The Annals of Probability
1996, Vol. 24, No. 2, 1036–1048

ON THE CHEMICAL DISTANCE FOR SUPERCRITICAL BERNOULLI PERCOLATION

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We prove large deviation estimates at the correct order for the graph distance of two sites lying in the same cluster of an independent percolation process. We improve earlier results of Gärtner and Molchanov and Grimmett and Marstrand and answer affirmatively a conjecture of Kozlov.

1. Introduction and statement of results. In this article we study for $d \geq 2$ independent (Bernoulli) bond percolation on the d-dimensional cubic lattice $\mathscr{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, where $\mathbb{E}^d = \{\{x, y\};\ \sum_{i=1,\dots,d} |x_i - y_i| = 1\}$ stands for the set of edges between nearest neighbors in $\mathbb{Z}^d$. That is, all bonds are open with probability p and closed with probability $1 - p$ independently of each other. The corresponding probability measure on $\{0, 1\}^{\mathbb{E}^d}$ is denoted by $\mathbb{P}$.

A *path* of $\mathscr{L}^d$ of length n (≥ 1) is a sequence $(x_0, x_1, \dots, x_n)$ of nearest-neighbor vertices. A path consisting of distinct vertices is called *self-avoiding*. We say that a path is open (closed) if all bonds between successive vertices of the path are open (closed). For $x \in \mathbb{Z}^d$ we denote by C_x the cluster of x, that is, the set of all vertices which are connected to x by a path, whose edges are all open. Throughout this paper we shall assume that p is strictly larger than the critical probability

$$p_c := \sup\{p\colon\ \theta(p) = 0\}, \tag{1.1}$$

where $\theta(p)$ is the probability that the cluster of the origin has infinite cardinality.

We shall write $x \leftrightarrow y$ to say that two sites, x and y, are in the same cluster. For such sites we denote by $D(x, y)$ the minimal length of an open path connecting x to y. This quantity is sometimes also called the chemical distance of x and y. On $\mathbb{Z}^d$ we shall use the distance induced by the norm

$$|y| := \sum_{i=1,\dots,d} |y_i|. \tag{1.2}$$

The main object of this paper is to prove the following large deviation bounds.

Received December 1994; revised August 1995.
[1]Research was partially supported by the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University.
[2]Research was partially supported by the Swiss National Foundation.
AMS 1991 *subject classifications*. 60K35, 82B43.
Key words and phrases. Graph distance, percolation, supercritical phase, large deviations.

THEOREM 1.1. *Let $p > p_c$. Then there exists a constant $\rho = \rho(p, d) \in [1, \infty)$ such that*

$$\limsup_{|y|\to\infty} \frac{1}{|y|} \log \mathbb{P}[0 \leftrightarrow y,\ D(0, y) > \rho|y|] < 0. \tag{1.3}$$

THEOREM 1.2. *Let $p > p_c$. Then, for any $y \in Z^d$,*

$$\limsup_{l\to\infty} \frac{1}{l} \log \mathbb{P}[0 \leftrightarrow y,\ D(0, y) > l] < 0. \tag{1.4}$$

Applying Theorem 1.1 together with the Borel–Cantelli lemma, we obtain the following comparison result for D and the usual distance.

COROLLARY 1.3. *Let $p > p_c$. Then $\mathbb{P}$-almost surely*

$$\limsup_{|y|\to\infty} \frac{1}{|y|} D(0, y) 1_{\{0\leftrightarrow y\}} \le \rho(p, d), \tag{1.5}$$

where $\rho(p, d)$ is the constant introduced in Theorem 1.1.

REMARK. It is easy to see that the probabilities in (1.3) and (1.4) decay at most exponentially; that is, we also have lower bounds of the same order. In the case of (1.3) we can pick for any $\rho \geq 1$ a fixed path γ with a length $|\gamma| \in [\rho|y|, \rho|y| + 2]$ joining 0 to y. Then $\mathbb{P}[0 \leftrightarrow y,\ D(0, y) > \rho|y|]$ is larger than the probability that γ is an isolated open path; that is, γ is open and all other bonds adjacent to some vertex of γ are closed. (Note that in this case γ is the only path joining 0 to y.) This probability obviously decays exponentially in $|y|$ and this yields for every $\rho \geq 1$ the claimed exponential lower bound for the probability considered in (1.3). The same argument works for Theorem 1.2.

Let us give some comments concerning our results. First of all, we would like to point out that the main difficulty of the proof of (1.3) and (1.4) is to derive bounds of the *correct* (i.e., here exponential) order. In fact, *polynomial* (respectively, *subexponential*) bounds have earlier been derived by various authors; cf. [6], [7] and [8].

Theorem 1.1 improves an earlier result of Gärtner and Molchanov (see Lemma 2.8 in [6]), where a polynomial upper bound is given for the decay of the probability in (1.3) for the case of site percolation with sufficiently high parameter. The improvement now is that we show that the true leading asymptotic behavior is in fact exponential and it holds in the whole supercritical regime. Corollary 1.3 has an important application in trapping problems, as discussed in [2]. In fact, this result is one of the key ingredients which enable us to derive asymptotic lower bounds for the survival probability of a random walk, which is killed by obstacles made of the closed bonds of a percolation process. Although we only treat the case of bond percolation, our calculations can obviously also be adapted to the site case. We refrained from treating both cases here for the sake of clarity.

Our result stated in Theorem 1.2 answers a question of Kozlov, which has appeared in the context of the study of the Darcy equation for random porous media [3]. Theorem 1.2 improves the subexponential large deviation upper bound of Grimmett and Marstrand; see the last equation in [7].

2. Renormalization. In this section we develop a renormalization technique for Bernoulli percolation. This is in the same spirit as the technique introduced in [11]; however, in our case the geometry of the renormalization is different and therefore an additional argument [cf. (2.16) and (2.18)] is needed in order to prove the required properties of the renormalized process.

Let us first introduce some additional notation. A *box* B is a subset of $\mathbb{Z}^d$ of the form $\{x \in \mathbb{Z}^d \mid r_i \le x_i \le s_i,\ 1 \le i \le d\}$, where $r,\ s \in \mathbb{R}^d$. We fix an integer $N > 1$. We shall see that only large values of N will be of interest; therefore, we shall implicitly assume in each definition involving N that N is at least so large that the definition makes sense. We now chop $\mathbb{Z}^d$ into disjoint boxes as follows: we set $B_0(N)$ to be the box $[-N,\ N]^d \cap \mathbb{Z}^d$ and define, for $\mathbf{i} \in \mathbb{Z}^d$,

$$B_{\mathbf{i}}(N) := \tau_{\mathbf{i}(2N+1)} B_0(N), \tag{2.6}$$

where $\tau_{\mathbf{b}}$ stands for the shift in $\mathbb{Z}^d$ with $\mathbf{b} \in \mathbb{Z}^d$. The boxes $(B_{\mathbf{i}}(N))_{\mathbf{i}\in\mathbb{Z}^d}$ define a partition of $\mathbb{Z}^d$. We now define the renormalized lattice as the graph with vertex set $(B_{\mathbf{i}}(N))_{\mathbf{i}\in\mathbb{Z}^d}$ and edge set $\{\{B_{\mathbf{i}}(N),\ B_{\mathbf{j}}(N)\},\ |\mathbf{j}-\mathbf{i}| = 1\}$. We shall identify this graph with a copy of $\mathscr{L}^d$, whose vertices we denote by bold letters in order to distinguish them from the vertices of the original lattice. We shall also need another family of boxes, namely,

$$B'_{\mathbf{i}}(N) := \tau_{\mathbf{i}(2N+1)} B_0(5N/4). \tag{2.7}$$

We now define for $N \in \mathbb{N}$, the following set of bonds:

$$\mathscr{E}(N) = \big\{\{ke^{(1)},\ (k+1)e^{(1)}\};\ k = 0, \dots, \lfloor N^{1/2} \rfloor\big\}, \tag{2.8}$$

where $e^{(1)}$ stands for the first unit vector in $\mathbb{Z}^d$. Finally, we set $\mathscr{E}_{\mathbf{i}} = \tau_{\mathbf{i}(2N+1)}\mathscr{E}(N)$.

Next we need the notion of a crossing cluster in a box (see [11]). We say that a cluster $\mathscr{C}$ contained in some box B' is a *crossing cluster* for $B \subseteq B'$, if for all d directions there is an open path contained in $\mathscr{C} \cap B$ joining the left face to the right face of the box B.

We now assume that $N \ge 10$ and introduce the events

$$\begin{aligned} R_{\mathbf{i}}^{(N)} := \{\exists\,!\ &\text{crossing cluster } \mathscr{C} \text{ in } B'_{\mathbf{i}}(N) \text{ for } B'_{\mathbf{i}}(N), \text{ all open paths} \\ &\text{contained in } B'_{\mathbf{i}}(N) \text{ of radius larger than } \tfrac{1}{10}N \text{ are connected} \\ &\text{to } \mathscr{C} \text{ within } B'_{\mathbf{i}}(N) \text{ and } \mathscr{C} \text{ is crossing for each subbox } B \subseteq \\ &B'_{\mathbf{i}}(N) \text{ of side length larger than } \tfrac{1}{10}N\}, \end{aligned} \tag{2.9}$$

$$S_{\mathbf{i}}^{(N)} := \{\text{there is at least one open bond in } \mathscr{E}_{\mathbf{i}}\}. \tag{2.10}$$

We now define a map ϕ_N from Ω to the space $\Omega' := \{0, 1\}^{\mathbb{Z}^d}$ (with the σ-field generated by the finite-dimensional cylinders) by

$$(\phi_N \omega)_{\mathbf{x}} := 1_{R_{\mathbf{x}}^{(N)} \cap S_{\mathbf{x}}^{(N)}}(\omega). \tag{2.11}$$

We denote the image measure of $\mathbb{P}$ under this map by $\mathbb{P}_N$. This defines a (dependent) site percolation process on the renormalized lattice. Sometimes we shall call the renormalized process the macroscopic process in order to distinguish it from the original (microscopic) bond percolation process. We shall call the sites of the macroscopic process white and black (instead of occupied and vacant); that is, the site $\mathbf{x} \in \mathbb{Z}^d$ is white if $(\phi_N \omega)_{\mathbf{x}} = 1$, otherwise it is black.

Next we need the notion of $*$-connectedness. We say that a subset of $\mathbb{Z}^d$ is $*$-connected if it is connected with respect to the adjacency relation

$$x \overset{*}{\sim} y \Leftrightarrow \max_{i=1,\dots,d} |x_i - y_i| = 1. \tag{2.12}$$

We are now able to state the main result of this section.

PROPOSITION 2.1.

(2.13) *If $\Gamma \subseteq \mathbb{Z}^d$ is a $*$-connected set of white sites, then there is a microscopic cluster contained in $\bigcup_{\mathbf{i}\in\Gamma} B'_{\mathbf{i}}(N)$, which is crossing for each box $B'_{\mathbf{i}}(N)$, $\mathbf{i} \in \mathbf{\Gamma}$.*

Moreover, for each $p > p_c$, there exists a function $\bar{p}$: $\mathbb{N} \to [0, 1)$ with $\lim_{N\to\infty} \bar{p}(N) = 1$, such that $\mathbb{P}_N$ stochastically dominates the law of an independent site percolation process with parameter $\bar{p}(N)$, meaning that, for any increasing event A,

$$\mathbb{P}_N(A) \geq \mathbb{P}^*_{\bar{p}(N)}(A), \tag{2.14}$$

where $\mathbb{P}^*_q$ denotes the law of an independent site percolation process with parameter q.

PROOF. Property (2.13) is an obvious consequence of the occurrence of the events $R_{\mathbf{i}}^{(N)}$ for all $\mathbf{i} \in \mathbf{\Gamma}$. To prove the second part, it is enough to check that (2.14) holds for any local increasing event A (local means that A depends only on finitely many sites). This will follow from the next proposition.

PROPOSITION 2.2. *We have, as $N \to \infty$,*

$$\alpha(N) := \sup_{L\geq 3} \sup_{\mathbf{z}\in B_0(L)} \operatorname{ess\,sup} \mathbb{P}_N[Y_{\mathbf{z}} = 0 \mid \sigma(Y_{\mathbf{x}}, \mathbf{x} \in B_0(L) \setminus \{\mathbf{z}\})] \to 0, \tag{2.15}$$

where Y denotes the coordinate process on Ω'; cf. (2.11).

PROOF. Let us first explain the outline of the proof. By Theorem 3.2 in [11] ($d \geq 3$) and Theorem 5 in [10] ($d = 2$), we can control the (unconditioned) probability of the event $\{Y_{\mathbf{z}} = 0\}$. This will be used to derive an iterative inequality for $\alpha(N)$ from which it will follow that $\alpha(N)$ is either always larger than a constant or tends to 0 for $N \to \infty$. To exclude the first possibility, we shall introduce a mixture of $\mathbb{P}_N$ with the measure corresponding to an independent site percolation process and show that if we replace $\mathbb{P}$ in the definition of $\alpha(N)$ by the mixed measure, then the iterative inequality is still satisfied. In the case of the independent measure, it is obvious that we are in the right regime and by a continuity argument we shall conclude this for $\mathbb{P}_N$, too.

Next we fix $N \geq 10$ and pick $L \geq 3$. We introduce for $\rho \in [0,\ 1]$ the following family of measures on Ω':

$$Q_N^\rho := \rho\,\mathbb{P}_N + (1-\rho)\,\mathbb{P}^*_{q(N)}, \tag{2.16}$$

where $q(N) := \mathbb{P}_N[Y_0 = 1]$ and $\mathbb{P}^*_q$ denotes the law of an independent site percolation process with parameter q. We set

$$\beta(\rho,\ L,\ N) := \sup_{\mathbf{z}\in B_0(L)} \operatorname{ess\,sup} Q_N^\rho[Y_{\mathbf{z}} = 0 \mid \sigma(Y_{\mathbf{x}},\ \mathbf{x} \in B_0(L) \setminus \{\mathbf{z}\})]. \tag{2.17}$$

We shall prove that there exist constants $c_1,\ c_2 > 0$, depending only on $(p,\ d)$, such that, for all $\rho \in [0,\ 1]$ and for all N with $1 - 4\cdot 3^d c_1 \exp(-c_2 N^{1/2}) > 0$,

$$\beta(\rho,\ L,\ N)(1 - 3^d \beta(\rho,\ L,\ N)) \leq c_1 \exp(-c_2 N^{1/2}). \tag{2.18}$$

Let us admit (2.18) for a moment and conclude the proof of the proposition. As a consequence of (2.18) we see that either

$$\beta \geq \frac{1}{2} 3^{-d}\Big(1 + \sqrt{1 - 4\cdot 3^d c_1 \exp(-c_2 N^{1/2})}\Big) \tag{2.19}$$

or

$$\beta \leq \frac{1}{2} 3^{-d}\Big(1 - \sqrt{1 - 4\cdot 3^d c_1 \exp(-c_2 N^{1/2})}\Big). \tag{2.20}$$

Next we claim that, for N and L fixed, $\beta(\rho,\ L,\ N)$ is a continuous function of ρ. Since the suprema in (2.17) run over a finite set and the conditional expectation in (2.17) depends only on finitely many sites, the continuity boils down to the fact that any macroscopic configuration $\nu \in \{0,\ 1\}^{B_0(L)}$ has strictly positive probability under $\mathbb{P}_N$. To see this, consider, for instance, the microscopic configurations where each bond between sites in $\bigcup_{\mathbf{x}\in B_0(L)} B_{\mathbf{x}}(N)$ is open, up to those lying in $\bigcup_{\mathbf{x}:\ \nu_{\mathbf{x}}=0} \mathscr{E}_{\mathbf{x}}$ (which are closed).

Because of this continuity argument we now see that, for fixed N and L, $\beta(0,\ L,\ N)$ and $\beta(1,\ L,\ N)$ satisfy both either (2.19) or (2.20). Now $\beta(0,\ L,\ N)$ is equal to $\mathbb{P}_N[Y_0 = 0]$ by definition, and by Theorem 3.2 in [11] and Theorem 5 in [10] this probability tends to 0 for $N \to \infty$. Therefore, for N large enough, $\beta(1,\ L,\ N)$ also satisfies (2.20). Since the r.h.s. of this inequality does not depend on L,

$$\alpha(N) \leq \frac{1}{2} 3^{-d}\Big(1 - \sqrt{1 - 4\cdot 3^d c_1 \exp(-c_2 N^{1/2})}\Big), \tag{2.21}$$

which implies the claim of the proposition.

So, let us now show (2.18). For this we pick $\mathbf{z} \in B_0(L)$ and introduce the sets

$$\mathscr{N}(\mathbf{z}) := \{\mathbf{x} \in \mathbb{Z}^d;\ x \overset{*}{\sim} z\}, \qquad \mathscr{M}^1 := \mathscr{N}(\mathbf{z}) \cap B_0(L),$$

$$\mathscr{M}^2 := B_0(L) \setminus (\mathscr{M}^1 \cup \{\mathbf{z}\}).$$

Note that by the choice of L these sets are nonempty. The σ-field $\sigma(Y_{\mathbf{x}},\ \mathbf{x} \in B_0^L \setminus \{\mathbf{z}\})$ is atomic and the atoms are of the form $\nu^1 \cap \nu^2$ with $\nu^i \in \{0,\ 1\}^{\mathscr{M}^i}$.

LEMMA 2.3. *There exist strictly positive constants* c_3, c_4, c_6, c_7, *depending only on* $(p,\ d)$, *such that*

$$Q_N^\rho[\{Y_{\mathbf{z}} = 0\} \cap \nu^1 \cap \nu^2] \le c_3 e^{-c_4 N} Q_N^\rho[\nu^2] + c_6 e^{-c_7 N^{1/2}} Q_N^\rho[\nu^1 \cap \nu^2]. \tag{2.22}$$

PROOF. It is enough to verify (2.22) for $\rho = 0$ and $\rho = 1$. Then by convexity it holds for any $\rho \in [0,\ 1]$. We begin with the case $\rho = 1$. We have

$$Q_N^1[\{Y_{\mathbf{z}} = 0\} \cap \nu^1 \cap \nu^2] \le \mathbb{P}[R_{\mathbf{z}}^c \cap \nu^2] + \mathbb{P}[S_{\mathbf{z}}^c \cap \nu^1 \cap \nu^2], \tag{2.23}$$

where we have dropped the N dependence of the events $R_{\mathbf{z}}$ and $S_{\mathbf{z}}$ [defined in (2.9) and (2.10)] and with a slight abuse of notation we have identified ν^i with $\phi_N^{-1}(\nu^i)$ [see (2.11)].

From Theorem 3.2 in [11] and Theorem 5 in [10], we know that

$$\mathbb{P}[R_{\mathbf{z}}^c] \le c_3 \exp(-c_4 N). \tag{2.24}$$

By the definition of the event S, it is obvious that, for certain strictly positive c_5, c_7,

$$\exp(-c_5 \lfloor N^{1/2} \rfloor) = \mathbb{P}[S_{\mathbf{z}}^c] \le c_6 \exp(-c_7 N^{1/2}). \tag{2.25}$$

Using the fact that, under $\mathbb{P}$, $R_{\mathbf{z}}$ is independent of ν^2 and $S_{\mathbf{z}}$ is independent of ν^1 and ν^2, we obtain (2.22) for $\rho = 1$. The case $\rho = 0$ is obvious [using (2.24) and (2.25)]. $\square$

We now obtain by the lemma above

$$Q_N^\rho[Y_{\mathbf{z}} = 0 \mid \nu^1 \cap \nu^2] \le c_3 \exp(-c_4 N) \frac{1}{Q_N^\rho[\nu^1 \mid \nu^2]} + c_6 \exp(-c_7 N^{1/2}). \tag{2.26}$$

The next step is to derive a lower bound on $Q_N^\rho[\nu^1 \mid \nu^2]$. It is convenient to introduce for $i \in \{0,\ 1\}$ the sets

$$C^i(\nu^1) := \{\mathbf{x} \in \mathscr{M}^1;\ \nu_{\mathbf{x}}^1 = i\}.$$

Using $\{Y_{\mathbf{x}} = 0\} \supseteq S_{\mathbf{x}}^c$, we have first, for $\rho = 1$ and $\rho = 0$,

$$\begin{aligned} Q_N^\rho[\nu^1 \mid \nu^2] &= Q_N^\rho\Biggl[\bigcap_{\mathbf{x} \in C^0(\nu^1)} \{Y_{\mathbf{x}} = 0\} \cap \bigcap_{\mathbf{x} \in C^1(\nu^1)} \{Y_{\mathbf{x}} = 1\} \cap \nu^2\Biggr] \Bigg/ Q_N^\rho[\nu^2] \\ &\ge \bigl(\exp(-c_5 N^{1/2})\bigr)^{3^d} Q_N^\rho\Biggl[\bigcap_{\mathbf{x} \in C^1(\nu^1)} \{Y_{\mathbf{x}} = 1\} \mid \nu^2\Biggr] \end{aligned} \tag{2.27}$$

[note that $|\mathscr{N}(\mathbf{z})| = 3^d - 1$]. By convexity [and by rewriting (2.27) with absolute probabilities], we see that (2.27) extends to all $\rho \in [0, 1]$. We now proceed as follows:

$$
\begin{aligned}
Q_N^\rho\Big[\bigcap_{\mathbf{x}\in C^1(\nu^1)} \{Y_{\mathbf{x}} = 1\} \,|\, \nu^2\Big] &\geq \Big(1 - \sum_{\mathbf{x}\in C^1(\nu^1)} Q_N^\rho[Y_{\mathbf{x}} = 0 \,|\, \nu^2]\Big)_+ \\
&\geq \Big(1 - \sum_{\mathbf{x}\in \mathscr{M}^1} Q_N^\rho[Y_{\mathbf{x}} = 0 \,|\, \nu^2]\Big)_+ \\
&\geq (1 - 3^d \beta(\rho, L, N))_+,
\end{aligned}
\tag{2.28}
$$

where $\beta(\rho, L, N)$ is the quantity defined in (2.17). Combining (2.26), (2.27) and (2.28), we see that, for any ν^1 and ν^2,

$$
Q_N^\rho[Y_{\mathbf{z}} = 0 \,|\, \nu^1 \cap \nu^2] \leq \frac{c_3 \exp(-c_4 N + 3^d c_5 N^{1/2})}{(1 - 3^d\beta(\rho, L, N))_+} + c_6 \exp(-c_7 N^{1/2}). \tag{2.29}
$$

Since the r.h.s. of the last inequality is independent of ν^1, ν^2 and z, $\beta(\rho, L, N)$ is itself bounded by the expression on the r.h.s. of (2.29). Note that, if $1 - 3^d\beta(\rho, L, N) \leq 0$, then (2.18) is trivially satisfied. Therefore, we can assume $1 - 3^d\beta(\rho, L, N) > 0$. In this case, by using the bound given in (2.29), we can easily verify (2.18) for certain positive c_1 and c_2. $\square$

3. Construction of a short path. The heart of the proof of our main theorems is a deterministic construction of a path between two sites in the same cluster. The construction uses the renormalization of the previous section and involves microscopic and macroscopic arguments at the same time.

Let us first introduce some additional notation. In the whole section we consider a fixed microscopic configuration $\omega \in \Omega$. We also fix some integer N and look at the induced macroscopic configuration $\phi_N(\omega) \in \Omega'$. We denote by $\mathscr{C}^*$ the set of all $*$-connected macroscopic black clusters; that is, the elements of $\mathscr{C}^*$ are the $*$-connected components of the set of black sites of $\mathbb{Z}^d$. For $\mathbf{i} \in Z^d$ we denote by $\mathbf{C}_{\mathbf{i}}^*$ the element of $\mathscr{C}^*$ containing $\mathbf{i}$. We use the convention that $\mathbf{C}_{\mathbf{i}}^* = \varnothing$, if $\mathbf{i}$ is white.

For a finite subset $\Lambda \subseteq \mathbb{Z}^d$ we introduce different types of boundaries, namely,

$$
\partial^{\text{out}}\Lambda := \big\{\mathbf{i} \in \Lambda^c\colon\ \exists\ \mathbf{j} \in \Lambda,\ \{\mathbf{i}, \mathbf{j}\} \in \mathbb{E}^d\big\}, \tag{3.30}
$$

$$
\partial^{\text{in}}\Lambda := \big\{\mathbf{i} \in \Lambda\colon\ \exists\ \mathbf{j} \in \Lambda^c,\ \{\mathbf{i}, \mathbf{j}\} \in \mathbb{E}^d\big\}, \tag{3.31}
$$

which are called the *outer* (respectively the *inner*) *boundaries* of Λ. We shall use the convention that for a white site $\mathbf{i} \in \mathbb{Z}^d$ we define $\partial^{\text{out}}\mathbf{C}_{\mathbf{i}}^* = \{\mathbf{i}\}$.

Observe that for any finite set $\Lambda \subseteq \mathbb{Z}^d$ there are only finitely many connected components of Λ^c and exactly one of them has infinite cardinality. We denote these components by $\Lambda_1^c, \dots, \Lambda_k^c$ and assume that Λ_1^c is the infinite component. We call the components $\Lambda_2^c, \dots, \Lambda_k^c$ *holes*. If Λ^c is connected, then we say that

Λ has no holes. We set

$$\widehat{\Lambda} := \Lambda \cup \Lambda_2^c \cup \cdots \cup \Lambda_k^c. \tag{3.32}$$

We define the *external outer* (respectively, *external inner*) *boundary* of Λ as

$$\partial_{\text{ext}}^{\text{out}}\Lambda := \partial^{\text{out}}\widehat{\Lambda}, \tag{3.33}$$

$$\partial_{\text{ext}}^{\text{in}}\Lambda := \partial^{\text{in}}\widehat{\Lambda}. \tag{3.34}$$

By Lemma 1.1 in [11] we have the following property, which is crucial for the remainder of our discussion:

(3.35) For any finite $*$-connected set Λ, the external boundaries $\partial_{\text{ext}}^{\text{out}}\Lambda$ and $\partial_{\text{ext}}^{\text{in}}\Lambda$ are $*$-connected.

The last definition we need is the notion of surrounding sets. For two subsets U, V of $\mathbb{Z}^d$ we say that U surrounds V if $V \subseteq \widehat{U}$. An equivalent definition is that any self-avoiding path of infinite length starting at some point of V hits U. In particular, any finite set is surrounded by itself as well as by its outer (respectively, inner) external boundary.

We now come to the main result of this section. We consider two sites x, $y \in \mathbb{Z}^d$. Let $\mathbf{a}(x)$ and $\mathbf{a}(y)$ be the unique sites of the renormalized lattice such that $x \in B_{\mathbf{a}(x)}$ and $y \in B_{\mathbf{a}(y)}$. For notational convenience we have dropped the N-dependence, since N will be fixed during the whole section. We set $n := |\mathbf{a}(x) - \mathbf{a}(y)|$ and choose a macroscopic path $\mathbf{A} = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n)$ with $\mathbf{a}_0 = \mathbf{a}(x)$ and $\mathbf{a}_n = \mathbf{a}(y)$. We also define the set of $*$-connected black clusters intersecting $\mathbf{A}$; that is,

$$\mathbb{C} := \{\mathbf{C}_{\mathbf{a}}^*,\ \mathbf{a} \in \mathbf{A}\} = \{\mathbf{C}^* \in \mathscr{C}^*,\ \mathbf{C}^* \cap \mathbf{A} \neq \varnothing\}. \tag{3.36}$$

Our main result now is the following proposition.

PROPOSITION 3.1. *If x and y are in the same microscopic cluster, then there exists a microscopic self-avoiding open path γ joining x to y, such that γ is contained in*

$$W := \bigcup_{\mathbf{a} \in \mathbf{A}} \Bigg(\bigcup_{\mathbf{b} \in \bar{\mathbf{C}}_{\mathbf{a}}^*} B'_{\mathbf{b}} \Bigg), \tag{3.37}$$

where $B'_{\mathbf{b}}$ is the box defined in (2.7) *and $\bar{\mathbf{C}}_{\mathbf{a}}^* := \mathbf{C}_{\mathbf{a}}^* \cup \partial^{\text{out}}\mathbf{C}_{\mathbf{a}}^*$ (note that $\partial^{\text{out}}\mathbf{C}_{\mathbf{a}}^* := \{\mathbf{a}\}$ if $\mathbf{a}$ is white). In particular, we have*

$$D(0,\ y) \leq |W|. \tag{3.38}$$

PROOF. It is enough to construct a not necessarily self-avoiding path lying in W, since we can always extract from this path a self-avoiding one. Since x and y are in the same microscopic cluster, there is an open path σ starting at y with the endpoint x. This path is of course not necessarily in W, but we will show that we can modify σ in such way that the modified path has all vertices in W.

If x and y are both in the same box $B'_{\mathbf{a}}$ (this is only possible if $n \leq 2$) and moreover the site $\mathbf{a}$ is white, then our claim directly follows from the occurrence of the event $R_{\mathbf{a}}$. Therefore, in what follows, we shall assume that ω is not such a configuration. It is convenient to prove first the following statement.

LEMMA 3.2. *Assume that there is no cluster in* $\mathbb{C}$ *which surrounds both* $\mathbf{a}(x)$ *and* $\mathbf{a}(y)$. *Then there exists a white vertex* $\mathbf{e} \in \mathbf{A}$ *with the following properties*:

(3.39) $\mathbf{e}$ *is not surrounded by any cluster in* $\mathbb{C}$;

(3.40) *the crossing cluster of* $B'_{\mathbf{e}}$ *is connected to* x *by a path contained in* W.

PROOF. If there is no cluster in $\mathbb{C}$ which surrounds $\mathbf{a}(x)$, then $\mathbf{e} = \mathbf{a}(x)$ satisfies both of the above conditions. Indeed, in this case $\mathbf{a}(x)$ is white and σ leaves the box $B'_{\mathbf{e}}$ (the case where x and y are both in $B'_{\mathbf{e}}$ was already excluded), so (3.40) follows from the occurrence of $R_{\mathbf{a}(x)}$.

We denote by $\mathbb{S}$ the subset of $\mathbb{C}$ consisting of all clusters which surround $\mathbf{a}(x)$. By the previous discussion we can assume that $\mathbb{S}$ is not empty. We now introduce the following order on $\mathbb{S}$: for $\mathbf{S}_1, \mathbf{S}_2 \in \mathbb{S}$ we set

$$\mathbf{S}_1 \leq \mathbf{S}_2 \;\Leftrightarrow\; \mathbf{S}_1 \subseteq \widehat{\mathbf{S}_2}. \tag{3.41}$$

It is easy to see that for this order $\mathbb{S}$ is a totally ordered finite set and we denote by $\mathbf{S}$ its maximal element. We then have $\mathbf{a}(x) \in \widehat{\mathbf{S}}$ and $\mathbf{S}$ is not surrounded by any other cluster in $\mathbb{C}$. Moreover, because of the assumption that no element of $\mathbb{C}$ surrounds $\mathbf{a}(x)$ and $\mathbf{a}(y)$ simultaneously, we see that $\mathbf{a}(y) \notin \widehat{\mathbf{S}}$ and therefore the path $\mathbf{A}$ leaves $\widehat{\mathbf{S}}$. We denote by $\mathbf{e}$ the "last" vertex of $\mathbf{A}$ [recall that $\mathbf{A}$ is a directed path going from $\mathbf{a}(x)$ to $\mathbf{a}(y)$], which belongs to $\partial^{\text{out}}_{\text{ext}}\mathbf{S}$. Then $\mathbf{e}$ is clearly white and not surrounded by any cluster in $\mathbb{C}$.

Next we show that $\mathbf{e}$ satisfies (3.40). For this we consider Σ, the set of sites in the renormalized lattice which correspond to the boxes $B_{\mathbf{i}}$ visited by σ. Then Σ is a $*$-connected finite set containing $\mathbf{a}(x)$ and $\mathbf{a}(y)$. Since $\mathbf{S}$ surrounds $\mathbf{a}(x)$ but not $\mathbf{a}(y)$, we see that Σ has nonempty intersection with $\widehat{\mathbf{S}}$ and also with $\widehat{\mathbf{S}}^c$. Therefore, we also have $\Sigma \cap \partial^{\text{out}}_{\text{ext}}\mathbf{S} \neq \varnothing$; that is, the microscopic path σ enters the union of the boxes $B_{\mathbf{i}}$, $\mathbf{i} \in \Sigma \cap \partial^{\text{out}}_{\text{ext}}\mathbf{S}$. Recall that σ is a directed path going from y to x.

Consider now $\sigma \cap \bigcup_{\mathbf{i} \in \partial^{\text{out}}_{\text{ext}}\mathbf{S}} B_{\mathbf{i}}$ and let u be the vertex in this set with the largest index. Let $\mathbf{u}$ be the vertex in $\partial^{\text{out}}_{\text{ext}}\mathbf{S}$ with $u \in B_{\mathbf{u}}$. Then it follows from the occurrence of $R_{\mathbf{u}}$ and from the fact that σ is not entirely contained in $B'_{\mathbf{u}}$, that u is a vertex of the crossing cluster of $B'_{\mathbf{u}}$. We also see, using (2.13) and (3.35), that the crossing cluster of $\mathbf{u}$ is connected to the crossing cluster of $\mathbf{e}$ by an open microscopic path in $\bigcup_{\mathbf{i} \in \partial^{\text{out}}_{\text{ext}}\mathbf{S}} B'_{\mathbf{i}}$ and therefore in W.

So we have to show that u is connected to x by an open path contained in W. This path is constructed as follows: we follow the path σ from u (in the direction of increasing index) until we arrive at x or hit a box corresponding to a white macroscopic site. If we arrive at x before hitting such a box, then the piece of σ connecting u to x is in W and we are finished. Otherwise the path σ enters a white box corresponding to a hole $\mathbf{H} \subseteq \mathbf{S}^c$, since, by the definition of u, σ will never hit after u a box corresponding to a site at the external outer boundary of $\mathbf{S}$. In the following we have to distinguish between several types of configurations, to which we shall also refer later:

(i) If $\mathbf{a}(x) \notin \widehat{\mathbf{H}}$, then we modify σ as follows. We denote by v_f and by v_l the first and last vertex of σ which is in a box of $\mathbf{H}$. The piece of σ between u and v_f is by construction in the union of boxes corresponding to sites of $\mathbf{S}$ and therefore in W. Now v_f and v_l are in boxes which correspond to sites of $\partial^{\text{in}}_{\text{ext}}\mathbf{H}$ and because of (2.13) and (3.35) they are connected by an open microscopic path in $\bigcup_{\mathbf{i}\in\partial^{\text{in}}_{\text{ext}}\mathbf{H}} B'_{\mathbf{i}}$, which is a subset of W, since $\partial^{\text{in}}_{\text{ext}}\mathbf{H} \subseteq \partial^{\text{out}}\mathbf{S}$. So we can replace σ between v_f and v_l by an open path in W.

(ii) If $\mathbf{a}(x) \in \widehat{\mathbf{H}}$, then we define v_f as before. Now our macroscopic path $\mathbf{A}$ intersects $\partial^{\text{in}}_{\text{ext}}\mathbf{H}$. Let $\mathbf{h}$ be a site of $\widehat{\mathbf{H}} \cap \partial^{\text{in}}_{\text{ext}}\mathbf{H}$. By the same argument as before, we can connect v_f to the crossing cluster of $B'_{\mathbf{h}}$ by a path in W.

(iia) If there is no cluster in $\mathbb{C}$ which surrounds $\mathbf{a}(x)$, then we can connect $\mathbf{h}$ to $\mathbf{a}(x)$ [note that $\mathbf{a}(x)$ is white in this case] by a $*$-connected white macroscopic path by just following $\mathbf{A}$ and the external outer boundaries of the black clusters which eventually intersect $\mathbf{A}$ between $\mathbf{a}(x)$ and $\mathbf{h}$. This implies again the existence of a microscopic path with the required properties.

(iib) If $\mathbf{a}(x)$ is surrounded by another black cluster $\mathbf{S}'$ in $\mathbb{C}$, then we can construct in the same way as in the case (iia) a path in W which connects u to the crossing cluster of that box $B'_{\mathbf{v}}$, $v \in \partial^{\text{out}}_{\text{ext}}\mathbf{S}'$, where σ enters for the last time a box corresponding to a site of this boundary. But now we are in the same situation as at the beginning of our construction and we can proceed in the same way to arrive at x or at the boundary of the next surrounding cluster of $\mathbf{a}(x)$ and so on. This finishes the proof of the lemma. $\square$

We can now proceed with the proof of Proposition 3.1. We shall have to distinguish between two cases:

Case I [*There is no cluster in* $\mathbb{C}$ *which surrounds both* $\mathbf{a}(x)$ *and* $\mathbf{a}(y)$]. In this case we can directly apply the previous lemma to see that there are two vertices $\mathbf{e}(x), \mathbf{e}(y) \in \mathbf{A}$ which are both not surrounded by any cluster of $\mathbb{C}$ and the crossing clusters of $B'_{\mathbf{e}(x)}$ (resp. $B'_{\mathbf{e}(y)}$) are connected to x (resp. y) by open microscopic paths lying in W.

Next we show that $\mathbf{e}(x)$ and $\mathbf{e}(y)$ are in the same $*$-connected white cluster. Our claim then immediately follows from (2.13). The argument is the following: if all vertices of $\mathbf{A}$ between $\mathbf{e}(x)$ and $\mathbf{e}(y)$ are white, then there is nothing to prove. Otherwise let $\mathbf{a}_1$ be the last white vertex of $\mathbf{A}$ after $\mathbf{e}(x)$. Since neither $\mathbf{e}(x)$ nor $\mathbf{e}(y)$ is surrounded by any black cluster in $\mathbb{C}$, $\mathbf{a}_1$ belongs to the external outer boundary of some black cluster and we can connect $\mathbf{a}_1$ by a $*$-connected macroscopic path to the "last" point of $\mathbf{A}$, which belongs to this boundary. We repeat the construction for this point instead of $\mathbf{e}_1$, and so on, until we arrive at the first white vertex of $\mathbf{A}$ before $\mathbf{e}(y)$ and then we are done.

Case II [*There is a cluster in* $\mathbb{C}$ *which surrounds both* $\mathbf{a}(x)$ *and* $\mathbf{a}(y)$]. In this case we are only interested in the smallest [with respect to the total order defined in (3.41)] of all these clusters, which we denote by $\mathbf{T}$.

Assume first that both $\mathbf{a}(x)$ and $\mathbf{a}(y)$ belong to $\mathbf{T}$. If σ is contained in the union of boxes corresponding to sites of $\mathbf{T}$, then our claim is immediate. Otherwise we follow σ (going from y to x) until it enters a white box. This box belongs to a (possibly infinite) connected component of $\mathbf{T}^c$ and we know that σ leaves this component, since $\mathbf{a}(x) \in \mathbf{T}$. Thus we can modify σ in the same way as we did in the proof of Lemma 3.2 [in case (i)] using the $*$-connectedness of the external outer boundary of $\mathbf{T}$ (resp., of the external inner boundary) of each hole of $\mathbf{T}$.

Consider now the case where exactly one of the sites $\mathbf{a}(x)$ and $\mathbf{a}(y)$, say $\mathbf{a}(y)$, belongs to $\mathbf{T}$. Then $\mathbf{a}(x)$ is in a hole $\mathbf{H}(x) \subseteq \mathbf{T}^c$. Using exactly the same construction as in (ii) in the proof of Lemma 3.2, we can connect x by a microscopic open path in W to the crossing cluster of the box, where σ leaves $\mathbf{H}(x)$ for the last time. Then the next box visited by σ lies in $\mathbf{T}$ and we can proceed in the same way as in the previous case, where $\mathbf{a}(x)$ and $\mathbf{a}(y)$ were both in $\mathbf{T}$.

The last case to look at is the situation where $\mathbf{a}(x)$ and $\mathbf{a}(y)$ are both in holes of $\mathbf{T}$, which we denote by $\mathbf{H}(x)$ and $\mathbf{H}(y)$, respectively. If $\mathbf{H}(x) \neq \mathbf{H}(y)$, then we use the same construction as before to connect x and y to sites which are boxes corresponding to sites of $\mathbf{T}$.

Consider now the case $\mathbf{H}(x) = \mathbf{H}(y) = \mathbf{H}$. By analogy to Case I we denote by $\mathbf{e}(x)$ the first vertex of $\mathbf{A}$ after $\mathbf{a}(x)$ which has the property that $\mathbf{e}(x)$ is white and not surrounded by any (black) cluster contained in $\mathbf{H}$. Similarly, let $\mathbf{e}(y)$ be the last vertex of $\mathbf{A}$ before $\mathbf{a}(y)$ with this property. By exactly the same reasoning as in case (ii) in the proof of Lemma 3.2, we know that there are microscopic open paths in W which connect x to the crossing cluster of $\mathbf{e}(x)$ [resp. y to the crossing cluster of $\mathbf{e}(y)$]. Since $\mathbf{A}$ intersects $\mathbf{T}$, there are vertices $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1$ is the first and $\mathbf{v}_2$ the last vertex of A which belongs to $\partial^{\text{in}}\mathbf{H} = \partial^{\text{in}}_{\text{ext}}\mathbf{H}$. Therefore, we can connect the crossing clusters of $\mathbf{v}_1$ and $\mathbf{v}_2$ traveling along white boxes corresponding to sites in $\partial^{\text{in}}_{\text{ext}}\mathbf{H}$. Finally, by analogous arguments as in Case I (considering only clusters in $\mathbf{H}$), we can connect $\mathbf{e}(x)$ to $\mathbf{v}_1$ and $\mathbf{e}(y)$ to $\mathbf{v}_2$ by macroscopic open paths and this implies our claim. □

4. Proof of the theorems. We can now combine the results of the previous two sections to give the proofs of Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. Let $N \geq 10$ and consider the renormalized lattice as described in Section 2. For $y \in \mathbb{Z}^d$ denote by $\mathbf{a}(y)$ the unique site such that $y \in B_{\mathbf{a}(y)}$ and set $n := |\mathbf{a}(y)|$. Fix a macroscopic path $\mathbf{A}$ of length n joining $\mathbf{0}$ to $\mathbf{a}(y)$. We denote by $\mathbf{a}_0, \ldots, \mathbf{a}_n$ the vertices of this path [where $\mathbf{a}_0 = \mathbf{0}$ and $\mathbf{a}_n = \mathbf{a}(y)$].

By Proposition 3.1 we know that we have, for any $\rho \geq 1$,

$$\{0 \leftrightarrow y,\ D(0, y) > \rho|y|\} \subseteq \{|W| > \rho|y|\}. \tag{4.42}$$

Observe that there is a constant $c = c(d) > 0$ such that

$$|W| \le N^d c\biggl(n + 1 + \sum_{\mathbf{C}^* \in \mathbb{C}} |\mathbf{C}^*|\biggr). \tag{4.43}$$

Using (2.14), we obtain

$$\mathbb{P}[0 \leftrightarrow y,\ D(0, y) > \rho|y|] \le \mathbb{P}_N\biggl[n + 1 + \sum_{\mathbf{C}^* \in \mathbb{C}} |\mathbf{C}^*| > \rho c N^{-d}|y|\biggr] \tag{4.44}$$

$$\le \mathbb{P}^*_{\bar{p}(N)}\biggl[n + 1 + \sum_{\mathbf{C}^* \in \mathbb{C}} |\mathbf{C}^*| > \rho c N^{-d}|y|\biggr]. \tag{4.45}$$

To estimate the last probability, we use a construction described by Fontes and Newman [5]. The main idea is to introduce *preclusters* $(\widetilde{\mathbf{C}}^*_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$. These are independent random subsets of $\mathbb{Z}^d$ with the property that the distribution of $\widetilde{\mathbf{C}}^*_{\mathbf{i}}$ is that of $\mathbf{C}^*_{\mathbf{0}}$ for all $\mathbf{i} \in Z^d$. Then we know by Lemma 1.3 in [4] that the r.h.s. of (4.44) is smaller than

$$\mathbb{P}^*_{\bar{p}(N)}\biggl[\frac{1}{n+1}\sum_{i=0}^{n}\Bigl(|\widetilde{\mathbf{C}}^*_{\mathbf{a}_i}| + 1\Bigr) > \rho c N^{-d}\frac{|y|}{n+1}\biggr] \tag{4.46}$$

and in the brackets we now have a sum of i.i.d. random variables. By the results of Menshikov [9] and Aizenman and Barsky [1], we know that, for $\bar{p}(N)$ large enough, we have, for some $h > 0$,

$$\mathbb{E}^*_{\bar{p}(N)}[\exp\{h(|\mathbf{C}^*_{\mathbf{0}}| + 1)\}] < \infty. \tag{4.47}$$

We choose $N = N(p, d)$ such that $\bar{p}(N)$ is in this regime. Now N is fixed and, for $|y|$ large enough, we have $|y|/(n+1) \ge N$. We next choose $\rho = \rho(p, d)$ such that $\mathbb{E}[|\mathbf{C}^*_{\mathbf{0}}|+1] < \rho c N^{-d+1}$. By Cramér's theorem, the probability in (4.46) has exponential decay in n (therefore also in $|y|$) and this proves our claim. □

PROOF OF THEOREM 1.2. We start again with $y \in Z^d$ fixed and the macroscopic path $\mathbf{A}$ of length $n = |\mathbf{a}(y)|$ joining $\mathbf{0}$ to $\mathbf{a}(y)$. We have again

$$\{0 \leftrightarrow y,\ D(0, y) > l\} \subseteq \{|W| > l\} \tag{4.48}$$

and therefore by the same argument as before

$$\begin{aligned}\mathbb{P}[0 \leftrightarrow y,\ D(0, y) > l] &\le \mathbb{P}^*_{\bar{p}(N)}\biggl[\sum_{i=0}^{n}(|\mathbf{C}^*_{\mathbf{a}_i}| + 1) > lcN^{-d}\biggr]\\ &\le (n+1)\mathbb{P}^*_{\bar{p}(N)}\bigl[|\mathbf{C}^*_{\mathbf{0}}| + 1 > lcN^{-d}/(n+1)\bigr].\end{aligned} \tag{4.49}$$

By picking N as in (4.47) and using Chebyshev's inequality, we obtain our claim. □

Acknowledgments. We are grateful to Professor Harry Kesten, who told us about the problem treated in Theorem 1.2, and to an anonymous referee who pointed out an error in an earlier version of this paper. P.A. would like to thank Professor Alain-Sol Sznitman for helpful discussions.

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