (by induction) $L_n\leqq L\cdot W_L$. It follows that

$$\lim_{n\to\infty} L_n=L\cdot W_L \quad [\text{i.e. } =\xi_{\min}, \text{ in the sense of (2.7)}]\,. \tag{2.11}$$

Before we continue let us formalize the notion which is implicit in this construction.

Definition 2.1. For a given pair of initial points $y_0<y_1$, a sequence of bonds $\{x_i,y_i\}$ defined for $i>1$ is called *nested* if for all $n\geqq 1$,

$$y_{n-1}<x_{n+1}\leqq y_n<y_{n+1} \tag{2.12}$$

(see Fig. 2). Nested sequences are *ordered* lexicographically by the sequence $y_2,x_2,y_3,x_3,\dots$. A nested sequence of occupied bonds is called *maximal* for the given initial $\{y_0,y_1\}$, if it is maximal in the above order among all the nested sequences of occupied bonds with the same initial values.

In the above construction, the variables L_n form the values y_n of the (unique) maximal nested sequence of occupied bonds, with $y_0=0$ and $y_1=L$.

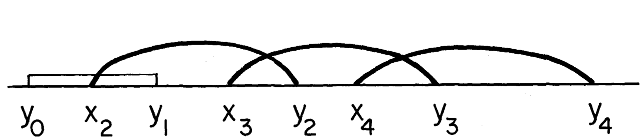

Fig. 2. A nested sequence of bonds

We shall learn about the limit in (2.11) by considering the increments:

$$U_n = L_n - L_{n-1}. \tag{2.13}$$

Their joint distribution is given by the following lemma.

Lemma 2.3. *For each L the conditional distribution of U_{n+1} given the previous U_i's is*

$$\mathrm{Prob}(U_{n+1} \leqq u \mid U_1 = L, U_2, \ldots, U_n) = (1 + U_n/u)^{-\beta}. \tag{2.14}$$

Proof. Let us consider the information acquired from the successive specification of the values of U_k, or equivalently of L_k. At the n^{th} step, the new information is that: i) there is no occupied bond in $(L_{n-2}, L_{n-1}] \times (L_n, \infty) \subset \mathbb{R}_0^2$, and ii) the interval $(L_{n-2}, L_{n-1}]$ is linked to arbitrarily small left-neighborhoods of L_n.

It is easy to see (e.g. inductively) that the information contained in the values of $\{U_1, \ldots, U_n\}$, reflects on the distribution of the occupied bonds only in the region $([0, L_{n-1}] \times \mathbb{R}) \cap \mathbb{R}_0^2 \left(\text{where } L_k = \sum_1^k U_j\right)$. Conditioned on these values, or in effect just on L_{n-1} and L_n, the event $\{U_{n+1} \leqq u\}$ corresponds to the statement that there is no occupied bond in $(L_{n-1}, L_n] \times (L_n + u, \infty)$. Since these two subsets of $\mathbb{R}_0^2$ are disjoint, the two sets of variables are *conditionally independent* – once the values of L_{n-1} and L_n are specified.

The quantity in the left-hand side of (2.14) can therefore be as easily calculated as the one in (2.1). We get

$$\begin{aligned}\mathrm{Prob}(U_{n+1} \leqq u \mid U_1 = L, U_2, \ldots, U_n) &= \exp\left(-\beta \int_{L_{n-1}}^{L_n} dy \int_{L_n+u}^{\infty} dz\, 1/|y-z|^2\right) \\ &= \exp\left(-\beta \int_{-U_n}^{0} dy \int_{u}^{\infty} dz\, 1/|y-z|^2\right) \\ &= [u/(u+U_n)]^{\beta}. \quad \square\end{aligned}$$

Proof of Lemma 2.2. We learn from (2.14) that the ratios

$$U_{i+1}/U_i \equiv \exp(-R_i) \tag{2.15}$$

form a collection of i.i.d. random variables. Using them to express U_n, L_n, and then W, one gets:

$$W = \lim_{n\to\infty} (L_n/L_1) = \sum_1^{\infty} (U_j/U_1) = 1 + \sum_1^{\infty} \exp(-[R_1 + \ldots + R_j]). \tag{2.16}$$

We next show that for $\beta < 1$, the summands in (2.16) decrease exponentially in j (with probability one) so that W is finite.

The distribution of the variables R_i is given by

$$\mathrm{Prob}(R_i < r) = \mathrm{Prob}(U_{i+1}/U_i > e^{-r}) = 1 - [e^r + 1]^{-\beta}. \tag{2.17}$$

For future use we note that by (2.17), each R_i can be represented as $R = \log(e^{V/\beta} - 1)$, where V has an exponential distribution (with mean 1).

The probability density of each R_i is

$$f_\beta(r) = \beta e^{-\beta r}/(1 + e^{-r})^{\beta+1}, \qquad -\infty < r < \infty, \tag{2.18}$$

which has all moments finite, so that by the law of large numbers

$$j^{-1}(R_1+\ldots+R_j)\to E(R)=\int_{-\infty}^{+\infty} r f_\beta(r)dr \tag{2.19}$$

with probability one. Thus (2.16) will be convergent if $E(R)>0$.

For $\beta=1$, we may calculate

$$\int_{-\infty}^{+\infty} r f_1(r)dr=[-\log(1+e^{-r})-r/(1+e^r)]_{-\infty}^{+\infty}=0. \tag{2.20}$$

By the comment below (2.17), $E(R)$ is strictly decreasing in β. Hence $E(R)>0$ for $\beta<1$, and the proof is complete. □

We now complete the proof of Proposition 2.1 by presenting the part of the argument which we have previously postponed.

Proof of Lemma 2.1. i) Since the events $A_{L,k}$ are increasing in k, $\mathrm{Prob}(A_L) = \lim_{k\to\infty}\mathrm{Prob}(A_{L,k})$. For the same reason, the complementary sets obey the relation

$$\mathrm{Prob}(A_L{}^c)=\mathrm{Prob}(A_L{}^c\,|\,A_{L,k}{}^c)\cdot\mathrm{Prob}(A_{L,k}{}^c), \tag{2.21}$$

where $\mathrm{Prob}(\cdot\,|\,\cdot)$ denotes a conditional probability.

If it could be shown that

$$\mathrm{Prob}(A_L\,|\,A_{L,k}{}^c)\geqq\delta \tag{2.22}$$

with a k-independent constant $\delta>0$, then all the above relations would imply:

$$\mathrm{Prob}(A_L{}^c)\leqq(1-\delta)\lim_{k\to\infty}\mathrm{Prob}(A_{L,k}{}^c)=(1-\delta)\,\mathrm{Prob}(A_L{}^c), \tag{2.23}$$

and hence $\mathrm{Prob}(A_L{}^c)=0$, or $\mathrm{Prob}(A_L)=1$, which immediately implies (2.5). We shall now prove that (2.22) does indeed follow from (2.6).

ii) To bound the conditional probability in (2.22) we shall make use of the following inequalities. For sets A, B, C with $1-\mathrm{Prob}(C)\leqq 1/2\ \mathrm{Prob}(B)$, one has $\mathrm{Prob}(B\cap C)\geqq\mathrm{Prob}(B)-\mathrm{Prob}(C^c)\geqq 1/2\,\mathrm{Prob}(B)$, so that

$$\mathrm{Prob}(C\,|\,B)\geqq 1/2,$$

and hence

$$\mathrm{Prob}(A\,|\,B)\geqq\mathrm{Prob}(A\,|\,B\cap C)\cdot\mathrm{Prob}(C\,|\,B)\geqq 1/2\,\mathrm{Prob}(A\,|\,B\cap C). \tag{2.24}$$

For us $A=A_L$, $B=A_{L,k}{}^c$, and $C=C_{V,kL}$ is the event that for some V – to be characterized next, there is no occupied bond between the sets $[-kL,kL]$ and $\mathbb{R}\backslash[-V,V]$ (see Fig. 3).

Let us note that for each N, and in particular $N=kL$, and $V>N$,

Prob(there is an occupied bond connecting $[-N,N]$ with $\mathbb{R}\backslash[-V,V]$)

$$=1-\exp\left[-\int_{[-N,N]}dx\int_{\mathbb{R}\backslash[-V,V]}dy\,\varrho(x,y)\right]\leqq\int_{[-N,N]}dx\int_{\mathbb{R}\backslash[-V,V]}dy\,\beta/|x-y|^2$$

$$\leqq 4\beta N/(V-N)\xrightarrow[\text{as } V\to\infty]{} 0. \tag{2.25}$$

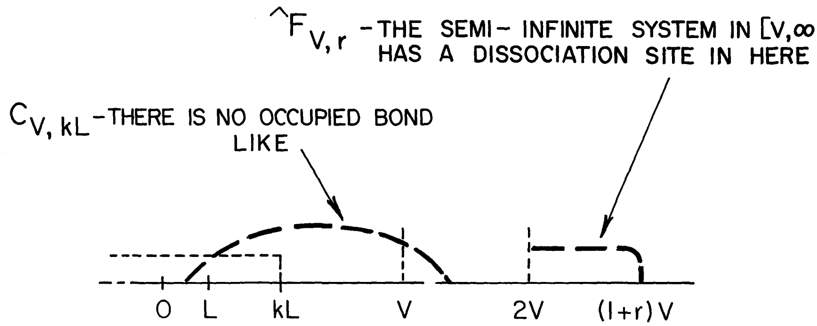

Fig. 3. The events used in the proof of Lemma 2.1

This proves that with V large enough (depending on L and k) the set $C_{V,kL}$ would satisfy the condition which led to (2.24).

iii) It remains now to show that for some $\delta>0$,

$$\mathrm{Prob}(A_L \mid A_{L,k}{}^c \cap C_{V,kL}) \geqq 2\delta, \quad \text{for each } L,\, k \text{ and large enough } V. \tag{2.26}$$

For the lower bound, we note that A_L and in fact $A_{L,(1+r)V/L}$, are implied by (i.e. include) the intersection of the following four events, where r can be any number in $(1,\infty)$ but we shall choose the one of (2.6):

$\hat{}F_{V,r}$ – the configuration obtained by "shifting the origin to V" satisfies the condition of $F_{V,r}$,

$G_{V,r}$ – none of the bonds linking a site in the interval $[-(r+1)V,\, V]$ with a site in $[2V,\infty)$ is occupied,

and their reflections (obtained by replacing $x\to -x$) – $\hat{}F^*_{V,r}$ and $G^*_{V,r}$.

The events $\hat{}F_{V,r}$, $\hat{}F^*_{V,r}$, and $G_{V,r}\cap G^*_{V,r}$ refer to the absence of "occupied" bonds in three disjoint subsets of $\mathbb{R}^2_0$. Furthermore, given that $C_{V,kL}$ occurred, these events are conditionally independent of $A_{L,k}{}^c$, since they refer only to the set of bonds which do not have an end point in $[-kL, kL]$ (all of the latter type which reach beyond $[-V, V]$ are known to be unoccupied). Using the independence of the Poisson process considered here, we get

$$\begin{aligned}
&\mathrm{Prob}(G_{V,r}\cap G^*_{V,r} \mid C_{V,kL}\cap A_{L,k}{}^c)\\
&=\exp\left(-2\beta \int_{-(1+r)V}^{V} dx \int_{2V}^{\infty} dy\, 1/|x-y|^2 + \beta \int_{-(1+r)V}^{-2V} dx \int_{2V}^{(1+r)V} dy\, 1/|x-y|^2\right.\\
&\quad\left. +2\beta \int_{-kL}^{kL} dx \int_{2V}^{\infty} dy\, 1/|x-y|^2\right) \geqq [8(1+r)]^{-\beta}.
\end{aligned} \tag{2.27}$$

Therefore, assuming (2.6),

$$\begin{aligned}
\mathrm{Prob}(A_L \mid C_{V,kL}\cap A_{L,k}{}^c) &\geqq \mathrm{Prob}(A_{L,(1+r)V/L} \mid C_{V,kL}\cap A_{L,k}{}^c)\\
&\geqq [\mathrm{Prob}(\hat{}F_{V,r} \mid C_{V,kL}\cap A_{L,k}{}^c)]^2 \cdot [8(1+r)]^{-\beta}\\
&= \mathrm{Prob}(F_{V,r})^2 \cdot [8(1+r)]^{-\beta} \geqq g^2 \cdot [8(1+r)]^{-\beta},
\end{aligned} \tag{2.28}$$

which proves that (2.26), and thus (2.23), are satisfied. □

Remarks. Lemmas 2.2 and 2.1 show that for $\beta<1$ the line is almost surely covered by dissociated intervals. One may use the above analysis to prove also that, conversely – for $\beta \geqq 1$ a.s. there are no such intervals (with probability one, the sum in (2.16) diverges).

The above statements are closely related to certain results concerning one-dimensional "Lévy dust" [15], and to properties of the "Swiss cheese" in self similar models in $d \geqq 1$ dimensions [16].

Before concluding the discussion of the continuum bond model, let us extract from the proof of Lemma 2.1, where we were just concerned with showing that dissociation occurs with probability one, an explicit bound on its "rate of convergence."

iv) *An Explicit Bound for the Probability of Dissociation in a Finite Volume*

The following result represents the bound to which the arguments used in the proof of Lemma 2.1 lead, for the probability that a given site is not yet covered by a dissociated interval in a large but finite volume. While at first glance the result might not seem to be a very strong improvement over the mere statement that this probability tends to zero, it can be used as a starting point for the derivation of the correct power law for the connectivity function in discrete models (see Proposition 5.2 in Sect. 5).

Corollary 2.1. *If, in a continuum (independent) bond model with* $\varrho(x,y) \leqq \beta/|x-y|^2$, *where* β *is finite,*

$$\mathrm{Prob}(F_{L,r}) \geqq g \quad \textit{for all} \quad L>0 \tag{2.29}$$

with some $1<r<\infty$ *and* $g>0$ *(independent of* L*), then*

$$\mathrm{Prob}(A_{L,k}{}^c) \leqq C' \exp[-(\lambda \ln k)^{1/2}] \quad \textit{for all} \quad L>0 \quad \textit{and} \quad k \geqq k_0, \tag{2.30}$$

with some constants k_0, C', *and* λ *(which are given explicitly below in terms of* β, r, *and* g*).*

Proof. The values of the constants for which (2.30) will be proven are

$$\lambda = 1/2\, g^2[8(1+r)]^{-\beta}, \qquad k_0 = \exp\{\lambda^{-1}(\ln[4(1+r)])^2\}$$

and

$$C' = \max(C, e^{\lambda}), \quad \text{where} \quad C = 32\beta(1+r). \tag{2.31}$$

We shall prove the corollary by showing that for any $\mathbf{k} \geqq k_0$ and L (which is fixed in our discussion), the assumption that

$$\mathrm{Prob}(A_{L,\mathbf{k}}{}^c) \geqq C \exp[-(\lambda \ln \mathbf{k})^{1/2}], \tag{2.32}$$

implies

$$\mathrm{Prob}(A_{L,\mathbf{k}}{}^c) \leqq e^{\lambda} \exp[-(\lambda \ln \mathbf{k})^{1/2}]. \tag{2.33}$$

For the value of L in the assumed (2.32), let us define $f(k) \equiv \mathrm{Prob}(A_{L,k}{}^c)$. $f(k)$ is decreasing in k, hence (2.32) implies

$$f(k) \geqq C \exp[-(\lambda \ln \mathbf{k})^{1/2}] \quad \text{if} \quad 1 \leqq k \leqq \mathbf{k}. \tag{2.34}$$

The analysis made in the proof of Lemma 2.1, with A in (2.24) taken as $A=A_{L,(1+r)V/L}$ implies that for any choice of V such that

$$1/2f(k)\geqq 4\beta kL/(V-kL) \qquad (\geqq \operatorname{Prob}(C_{V,kL}{}^c)) \tag{2.35}$$

the following is satisfied

$$f((1+r)V/L)/f(k)\leqq 1-1/2g^2[8(1+r)]^{-\beta}\equiv 1-\lambda\leqq e^{-\lambda}, \tag{2.36}$$

where the main inequality is based on (2.24) and (2.28).

We choose now

$$V=2kL+16\beta kL\exp[(\lambda\ln\mathbf{k})^{1/2}]/C. \tag{2.37}$$

Noting that $V-kL\geqq V/2$, and making use of (2.34), it is easy to see that the condition in (2.35) is satisfied, provided $k\leqq\mathbf{k}$. Furthermore, with our choice of C and k_0, in (2.31), if $\mathbf{k}\geqq k_0$ (as we have assumed in (2.32)) then the first term in the right-hand side of (2.37) is not larger than the second, and one has:

$$(1+r)V/L\leqq(1+r)32\beta k\exp[(\lambda\ln\mathbf{k})^{1/2}]/C=k\cdot\exp[(\lambda\ln\mathbf{k})^{1/2}]. \tag{2.38}$$

Let us then substitute (2.38) in (2.36), apply the monotonicity of $f(\cdot)$, and iterate. The result is the inequality:

$$f(k)\leqq e^{-n\lambda}f(k\exp[(-n\lambda\ln\mathbf{k})^{1/2}]), \tag{2.39}$$

which holds if $k\leqq\mathbf{k}$, $\mathbf{k}\geqq k_0$, and if n is such that the argument of $f(\cdot)$ in the right-hand side is not less than 1. Using (2.39) with $k=\mathbf{k}$ and $n=\mathbf{n}=$ (the greatest integer $\leqq(\lambda^{-1}\ln\mathbf{k})^{1/2}$), and the fact that $f(\cdot)\leqq 1$ since it is a probability, we obtain

$$f(\mathbf{k})\leqq e^{-\mathbf{n}\lambda}\leqq\exp\{-[(\lambda^{-1}\ln\mathbf{k})^{1/2}-1]\lambda\}=e^{\lambda}\exp[(-\lambda\ln\mathbf{k})^{1/2}]. \tag{2.40}$$

This is just (2.33), and hence the proof is completed. □

3. The Absence of Percolation for $\beta<1$

We now turn our attention to discrete percolation models, like the one described in the introduction. In this section it will be shown that systems for which the bond occupation probability is asymptotically dominated by $\beta/|x-y|^2$, with $\beta<1$, dissociate – in the sense introduced in our discussion of the continuum model. In particular, under this condition there is no percolation, regardless of the short distance behavior of $K_{x,y}$ (assuming, of course, that K is nowhere 1).

Once such a result is known for independent models, its validity can be immediately extended to systems which these dominate in the Fortuin-Kasteleyn-Ginibre sense [17]. However, we choose to give the proof directly for general systems. This helps us in keeping track of the assumptions which are really relevant for our argument. In particular, the explicit argument is essential for the development of the "renormalized" version of the criterion for dissociation, which is presented in the next section.

In a general bond percolation model, $n_{\{x,y\}}$ need not be independent of the other bond variables, nor have a translation invariant distribution. Considering the conditional occupation probabilities – conditioned in each case on completely specified values of all the other bonds, we denote

$$K^+_z = \sup_{x,\{m_b\}} \{\mathrm{Prob}(n_{\{x,x+z\}}=1 \mid n_b=m_b \text{ for } b \neq \{x,x+z\})\} \tag{3.1}$$

and

$$\beta^+ = \limsup_{|z|\to\infty} K^+_z|z|^2 . \tag{3.2}$$

Thus, for each $\beta^* > \beta^+$ there is a uniform asymptotic upper bond on the probability that $\{x, y\}$ is occupied,

$$\mathrm{Prob}(n_{\{x,y\}}=1 \mid \ldots) \leqq \beta^*/|x-y|^2, \quad \text{if} \quad |x-y| > D(\beta^*), \tag{3.3}$$

where ... represents any conditioning which does not involve the bond $\{x, y\}$ directly, and $D(\beta^*) < \infty$.

M will continue to denote here the probability that the origin belongs to an infinite cluster. The following proposition is our first criterion for the absence of percolation. Its proof is given towards the end of this section after a number of preliminary results.

Proposition 3.1. *In a bond percolation model on* $\mathbb{Z}$, *if:*

i) $$\sup\{K^+_z\} \equiv p^+ < 1,$$

and (3.4)

ii) $$\beta^+ < 1,$$

then with probability one $\mathbb{Z}$ *is entirely covered by finite dissociated intervals. In particular*

$$M=0. \tag{3.5}$$

Let us note that while it is difficult to find a continuum notion of bond percolation utilizing only point contacts, the notion of dissociation makes sense for both the continuum and the lattice.

The proof of Proposition 3.1 will be structured along the lines of the proof of Proposition 2.1, and will consist of the adaptation of the analysis of Sect. 2. The main points at which the systems considered here differ from those in the preceding section, in addition to the discreteness of the lattice, are:

i) no assumption of independence for the bond occupation variables,

and

ii) the bound (3.3) applies only asymptotically.

To facilitate the adjustment of a continuum analysis to discrete models, let us note that if $\beta^+ < 1$, then there is a finite distance, which we shall denote D^+, beyond which (3.3) can be replaced by:

$$\begin{aligned} \mathrm{Prob}(n_{\{x,y\}}=0 \mid \ldots) &\geqq (1-K^+_{x-y}) \\ &\geqq \exp\left(-\iint_{\substack{x\leqq u<x+1\\ y\leqq v<y+1}} du\,dv\,\beta'/|u-v|^2\right), \quad \text{if} \quad |x-y|>D^+ \end{aligned} \tag{3.6}$$

with $\beta' = (1+\beta^+)/2$, which is still less than 1.

Taking their natural extensions, we shall adopt here the notions of the dissociation events A_L and $F_{L,k}$, of maximal nested sequences, etc., introduced in the preceding section. Just as over there, the main step is the derivation of the uniform lower bound for the semi-infinite dissociation problem – which is expressed in the following statement.

Lemma 3.1. *For a random system of bonds on the semi-infinite lattice $\mathbb{Z}_+$, if the two assumptions made in Proposition 3.1 are satisfied, then for each $k \in (1, \infty)$ and $L > 0$,*

$$\text{Prob}(F_{L,k}) \geqq g^+(k) \tag{3.7}$$

with a value $g^+(k) > 0$ which depends on the measure only through the parameters β^+, D^+, and p^+.

For the proof of Lemma 3.1, which is given following Lemma 3.3 below, we shall again consider a maximal nested sequence of occupied bonds and study the random variables W_L, L_n, and U_n, defined by (2.7), (2.10), and (2.13). (Although it does not always show in our notation, all these variables depend on the parameter L.) Our key observation (to be compared to Lemma 2.3) is

Lemma 3.2. *If a measure satisfies* (3.6) *with some value of D^+ and $\beta' > 0$, then for each L, the conditional distribution of U_{n+1} given the previous U_i's satisfies*

$$\text{Prob}(U_{n+1} \leqq u \mid U_1 = L, U_2, \ldots, U_n) \geqq (1 + U_n/u)^{-\beta}, \quad \textit{for} \quad u \geqq D^+. \tag{3.8}$$

Proof. Conditioned on the values of $U_1, \ldots, U_n$, the event $\{U_{n+1} \leqq u\}$ is that none of the bonds in the set $\{(x, y) \mid x \in (L_{n-1}, L_n],\ y \in (L_n + u, \infty)\}$ is occupied $\left(\text{where } L_j = \sum_1^j U_i\right)$. The specification of the values of $U_1, \ldots, U_n$ on which the conditioning is made amounts to some explicit information on the occupation of bonds in a collection which is disjoint from the above set. The left-hand side of (3.8) admits a telescopic decomposition into a product of probabilities for the bonds in the set to be unoccupied, conditioned on successively enhanced information. Using the definition (3.1) of $K^+{}_{x-y}$, and (3.6), we get

$$\text{LHS} \geqq \prod_{\substack{x \in (L_{n-1}, L_n] \\ y \in (L_n + u, \infty)}} (1 - K^+{}_{x-y}) \geqq \exp\left(-\beta' \int_{L_{n-1}}^{L_n} du \int_{L_n+u}^{\infty} dv\, 1/|u-v|^2\right) = (1 + U_n/u)^{-\beta'}. \tag{3.9}$$

Since we have tacitly assumed here that u is an integer, let us note for a future purpose that by trivial arguments (3.8) extends now to all real values $u \geqq D^+$. □

In our application of the above result we shall make use of a comparison principle, which has already been derived within the context of queuing theory [18]. We include a proof for the sake of completeness.

Lemma 3.3. *Suppose U_i, U'_i are two sequences of random variables satisfying, for each n:*

i) $$\text{Prob}(U_{n+1} \leqq u \mid U_1 = u_1, \ldots, U_n = u) \geqq \text{Prob}(U'_{n+1} \leqq u \mid U'_1 = u_1, \ldots, U'_n = u_n) \tag{3.10}$$

for all u, u_i,

ii) *the right-hand side of* (3.10) *is nonincreasing in each* u_i,

then for each n and every coordinatewise nondecreasing function on $\mathbb{R}^n$, *such that* $f(U'_1, \ldots, U'_n)$ *has finite absolute expectation,*

$$E(f(U_1, \ldots, U_n)) \leqq E(f(U'_1, \ldots, U'_n)). \tag{3.11}$$

Proof. We proceed by induction on n. The inequality (3.11) is elementary for $n=1$. For $n=N+1$, we express the right-hand side of (3.11) as $E(\varphi'(U'_1, \ldots, U'_N))$ and the left-hand side as $E(\varphi(U_1, \ldots, U_N))$, where

$$\varphi'(u'_1, \ldots, u'_N) = E(f(U'_1, \ldots, U'_{N+1}) \mid U'_1 = u'_1, \ldots, U'_N = u'_N),$$

and φ is defined similarly for $U_1, \ldots, U_N$. Note that since both f and the right-hand side of (3.10) are nondecreasing in each coordinate – so is the function φ'. On the other hand, (3.10) and the fact that the function f is nondecreasing in its last coordinate imply that $\varphi \leqq \varphi'$ pointwise. Thus

$$\begin{aligned} E(f(U_1, \ldots, U_{N+1})) &= E(\varphi(U_1, \ldots, U_N)) \leqq E(\varphi'(U_1, \ldots, U_N)) \\ &\leqq E(\varphi'(U'_1, \ldots, U'_N)) = E(f'(U'_1, \ldots, U'_{N+1})), \end{aligned} \tag{3.12}$$

where the second inequality follows from the induction hypothesis. □

We are now ready for the proof of Lemma 3.1, which is the only place in the analysis where the main assumption $\beta^+ < 1$ plays a role. The corresponding step in the previous section's discussion of the self similar continuum model was Lemma 2.2 (which in fact proved more than the required inequality (2.6)). Its proof will be used in the following argument.

Proof of Lemma 3.1. Let us now assume that a given system satisfies the assumptions listed in Proposition 3.1. Our goal is to derive the lower bound (3.7).

For a given $L > 0$ and $k > 1$, we shall consider the nondecreasing sequence L_n (of the sites y_n reached by a maximal nested sequence of occupied bonds) discussed above, with its associated sequence of increments U_n and the limit $L \cdot W_L$. As in (2.8), we have

$$\mathrm{Prob}(F_{L,k}) = \mathrm{Prob}(W_L < k). \tag{3.13}$$

Let N^* be the index of the first increment U_n which does not exceed D^+, or $+\infty$ if there is no such n, and let L^*_n be the sequence obtained from L_n by stopping it at N^*. U^*_n and W^*_L will denote the increments and the limit of L^*_n. It is easy to see that the bound (3.8) which holds for the sequence U_n as long as $u \leqq D^+$, is satisfied by U^*_n without that restriction. (The argument is elementary in either case, but it depends on whether $n \geqq N^*$ or not. That event is measurable with respect to both $\{U_1, \ldots, U_n\}$ and $\{U^*_1, \ldots, U^*_n\}$.)

We may now use the above observation, and the comparison principle of Lemma 3.3. (aided by simple "approximation arguments" – needed because Lemma 3.3 directly treats only functions of finitely many variables), to compare the sequence U^*_n with the process U'_n which corresponds to the self similar continuum model, satisfying the equality (2.14), with the value of the parameter β set at $\beta' = (1+\beta^+)/2$ (<1). Since for both models the corresponding quantities W^*_L and

W' satisfy the first two equalities of (2.16), and hence are increasing functions of the U^*_n or U'_n variables, we have

$$\text{Prob}(W^*_L<k) \geqq \text{Prob}(W'<k) \qquad (>0 \text{ for } \beta^+<1), \tag{3.14}$$

where the right-hand side is independent of L (and depends only on β^+ and k).

It remains now only to make the observation that if $W^*_L<\infty$, then $N^*<\infty$, i.e. the increments U_n reach a smaller value than $D^+ + 1$, and that if this occurs then there is a uniformly positive probability for the sequence L_n to stop right there, in which case $W_L = W^*_L$.

To be more specific, the latter event would certainly occur if none of the sites in $(L_{N^*-1}, L_{N^*}]$ is connected by an occupied bond to (L_{N^*}, ∞). For each site the probability that none of its bonds which reach to the right of it is occupied, conditioned on any information about the other bonds, is at least

$$\delta = \sup_x \left\{ \prod_{y>x} (1-K^+_{x-y}) \right\} \geqq (1-p^+)^{D^+} \exp\left(-\int_{D^+}^{\infty} dx\beta'/x^2 \right). \tag{3.15}$$

Hence,

$$\text{Prob}(W_L<k) \geqq \text{Prob}(W^*_L<k)\delta^{(D^+ +1)}. \tag{3.16}$$

Combining the above results, we see that (3.7) is valid with the explicit bound:

$$\text{Prob}(F_{L,k}) \geqq g_{\beta'}(k)\delta^{(D^+ +1)} \equiv g^+(k), \tag{3.17}$$

where $g_\beta(k) = \text{Prob}(W<k)$ with W being the variable studied in Lemma 2.2, with the given value of β. This proves Lemma 3.1. □

It remains now to show that the above result implies Proposition 3.1, which is the main result of this section.

Proof of Proposition 3.1. Proposition 3.1 requires the extension of Lemma 3.1 in two directions: the proof of the *two sided* dissociation, and the conversion of uniformly positive probability into "probability one" statements. The arguments which accomplish both were given in the proof of Lemma 2.1. Instead of repeating them here, we shall just list the necessary adjustments, which address the differences listed below the statement of Proposition 3.1.

i) In (2.25) replace the continuum expression by

$$\begin{aligned} &\text{Prob(there is an occupied bond connecting } [-N,N] \text{ with } \mathbb{R}\backslash[-V,V]) \\ &\leqq \sum_{\substack{-N\leqq x\leqq N \\ y\in\mathbb{R}\backslash[-V,V]}} K_{x,y} \leqq [1+o(1)] \sum_{\substack{-N\leqq x\leqq N \\ y\in\mathbb{R}\backslash[-V,V]}} \beta^+/|x-y|^2 \\ &\leqq [1+o(1)]4\beta^+N/|V-N|_+ \xrightarrow[\text{as } V\to\infty]{} 0. \end{aligned} \tag{3.18}$$

ii) Make the observation that it suffices to prove (2.26) with the additional restriction $V>D^+$ (which would allow one to apply (3.6) in the next remark).

iii) In the last part of the argument, instead of the independence argument used there for the Poisson process, use the decomposition

$$\begin{aligned} \text{Prob}(A_L \mid C_{V,kL} \cap A_{L,k}{}^c) &\geqq \text{Prob}({}^\frown F_{V,r} \cap {}^\frown F^*_{V,r} \cap G_{V,r} \cap G^*_{V,r} \mid C_{V,kL} \cap A_{L,k}{}^c) \\ &= \text{Prob}({}^\frown F_{V,r} \mid C_{V,kL} \cap A_{L,k}{}^c) \cdot \text{Prob}({}^\frown F^*_{V,r} \mid C_{V,kL} \cap A_{L,k}{}^c \cap {}^\frown F_{V,r}) \\ &\quad \times \text{Prob}(G_{V,r} \cap G^*_{V,r} \mid C_{V,kL} \cap A_{L,k}{}^c \cap {}^\frown F_{V,r} \cap {}^\frown F^*_{V,r}). \end{aligned} \tag{3.19}$$

Each of the above factors is a conditional probability of an event which depends only on a subset of the set of all the bonds, and the conditioning is on an event which is determined in some other subset. In each case, the events are such that when a bond belongs to both sets it is conditioned to be unoccupied.

To bound the first two factors we note that each of them refers to an event which is determined by the bonds of a (shifted) semi-infinite lattice $\mathbb{Z}_+$ (or $\mathbb{Z}_-$) with a probability measure which is induced by conditioning on an event which is determined in the complementary set. These measures satisfy the assumptions required in Lemma 3.1. Thus each of the first two terms is not less than $g^+(r)$.

For the last factor we shall use a finer telescopic decomposition (into a product in which each factor is associated with a bond), and repeatedly apply (3.6). One obtains an integral expression – like the one in (2.27), which yields a lower bound of $[8(1+r)]^{-\beta'}$.

Thus we get

$$\mathrm{Prob}(A_L \mid C_{V,kL} \cap A_{L,k}{}^c) \geqq g^+(r)^2 \cdot [8(1+r)]^{-\beta'}, \tag{3.20}$$

which is used in the same way as (2.28). (For an extension which will be made in Sect. 4, it is useful to note that in this part of the argument no independent use was made of the main condition $\beta^+ < 1$; the results of Lemma 3.1 and $\beta^+ < \infty$ would have sufficed.) $\square$

To summarize, we saw here that the dissociation which we first discussed in the context of the self similar continuum models occurs also in discrete models (including systems which are not independent) if $\beta^+ < 1$. It is easy to extend one of the remarks made at the end of the previous section, and to demonstrate that for independent lattice models the complete dissociation does not occur if $\liminf\limits_{|x-y|\to\infty} K_{x-y}|x-y|^2 > 1$.

The lack of dissociation does not imply percolation since, for example, percolation is impossible in translation invariant independent models satisfying (1.4). However, it was demonstrated in ref. [5] that, in the independent case, if $\beta > 1$ then there is percolation if also the short range parameter p is sufficiently close to 1.

In the next section we shall see a "renormalized" version of the criterion (3.4) for the lack of percolation.

4. Discontinuity of the Percolation Density

i) Statement of the Main Result

We are now ready for the derivation of the main result of this paper, which includes as a particular case the proposition discussed in the introduction. In essence, we shall show that the criterion derived in Sect. 3, that percolation requires $\beta^+ \geqq 1$, can be replaced by a stronger requirement of the type – $\beta^+ M^2 \geqq 1$. For the independent translation invariant models, that is exactly what will be proven. However, in its simplest form this statement is not valid in complete generality. To obtain a correct and provable criterion we shall qualify it in two ways: i) the percolation density M will be replaced by M^+, a similar quantity which incorporates the possible long range dependence effects, and ii) the result will be restricted to models which have a strong FKG type property.

The Fortuin-Kasteleyn-Ginibre (FKG) condition has been formulated [17] in terms of a partial order on the space of configurations, which we define by the pointwise domination of the occupation variables. I.e. a bond configuration (n) is said to dominate another one $(n \gg n')$ if its set of occupied bonds includes the other's. This partial order induces the notions of increasing functions (for which $n \gg n' \Rightarrow f(n) \geqq f(n')$), increasing events (those whose indicator functions are increasing) and decreasing events.

A rather useful class of probability measures, introduced (in reliability theory) by Esary, Proschan, and Walkup [19], consists of those for which any two increasing functions are positively correlated. We shall refer to the latter as the *standard FKG property*. It is implied by a sufficiency criterion, presented in ref. [17], which defines a more restrictive class. The class of measures for which our result will be proven is characterized by the condition which we introduce next as the *strong FKG property*. While this class appears to fall between the previous two, it was actually shown by van den Berg and Burton [20] to be identical to the class satisfying the sufficiency condition of [17]. (We thank F. den Hollander for calling our attention to this fact.)

Definition 4.1. We say that a probability measure (of a random bond system) has the *strong FKG property* if for every subset A of bonds, the conditional probabilities with respect to the corresponding σ-algebra B_A satisfy $\mathrm{Prob}(R \cap R' \mid B_A) \geqq \mathrm{Prob}(R \mid B_A) \cdot \mathrm{Prob}(R' \mid B_A)$ for any increasing events, R and R'. This means that for "almost every" specification of the occupation variables for the bonds in A, the conditional distribution of the remaining bonds has the standard FKG property.

It is not too difficult to see that an equivalent statement of the above defined property is that for any $A \subset \mathbb{R}_0^2$, any increasing event R, and any decreasing event N:

$$\mathrm{Prob}(R \mid N, B_A) \leqq \mathrm{Prob}(R \mid B_A). \tag{4.1}$$

The collection of measures which have the strong FKG property includes, of course, the independent models discussed in the introduction. Examples of dependent models of special interest are the percolation systems which correspond to the Ising and Potts models, which are discussed in the companion work [3].

To introduce the quantity M_+, let

$$M_H = \sup_{x,\{m\}} \mathrm{Prob}(x \text{ is connected by a path of occupied bonds to } \mathbb{R}\backslash[x-H, x+H] \mid n_b = m_b \text{ for bonds with both ends in } \mathbb{R}\backslash[x-H, x+H]). \tag{4.2}$$

In essence, M_H is the finite volume approximant of the percolation probability, evaluated for the most favorable boundary conditions (in case there is dependence). It is clearly a decreasing function of H. We define

$$M_+ = \lim_{H \to \infty} M_H. \tag{4.3}$$

Remarks. For independent translation-invariant models $M_+ = M$ (i.e. $= P_\infty$). For translation-invariant systems with the strong FKG property it can be shown that

M_+ is the percolation density (P_∞) of the state obtained with the "wired boundary conditions" (this concept being further discussed in [3]).

Following is the main result of this section. (Its proof is given in the last part 4iv)).

Proposition 4.1. *If a one dimensional bond percolation model satisfies the "strong FKG" condition, is regular ($K^+{}_z<1$ for all z), and*

$$\beta^+ M_+{}^2 < 1 , \tag{4.4}$$

then with probability one its configurations have no infinite clusters (and hence $M=0$).

Henceforth, we shall denote

$$\beta_R = \beta^+ M_+{}^2 . \tag{4.5}$$

The main idea of the proof is to "trim away" bonds which can belong only to "dangling ends" of an infinite cluster. This will be done in a way which does not destroy the existence of an infinite cluster, if there is one, but which reduces the density of the relevant bonds. This operation "renormalizes" the effective value of "β" into β_R $(+\varepsilon)$. The resulting observation will be that if $\beta_R<1$, then the "trimmed" system dissociates, and thus there can be no percolation in the original model. The proof is based on the arguments developed in the previous sections, with some modifications. These are required since our procedure does not reduce the quantity β^+, which is based on the worst case analysis, and hence we cannot simply apply Proposition 3.1 to the "trimmed" system.

ii) Anchored Bonds

The first step is the introduction of the notion of H-achored sites and bonds, which will be used with H large enough so that M_H is sufficiently close to M_+.

Definition 4.2. We shall say that a site x is *H-anchored* if it is doubly connected to $\mathbb{R}\backslash[x-H, x+H]$, i.e. if there are two disjoint sequences of occupied bonds which form connected paths linking x with the exterior of its H-neighborhood.

The pair of sites $\{x, y\}$ will be said to be *connected by H-anchored bond* (or, equivalently, we will simply say that there is an H-anchored bond $\{x, y\}$) if the bond $\{x, y\}$ is occupied and each of the two sites is H-anchored (see Fig. 4).

Let us remark here that if there is percolation in the original random bond system then the anchored bonds also percolate. Percolation is after all equivalent

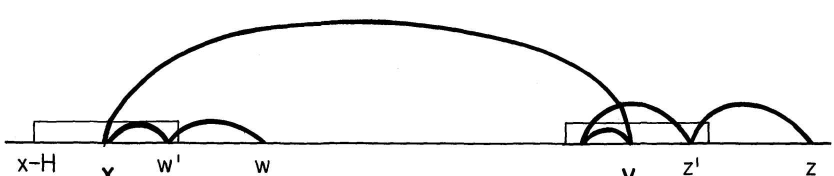

Fig. 4. An H-anchored bond $\{x, y\}$ with its left and right anchors – w and z

to the existence of infinitely long, nonrepeating, connected paths of occupied bonds. Except possibly for the first few, all the bonds in such a path would automatically be anchored. Our strategy for the proof of Proposition 4.1 will be to derive the stronger statement that under its assumptions, and with a proper choice of H, the system of the H-anchored bonds dissociates.

First let us present the main observation which explains why $\beta^+ M_+{}^2$ may be regarded as the renormalized value of β.

Lemma 4.1. *If a bond percolation model has $\beta^+ < \infty$, then for each $H < \infty$ and $\varepsilon > 0$ there is some $D_\varepsilon < \infty$ (depending also on H) such that for all pairs of sites with $|x-y| \geqq D_\varepsilon$:*

$$\mathrm{Prob}(x \text{ and } y \text{ are connected by an } H\text{-anchored bond} \mid A) \leqq (\beta^+ M_H{}^2 + \varepsilon)/|x-y|^2, \tag{4.6}$$

where A is any event determined by the occupation variables of a collection of bonds for which both sites are at distances greater than H from both x and y.

Proof. If the two sites x and y are connected by an H-anchored bond and $|x-y| > H$ then *either*:

i) x belongs to a connected path of occupied bonds which reaches beyond the site's H-neighborhood without using for that purpose any of the bonds which touch $[y-H, y+H]$; the same holds for y (without using any of the bonds which touch $[x-H, x+H]$), and furthermore the bond $\{x, y\}$ is occupied.

ii) $\{x, y\}$ is occupied and there is another occupied bond $\{u, v\}$ with $u \in [x-H, x+H]$ and $v \in [y-H, y+H]$.

Adding the probabilities of the above two events, computed by successive conditioning in the above indicated order, we get the following bound for the left-hand side of (4.6):

$$\text{left-hand side} \leqq (\beta^+ + \delta)/|x-y|^2 \cdot [M_H{}^2 + (2H+1)^2(\beta^+ + \delta)/(|x-y| - 2H)^2], \tag{4.7}$$

whenever $|x-y| - 2H \geqq D(\beta^+ + \delta)$ (defined in (3.3)). It is now obvious that by choosing $\delta = \varepsilon/(2M_H{}^2)$, and then D_ε large enough so that the second term in the sum in (4.7) is sufficiently small for $|x-y| \geqq D_\varepsilon$, one arrives at (4.6). □

In view of the bound (4.6) on the density of the anchored bonds, one could be tempted to hope that the quantity β^+ *for the system of anchored bonds* is bounded by $\beta_R + \varepsilon$ (for large H), in which case our main result (that $\beta_R < 1$ implies dissociation) would follow immediately by an application of Proposition 3.1. That, however, is not true. While the anchored bonds are "sparser," they are strongly correlated (due to the site conditions). In fact, it is easy to see that the quantity β^+ for the system of anchored bonds is the same as for the original model. Alternatively, one could try to repeat the arguments which led to Proposition 3.1, for which the requirement $\beta^+ < 1$ which is based on the "worst case" conditioning was somewhat too strict. That is the route we shall follow, with modifications which will be explained below.

iii) The Main Bound on the Dissociation of Anchored Bonds

We now extend to the system of anchored bonds the notions of dissociation, and the events $A_{L,k}$ and $F_{L,k}$ defined in Sect. 2iii). To avoid confusion, we shall mark with the tilde (˜) the symbols denoting these events for the system of anchored bonds. As in the analysis of the previous two sections, the main technical estimate needed for the proof of the dissociation is the following lemma.

Lemma 4.2. *In any one dimensional bond percolation model which satisfies the "strong FKG" condition, is regular and has $\beta^+ < \infty$, we may compare the H-anchored bond events $\tilde{F}_{L,k}$ to the corresponding ordinary bond events $F_{L,k}$ in a related independent bond model as follows:* i) *Given any $H < \infty$, $\beta' > \beta^+ M_H{}^2$ and $k > 1$, there exist some $R < \infty$ and $\varepsilon > 0$ so that*

$$\operatorname{Prob}(\tilde{F}_{L,k}) \geqq \varepsilon \operatorname{Prob}'(F_{L,k}) \quad \textit{for all} \quad L \geqq R. \tag{4.8a}$$

Here the right-hand side probability refers to the self similar independent model with parameter β' whose nearest neighbor bond density K_1 is set to zero; the other bond densities for this model are given by (2.1) *with $\mu = \beta'$ and $s = 2$. Hence,* ii) *if $\beta^+ M_+{}^2 < 1$ and H is chosen so that $\beta^+ M_H{}^2 < 1$, then for each $k > 1$, there is some $\tilde{g}(k) > 0$ so that*

$$\operatorname{Prob}(\tilde{F}_{L,k}) \geqq \tilde{g}(k) \quad \textit{for all} \quad L > 0. \tag{4.8b}$$

Remark. The proof of Lemma 4.2, which is given after the proof of Lemma 4.5, shows that (4.8a) remains valid when the hypothesis $\beta' > \beta^+ M_H{}^2$ is weakened by modifying the definitions of β^+ and M_H as follows: replace the sup over all x in (3.1) and (4.2) by the sup over $x \geqq -2H$. This implies that if B denotes the σ-algebra generated by the occupation variables of some subset of bonds which lie entirely within $(-\infty, -2H)$, then under the hypotheses of Lemma 4.2, both (4.8a) and (4.8b) remain valid when the probabilities are conditioned on B.

For the proof of Lemma 4.2 we shall use the "maximal nested sequence" technique developed in the earlier sections. However, the anchored bonds have the rather inconvenient feature that the existence of one carries implications which extend beyond the interval delineated by its endpoints. We find it therefore necessary to apply this technique to intervals which are defined by the effective spans of the anchored bonds. We shall now introduce this concept, present some of its key properties, and then consider the corresponding nested sequences.

First let us denote by $C_H(x)$, where $x \in \mathbb{Z}$, the cluster of sites in $[x-H, x+H]$ which are connected to x by paths of occupied bonds lying entirely in $[x-H, x+H]$. In the following definition we implicitly introduce a new parameter J, which along with H will always be defined in the given context. We shall always assume that $J > 3H$ (in fact J/H will be chosen very large).

Definition 4.3. i) For an anchored bond $\{x, y\}$ (with $x < y$) we define its left and right *anchors* as the maximal values of w and, correspondingly, z such that w is directly connected (i.e. by an occupied bond) with a site in $C_H(x)$ and z is directly connected with a site in $C_H(y)$. We denote the maximal sites in $C_H(x)$ (within the order of $\mathbb{Z}$) to which w and z are directly connected by w' and z', correspondingly (see Fig. 4).

ii) The *span* of an anchored bond $\{x, y\}$ (with $x<y$) is defined as the interval $[r, s]$ whose left endpoint is $r=x$, and right endpoint is

$$s=\begin{cases} y & \text{if } \ y-x\leqq J \\ \max\{y, w, z\}+2H & \text{if } \ y-x>J\,. \end{cases} \tag{4.9}$$

The following lemma shows that the "density" of the spans is not that different than that of the anchored bonds themselves.

Lemma 4.3. *Given a bond percolation model with $\beta^+<\infty$, and given some $H<\infty$ and $\varepsilon>0$, then there is a choice of values of J and S for which the spans of anchored bonds (defined with these values of H and J) satisfy the following bound. For every site $x\in\mathbb{Z}$ and set $A\subset\mathbb{Z}$, with* $\operatorname{dist}\{x, A\}\geqq S$*:*

Prob(*there is an H-anchored bond whose span $[r, s]$, has s in the set A and $r=x$*)

$$\leqq \sum_{y;\, \operatorname{dist}\{y, A\}\leqq J} (\beta^+ M_H{}^2+\varepsilon)/|y-x|^2\,. \tag{4.10}$$

Explicit choices for the parameters are: $J=3H+(81/2)(2H+1)(\beta^+ +1)^2\varepsilon^{-1}$, and $S=2J+D_{\varepsilon/2}+2D(\beta^+ +1)$ (the last two terms being defined next to (4.6) *and* (3.3)*).*

Proof. A necessary requirement for the occurrence of the event in the left-hand side of (4.10), is that at least one of the following three conditions is satisfied (see (4.9)):

i) There is an anchored bond $\{x, y\}$ with some y $(>x)$ such that $\operatorname{dist}\{y, A\}\leqq J$.

ii) For some $\{y, w, w'\}$ – such that $y-x>J$, $|x-w'|\leqq H$, $w+2H\in A$, and $w-w'>J-H$, the two (regular-) bonds $\{x, y\}$ and $\{w', w\}$ are occupied. (That is a necessary condition for the existence of an anchored bond $\{x, y\}$ whose span is generated by the bond's left anchor, w, and reaches A.)

iii) For some $\{y, z, z'\}$ – with $y-x>J$, $|y-z'|\leqq H$, $z+2H\in A$, and $z-z'>J-3H$, the two bonds $\{x, y\}$ and $\{z, z'\}$ are occupied. (Note that the cases where A is reached through the right anchor, z, but the condition $z-z'>J-3H$ is violated, are already covered by i).)

Let P_1, P_2, and P_3 be the probabilities of the above events. We shall estimate each of them by the sum of the probabilities of the relevant bond events to occur.

We obtain:

$$P_1\leqq \sum_{y;\, \operatorname{dist}\{y, A\}\leqq J} (\beta^+ M_H{}^2+\varepsilon/2)/|y-x|^2\,, \tag{4.11}$$

where we assumed that $\operatorname{dist}\{x, A\}-J\geqq D_{\varepsilon/2}$, so that Lemma 4.1 can be applied

$$\begin{aligned} P_2 &\leqq \sum_{y;\, y-x>J}\ \sum_{\substack{w'\in[x-H, x+H]\\ w\in A-2H}} K^+{}_{y-x}K^+{}_{w-w'} \\ &\leqq [(\beta^+ +1)/J]\cdot(2H+1)\sum_{w\in A-2H}(\beta^+ +1)/|w-x-H|^2\,. \end{aligned} \tag{4.12}$$

In applying the bound (3.3) to K^+, we used here the fact that $J\geqq D(\beta^+ +1)$ and $\operatorname{dist}\{x, A\}-3H\geqq D(\beta^+ +1)$.

And

$$\begin{aligned}P_3 &\leqq \sum_{z\in A-2H} \sum_{y;\,y-x>J} \sum_{\substack{z';\,|y-z'|\leqq H\\ \text{and } z-z'>J-3H}} K^+{}_{y-x}K^+{}_{z-z'}\\ &\leqq \sum_{z\in A-2H} \sum_{y;\,y-x>J} \sum_{z';\,|y-z'|\leqq H} K^+{}_{y-x}(\beta^+ +1)\cdot 4/|z-x-H|^2\\ &\quad + \sum_{z\in A-2H} \sum_{z';\,z-z'>J-3H} \sum_{y;\,|y-z'|\leqq H} K^+{}_{z-z'}(\beta^+ +1)\cdot 4/|z-x-H|^2\\ &\leqq 2(2H+1)[(\beta^+ +1)/(J-3H)] \sum_{z\in A-2H} (\beta^+ +1)\cdot 4/|z-x-H|^2, \end{aligned} \tag{4.13}$$

where the second inequality is based on the observation that at least one of the lengths $|y-x|$ and $|z-z'|$ has to be there not less than $|z-x-H|/2$. We also used the fact $J-3H \geqq D(\beta^+ +1)$ and $\operatorname{dist}\{x,A\}-3H \geqq 2D(\beta^+ +1)$.

For J, and then S, large enough (as specified in the statement of the lemma) the terms P_2 and P_3 are only small corrections to P_1, together adding a term $\varepsilon/2$ to the $\varepsilon/2$ in (4.11). Thus we obtain:

$$\text{left-hand side of (4.10)} \leqq P_1+P_2+P_3 \leqq \sum_{y;\,\operatorname{dist}\{y,A\}\leqq J} (\beta^+ M_H{}^2+\varepsilon)/|y-x|^2, \tag{4.14}$$

which proves (4.10). □

A slight variation of the above proof yields the following result.

Lemma 4.4. *If a bond percolation model has the (standard) FKG property, and $\beta^+ < \infty$, then for each $H<\infty$ and $\varepsilon>0$ there is a choice of J and S (specified in Lemma 4.3) with which for any two subsets $A, B \subset \mathbb{Z}$, with $\operatorname{dist}\{A,B\} \geqq S$ –*

$$\begin{aligned}&\operatorname{Prob}(\textit{there is an } H\textit{-anchored bond } \{x,y\}\\ &\quad \textit{whose span } [r,s] \textit{ has } r\in A \textit{ and } s\in B)\\ &\leqq 1-\exp\left[-\sum_{\substack{x\in A\\ y;\,\operatorname{dist}\{y,B\}\leqq J}} (\beta^+ M_H{}^2+2\varepsilon)/|y-x|^2\right].\end{aligned} \tag{4.15}$$

(The relevance of the value of J here is through the definition of the span of an anchored bond.)

Proof. The main idea is to exponentiate the bounds which led to (4.10). For this purpose we shall focus on the probability that the sets A and B are not "linked," in the sense seen in the left-hand side of (4.15), and use the (standard) FKG property.

For the given sets A and B we shall again distinguish between three ways in which the span of an anchored bond may link them, as was done in the proof of Lemma 4.2. Let us take note of the fact that the probability of such an event to occur was bounded in (4.11)–(4.14) by the sum of the probabilities of a collection of bond events which are all *increasing* – in the FKG sense. (Even though the event in the left-hand side of (4.15) itself is not monotone, if the set B is not a semi-infinite interval.)

By the assumed FKG property, the probability that none of the events which were added in (4.11)–(4.14) occurs is not less than the product of the corresponding individual probabilities. Thus, for example, the first inequality in (4.14) can now be replaced by

$$\text{left-hand side of (4.10)} \leqq 1-(1-P_1)\cdot(1-P_2)\cdot(1-P_3). \tag{4.16}$$

Similarly, for our analog of P_1 we have, instead of (4.11)

$$1-P_1 \geqq \prod_{\substack{x\in A\\ y;\,\mathrm{dist}\{y,B\}\leqq J}} [1-(\beta^+ M_H{}^2+\varepsilon/2)/|x-y|^2]$$

$$\geqq \exp\left[-\sum_{\substack{x\in A\\ y;\,\mathrm{dist}\{y,B\}\leqq J}} (\beta^+ M_H{}^2+\varepsilon)/|x-y|^2\right], \tag{4.17}$$

where in the second step we used the inequality $1-x \geqq e^{-x(1+2x)}$ for $0<x<1/2$, and assumed that $\varepsilon \leqq 2$ and $2(\beta^+ +1)^2/[\mathrm{dist}\{A,B\}-J]^2 \leqq \varepsilon/2$. The last condition is satisfied with our choice of J and S, and the assumption made on $\mathrm{dist}\{A,B\}$.

Equation (4.15) is proven by taking a similar approach to all the terms which contribute to (4.14), or actually to its analog (with the sets $\{x\}$ and A replaced by A and B) which is relevant here. In fact, just as in the proof of the previous lemma, the terms P_2, P_3 represent only a small perturbation to the expression seen in (4.17). □

In Sect. 2iii) we defined the concept of a maximal nested sequences of bonds, and used it in the proof of the analog of Lemma 4.2. It turns out that for a reason which was mentioned above, the spans of the anchored bonds offer a better notion for the renormalized version of that analysis than the anchored bonds themselves. The main place where this difference is manifested is the derivation of the following estimate, in which we implicitly refer to an obvious extension of Definition 2.1.

Lemma 4.5. *Under the assumptions of Lemma 4.2, for each H and $\beta' > \beta^+ M_H{}^2$ there are J and R with which the following condition is satisfied: if the random variables $\{r_i, s_i\}_{i>2}$ form a maximal nested sequence of the spans of occupied anchored bonds, for some given initial values $\{s_0, s_1\}$, then for all $n>1$ and $u \geqq R$, their distribution satisfies:*

$$\mathrm{Prob}(s_n \geqq s_{n-1}+u \mid \{r_i, s_i\}_{i=2,\ldots,n-1}) \leqq 1-[1+(s_{n-1}-s_{n-2})/u]^{-\beta'}, \tag{4.18}$$

when the conditioning is on a set of values with $s_i - s_{i-1} \geqq R$ for all $1<i\leqq n-1$.

Proof. For the given H and β' let us set $\varepsilon = (\beta' - \beta^+ M_H{}^2)/4$, and choose the values of J and S which are referred to in Lemmas 4.3 and 4.4. R will be chosen later, but it will satisfy $R > \max\{S, J\}$.

Let us now start the argument by describing an efficient method for keeping track at the level of the original bond system of the information provided by specifying that a given interval is the span of some occupied anchored bond, and that furthermore a given sequence of such intervals is maximal.

Any given interval $[r,s]$, of length $>J+2H$, can form the span of a number of anchored bonds $\{x,y\}$ – from the same configuration of the original bond system. Of course, x must equal r so that only y can vary. Let us therefore always choose out of this collection of anchored bonds the bond $\{x,y\}$ with the maximal value of y. From now on, whenever we have to discuss the conditional probability – conditioned on the existence of an anchored bond whose span is a given interval $[r,s]$, we shall consider the further conditioning on:

i) the values of $\{x, y, w, w', z, z'\}$ for the maximal anchored bond (in the sense described above) spanning $[r,s]$ and

ii) the occupation numbers $\{n_b\}$ for all the bonds which lie entirely in the H-neighborhood of x (i.e. in $[x-H, x+H]$) and for all the bonds lying in the H-neighborhood of y, as well as $n_{\{x,y\}}$, $n_{\{w,w'\}}$, and $n_{\{z,z'\}}$ (all $=1$).

When the conditioning is on a collection of intervals $\{r_i, s_i\}$ being spanned, we shall specify i) and ii) for each one of them. It is easy to see that once the information listed above is specified, it determines that each of the intervals is indeed the span of some anchored bond. The resulting conditional distribution of the bond variables $\{n_{b'}\}$ is constrained in two ways: first, by the given values for the above described collection of bonds, and secondly, by the requirement that certain sites and bonds be the "latest" ones with the required connections. The first set of constraints will be rather easy to incorporate explicitly. The second would not have been so, had it not been for the fact that it is of a definitely negative nature – in the sense of FKG, and therefore (by (4.1)) it can only lower the conditional probabilities of increasing events.

Turning to the case in (4.18), let us note that the conditioning there is also on the added information that the given collection of intervals forms a *maximal* nested sequence of spans. That forms another addition to the complete information about the specified collection of bonds, which however is also negative.

This refined conditioning may be used in estimating the left-hand side of (4.18) which is certainly bounded by the supremum of the more explicitly conditioned probabilities. Since the reader who has read the last two lemmas has already seen the relevant expressions, we shall not repeat those here but just describe the considerations which permit us to control the effects of the conditioning.

The event $\{s_n \geqq s_{n-1}+u\}$ will occur precisely if there is an anchored bond whose span has one end in the interval $I_1=[s_{n-2}+1, s_{n-1}]$ and the other in $I_2=[s_{n-1}+u, \infty)$. The indicator function of this event can be easily seen to form an increasing function of the variables $\{n_b\}$ (once the values of s_{n-1} and s_{n-2} are given). Therefore, by the above comments, and "the strong FKG" property of the measure (used here for the first time in our argument), the negative information coming from the various maximality conditions can be ignored in the calculation of an upper bound. Thus, the only effect of the conditioning we need to concern ourself with is the statement that a specified collection of bonds are all occupied.

The restriction that for $i=2, \ldots, n-1$: $s_i - s_{i-1} \geqq R > J$ (and hence $s_i - r_i > J$), implies that all the bonds which in our refinement are associated with $[r_i, s_i]$ are contained in the interval $(-\infty, s_i - H]$ (see (4.9)). Therefore the specified bonds have no site in I_2, and altogether only a uniformly bounded number of sites in I_1. The H-neighborhoods of these sites cover not more than $(12H+4)$ lattice points (see item i) immediately below). The probability that the two sets I_1 and I_2 are linked by the span of some anchored bond may now be bounded using the arguments of Lemmas 4.1 and 4.2, with the following corrections:

i) The effect on the term P_1 of the bonds already specified as occupied will be overestimated by regarding as anchored each site in I_1 which is within the distance $2H$ to either x_{n-1} or y_{n-1}, or within the distance H to either w_{n-1} or z_{n-1}. That will convert the factors $\beta^+ M_H{}^2$ into $\beta^+ M_H$, but only for a fraction $(\leqq (12H+4)/R)$ of terms in our analog of (4.15).

ii) In the exponentiated version of (4.12) and (4.13) omit the terms for which the bond $\{x, y\}$ is one of the bonds specified to be occupied. The contribution of the corresponding events in the lists ii) and iii) of the proof of Lemma 4.3 will be separately taken into account by estimating the probability that one of the $(12H+4)$ above mentioned sites (all below s_{n-1}) has an occupied bond directly linking it with I_2-2H. For each such bond the conditional probability is still bounded by the function K^+.

It is easy to see that if R is chosen so that $[J+H(\beta^+ +1)]/R \leqq C\varepsilon$, with a small enough constant C, then the above corrections will still represent only small perturbations of P_1. Hence, under the assumptions made in the statement of the lemma we have the following analog of (3.9):

$$1-\text{left-hand side of (4.18)} \geqq \exp\left(-\int_{I_1} dx \int_{I_2} dy (\beta^+ {M_H}^2 + 3\varepsilon)/|x-y|^2\right). \quad (4.19)$$

Evaluating the integral, and noting that $\beta^+ {M_H}^2 + 3\varepsilon \leqq \beta'$, one arrives at (4.18). □

The above result contains the main estimate needed for the proof of Lemma 4.2.

Proof of Lemma 4.2. Let us start by choosing J and R as in Lemma 4.5. The goal is to prove the existence of a lower bound, which is independent of L, on the probability that there is a site ξ in (L, kL) for which there is no H-anchored bond $\{x, y\}$ with $x \in [0, \xi]$ and $y \in (\xi, \infty)$.

Let us note that if the end points of an anchored bond satisfy the last two conditions then so do the end points of its span! Therefore, to prove the first claim made in Lemma 4.2 it suffices to prove a bound like (4.8a) for the analogous event which refers to the *spans of anchored bonds* instead of the anchored bonds themselves. The argument used for the proof of Lemma 3.1 implies, mutatis mutandis, that such a bound follows from Lemma 4.5. The bound (4.8b) follows from (4.8a) by choosing $\beta' < 1$ and using Lemma 3.1 (or Lemma 2.2). □

iv) Proof of the Main Result

Proof of Proposition 4.1. Given that $\beta_R < 1$, then by (4.3) we may choose H large enough so that the finite volume approximant of β_R also satisfies this condition, i.e.: $\beta^+ {M_H}^2 < 1$. We shall see that for such values of H, with probability one the system of the H-anchored bonds dissociates.

The key result needed for the proof of the dissociation is provided by Lemma 4.2, which was proven at the end of the preceding discussion. The technique for that application of (4.8b) was presented in the proofs of Lemma 2.1 and Proposition 3.1. We shall now assume that the reader is thoroughly familiar with the arguments presented there, and just briefly review the steps which require any adjustment.

The basic idea is to prove that for all L

$$\text{Prob}(\tilde{A}_L) = 1\,, \quad (4.20)$$

by deriving a uniform lower bound on $\text{Prob}(\tilde{A}_L \mid \tilde{A}_{L,k}{}^c)$. To do that, we further condition on the event $\tilde{C}_{V,kL}$, whose purpose is to "contain" the "positive" information about the bonds in $[-kL, kL]$, which is implied by the statement that the event $\tilde{A}_{L,k}$ does not occur. Because of the nature of the anchored bonds, our definition of $\tilde{C}_{V,N}$ (which interests us for $N=kL$) will deviate slightly from the definition of $C_{V,N}$ found below (2.24). We take it here to be the event that there is no *pair* of occupied bonds $\{a,b\}$ and $\{c,d\}$ with $a\in[-N,N]$, $|b-c|\leqq H$, and $|d|\geqq V-2H$. Note that by taking $b=c=d$, the occurrence of $\tilde{C}_{V,N}$ implies there is no occupied bond $\{a,d\}$ with $|a|\leqq N$ and $|d|\geqq V-2H$. Instead of (3.18) we have the equally useful bound (obtained by elementary arguments as in (4.13)):

$$\begin{aligned} 1-\operatorname{Prob}(\tilde{C}_{V,N}) &\leqq \sum_{\substack{a,b,c,d \\ |a|\leqq N,\ |c-b|\leqq H,\ |d|\geqq V-2H}} K^{+}{}_{b-a}K^{+}{}_{d-c} \\ &\leqq 2(2N+1)(2H+1)\Big(\sum_{w} K^{+}{}_{w}\Big)\Big(\sum_{|z|\geqq|V-N-3H|/2} K^{+}{}_{z}\Big) \\ &\leqq O(N/|V-N-3H|) \longrightarrow 0, \quad \text{as} \quad V/N \longrightarrow \infty . \end{aligned} \tag{4.21}$$

As explained in the proof of Lemma 2.1, for V large enough we have

$$\operatorname{Prob}(\tilde{A}_L\,|\,\tilde{A}_{L,k}{}^c)\geqq 1/2\cdot\operatorname{Prob}(\tilde{A}_L\,|\,\tilde{A}_{L,k}{}^c\cap\tilde{C}_{V,kL}). \tag{4.22}$$

We evaluate the conditional probability in the right-hand side (4.22) by further conditioning on all the occupation variables of all the regular bonds which lie entirely in $(-V+2H, V-2H)$, noting that the event $\tilde{A}_{L,k}{}^c\cap\tilde{C}_{V,kL}$ is determined by the bonds in that interval together with purely "negative" information from outside that interval.

For a lower bound on the last expression in (4.22), we shall use the natural analog of the first inequality of (3.19), and resort to the strong FKG property of the measure in order to bound below the expression found there by the product of three "simple" conditional probabilities. (The strong FKG property allows us here to avoid the telescopic conditioning which was used in (3.19)).

The resulting estimate is

$$\begin{aligned} &\operatorname{Prob}(\tilde{A}_L\,|\,\tilde{A}_{L,k}{}^c\cap\tilde{C}_{V,kL}) \\ &\quad\geqq \inf[\operatorname{Prob}({}^{\wedge}\tilde{F}_{V,r}\,|\,B)\cdot\operatorname{Prob}({}^{\wedge}\tilde{F}^*{}_{V,r}\,|\,B)\cdot\operatorname{Prob}(G_{V,r}\cap G^*{}_{V,r}\,|\,B)], \end{aligned}$$

where B denotes the σ-algebra generated by the occupation variables of the regular bonds which lie entirely in $(-V+2H, V-2H)$; and the infimum is over all possible configurations of those bonds. The event ${}^{\wedge}\tilde{F}_{V,r}$ is the H-anchored bond analogue of ${}^{\wedge}F_{V,r}$, while $G_{V,r}$ is just the regular bond event defined following (2.26). (Similar statements hold for the reflected events, which are denoted by asterisks.) Lemma 4.2 can be applied to the conditioned measures of the first two factors on the right-hand side of this estimate (see the remark following Lemma 4.2), while the third factor can be estimated as in (3.20) (or (2.28)) to yield the result that, just as in (3.20), for every $r>1$:

$$\operatorname{Prob}(\tilde{A}_L\,|\,\tilde{A}_{L,k}{}^c\cap\tilde{C}_{V,kL})\geqq\tilde{g}(r)^2\cdot[8(1+r)]^{-(\beta^{+}+1)}. \tag{4.23}$$

(Note that $\tilde{A}_L$ could have also been replaced here by $\tilde{A}_{L,(1+r)V/L}$.)

As was explained in the first part of the proof of Lemma 2.1, the inequalities (4.22) and (4.23) imply that (4.20) is satisfied for each L.

It follows that

$$\operatorname{Prob}(\bigcap \tilde{A}_L)=1, \tag{4.24}$$

i.e. with probability one the system of H-anchored bonds dissociates (see the discussion at the beginning of Sect. 2iii)).

As has already been remarked, the existence of an infinite cluster implies that *for any* H there is an infinite chain of bonds which are all occupied and H-anchored. The point is that in any infinite path of occupied bonds, after the first few (which get one out of the H-neighborhood of the starting point) all the bonds are anchored.

The dissociation proven above was established only for H large enough (so that $\beta^+ M_H{}^2 < 1$). However the preceding comment shows that the dissociation for any H implies the absence of ordinary percolation in the original system of bonds. □

In the above proof we followed the lines of the proof of Lemma 2.1. Let us recall that we have already seen how to extract from this argument a more explicit bound. Using the technique of Corollary 2.1 (at the end of Sect. 2), we get the following result, which will be used in the next section.

Corollary 4.1. *If a bond percolation model satisfies the assumptions of Proposition 4.1 (in particular, it has $\beta_R < 1$) then its two point connectivity function (defined in the introduction) satisfies:*

$$\tau(x, y) \leqq C \exp[-(\hat{\lambda} \ln|x-y|)^{1/2}] \tag{4.25}$$

with some $C < \infty$ and $\hat{\lambda} > 0$.

Proof. Let us first choose H such that $\beta^+ M_H{}^2 < 1$, as in the proof of Proposition 4.1. The analysis given in the proof of Corollary 2.1 shows that the arguments used above actually imply that

$$\mathrm{Prob}(\tilde{A}_{L,k}{}^c) \leqq \tilde{C} \exp[-(\tilde{\lambda} \ln k)^{1/2}] \quad \text{for all} \quad L > 1 \quad \text{and} \quad k > 1 \tag{4.26}$$

with some $\tilde{C} < \infty$ and $\tilde{\lambda} > 0$ [because of the slight difference in the "constants" which appear in (4.21) and in (2.25), the values of $\tilde{C}$ and $\tilde{\lambda}$ differ in some insignificant details from those indicated by (2.31)].

As an aside, let us comment here that it would be most natural to introduce at this point the modified connectivity function,

$$\tilde{\tau}_H(x, y) = \mathrm{Prob}(x \text{ and } y \text{ are connected by a path of occupied } H\text{-anchored bonds}). \tag{4.27}$$

Since $\tilde{\tau}_H(0, x) \leqq \mathrm{Prob}(\tilde{A}_{1,x}{}^c)$, (4.26) directly implies that the function $\tilde{\tau}$ obeys the bound (4.25). It would be a mild addition to assume that there is a uniform lower bound, $\delta_H > 0$, on the probability that any given site is H-anchored. In such case $\tilde{\tau}_H(x, y) \geqq \delta_H{}^2 \tau(x, y)$ (by FKG) and thus (4.25) is satisfied as claimed. However, since the assumptions made in Proposition 4.1 do not imply such a lower bound, we shall give an argument which avoids it.

In order to bound $\tau(x, y)$ directly, let us consider the event that there is a path of occupied (regular) bonds, which does not repeat any bond, connecting x with y ($> x$). Let V be a number in the range: $H < V < |y-x| - H$. Any point which the above path visits in the set $N_x = [x-V, x+V] \backslash [x-H, x+H]$ is automatically anchored. The only way the path can avoid N_x is by including an occupied bond which reaches directly from $[x-H, x+H]$ to $\mathbb{R} \backslash [x-V, x+V]$. Similar considerations apply, of course, to N_y. On the other hand, the existence of a pair of connected sites $a \in N_x$ and $b \in N_y$, which are anchored, and hence also connected by a path of anchored bonds, is inconsistent with the x-translate of the event $\tilde{A}_{V,k}$ with $k=(y-x-V)/V$. Thus, for each V in the above range:

$$\begin{aligned}\tau(x,y) \leqq &\ \mathrm{Prob}(\text{there is an occupied bond} \\ &\quad \text{linking } [x-H, x+H] \text{ with } \mathbb{R}\backslash[x-V, x+V]) \\ &\quad +\mathrm{Prob}(\text{there is an occupied bond} \\ &\quad \text{linking } [y-H, y+H] \text{ with } \mathbb{R}\backslash[y-V, y+V]) \\ &\quad +\mathrm{Prob}(x\text{-translate of } \tilde{A}_{V,(y-x-V)/V^c}) \\ &\leqq 4(2H+1)(\beta^+ +1)/(V-H-1)+\tilde{C}\exp(-\{\tilde{\lambda}\ln[(y-x-V)/V]\}^{1/2}),\end{aligned} \tag{4.28}$$

where we also assumed that $H \geqq D(\beta^+ +1)$.

For each x and y we now choose, for the above bound, $V=\exp\{[\tilde{\lambda}\ln(y-x)]^{1/2}\}$. Elementary estimates show that with this choice, (4.28) implies (4.25) (with any $\hat{\lambda}<\tilde{\lambda}$). □

5. The Asymptotic Behavior of the Connectivity Function

In this section we shall consider the connectivity function: $\tau(x,y)=\mathrm{Prob}(x$ and y are connected by a path of occupied bonds).

Taking into account the possibility that the two sites are directly connected, one has the trivial bound,

$$\tau(x,y) \geqq K_{x,y} \equiv \mathrm{Prob}(n_{x,y}=1). \tag{5.1}$$

The first part of Proposition 1.2, which will be proven below, following Lemma 5.4, implies that for independent, translation invariant models with

$$K_{x,y} \approx \mu/|x-y|^s \tag{5.2}$$

there is also a converse bound,

$$\tau(x,y) \leqq C/|x-y|^s \quad \text{(with a } \beta \text{ dependent constant)}, \tag{5.3}$$

which differs from (5.1) by only a finite multiplicative constant, and holds throughout the regime $\beta<\beta_c$. The second part of Proposition 1.2 gives a lower bound on the decay of $\tau(x,y)$ at the critical point.

Although Proposition 1.2 is stated for independent percolation models, much of the analysis does not require independence. At the end of this section we present some results for models with dependent bond variables.

A key ingredient used in obtaining Proposition 1.2 is an inequality of Hammersley which is stated in the following proposition. Its proof may be found, for example in refs. [21, 7].

Proposition 5.1. *In an independent percolation model, for any subset of sites Λ, if $x \in \Lambda$ and $y \notin \Lambda$, then:*

$$\tau(x,y) \leqq \sum_{u \in \Lambda,\, v \notin \Lambda} \tau(x,u) K_{u,v} \tau(v,y). \tag{5.4}$$

The rest of the analysis of this section consists of the discussion of the consequences of the inequality (5.4) for one dimensional models.

Hammersley's inequality (5.4) is the percolation version of the "Simon inequality" satisfied by Ising model correlation functions [12]. In ref. [12], this type of inequality was shown to be very useful for the derivation of nonperturbative results for finite range systems. The arguments presented here show (following ref. [13]) how it may be applied in the case of long range models.

We shall now restrict our attention to translation invariant models, and henceforth assume that $\tau(x, y)$ is a function which obeys (5.4), and that $\tau(x, y) = \tau(0, y-x)$. We shall also not distinguish between $K_{x,y}$ and K_{y-x}. Let us now present two different applications of (5.4), obtained with different choices of Λ.

Lemma 5.1. *Let a positive function $\tau(0, x)$ satisfy (5.4) with positive K_x such that*

$$\sum_x K_x \cdot |x|^\delta < \infty, \quad \text{for some} \quad \delta > 0. \tag{5.5}$$

If there is some $L < \infty$, such that the following quantity is less than one,

$$\gamma_L \equiv \sum_{|u| \leqq L,\ |v| > L} \tau(0, u) K_{u,v} < 1, \tag{5.6}$$

then for some (small enough) $\varepsilon > 0$:

$$\sum_x \tau(0, x) |x|^\varepsilon < \infty, \tag{5.7}$$

and hence also $\tau(0, x) \leqq C/|x|^\varepsilon$, with a finite C (e.g. $C =$ left-hand side of (5.7)).

Proof. Let us denote $d(x-y) = \ln(|x-y|+1)$, and observe that it satisfies the triangle inequality. (In fact it is a metric for $\mathbb{R}$. Its relevance for a related problem was pointed out by Gross [22].) As a consequence of (5.5) we have

$$\sum_{|u| \leqq L,\ |v| > L} \tau(0, u) e^{\delta \cdot d(u)} K_{v-u} e^{\delta \cdot d(v-u)} < \infty. \tag{5.8}$$

Therefore, by the triangle inequality, the dominated convergence theorem, and (5.6), for small enough $\varepsilon > 0$ the following quantity (for which $\gamma_L(0) \equiv \gamma_L$) is also less than one,

$$\gamma_L(\varepsilon) = \sum_{|u| \leqq L,\ |v| > L} \tau(0, u) K_{v-u} e^{\varepsilon \cdot d(v)} < 1. \tag{5.9}$$

Consider now, for any finite volume cutoff M,

$$\chi_\varepsilon^{(M)} = \max_{x \in [-M, M]} \sum_{y \in [-M, M]} \tau(x, y) e^{\varepsilon \cdot d(y-x)}. \tag{5.10}$$

For each x and y such that $|y-x| > L$ we shall use (5.4) with $\Lambda = [x-L, x+L]$, and apply the triangle inequality for $d(\cdot)$. One gets

$$\tau(x, y) e^{\varepsilon \cdot d(y-x)} \leqq \sum_{\substack{u, v; \\ |u-x| \leqq L,\ |v-x| > L}} \tau(x, u) K_{u,v} e^{\varepsilon \cdot d(v-x)} \tau(v, y) e^{\varepsilon \cdot d(y-v)}, \tag{5.11}$$

and thus, upon summation

$$\chi_\varepsilon^{(M)} \leqq \sum_{|x| < L} \tau(0, x) e^{\varepsilon \cdot d(x)} + \chi_\varepsilon^{(M)} \cdot \gamma_L(\varepsilon), \tag{5.12}$$

or equivalently,

$$\chi_\varepsilon^{(M)} \leqq \sum_{|x| < L} \tau(0, x) e^{\varepsilon \cdot d(x)} / [1 - \gamma_L(\varepsilon)]. \tag{5.13}$$

Let us observe that the last bound is uniform in M. Taking the limit $M \to \infty$ we obtain (5.7). □

Lemma 5.2. *If in addition to the assumptions made in Lemma 5.1,*

$$K_x \leqq \mu/|x|^s \quad \text{for} \quad x > x_0, \tag{5.14}$$

with some $s>1$, $\mu<\infty$, and $x_0<\infty$, then

$$\tau(0,x) \leqq C/|x|^s \tag{5.15}$$

for some finite C.

Proof. For each $x > 3x_0$ let us now apply (5.4) with $\Lambda = [-x/6, x/6] \cap \mathbb{Z}$. We get

$$\tau(0,x) \leqq \sum_{\substack{|u| \leqq x/6 \\ |v| > x/6}} \tau(0,u) K_{u-v} \tau(v,x). \tag{5.16}$$

Let us denote $T(x) = \sup\{\tau(0,y) \mid |y| \geqq x\}$, and $\chi = \sum_x \tau(0,x)$. In (5.16) we shall distinguish between terms with $|v| > x/2$, in which case $K_{u-v} < \mu/|x/3|^s$, and terms with $|v| \leqq x/2$, for which $\tau(v,x) \leqq T(x/2)$. Altogether, we have

$$\begin{aligned} \tau(0,x) &\leqq \chi^2 \cdot \mu/[x/3]^s + \sum_{\substack{|u| \leqq x/6 \\ |v| > x/6}} \tau(0,u) K_{u-v} \cdot T(x/2) \\ &\leqq C'/|x|^s + \gamma_{x/6} \cdot T(x/2). \end{aligned} \tag{5.17}$$

Let α be any number in $(0, 2^{-s})$. The conclusion of Lemma 5.1 implies that under the assumptions made here $\gamma_x \to 0$, when $x \to \infty$. Therefore, there is some $L_0 < \infty$ such that for all $x \geqq L_0$, $\gamma_{x/6} \leqq \alpha$. Equation (5.17) implies that for all $x > \hat{L} \equiv \max\{L_0, 3x_0\}$:

$$T(x) \leqq C'/|x|^s + \alpha \cdot T(x/2). \tag{5.18}$$

Iterating (5.18) n times, with n the smallest integer for which $x/2^n \leqq \hat{L}$, we get for all $x > \hat{L}$,

$$\begin{aligned} T(x) &\leqq C'[1 + 2^s \cdot \alpha + \ldots + (2^s \cdot \alpha)^{n-1}]/|x|^s + \alpha^n T(x/2^n) \\ &\leqq [C'/(1 - 2^s \cdot \alpha) + \hat{L}^s T(\hat{L}/2)]/|x|^s, \end{aligned} \tag{5.19}$$

for which we used: $\alpha^n \leqq 2^{-n \cdot s} = x^{-s}(x/2^n)^s \leqq (\hat{L}/x)^s$, and $x/2^n > \hat{L}/2$.

Since $\tau(0,x) \leqq T(x) \leqq 1$, (5.19) easily implies that for all x:

$$\tau(0,x) \leqq [C'/(1 - 2^s \cdot \alpha) + \hat{L}^s]/|x|^s, \tag{5.20}$$

where (5.19) was extended to short distances by the replacement of the factor $T(\hat{L}/2)$ by 1. This proves (5.15). □

The above results show that an upper bound like (5.3) is satisfied whenever the quantity γ_L is less than 1 for *some* L. We shall now present two sufficiency conditions for this to happen.

Lemma 5.3. *If $\tau(0,x)$ and K_x are positive functions such that $\chi \equiv \sum_x \tau(0,x) < \infty$ and $\|K\| \equiv \sum_x K_x < \infty$, then γ_L, as defined in (5.6), satisfies $\gamma_L < 1$ for all large enough L.*

Proof. Under the above assumptions, the sum in the right-hand side of (5.6) is convergent even without the restrictions on u and v, being equal to $\chi \cdot \|K\|$. Since γ_L

is bounded by the tail of this convergent sum, obtained by imposing the restriction $|v|>L$, we see (by the dominated convergence theorem) that γ_L tends to zero as $L\to\infty$. □

Lemma 5.4. *If the function K_x satisfies the condition* (5.14), *and τ obeys:*

$$\begin{aligned} &\lim_{x\to\infty}\tau(0,x)=0 && \textit{in case}\quad s>2,\\ &\textit{for}\quad x>\hat{x},\quad \tau(0,x)\leqq\lambda/(1+\ln x)\quad \textit{with}\quad \lambda<(2\mu)^{-1} && \textit{in case}\quad s=2,\\ &\textit{for}\quad x>\hat{x},\quad \tau(0,x)\leqq\lambda/|x|^{2-s}\qquad \textit{with}\quad \lambda<(a_s\mu)^{-1} && \textit{in case}\quad 2>s>1, \end{aligned}\tag{5.21}$$

with some finite $\hat{x}$, and the constants a_s given by (5.24) *below, then*

$$\limsup_{L\to\infty}\gamma_L<1\,.\tag{5.22}$$

Proof. Let us first decompose the summation in (5.6) as follows:

$$\begin{aligned}\gamma_L&=\sum_{|u|\leqq L,\,|v|>L}\tau(0,u)K_{u-v}\\ &\leqq\sum_{|u|<L-x_0,\,|v|>L}\tau(0,u)\mu/|v-u|^s+2\sum_{L-x_0\leqq u\leqq L,\,|v|>L}\tau(0,u)K_{u-v}\,,\end{aligned}$$

where $x_0\geqq1$ is large enough so that (5.14) is satisfied. Summing over v we get:

$$\begin{aligned}\gamma_L&\leqq2\mu/(s-1)\cdot L^{-(s-1)}+2\mu/(s-1)\\ &\quad\times\int_{[0,L-1]}du\,T(u)[1/(L-u)^{s-1}+1/(L+u)^{s-1}]+2T(L-x_0)\cdot(1+x_0)\,\|K\|\,.\end{aligned}\tag{5.23}$$

To prove the claim for the cases $s\geqq2$, one may split the domain of integration into two parts and bound $T(u)$ by the corresponding expression in (5.21) for $u\in(D,L-1]$, and by 1 for $u\in[0,D)$. With these bounds the integrals can be readily evaluated. Our choice of D, as a function of L, is: i) for $s>2$, $D=L-R$, with R chosen so that $2\mu/[(s-1)(s-2)R^{s-2}]<1$, ii) for $s=2$ (which we regard as the most interesting case) we take $D=L/\ln L$.

For $1<s<2$ a direct substitution of (5.21) in (5.23) shows that (5.22) is satisfied, if a_s is chosen as follows

$$a_s=2(s-1)^{-1}\int_{[0,1]}dx\,x^{-(2-s)}[(1-x)^{-(s-1)}+(1+x)^{-(s-1)}]\quad(<\infty).\quad\square\tag{5.24}$$

Proof of Proposition 1.2. i) For $\beta<\beta_c$. The recent result of reference [14] shows that $\chi<\infty$ for all translation invariant independent percolation models at $\beta<\beta_c$ (where the critical point is defined by the onset of percolation). Therefore the desired upper bound (1.8) for $\tau(x,y)$ follows from Proposition 5.1 and Lemmas 5.2 and 5.3.

ii) For $\beta=\beta_c$. We first consider the case $s<2$. Lemma 5.4 shows that were (1.9) not valid, then $\tau(0,x)$ would be $O(1/|x|^s)$ and hence χ would be finite at $\beta=\beta_c$; but this would contradict the result of ref. [7] that $\chi\to\infty$ as $\beta\to\beta_c(-0)$. Thus $\tau(0,x)$ obeys the lower bound (1.9) for $s<2$.

For $s=2$, we may restrict our attention to the maximal irreducible sublattice containing the origin (see Remark ii) following Proposition 1.2) on which $M>0$ at the critical point (by Proposition 1.1). Applying the uniqueness of the infinite cluster proven in ref. [11], and an FKG inequality for the positive events that 0 and x belong to the infinite cluster, we conclude that $\tau(0,x) \geqq M^2$ for any x in the sublattice. This yields (1.9) for $s=2$. □

We conclude this section with a proposition applicable to certain dependent bond models.

Proposition 5.2. *If a translation invariant one-dimensional bond percolation model is regular and satisfies:* i) *the "strong FKG" condition,* ii) *the bound* (5.14) *on* K_x, *with* $s \geqq 2$, *and* iii) *the Hammersley-Simon inequality* (5.4), *then whenever* $M_+ = 0$:

$$\tau(0,x) \leqq C/|x|^s \quad \text{with some} \quad C < \infty . \tag{5.25}$$

Proof. The upper bound provided by Corollary 4.1 (which in itself seems to be a very weak result) implies that the $s=2$ or $s>2$ case of (5.21) is valid, and hence, by Lemma 5.4, $\tau(0,x) = O(1/|x|^s)$. □

Remark. The above bound provides an extension of only one of the results proven in Proposition 1.2 for independent models. Let us summarize what auxiliary information would permit an immediate extension of the other results to a model with dependent bond variables, which satisfies the assumptions of Proposition 5.2.

i) An extension of (5.25) to $s<2$ would be implied by the finiteness of χ for all $\beta < \beta_c$ (proven for the independent case in [14]).

ii) The lower bound (1.9), on the critical point behavior of $\tau(0,x)$ at the critical point, would follow, for the case $s<2$, from the additional information that $\chi = \infty$ at $\beta = \beta_c$. (Such a result would be implied by a slight improvement of the inequality (5.4), as in [23], which is in fact valid for independent percolation [7].)

iii) For $s=2$, the knowledge that $\chi = \infty$ at $\beta = \beta_c$ would imply that at $\beta = \beta_c$, $\tau(0,x)$ does not decay faster than $[2\mu \ln x]^{-1}$. A uniform lower bound, like the one derived here, would be implied by the uniqueness of the infinite cluster (see [24, 25, 11] for related discussions).

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