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Conformal Invariance of Voronoi Percolation

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Abstract: It is proved that in the Voronoi model for percolation in dimension 2 and 3, the crossing probabilities are asymptotically invariant under conformal change of metric.

To define Voronoi percolation on a manifold M, you need a measure μ, and a Riemannian metric ds. Points are scattered according to a Poisson point process on (M, μ), with some density λ. Each cell in the Voronoi tessellation determined by the chosen points is declared *open* with some fixed probability p, and *closed* with probability $1-p$, independently of the other cells. The above conformal invariance statement means that under certain conditions, the probability for an open crossing between two sets is asymptotically unchanged, as $\lambda \to \infty$, if the metric ds is replaced by any (smoothly) conformal metric ds'. Additionally, it is conjectured that if μ and μ' are two measures comparable to the Riemannian volume measure, then replacing μ by μ' does not effect the limiting crossing probabilities.

1. Introduction

Let γ be a simple closed curve in $\mathbb{C} = \mathbb{R}^2$, and let D be the closed topological disk which it bounds. Pick two disjoint arcs $\gamma_1, \gamma_2 \subset \gamma$. Let $\epsilon > 0$ be small, and let $\epsilon\mathbb{Z}^2$ denote the square grid rescaled by ϵ. Fix some $p \in [0, 1]$ and declare each edge in $\epsilon\mathbb{Z}^2$ to be *open* with probability p, and *closed* with probability $1 - p$, independently of the other edges. This is just the standard bond percolation model on the square grid; for background and history, see [9]. Let $PC_{\epsilon,p}(D, \gamma_1, \gamma_2)$ be the probability that there is a path of open edges in the subgraph of $\epsilon\mathbb{Z}^2$ lying in D that connects a vertex which has an edge crossing γ_1 to a vertex which has an edge crossing γ_2. This is called the *crossing probability* for (D, γ_1, γ_2) in the bond percolation model with parameters ϵ, p.

The main interest is in the limit as $\epsilon \to 0$. H. Kesten [10] proved that the critical probability p_c (the least p above which there is an infinite open connected component with probability 1) for bond percolation on the square lattice is $1/2$, and that

$$0 < \liminf_{\epsilon\to 0} PC_{\epsilon,p_c}(D,\gamma_1,\gamma_2) \le \limsup_{\epsilon\to 0} PC_{\epsilon,p_c}(D,\gamma_1,\gamma_2) < 1.$$

Although not proved yet, it is widely believed that the limit

$$PC_{0,p}(D,\gamma_1,\gamma_2) = \lim_{\epsilon\to 0} PC_{\epsilon,p}(D,\gamma_1,\gamma_2),$$

exists for $p = p_c$. It is known that for $p \neq p_c$ the limits exist, and $PC_{0,p}(D,\gamma_1,\gamma_2)$ is 0 if $p < p_c$ and 1 if $p > p_c$.

Aizenman, Langlands, Pouliot and Saint-Aubin have conjectured that the limits $PC_{0,p}(D,\gamma_1,\gamma_2)$ are conformally invariant. More precisely,

Conjecture 1.1 ([12]**).** *Let $f : D \to \hat{D}$ be a homeomorphism of D onto another topological disk $\hat{D} \subset \mathbb{C}$, and suppose that f is conformal in the interior of D. Then*

$$PC_{0,p_c}(D,\gamma_1,\gamma_2) = PC_{0,p_c}\big(f(D), f(\gamma_1), f(\gamma_2)\big).$$

This conjecture motivated the current work. In [11] numerical data from computer simulations has been collected, estimating the crossing probabilities of rectangles. The discussion of these results led to the above conjecture. Subsequently, J. L. Cardy [6] found a heuristic argument supporting this conjecture, and derived (using arguments outside the scope of mathematics) a formula for the limiting crossing probabilities, in terms of the cross ratio of the images of the endpoints of γ_1, γ_2 under the conformal map from D to the unit disk. Cardy's formula matched the numerical data quite well. Later, Langlands et. al. [12] have obtained more precise numerical data, giving further support to the conjecture and to Cardy's formula.

Although the current work does not settle the conjecture, it does prove a related conformal invariance property, which, in our view, is not less important. In order to discuss it, the Voronoi percolation model must be introduced. The precise definitions are given in Sect. 2, but a loose description will be given here.

Let M be a smooth manifold, and let ds be a Riemannian metric on M. Let μ be a measure on M that is comparable to vol, the Riemannian volume measure on M. The most interesting case is μ = vol. Take some parameters $p \in [0,1], \lambda > 0$. Now let ω be a Poisson point process on (M,μ), with density λ. Each cell in the Voronoi tiling with nuclei ω is declared *open* with probability p, and *closed* otherwise. Then one looks at crossing probabilities inside the union of all open tiles. The measure μ plays a role in the choice of the nuclei ω, and the metric ds is instrumental in defining the Voronoi tessellation. Our main result is that, in dimension d = 2 or 3, asymptotically, the crossing probabilities are unchanged if the metric ds is replaced by any other smoothly conformal metric.

Note that the effect of a mapping f is to change both the measure μ and the metric ds. The main advantage of the Voronoi percolation model is that it permits a separate treatment of the effects of the change in μ and the change in ds. We conjecture that in two dimensions μ may be changed to any comparable measure, without effecting the limiting crossing probabilities. It is shown that this Density Invariance Conjecture and our main result imply the analog of Conjecture 1.1 in the Voronoi model. Although this seems almost tautological at first sight, there is some work involved in dealing with some sticky boundary issues. Some numerical evidence supporting the Density Invariance Conjecture in dimension two are presented here. The simulations also suggest that the limiting crossing probabilities for Voronoi percolation in dimension 2 are the same as in the $\mathbb{Z}^2$ model.

The impression that one might get from Conjecture 1.1 is that the conformal invariance has something to do with analyticity, since conformal maps are analytic. In fact, as the physics literature suggests [2], this impression is erroneous. Our main result shows that the conformal invariance is much more general, and holds outside the realm of analytic maps and dimension 2.

The Voronoi percolation model has been introduced into the mathematical literature by M. Q. Vahidi-Asl and J. C. Wierman [15], in the context of first passage percolation. Here are some useful properties of this model:

- Rotation invariance.
- Duality: in dimension 2 and $p = 1/2$, the union of open tiles has the same stochastic behavior as the set of closed tiles. Based on this, A. Zvavitch [16] has shown that there is no unbounded open cluster (component) for Voronoi percolation with $p = 1/2$ in $\mathbb{R}^2$.
- Generality: the model makes sense in the setting of Riemannian manifolds. In particular, the theory of Voronoi percolation in the hyperbolic plane is interesting [4].
- Separation of measure and metric, as discussed above.
- Gradual refinement: one may pass from a configuration to a denser configuration by inserting new random points one by one. In contrast, when refining a grid, it is necessary to make drastic changes.

The reader may wish to look into the work of M. Aizenman [1], who constructs a continuous limit of percolation models using Voronoi percolation.

The plan of the paper is as follows. Sect. 2 gives precise definitions, and the statement of the main results. A brief outline of the proof is sketched in Sect. 3, while Sects. 4 through 9 provide the details. Of these, Sects. 4 through 6 are geometric in nature, and Sects. 7 through 8 are probabilistic. Sect. 9 assembles the pieces together and completes the proof. Finally, Sect. 10 introduces the Density Invariance Conjecture, presents numerical evidence for it in dimension two, and shows that it implies the analog of Conjecture 1.1 in the Voronoi percolation setting.

2. The Voronoi Percolation Model and Statement of the Main Result

Throughout the paper, M will be a smooth Riemannian manifold, d will be the dimension of M, and ds will denote the Riemannian metric on M. Let $d_0(\cdot,\cdot)$ be the distance function associated with (M, ds). Also associated with ds is the natural volume measure, vol. Let μ be measure on M *comparable* to vol, which means that there is a constant $c > 0$ such that $c^{-1}\,\mathrm{vol}(A) \leq \mu(A) \leq c\,\mathrm{vol}(A)$ for every measurable $A \subset M$.

Given parameters $\lambda > 0$, $p \in [0, 1]$, one defines the Voronoi percolation process on (M, ds, μ, λ, p), as follows. Let Ω be the space of all subsets ω of M such that the intersection of ω with any compact subset of M is finite. There is a (Borel) probability measure P_λ on Ω given by the Poisson point process on (M, μ) with density λ. The measure P_λ is characterized by the formula,

$$P_\lambda\big(|\omega \cap A| = k\big) = \frac{\big(\lambda\mu(A)\big)^k}{k!}\,\exp\big(-\lambda\mu(A)\big), \tag{2.1}$$

for every measurable A (with finite measure) and every integer k, and by the requirement that $|\omega \cap A_1|, \ldots, |\omega \cap A_n|$ are independent random variables when $A_1, \ldots, A_n$ are disjoint measurable sets. Here, and below, for any set X, the cardinality of X will be denoted $|X|$.

The elements $\omega \in \Omega$ are called *configurations*. Let ω be some configuration. Given any $z \in \omega$, its *Voronoi tile* $\mathbf{T}(z) = \mathbf{T}(\omega, ds, z)$ is the set of all points $w \in M$ such that $d_0(w, z) \leq d_0(w, z')$ for all $z' \in \omega$. The collection of all Voronoi tiles is the *Voronoi tiling* of ω, and will be denoted $\mathbf{T}(\omega, ds)$. It is indeed a tiling of M, except for the trivial case (which will henceforth be ignored) where $\omega = \emptyset$.

In Voronoi percolation, each tile of $\mathbf{T}(\omega, ds)$ is declared *open* with probability p, and *closed* with probability $1 - p$, independently, and one studies the connected components of the union of all open tiles. We now make an equivalent, slightly different and more precise, formulation. Let $\hat{\Omega} = \Omega \times \Omega$. Then $P_{\lambda,p}$ is defined to be the product measure $P_{p\lambda} \times P_{(1-p)\lambda}$ on $\hat{\Omega}$. Given $\omega = (\omega_o, \omega_c) \in \hat{\Omega}$, the set ω_o will be called the set of open nuclei, and ω_c is the set of closed nuclei. The projection map $\pi : \hat{\Omega} \to \Omega$ is defined by $\pi(\omega_o, \omega_c) = \omega_o \cup \omega_c$. If $\pi\hat{\omega} = \omega$, then $\hat{\omega}$ will be called a *coloring* of ω. The elements of $\hat{\Omega}$ are called *colored configurations*.

Let $\tau \in \hat{\Omega}$ be distributed according to P_λ, and let τ_o be a random subset of τ, chosen so that for any $x \in \tau$ the probability for $x \in \tau_o$ is p, and for different $x, x' \in \tau$ the events $x \in \tau_o$, $x' \in \tau_o$ are independent. Then it is not hard to verify that $(\tau_o, \tau - \tau_o)$ is distributed according to $P_{\lambda,p}$. This means that a legitimate way of generating a $P_{\lambda,p}$-random $\hat{\omega}$ is by first selecting a P_λ random ω and then selecting an appropriate random coloring of it. We shall make use of these two distinct ways of generating a $P_{\lambda,p}$-random colored configuration.

Given a colored configuration $\omega \in \hat{\Omega}$, the tiles in $\mathbf{T}(\hat{\omega}, ds) = \mathbf{T}(\pi\hat{\omega}, ds)$ which belong to open nuclei are called *open* tiles, and the other tiles are *closed* tiles.

We soon define the *crossing* events and the crossing probabilities. Perhaps the cleanest situation to deal with is one in which there is no boundary: M is compact (and boundaryless), and one is looking for percolation in homotopy classes; that is, the “crossing” event is the event that there is a closed curve, contained in the union of open tiles, which is in a prescribed homotopy class. However, this is not the situation prevalent in the literature. The definitions below are not the most natural ones, with respect to the way the boundary is dealt with. They have been adopted because they make the proofs easier (that is, possible), and since we feel that it is better to leave the boundary issues to future investigations.

Let M' be a compact d-dimensional set in M, which has smooth boundary, and let $\mathfrak{S}_1, \mathfrak{S}_2 \subset M'$ be two open disjoint sets, with smooth boundary. Given $\omega = (\omega_o, \omega_c) \in \hat{\Omega}$, let $\mathbf{T}'_O(\omega, ds)$ be the union of open tiles of $\mathbf{T}(\omega, ds)$ which have nuclei in M', and let $\mathcal{C} = \mathcal{C}(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds) \subset \hat{\Omega}$ be the event that there is a connected component of $\mathbf{T}'_O(\omega, ds)$ which intersects both $\omega_o \cap \mathfrak{S}_1$ and $\omega_o \cap \mathfrak{S}_2$. If $\omega \in \mathcal{C}$, we say that there is a crossing from $\mathfrak{S}_1$ to $\mathfrak{S}_2$ in (M, M', ω, ds).

Now suppose that $u : M \to \mathbb{R}$ is a smooth function, and consider the metric $e^u ds$, which is conformal to our original metric ds.

2.1. Conformal invariance theorem for percolation. *Suppose that* $d = \dim(M) = 2$ *or* 3. *Let* $I \subset (0, 1)$ *be a compact interval. Then*

$$\lim_{\lambda\to\infty} P_{\lambda,p}\Big(\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right) - \mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, e^u ds\right)\Big) = 0,$$

uniformly for $p \in I$.

This means that the set of configurations $\omega \in \hat{\Omega}$ for which there is a crossing with respect to the metric ds, but not with respect to the conformal metric $e^u ds$ has measure tending to 0 as $\lambda \to \infty$, and the convergence is uniform in p as long as p is kept away from 0 and 1. In particular, when λ is large, the probability of $\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right)$ is approximately the same as the probability of $\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, e^u ds\right)$, and the same is true for intersections of such events.

Actually, the theorem is true even with $I = [0, 1]$. To prove this one needs to show that for some constant $\delta > 0$, we have

$$\lim_{\lambda\to\infty} P_{\lambda,p}\Big(\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right)\Big) = 0,$$

uniformly for $p \in [0, \delta]$, and

$$\lim_{\lambda\to\infty} P_{\lambda,p}\Big(\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right)\Big) = 1,$$

uniformly for $p \in [1 - \delta, 1]$. These facts, which are actually valid in greater generality, are not hard. (The analogous statements in the discrete setting are certainly well known.) But because the methods involved are almost disjoint from those of this paper, and for the sake of keeping the size of the article reasonable, the proof will be delayed to some future work.

The point about the limit in Theorem 2.1 being uniform is that one may let p depend on λ and tend to p_c as $\lambda \to \infty$, and still the theorem applies. Any value can be prescribed for $\lim_{\lambda\to\infty} P_{\lambda,p}\Big(\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right)\Big)$, if p is an appropriate function of λ. This issue is even more important in dimension 3, since it has not been proved in any model that the limit $\lim_{\lambda\to\infty} P_{\lambda,p_c}\Big(\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right)\Big)$ is not always 1 or 0.

We now discuss a variant of the theorem involving percolation in homotopy classes. Let α be a collection of homotopy classes of M' and let $\mathcal{C}(M, M', \alpha, ds) \subset \hat{\Omega}$ denote the event that there is a path in $\mathbf{T}'_O(\omega, ds)$ which realizes a homotopy class in α.

2.2 Conformal invariance theorem for percolation in homotopy classes. *Suppose that* $d = \dim(M) = 2$ *or* 3. *Let* $I \subset (0, 1)$ *be a compact interval. Then*

$$\lim_{\lambda\to\infty} P_{\lambda,p}\Big(\mathcal{C}\left(M, M', \alpha, ds\right) - \mathcal{C}\left(M, M', \alpha, e^u ds\right)\Big) = 0,$$

uniformly for $p \in I$.

The same proof applies to both theorems.

3. Brief Outline of the Proof of Theorems 2.1 and 2.2

Consider a configuration $\omega \in \Omega$, and the Voronoi tilings $\mathbf{T}_0, \mathbf{T}_1$ produced by using the two metrics ds and $e^u ds$. A situation where there are two neighboring tiles in $\mathbf{T}_0$ and the corresponding tiles in $\mathbf{T}_1$ do not neighbor is called a *defect*. The first step is to analyse the geometry of configurations that are defect prone. We shall find that for compact sets in dimension d, in configurations with approximately n^d cells, the typical number of defects is in the order of n^{d-2}. In particular, for $d = 2$, the expected number of defects is finite.

It turns out that the best way to deal with the defects is to think of a typical configuration as a defect-free configuration, with defects added on top of it by an independent (sort of) Poisson process, which has small density. In practice, much effort is required to make this philosophy work.

In dimensions 2 and 3, defects turn out to be rare enough so that they do not effect percolation. The effect of the defects added on top of a defect-free configuration is majorized by changing the status of all tiles intersecting sufficiently large spherical shells about the location of the defect, from open to closed, say. We shall need quite delicate tail estimates for the number of tiles intersecting such spherical shells. Using these estimates, and a second moment argument, it will follow that (for $d = 2, 3$), with high probability, these haphazard defects will not destroy percolation.

Almost all of the proof does not assume $d = 2, 3$, only at the very end we shall apply this restriction. Perhaps this might be valuable in the future, in extending the results to higher dimensions. From time to time, remarks will be made, hinting how the proof may be simplified if one restricts to the case $M = \mathbb{R}^2$.

4. The Geometry of Defects

We consider some fixed configuration $\omega \in \Omega$. Recall that $d_0(\cdot,\cdot)$ denotes the metric on M corresponding to the Riemannian metric ds, and let $d_1(\cdot,\cdot)$ be the metric corresponding to the conformal Riemannian metric $e^u ds$. Let $\mathbf{T}_0(\omega)$ be the Voronoi tessellation for ω with respect to $d_0(\cdot,\cdot)$, and let $\mathbf{T}_1(\omega)$ be the Voronoi tessellation obtained by using the metric $d_1(\cdot,\cdot)$. A *defect* is a pair of points $p_1, p_2 \in \omega$ such that the Voronoi tiles $\mathbf{T}_0(p_1), \mathbf{T}_0(p_2)$ are adjacent, but the corresponding tiles $\mathbf{T}_1(p_1), \mathbf{T}_1(p_2)$ are not. That is, $\mathbf{T}_0(p_1) \cap \mathbf{T}_0(p_2) \neq \emptyset = \mathbf{T}_1(p_1) \cap \mathbf{T}_1(p_2)$.

Lemma 4.1. *Let K be a compact subset of M. There is a constant $C = C(u,K) > 0$ with the following property. Suppose that $q, p_1, p_2 \in K$ and $r \in \big(0, C^{-1}\big)$, satisfy $d_0(q,p_1) = d_0(q,p_2) = r$. Then there is a $q_1 \in M$ satisfying*

(1) $d_0(q_1, q) < Cr^2$,
(2) $d_1(q_1, p_1) = d_1(q_1, p_2)$, *and*
(3) $|d_1(q_1,p_1) - d_1(q_1,p)| < Cr^3$, *for any $p \in M$ satisfying $d_0(q,p) = r$.*

One fact the lemma tells us is that a small ball in one metric is very close to a ball in a conformal metric. In general, the two balls should be allowed to have different centers, in order to obtain the correct order of approximation.

In the particular situation where (M, ds) is a domain in the plane with the Euclidean metric and the metric $e^u ds$ is the pullback to M of the Euclidean metric under a conformal map $f : M \to \mathbb{C}$, the lemma is significantly easier. One may take q_1 as the center of the circle which is the image of the circle $|z - q| = r$ under a Möbius transformation which agrees with f at q, p_1, p_2. Then C is bounded by a constant times the maximum modulus of the Schwarzian derivative of f near q.

Proof. Since the restriction of u to K is bounded, and the lemma is not effected if we add a bounded constant to u, we assume without loss of generality that $u(q) = 0$.

Set $u_t = tu$, and let $d_t(\cdot,\cdot)$ be the metric induced by the Riemannian metric $e^{u_t} ds$. Then d_t is a one parameter family of metrics, interpolating between d_0 and d_1. We shall first solve the differential problem; that is, a tangent vector v will be found such that for a path $q(t)$ in M satisfying $q(0) = q, q'(0) = v$, the conditions

(1') $|v| < Cr^2$,
(2') $\frac{d}{dt} d_t\big(q(t), p_1\big) = \frac{d}{dt} d_t\big(q(t), p_2\big)$ at $t = 0$, and
(3') $\big|\frac{d}{dt} d_t\big(q(t), p_1\big) - \frac{d}{dt} d_t\big(q(t), p\big)\big| < Cr^3$ at $t = 0$, for every $p \in M$ satisfying $d_0(q,p) = r$,

are satisfied. It will then be quite easy to get the original statement from the differential statement.

Our first goal is to estimate the derivative $\frac{\partial}{\partial t} d_t(q, p_1)$ at $t = 0$. Since r is as small as we wish, we may assume that any geodesic segment joining two points whose distance is at most $2r$ is unique, in any of the metrics d_t. Let γ_t be the geodesic segment for the metric d_t joining q and p_1, and suppose that each γ_t is parameterized according to arc-length. Set $g(x, y) = \text{lenght}_{d_x}(\gamma_y)$, the length of γ_y in the metric d_x. Then g is smooth (since geodesics can be obtained by solving an ODE on the tangent bundle), and

$$d_t(q, p_1) = g(t, t). \tag{4.1}$$

Because the curve γ_0 is length minimizing in the metric d_0, the equation

$$\frac{\partial}{\partial y} g(0, 0) = 0, \tag{4.2}$$

holds. Therefore, (4.1) implies

$$\frac{\partial}{\partial t}\bigg|_{t=0} d_t(q, p_1) = \frac{\partial}{\partial x} g(0, 0) = \frac{\partial}{\partial x}\bigg|_{x=0} \text{lenght}_{d_x}(\gamma_0). \tag{4.3}$$

Because γ_0 is parameterized according to arclength, we have,

$$\text{lenght}_{d_x}(\gamma_0) = \int_0^r e^{xu(\gamma_0(s))}\, ds.$$

Together with (4.3), this gives,

$$\frac{\partial}{\partial t}\bigg|_{t=0} d_t(q, p_1) = \int_0^r u\big(\gamma_0(s)\big)\, ds. \tag{4.4}$$

Using local coordinates, and $u(q) = 0$, the following estimates are obtained,

$$\begin{aligned} \gamma_0(s) &= q + s\gamma_0'(0) + O(s^2), \\ u\big(\gamma_0(s)\big) &= s\nabla u(q) \cdot \gamma_0'(0) + O(s^2). \end{aligned}$$

Substituting this into (4.4) yields,

$$\frac{\partial}{\partial t}\bigg|_{t=0} d_t(q, p_1) = \frac{1}{2} r^2 \nabla u(q) \cdot \gamma_0'(0) + O(r^3). \tag{4.5}$$

If v is any tangent vector at q, and $q(t)$ is a path in M with $q'(t) = v$, then we have

$$\frac{d}{dt}\bigg|_{t=0} d_0\big(q(t), p_1\big) = -v \cdot \gamma_0'(0),$$

because $-\gamma_0'(0)$ is the gradient of the d_0-distance from p_1 at q. Hence, it follows from (4.5) that

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} d_t\big(q(t), p_1\big) &= \frac{d}{dt}\bigg|_{t=0} d_0\big(q(t), p_1\big) + \frac{d}{dt}\bigg|_{t=0} d_t\big(q, p_1\big) \\ &= -v \cdot \gamma_0'(0) + \frac{1}{2} r^2 \nabla u(q) \cdot \gamma_0'(0) + O(r^3). \end{aligned} \tag{4.6}$$

Suppose that v has the form

$$v = \frac{1}{2} r^2 \nabla u(q) + O(r^3). \tag{4.7}$$

Then we get $\frac{d}{dt} d_t\big(q(t), p_1\big) = O(r^3)$. The same would be equally true if p_1 is replaced by any p such that $d_0(q,p) = r$, because the above expression for v does not depend on p_1. Consequently, (1') and (3') would be satisfied for an appropriately chosen $C = C(u, K)$. So all that remains for the solution of the differential problem is to find the $O(r^3)$ term in the expression for v, which would guarantee (2').

Let β_t be the geodesic segment joining q and p_2 in the metric d_t, parametrized according to arc-length. We use the expression (4.4) and the corresponding expression with β and p_2 replacing γ and p_1, to get,

$$\begin{aligned} &\frac{\partial}{\partial t}\bigg|_{t=0} \Big(d_t(q,p_1) - d_t(q,p_2) \Big) \\ &\qquad = \int_0^r \Big(u\big(\gamma_0(s)\big) - u\big(\beta_0(s)\big) \Big)\, ds \\ &\qquad = \int_0^r \nabla u\big(\beta_0(s)\big) \cdot \big(\gamma_0(s) - \beta_0(s)\big)\, ds + \int_0^r O\Big(|\gamma_0(s) - \beta_0(s)|^2 \Big)\, ds. \end{aligned} \tag{4.8}$$

Let $\alpha_w(s)$ denote the geodesic starting at $\alpha_w(0) = q$ with initial direction $\alpha'_w(0) = w$. Then $\alpha_w(s)$, $\alpha'_w(s)$ and $\alpha''_w(s)$ are smooth functions of w and s. Consequently, for $s \in [0, r]$,

$$\gamma_0(s) - \beta_0(s) = O\big(r|\gamma'_0(0) - \beta'_0(0)|\big), \tag{4.9}$$

$$\gamma''_0(s) - \beta''_0(s) = O\big(|\gamma'_0(0) - \beta'_0(0)|\big). \tag{4.10}$$

Using this in (4.8), gives,

$$\begin{aligned} &\frac{\partial}{\partial t}\bigg|_{t=0} \Big(d_t(q,p_1) - d_t(q,p_2) \Big) \\ &\quad = \int_0^r \nabla u(q) \cdot \big(s\gamma'_0(0) - s\beta'_0(0) \big)\, ds \\ &\qquad + \int_0^r \nabla u(q) \cdot \big(\gamma_0(s) - s\gamma'_0(0) - \beta_0(s) + s\beta'_0(0) \big)\, ds \\ &\qquad + \int_0^r \Big(\nabla u\big(\beta_0(s)\big) - \nabla u(q) \Big) \cdot \big(\gamma_0(s) - \beta_0(s) \big)\, ds \\ &\qquad + O\left(r^3 |\gamma'_0(0) - \beta'_0(0)|^2 \right) \\ &\quad = \frac{1}{2} r^2 \nabla u(q) \cdot \big(\gamma'_0(0) - \beta'_0(0) \big) \\ &\qquad + \int_0^r \nabla u(q) \cdot \big(\gamma_0(s) - s\gamma'_0(0) - \beta_0(s) + s\beta'_0(0) \big)\, ds \\ &\qquad + O\left(r^3 |\gamma'_0(0) - \beta'_0(0)| \right). \end{aligned} \tag{4.11}$$

Set

$$h(s) = \gamma_0(s) - s\gamma'_0(0) - \beta_0(s) + s\beta'_0(0).$$

Note that $h(0) = h'(0) = 0$, and $h''(s) = \gamma_0''(s) - \beta_0''(s)$, which is $O\big(|\gamma_0'(0) - \beta_0'(0)|\big)$, according to (4.10). Therefore,

$$h(s) = O\big(r^2|\gamma_0'(0) - \beta_0'(0)|\big), \tag{4.12}$$

for $s \in [0, r]$. Now use this in (4.11),

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_{t=0}\Big(d_t(q,p_1) - d_t(q,p_2)\Big) \\
&\quad = \frac{1}{2}r^2\nabla u(q)\cdot\big(\gamma_0'(0) - \beta_0'(0)\big) + \int_0^r \nabla u(q)\cdot h(s)\,ds + O\left(r^3|\gamma_0'(0) - \beta_0'(0)|\right) \\
&\quad = \frac{1}{2}r^2\nabla u(q)\cdot\big(\gamma_0'(0) - \beta_0'(0)\big) + O\left(r^3|\gamma_0'(0) - \beta_0'(0)|\right).
\end{aligned} \tag{4.13}$$

Let A be the $O\left(r^3|\gamma_0'(0) - \beta_0'(0)|\right)$ term, that is,

$$A = \frac{\partial}{\partial t}\Big|_{t=0}\Big(d_t(q,p_1) - d_t(q,p_2)\Big) - \frac{1}{2}r^2\nabla u(q)\cdot\big(\gamma_0'(0) - \beta_0'(0)\big). \tag{4.14}$$

Choose,

$$v = \frac{1}{2}r^2\nabla u(q) + A\,\frac{\gamma_0'(0) - \beta_0'(0)}{|\gamma_0'(0) - \beta_0'(0)|^2}, \tag{4.15}$$

and, as before, let $q(s)$ be a path satisfying $q(0) = q$ and $q'(0) = v$. Then

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_{s=0}\Big(d_0(q(s),p_1) - d_0(q(s),p_2)\Big) &= -v\cdot\gamma_0'(0) + v\cdot\beta_0'(0) \\
&= -\frac{1}{2}r^2\nabla u(q)\cdot\big(\gamma_0'(0) - \beta_0'(0)\big) - A \\
&= -\frac{\partial}{\partial t}\Big|_{t=0}\Big(d_t(q,p_1) - d_t(q,p_2)\Big),
\end{aligned}$$

by (4.14). Consequently,

$$\begin{aligned}
&\frac{d}{dx}\Big|_{x=0}\Big(d_x(q(x),p_1) - d_x(q(x),p_2)\Big) \\
&= \frac{\partial}{\partial s}\Big|_{s=0}\Big(d_0(q(s),p_1) - d_0(q(s),p_2)\Big) + \frac{\partial}{\partial t}\Big|_{t=0}\Big(d_t(q,p_1) - d_t(q,p_2)\Big) = 0,
\end{aligned}$$

which shows that (2') holds. Since A is $O\left(r^3|\gamma_0'(0) - \beta_0'(0)|\right)$, the definition (4.15) of v satisfies (4.7). Hence (1') and (3') are still satisfied, as we have seen above. This completes the solution of the differential problem.

To solve the original problem, for every point q^* and every $t \in [0, 1]$, define $v(q^*, t)$ as in (4.15), but with the metric d_t replacing d_0 and the point q^* replacing q. Let $q(t)$ be the solution of the initial value problem $q(0) = q, q'(t) = v\big(q(t), t\big)$, and set $q_1 = q(1)$. (Because $v\big(q(t), t\big) = O(r^2)$, $r < C^{-1}$, and $q \in K$, by an appropriate choice of the constant C it is guaranteed that this initial value problem has a solution in the interval $[0, 1]$. The essential point here is that $q(t)$ stays in a compact subset of M.) Then it is easy to see that (1) and (2) hold. Verifying (3) is just slightly harder, because (3') was obtained only for points p satisfying $d_0(q, p) = d_0(q, p_1)$, and these are generally not the points satisfying $d_t(q(t),p\big) = d_t(q(t),p_1\big)$. To deal with that, start with any p satisfying $d_0(q,p) = r$. At every point z let $w(z,t)$ be the direction at z of the geodesic for the metric d_t that goes from z to $q(t)$. Let $p(t)$ satisfy $p(0) = p$ and

$$p'(t) = \left(\frac{\partial}{\partial s}\bigg|_{s=t} \Big(d_s\big(q(s),p_1\big) - d_s\big(q(s),p(t)\big)\Big)\right) w\big(p(t),t\big).$$

Then $d_t\big(q(t),p(t)\big) = d_t\big(q(t),p_1\big)$ for all $t \in [0,1]$. Hence, $d_1\big(q_1,p(1)\big) = d_1(q_1,p_1)$. By the equivalent of (3') at t, $|p'(t)| = O\left(r^3\right)$. So $d_0\big(p(1),p\big) = O\left(r^3\right)$, which gives

$$d_1(q_1,p) - d_1(q_1,p_1) = d_1(q_1,p) - d_1\big(q_1,p(1)\big) = O\left(r^3\right).$$

This implies (3), and completes the proof. □

Notation. Suppose that q,z are points in M. If there is a unique shortest geodesic segment from q to z, in the metric d_0, then the direction at q of that geodesic will be denoted $N_q(z)$. When working in local coordinates, $N_q(z)$ can be thought of as a unit vector in $\mathbb{R}^d$. We may also think of $N_q(z)$ as a unit vector in T_qM, the tangent space to M at q.

Note that for any compact $K \subset M$ there is an $\epsilon > 0$ such that $N_q(z)$ is well defined when $q \in K$ and $d_0(q,z) < \epsilon$.

The following lemma will help us prove that defects are rare.

Lemma 4.2. *Let K be a compact subset of M. There is a constant $C = C(M,ds,u,K) > 0$ such that the following holds. Let $\omega \in \Omega$, and consider the two Voronoi tessellations, $\mathbf{T}_0 = \mathbf{T}_0(\omega)$, $\mathbf{T}_1 = \mathbf{T}_1(\omega)$, obtained by using the metrics d_0 and d_1. Suppose that $p_1,p_2 \in K \cap \omega$ form a defect (that is, $\mathbf{T}_0(p_1) \cap \mathbf{T}_0(p_2) \neq \emptyset = \mathbf{T}_1(p_1) \cap \mathbf{T}_1(p_2)$) and assume that $\mathbf{T}_0(p_1) \cap \mathbf{T}_0(p_2) \subset K$. Let q be the point in $\mathbf{T}_0(p_1) \cap \mathbf{T}_0(p_2)$ which maximizes $d_0\big(q,\omega - \{p_1,p_2\}\big) - d_0(q,p_1)$, and set $r = d_0(q,p_1)$, $r' = d_0\big(q,\omega-\{p_1,p_2\}\big)$. Let $Z = \{z_1,\ldots,z_k\}$ be the set of points $z \in \omega - \{p_1,p_2\}$ such that $d_0(q,z) = r'$. If $r < C^{-1}$, then*

(1) *$r' \le r + Cr^3$, and*
(2) *the vectors $\{N_q(p_1), N_q(p_2), N_q(z_1),\ldots,N_q(z_k)\}$ are affinely dependent.*

Proof. Take C to be larger than the constant in Lemma 4, and assume $r < C^{-1}$. Let q_1 be the point described in that lemma, and set $r_1 = d_1(q_1,p_1) = d_1(q_1,p_2)$. Since $q_1 \notin \emptyset = \mathbf{T}_1(p_1) \cap \mathbf{T}_1(p_2)$, there is a point $z_0 \in \omega - \{p_1,p_2\}$ with $d_1(q_1,z_0) < r_1$. We know that $d_0(q,z_0) \ge r$ and $d_0(q,q_1) = O(r^2)$. Hence, there is a point z_0' on the d_1-geodesic segment from z_0 to q_1 that satisfies $d_0(q,z_0') = r$. Then, according to 4. (3), $d_1(q_1,z_0') + O(r^3) \ge d_1(q_1,p_1) > d_1(q_1,z_0)$. But since $d_1(q_1,z_0) = d_1(q_1,z_0') + d_1(z_0',z_0)$, it follows that $d_1(z_0,z_0') = O(r^3)$, which implies $d_0(z_0,z_0') = O(r^3)$. Consequently, $d_0(q,z_0) = r + O(r^3)$. By construction, among all the points in $\omega - \{p_1,p_2\}$ the points in Z are closest to q. Therefore, $r' \le d_0(q,z_0) = r + O(r^3)$ for $z \in Z$, and (1) is established.

Let L be the set of points p in M such that $d_0(p,p_1) = d_0(p,p_2)$. If $q(t)$ is a smooth path in M which satisfies $q(0) = q$, then

$$\frac{d}{dt}\bigg|_{t=0} \Big(d_0\big(q(t),p_1\big) - d_0\big(q(t),p_2\big)\Big) = q'(t) \cdot \big(N_q(p_2) - N_q(p_1)\big).$$

Because $N_q(p_1) \neq N_q(p_2)$, it follows by the implicit function theorem that $L \cap W$ is a smooth $d-1$ manifold, for some open $W \subset M$ which contains q.

Let $w \in T_qM$ be any tangent vector at q which is orthogonal to $\big(N_q(p_2) - N_q(p_1)\big)$. Then there is a smooth path $q(t)$ in L such that $q(0) = q$ and $q'(0) = w$. Recall that q maximizes

$$d_0\big(p, \omega - \ p_1, p_2\ \big) - d_0(p, p_1), \tag{4.16}$$

among p in $\mathbf{T}_0(p_1) \cap \mathbf{T}_0(p_2)$. Since (4.16) is negative when $p \in L - \mathbf{T}_0(p_1) \cap \mathbf{T}_0(p_2)$, it follows that q maximizes (4.16) among p in $L \cap W$. Therefore, there must be some $z \in Z$ such that

$$0 \geq \left.\frac{d}{dt}\right|_{t=0} \Big(d_0\big(q(t), z\big) - d_0\big(q(t), p_1\big)\Big) = w \cdot \big(N_q(p_1) - N_q(z)\big).$$

This means that for every vector w tangent to L, $w \in T_qL$, there is some $j \in 1, \ldots, k$ with $w \cdot v_j \leq 0$, where v_j is the orthogonal projection of $N_q(p_1) - N_q(z_j)$ onto T_qL. Therefore, 0 is in the convex hull of $\{v_1, \ldots, v_k\}$ (see Eggleston [7, Ch. 1, §7]), and consequently $\{v_1, \ldots, v_k\}$ is linearly dependent. Hence, the linear span of $\{v_1, \ldots, v_k\}$ is contained in a $k - 1$ dimensional subspace of T_qL. Because each $N_q(p_1) - N_q(z_j)$ is a linear combination of v_j and $N_q(p_1) - N_q(p_2)$, it follows that the set $\{N_q(p_1) - N_q(p_2), N_q(p_1) - N_q(z_1), \ldots, N_q(p_1) - N_q(z_k)\}$ is contained in a k dimensional subspace of T_qM. This proves (2), and establishes the lemma. □

Recall that M' is a compact subset of M in which the crossing is considered. Let $M^* \subset M$ be some compact set that contains M' in its interior. We now define a potential defect to be a situation where some of the necessary conditions for a defect of Lemma 4.2 are satisfied.

Definition. *Let C be the constant in Lemma 4.2, with K taken to be M^*. Consider some configuration $\omega \in \Omega$. A potential defect is a situation where, there is an integer $k \geq 1$, and a point $q \in M$, and nuclei $p_1, p_2, z_1, \ldots, z_k \in \omega$, and numbers $r, r' > 0$ such that*

(1) $r < C^{-1}$,
(2) $r \leq r' \leq r + Cr^3$,
(3) $r = d_0(q, p_1) = d_0(q, p_2)$,
(4) $r' = d_0(q, z_1) = \cdots = d_0(q, z_k)$, *and*
(5) *the vectors* $\{N_q(p_1), N_q(p_2), N_q(z_1), \ldots, N_q(z_k)\}$ *are affinely dependent.*

The number r is called the span of the potential defect, and the point q is the navel of the potential defect.

5. Defects are Rare

This section will provide an estimate for the probability of having a defect or potential defect in a given region. The argument will be based on the necessary condition for defects given in Lemma 4.2. We start with the following almost obvious lemma.

Lemma 5.1. *Let $m \geq 3$ be some integer, and let X_m be the set of points $x = (z_1, \ldots, z_m) \in \big(\mathbb{R}^d\big)^m$ such that $\{z_1, \ldots, z_m\} \subset \mathbb{R}^d$ is affinely dependent and $|z_j| = 1$ for each $j = 1, \ldots, m$. Then X_m has finite $(md - d - 2)$-dimensional measure.*

Proof. Let Y be the space of tuples $y = (L, w, y_1, \dots, y_m, \theta)$, where $L \subset \mathbb{R}^d$ is an $(m-2)$-dimensional linear subspace, w is a unit vector orthogonal to L, $y_1, \dots, y_m$ are unit vectors in L, and $\theta \in [0, \pi/2]$. Then, clearly, Y is a compact $(md-d-2)$-dimensional smooth manifold with boundary, and therefore has finite $(md-d-2)$-dimensional measure. The map

$$\Phi(L, w, y_1, \dots, y_m, \theta) = (\cos\theta\, w + \sin\theta\, y_1, \dots, \cos\theta\, w + \sin\theta\, y_m),$$

takes Y onto X_m, and is a Lipschitz map. Therefore X_m has finite $(md-d-2)$-dimensional measure. □

Here is another nearly trivial lemma.

Lemma 5.2. *Let $W \subset \mathbb{R}^d$ be open, let ν be Lebesgue measure on W, let $\lambda > 0$, and let $\omega \subset W$ be a Poisson point process on (W, ν) with density λ. Let $m > 1$ be an integer, and let $\hat\omega_m \subset W^m$ be the set of points $(v_1, \dots, v_m) \in \omega^m$ such that $v_j \neq v_k$ for $j \neq k$. Let $\nu_m = \nu \times \cdots \times \nu$ be the product measure in W^m, and let $S \subset W^m$ be measurable. Then the probability that $\hat\omega_m$ will intersect S is at most $\lambda^m \nu_m(S)$.*

Proof. One first proves the lemma in the case that $S \subset W^m$ is a box disjoint from the diagonals $w_j = w_k$. The general case follows. Details are left to the reader. □

For an interval $I \subset \mathbb{R}$ and a set $W \subset M$, let $\mathcal{PD}(W, I)$ be the event that in W there is a navel of a potential defect whose span is in the interval I.

Proposition 5.3. *Let K be a compact subset of M, and let $W \subset K$ be open. Then,*

$$P_\lambda\Big(\mathcal{PD}\big(W, [0, \epsilon]\big)\Big) \le C\, \mathrm{vol}(W) \lambda^{d+2} \epsilon^{d^2+d+2},$$

for every $\epsilon \in \big[\lambda^{-1/d}, C^{-1}\big]$, where $C > 0$ is a constant which may depend on M, μ, ds, u, K, but not on W, λ, ϵ.

Lemma 5.4. *Let the situation be as in the proposition. There is a constant $C_0 = C_0(M, ds, K) > 0$, such that the following holds. Let $\epsilon \in [0, C_0]$, $\delta \in (0,1)$, and let k in the range $1, 2, \dots, d$. Let S be the set of all tuples $(p_1, p_2, z_1 \dots, z_k) \in M^{k+2}$ such that for some $q \in W$, $r \in [0, \epsilon]$, and $r' \in [r, r + \delta\epsilon)$, we have $r = d_0(q, p_1) = d_0(q, p_2)$, $r' = d_0(q, z_1) = \cdots = d_0(q, z_k)$, and the vectors $\{N_q(p_1), N_q(p_2), N_q(z_1), \dots, N_q(z_k)\}$ are affinely dependent. Then the $(k+2)d$-dimensional measure of S is $O(1)\, \mathrm{vol}(W) \delta \epsilon^{(k+1)d}$.*

Proof. Recall the sets X_m of Lemma 5. Let

$$Y = [0,1] \times [0, \delta) \times X_{k+2}.$$

From that lemma it follows that the $(k+1)d$-dimensional measure of Y is $O(\delta)$. Let $Y' = \epsilon Y$; that is, the set Y scaled by ϵ. Then the $(k+1)d$-dimensional measure of Y' is $O(1)\, \delta \epsilon^{(k+1)d}$. Consider the map

$$\Psi_0 : Y' \to \big(\mathbb{R}^d\big)^{k+2}$$

defined by

$$\Psi_0(r, \alpha, x_1, \dots, x_{k+2}) = \big(r\epsilon^{-1} x_1, r\epsilon^{-1} x_2, (r+\alpha)\epsilon^{-1} x_3, \dots, (r+\alpha)\epsilon^{-1} x_{k+2}\big).$$

Differentiation shows that Ψ_0 is Lipschitz in Y' with a Lipschitz constant which depends only on d. (Because $r, r+\alpha$ and the x_j's are $O(\epsilon)$.) Consequently, the $(k+1)d$-dimensional measure of $\Psi_0(Y')$ is also $O(1)\,\delta\epsilon^{(k+1)d}$.

We assume, with no loss of generality, that W is contained in a coordinate chart of M. This allows us to identify the tangent space T_zM with $\mathbb{R}^d$, for z in a neighborhood of W. Given a point $z \in W$ and a vector $v \in \mathbb{R}^d$ with $|v| \leq 15\epsilon$, let $\exp_z(v)$ denote the point $x \in M$ such that $d_0(z,x) = |v|$ and the tangent at z of the d_0-geodesic segment from z to x is $v/|v|$. (This is usually called the exponential map.) Since geodesics can be obtained by solving an ODE on the tangent bundle, the map $\exp_z(v)$ is smooth in z and v (that is, the geodesic flow is smooth). For each $z \in W$ and each $(v_1, v_2, \ldots, v_{k+2}) \in \Psi_0(Y')$, set

$$\Psi_1(z, v_1, v_2, \ldots, v_{k+2}) = \big(\exp_z(v_1), \ldots, \exp_z(v_{k+2})\big).$$

Since Ψ_1 is smooth, we find that the $(k+2)d$-dimensional measure of $\Psi_1\big(W \times \Psi_1(Y')\big)$ is $O(1)\,\mathrm{vol}(W)\delta\epsilon^{(k+1)d}$. The lemma follows, because $S = \Psi_1\big(W \times \Psi_0(Y')\big)$. □

Proof of 5.3. Let C_1 denote the constant of Lemma 4.2. Fix some small $\epsilon \geq \lambda^{-1/d}$. Let $\mathcal{A}_k$, $k = 1, 2, \ldots$ be the event that there is a point $q \in W$, an $r \in [0, \epsilon]$, an $r' \in [r, r+C_1r^3)$, and distinct points $p_1, p_2, z_1, \ldots, z_k \in \omega$ such that $d_0(q, p_1) = d_0(q, p_2) = r$, $d_0(q, z_j) = r'$, $j = 1, \ldots, k$, and the unit vectors $\{N_q(p_1), N_q(p_2), N_q(z_1), \ldots, N_q(z_k)\}$ are affinely dependent.

By definition, $\mathcal{PD}\big(W, [0,\epsilon]\big) \subset \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots$. Set $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_d$, and note that $\mathcal{A}_k \subset \mathcal{A}_d$ for $k > d$, because any subset of $\mathbb{R}^d$ whose cardinality is $d+2$ is affinely dependent. Therefore,

$$\mathcal{PD}\big(W, [0,\epsilon]\big) \subset \mathcal{A}. \tag{5.1}$$

Now fix some $k = 1, \ldots, d$, and consider $\mathcal{A}_k$. To estimate $P_\lambda(\mathcal{A}_k)$, apply Lemma 5.4 with $\delta = O(\epsilon^2)$, and then use Lemma 5. (Here the measure μ is only comparable, not equal to Lebesgue measure, but that is enough.) The combination of these two lemmas gives

$$P_\lambda(\mathcal{A}_k) \leq O(1)\,\mathrm{vol}(W)\lambda^{k+2}\epsilon^{(k+1)d+2}. \tag{5.2}$$

Since we take $\epsilon \geq \lambda^{-1/d}$, this is largest when $k = d$, and hence,

$$P_\lambda\Big(\mathcal{PD}\big(W, [0,\epsilon)\big)\Big) \leq P_\lambda(\mathcal{A}) \leq O(1)\,\mathrm{vol}(W)\lambda^{d+2}\epsilon^{(d+1)d+2},$$

which proves the proposition. □

Set,

$$L = L(\lambda) = \lambda^{-1/d}(\log\lambda)^{(d+1)/d^2}. \tag{5.3}$$

This quantity L will be an important length scale in the following sections. The two essential features of the choice of L is that it tends to zero faster than $\lambda^{-1/d}(\log\lambda)^{1/(d-1)}$, but slower than $\lambda^{-1/d}(\log\lambda)^{1/d}$.

Lemma 5.5 (no giant tiles). *Let $K \subset M$ be compact, and let $c > 0$ be some constant. Then the P_λ-probability that there will be a tile in $\mathbf{T}(\omega, ds)$ which intersects K and has diameter greater than cL tends to zero as $\lambda \to \infty$.*

Proof. Let $U \subset M$ be an open set which contains K and has compact closure. Suppose that λ is sufficiently large so that the distance from K to $M - U$ is greater than $4cL$. Let X be a maximal subset of U with the property that any distinct elements of it have distance at least $cL/9$. The cardinality of X satisfies

$$|X| = O(L^{-d}). \tag{5.4}$$

For any $x \in X$, let $\mathcal{E}_x \subset \Omega$ be the event that the ball $B_0(x, cL/9)$ does not intersect ω. Then we have

$$P_\lambda\big(\mathcal{E}_x\big) = e^{-\lambda L^d/O(1)}. \tag{5.5}$$

Let $z \in \omega$, and let $\mathbf{T}(z)$ be the tile with nucleus z in $\mathbf{T}(\omega, ds)$. Suppose that $\mathbf{T}(z)$ intersects K, and its diameter is greater than cL. Then there is some $y \in \mathbf{T}(z)$ such that $d_0(y, z) \geq cL/2$ and $d_0(y, K) \leq cL$. Consequently, the ball $B_0(y, cL/3)$ is disjoint from ω and contained in U. There will be some $x \in X$ with $d_0(x, y) < cL/6$. For that x, we shall have $\omega \in \mathcal{E}_x$. This shows that the event that there is some tile which intersects K and has diameter $\geq cL$ is contained in $\cup_{x \in X} \mathcal{E}_x$. Hence, we get from (5.4) and (5.5) that the probability of that event is at most

$$O(L^{-d})e^{-\lambda L^d/O(1)},$$

which tends to zero as $\lambda \to \infty$, by (5.3). □

Note that the number of tiles of $\mathbf{T}(\omega, ds)$ that are expected to intersect K is in the order of λ.

Theorem 5.6. *Suppose $K \subset M$ is compact, and has positive volume. Then the expected number of defects involving tiles in K is $O(1)\lambda^{(d-2)/d}$, when λ is large.*

The theorem will not be needed in the following, because we will need information about potential defects more than about actual defects. It is presented only for completeness.

Proof. Let $W \subset M$ be a set whose diameter is smaller than λ^{-1}, say, and let w_0 be some point in W. For any interval $[a, b]$, let $h_W(a, b)$ be the probability that there will be in W a navel of a defect with span in the range $[a, b]$. By Proposition (probdef),

$$h_W\left(0, \lambda^{-1/d}\right) \leq O(1)\,\mathrm{vol}(W)\lambda^{(d-2)/d}.$$

Now consider some $\epsilon \geq \lambda^{-1/d}$. If there is for a configuration ω a navel in W of a defect with span in the range $[\epsilon, 2\epsilon]$, then the ball $B_0(w_0, \epsilon/2)$ does not contain any point in ω. This latter event is independent of $\mathcal{PD}\big(W, [\epsilon, 2\epsilon]\big)$, and consequently,

$$\begin{aligned} h_W(\epsilon, 2\epsilon) &\leq P_\lambda\big(B_0(w_0, \epsilon/2) \cap \omega = \emptyset\big) P_\lambda\Big(\mathcal{PD}\big(W, [\epsilon, 2\epsilon]\big)\Big) \\ &\leq O(1)e^{-\lambda\epsilon^d/O(1)}\,\mathrm{vol}(W)\lambda^{d+2}\epsilon^{d^2+d+2}, \end{aligned}$$

again, using Proposition 5.3. Consider a tiling of K by sets $\{W_j\}$ with very minute diameters. Let $n(\omega)$ be the number of tiles in the tiling which contain a navel of a defect. Then

$$En(\omega) \le \sum_j h_{W_j}\left(0, \lambda^{-1/d}\right) + \sum_j \sum_{k=0}^{\infty} h_{W_j}\left(\lambda^{-1/d}2^k, \lambda^{-1/d}2^{k+1}\right)$$
$$\le O(1)\,\mathrm{vol}\,(K)\,\lambda^{(d-2)/d} \sum_{k=0}^{\infty} e^{-2^{kd}/O(1)} 2^{k(d^2+d+2)} = O(1)\lambda^{(d-2)/d}.$$

Since every defect has a navel which is not the navel of any other defect, for any specific configuration ω, the number of tiles W_j which meet a navel tends to a number at least as large as the number of defects of ω as the tiling W_j becomes very fine. Consequently, by the monotone convergence theorem, the expected number of defects is bounded by the limsup of $En(\omega)$, as the tiling W_j becomes finer. The theorem follows. □

Remark. In fact, the estimate in Theorem 5.6 is sharp.

6. The Size of Spherical Shells

This section is devoted to proving a tail estimate for the number of Voronoi cells in a random Voronoi tiling $\mathbf{T}(\omega, ds)$ which meet a union of spheres. (It is possible to do without this section if one is interested only in the case $M = \mathbb{R}^2$.) The precise statement which we shall need is as follows.

Proposition 6.1. *Let $M_0 \subset M$ be compact, let $K \subset M_0$ be a finite set, and let $a, R, \lambda > 0$, with λ large and $R^2 \le \lambda^{-1/d}$. For each $x \in K$, let $S(x)$ be the sphere of radius R about x. Given $\omega \in \Omega$, let $n(\omega) = n(K, R, \lambda, \omega)$ be the number of Voronoi tiles in the tiling $\mathbf{T}(\omega, ds)$ that have diameter $\le R$ and intersect $\cup_{x\in K} S(x)$. Then there is a constant $C = C(a, M, M_0, ds, \mu) > 0$ such that*

$$E_\lambda \exp\left(an(\omega)\right) \le \exp\left(CR^{d-1}\lambda^{(d-1)/d}|K|\right),$$

where E_λ denotes the expectation operator of (Ω, P_λ).

6.2 Lemma of Ball Unions. *Let $0 < C < \infty$, and let $A \subset \mathbb{R}^d$ be a union of open balls with centers on the unit sphere $S^{d-1} \subset \mathbb{R}^d$, and with radii bounded by C. Then the $(d-1)$-dimensional measure of ∂A is bounded by a constant which depends only on C and d.*

The proof is motivated by hyperbolic geometry, but does not use it.

Proof. Suppose first that A is a finite union of such balls,

$$A = \cup_{j=1}^n B(q_j, r_j).$$

Let X be the set of points $x \in \partial A$ such that x is on the boundary of exactly one of the balls $B(q_j, r_j)$. Then X has full $(d-1)$-measure in ∂A. We now define a map $f : X \to S^{d-1}$. Let $x \in X$, and suppose that j is the index such that $x \in \partial B(q_j, r_j)$. If $x \in X \cap S^{d-1}$, set $f(x) = x$. Otherwise, let B_x be the largest open ball which is contained in $B(q_j, r_j)$, is internally tangent to $B(q_j, r_j)$ at x, and is disjoint from S^{d-1}. See Fig. 1. Clearly, B_x is well defined, and there is precisely one intersection point of ∂B_x and S^{d-1}. Let $f(x)$ be that intersection point.

Note that $f : X \to S^{d-1}$ is a continuous map. Suppose that y is a point in $A \cap S^{d-1}$. Let B^y be the maximal open ball which is externally tangent to S^{d-1} at y and is contained

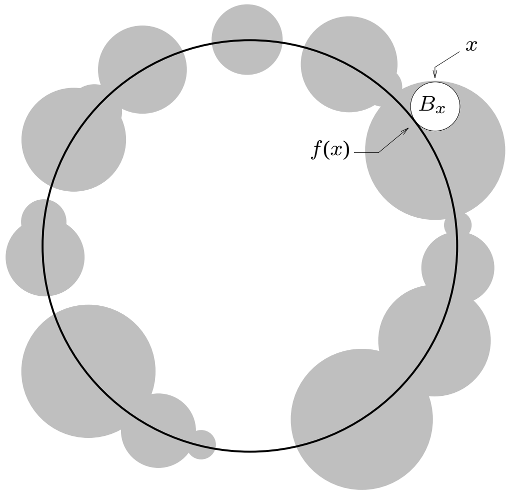


Fig. 1. The definition of the map $f(x)$

in A. If x is some point in $X \cap \partial B^y$, then the ball B^y is strictly contained in the ball $B(q_j, r_j)$ with $x \in \partial B(q_j, r_j)$. Consequently, x is the only point in $\partial A \cap B^y$, and $B_x = B^y$. It follows that for every $y \in S^{d-1}$ there is at most a unique $x \in X$ outside the unit ball $B(0,1)$ such that $f(x) = y$. The same argument applies to the points x inside the unit ball. Therefore, the map f is at most 2 to 1.

Consider one of the balls, $B(q_j, r_j)$, in the union making up A. It is enough to show that locally the map f does not contract distances too much in $X \cap \partial B(q_j, r_j)$. This can be done by inspecting the extreme cases where either r_j is small or at the points where $\partial B(q_j, r_j)$ is close to S^{d-1}. Alternatively, observe that the restriction of f to a component of $\partial B(q_j, r_j) \cap X - S^{d-1}$ is equal to an inversion in some $(d-1)$-dimensional sphere Z and that the center of Z cannot be too close to S^{d-1}. The case where A is a union of infinitely many balls follows by a limiting argument. The details are left to the reader. $\square$

Remark. In the lemma, one may replace the assumption that the centers of the balls making up A are on S^{d-1} by the assumption that the interior angle of the intersection of these balls with the unit ball be bounded away from 0.

For book-keeping, we introduce yet another tiling of M, $\mathbf{T}_B$, which will be non-random. The only important feature of $\mathbf{T}_B$ is that every one of its tiles has diameter $O\left(\lambda^{-1/d}\right)$ and volume at least $C\lambda^{-1}$ for some constant $C = C(ds) > 0$. For example, $\mathbf{T}_B$ may be constructed as follows. Take a set of points $B \subset M$ such that the distance between any two points in B is at least $\lambda^{-1/d}$, and B is maximal with this property, and let $\mathbf{T}_B = \mathbf{T}(B, ds)$, denote the corresponding Voronoi tiling.

Proof of 6.1. For $x \in K$, let $A(x)$ be the union of all open balls with radius at most R and center in $S(x)$ that do not intersect any tile of $\mathbf{T}_B$ which intersects ω.

Let $U(x) = U(x, \omega)$ denote the set of points in ω which are nuclei of tiles with diameter $\leq R$ that intersect $S(x)$, and suppose that $q \in U(x)$. Then there must be a point $z \in S(x)$ such that q is the closest point to z which is in ω. Hence q is on the boundary of an open ball with center in $S(x)$ which is disjoint from ω and has diameter $\leq R$. Recall that every tile of $\mathbf{T}_B$ has diameter $\leq C_1\lambda^{-1/d}$, where C_1 is some constant. It follows that q has distance at most $C_1\lambda^{-1/d}$ from $A(x) \cup S(x)$. Let $H(x,\omega)$ be the set of all the tiles of $\mathbf{T}_B$ which are at distance at most $C_1\lambda^{-1/d}$ from $A(x) \cup S(x)$, but do not intersect $A(x)$. We may conclude that $U(x) \subset \cup H(x,\omega)$. (If Q is a set of tiles, then $\cup Q$ denotes the union of the tiles in Q.)

Set

$$H = \bigcup_{x \in K} H(x,\omega),$$
$$n^* = |(\cup H) \cap \omega|.$$

Then $n^* = n^*(\omega) \geq n(\omega)$. In order to bound the tail of n^*, let us estimate from above the size of H. Assume first that the metric ds is a flat (Euclidean) metric. For each $z \in S(x)$ let $r(z)$ be the maximal $r \geq 0$ such that the open ball of radius r about z is disjoint from tiles of $\mathbf{T}_B$ which intersect ω ($r(z) = 0$ if z is in a tile of $\mathbf{T}_B$ which intersects ω). Set $r^*(z) = \min\{r(z), R\}$ and for $t \geq 0$,

$$A(x,t) = \bigcup_{z \in S(x)} B\bigl(z, r^*(z) + t\bigr).$$

Then $A(x) = A(x,0)$ and each tile in $H(x,\omega)$ is contained in $A\left(x, 2C_1\lambda^{-1/d}\right) - A(x)$. In order to bound the cardinality of $H(x,\omega)$, we estimate the volume of $A\left(x, 2C_1\lambda^{-1/d}\right) - A(x)$,

$$\begin{aligned} \operatorname{vol}\left(A\left(x, 2C_1\lambda^{-1/d}\right) - A(x)\right) &= \int_0^{2C_1\lambda^{-1/d}} \frac{d}{dt}\operatorname{vol} A(x,t)\,dt \\ &= \int_0^{2C_1\lambda^{-1/d}} \operatorname{vol}_{d-1}\partial A(x,t)\,dt, \end{aligned} \tag{6.1}$$

where vol_{d-1} denotes the $d-1$ dimensional measure. By the Lemma of Ball Unions (6.2), appropriately rescaled, we know that

$$\operatorname{vol}_{d-1}\partial A(x,t) \leq C_2 R^{d-1}$$

for some constant C_2, and all $t \leq R$. It follows then from (6.1) that

$$\operatorname{vol}\bigl(\cup H\bigr) \leq \sum_{x \in K} \operatorname{vol}\left(A\left(x, 2C_1\lambda^{-1/d}\right) - A(x)\right) \leq 2C_1C_2|K|R^{d-1}\lambda^{-1/d}. \tag{6.2}$$

We set

$$\beta = C_3|K|R^{d-1}\lambda^{(d-1)/d}, \tag{6.3}$$

with C_3 a large constant. Since μ is comparable to the measure induced by ds, we get from (6.2),

$$\mu\bigl(\cup H\bigr) \leq \beta\lambda^{-1}, \tag{6.4}$$

provided C_3 is large enough. Because the measure of a tile in $\mathbf{T}_B$ is at least $O(1)^{-1}\lambda^{-1}$, we also get,

$$|H| \leq \beta, \tag{6.5}$$

if C_3 is large enough.

To remove the assumption that ds is the Euclidean metric, observe that for $x \in M$ one may choose a Euclidean metric for a neighborhood of x such that for points at distance $O(R)$ from x distances are distorted by not more than an additive constant of $O(R^2)$. Since we have the assumption $R^2 \le \lambda^{-1/d}$, it is easy verify that the distortion will not influence the validity of the argument above, but may only change the constants.

It is true that the collection of tiles H depends on ω. Hence we cannot naively use the standard formula for the probability that $\omega \cap (\cup H)$ has a given cardinality in terms of λ and $\mu(\cup H)$. But note that H only depends on which tiles of $\mathbf{T}_B$ which contain a point of ω, and does not depend on the number of points in each such tile. Consider some tile T, and suppose that g is the number of points in $\omega \cap T$. Then the distribution of $g + 1$ dominates the distribution of g conditioned on $g \ge 1$. This can be thought of as a continuous instance of the BK inequality [5], but may also be verified directly. We conclude from this argument and the inequalities (6.4), (6.5) that for each m,

$$P_\lambda\big(n^* \ge m + \beta\big) \le \sum_{j=m}^{\infty} \frac{\beta^j}{j!} e^{-\beta}.$$

Consequently,

$$E \exp\big(an(\omega)\big) \le E \exp\big(an^*\big) \le e^{a\beta} \sum_{j=0}^{\infty} e^{aj} \frac{\beta^j}{j!} e^{-\beta} = \exp\big(a\beta - \beta + e^a \beta\big),$$

and the proposition follows. □

7. Clean Configurations

A *local potential defect* is a potential defect whose span is less than $1.1L$, where $L = L(\lambda) = \lambda^{-1/d} (\log \lambda)^{(d+1)/d^2}$ as in (5.3). This section will study the statistical properties of configurations that have no local potential defects. These will be called *clean* configurations. We shall continue to use the book-keeping tiling $\mathbf{T}_B$, which was introduced in Sect. 6. In the following, we assume that λ is sufficiently large, so that the diameter of any tile in $\mathbf{T}_B$ is less than $L/100$.

Let $\omega \in \Omega$ be some configuration. Its *local potential defect zone* $Z(\omega)$ is defined as follows. Let $Z_0(\omega)$ be the set of all navels of local potential defects and let $Z(\omega)$ be the set of all tiles of $\mathbf{T}_B$ which contain a point in $Z_0(\omega)$.

Let Q be any set of tiles of $\mathbf{T}_B$. Denote by $\mathcal{D}(Q)$ the event that $Z(\omega) = Q$, let $\mathcal{F}(Q)$ be the event that $Z(\omega) \supset Q$, and let $\mathcal{N}(Q)$ be the event that $Z(\omega) \cap Q = \emptyset$. A clean configuration is just a configuration in $\mathcal{D}(\emptyset)$. We would like to discuss the distribution of clean configurations, that is, to condition on $\mathcal{D}(\emptyset)$. Hence it would be useful to have $P_\lambda\big(\mathcal{D}(\emptyset)\big) > 0$. If M has finite volume, this is clear, since with positive, but very small, probability the configuration ω will contain only a single point, and then no potential defects are possible. (It will be shown below that the clean configurations are typically not so sparse.) Hence, we shall for simplicity now assume that M has finite volume. There are obvious and simple methods to extend the discussion to the infinite volume case.

Suppose that $\mathcal{A} \subset \hat{\Omega}$ is some event, and X is some subset of M. We say that $\mathcal{A}$ is *independent* of X, if whenever $\omega \in \mathcal{A}$ and $\omega' \in \hat{\Omega}$ differ only in points which are in

X, then also $\omega' \in \mathcal{A}$. We shall say that $\mathcal{A}$ *depends only on* X, if $\mathcal{A}$ is independent of $M - X$.

The next two lemmas relate the properties of random clean configurations to the properties of ordinary configurations.

7.1. First lemma of clean configurations. *Let Q be any set of tiles of $\mathbf{T}_B$, and let $\mathcal{A} \subset \hat{\Omega}$ be some event which depends only on $\cup Q$, the union of tiles in Q. Let Q_{2L} be the set of tiles of $\mathbf{T}_B$ with distance at most $2L$ to $\cup Q$. Then*

$$P_\lambda\bigl(\mathcal{A}|\mathcal{D}(\emptyset)\bigr) \leq \frac{P_\lambda(\mathcal{A})}{P_\lambda\bigl(\mathcal{N}(Q_{2L})\bigr)}.$$

In the proof, we shall need the FKG [8] inequality for Poisson point processes. An event $\mathcal{X} \subset \Omega$ is *increasing*, if $\omega' \in \mathcal{X}$ whenever $\omega \in \mathcal{X}$ and $\omega \subset \omega' \in \Omega$. A random variable $f : \Omega \to \mathbb{R}$ is increasing if $f(\omega') \geq f(\omega)$ whenever $\omega' \supset \omega$. Similarly *decreasing* events and random variables are defined. The FKG inequality for events states that $P_\lambda(\mathcal{X} \cap \mathcal{Y}) \geq P_\lambda(\mathcal{X})P_\lambda(\mathcal{Y})$ if either $\mathcal{X}, \mathcal{Y}$ are both increasing events, or both decreasing events. The FKG inequality for random variables states that $E(fg) \geq Ef\, Eg$, if f, g are both increasing random variables, or both are decreasing random variables. The proof of the FKG inequality for events in Poisson point processes may be found in the paper by R. Roy [14]. Although the setting there is a bit different, the proof is easily adapted to our situation. The FKG inequality for random variables can be obtained as a corollary of the inequality for events.

Proof. Let Y be the set of tiles of $\mathbf{T}_B$ which are not in Q_{2L}. Observe that $\mathcal{N}(Y)$ is independent of $\mathcal{A}$. Also note that $\mathcal{N}(Y)$ and $\mathcal{N}(Q_{2L})$ are both decreasing events, and therefore they are positively correlated, by the FKG inequality. These are the facts that enter into the following estimate:

$$\begin{aligned} P_\lambda\bigl(\mathcal{A}|\mathcal{D}(\emptyset)\bigr) &= \frac{P_\lambda\bigl(\mathcal{A} \cap \mathcal{N}(Q_{2L}) \cap \mathcal{N}(Y)\bigr)}{P_\lambda\bigl(\mathcal{N}(Q_{2L}) \cap \mathcal{N}(Y)\bigr)} \\ &\leq \frac{P_\lambda\bigl(\mathcal{A} \cap \mathcal{N}(Y)\bigr)}{P_\lambda\bigl(\mathcal{N}(Q_{2L}) \cap \mathcal{N}(Y)\bigr)} = \frac{P_\lambda(\mathcal{A})P_\lambda\bigl(\mathcal{N}(Y)\bigr)}{P_\lambda\bigl(\mathcal{N}(Q_{2L}) \cap \mathcal{N}(Y)\bigr)} \\ &\leq \frac{P_\lambda(\mathcal{A})P_\lambda\bigl(\mathcal{N}(Y)\bigr)}{P_\lambda\bigl(\mathcal{N}(Q_{2L})\bigr)P_\lambda\bigl(\mathcal{N}(Y)\bigr)} = \frac{P_\lambda(\mathcal{A})}{P_\lambda\bigl(\mathcal{N}(Q_{2L})\bigr)}. \end{aligned}$$

□

In order to effectively apply Lemma 7, we shall need an estimate for $P_\lambda\bigl(\mathcal{N}(Q)\bigr)$ when Q is a set of tiles in $\mathbf{T}_B$. Proposition 5.3 gives,

$$\begin{aligned} P_\lambda\bigl(\mathcal{N}(Q)\bigr) &\geq 1 - O(1)\,\mathrm{vol}(\cup Q)\lambda^{d+2}L^{d^2+d+2} \\ &= 1 - O(1)\,\mathrm{vol}(\cup Q)\lambda^{(d-2)/d}(\log\lambda)^{O(1)}. \end{aligned} \tag{7.1}$$

We shall need a different estimate for the case that $|Q|$, the number of tiles in Q, is large. For any set of tiles $Q \subset \mathbf{T}_B$, the event $\mathcal{N}(Q)$ is monotone decreasing. Therefore, the FKG inequality and (7.1) give,

$$P_\lambda\big(\mathcal{N}(Q)\big) \geq \prod_{T\in Q} P_\lambda\big(\mathcal{N}(T)\big) \geq \Big(1 - O\,(1)\,\lambda^{-2/d}(\log\lambda)^{O(1)}\Big)^{|Q|}$$
$$\geq \exp\Big(-O\,(1)\,\lambda^{-2/d}(\log\lambda)^{O(1)}|Q|\Big)\,, \tag{7.2}$$

because $1-\epsilon \geq e^{-2\epsilon}$ when $\epsilon > 0$ is small.

7.2. Second lemma of clean configurations. *Let Q be a set of tiles of $\mathbf{T}_B$, and let Q_{6L} be the set of all tiles of $\mathbf{T}_B$ with distance at most $6L$ from $\cup Q$. Let $\mathcal{A} \subset \hat\Omega$ be some event which is independent of $\cup Q_{6L}$. Then,*

$$P_{\lambda,p}\big(\mathcal{A}|\mathcal{D}(Q)\big) \leq \frac{P_{\lambda,p}\big(\mathcal{A}|\mathcal{D}(\emptyset)\big)}{P_\lambda\big(\mathcal{N}(Q_{6L})\big)}.$$

Proof. For $j = 1, 2$ let B_j be the set of all tiles T of $\mathbf{T}_B - Q$ such that the distance from T to $\cup Q$ is in the range $[3(j-1)L, 3jL)$. Also let B_3 be all the tiles of $\mathbf{T}_B$ which are not in $Q \cup B_1 \cup B_2$,

$$P_{\lambda,p}\big(\mathcal{A}|\mathcal{D}(Q)\big) = \frac{P_{\lambda,p}\big(\mathcal{A}\cap\mathcal{F}(Q)\cap\mathcal{N}(B_1)\cap\mathcal{N}(B_2)\cap\mathcal{N}(B_3)\big)}{P_\lambda\big(\mathcal{F}(Q)\cap\mathcal{N}(B_1)\cap\mathcal{N}(B_2)\cap\mathcal{N}(B_3)\big)}$$
$$\leq \frac{P_{\lambda,p}\big(\mathcal{F}(Q)\cap\mathcal{N}(B_1)\cap\mathcal{N}(B_3)\cap\mathcal{A}\big)}{P_\lambda\big(\mathcal{F}(Q)\cap\mathcal{N}(B_1)\cap\mathcal{N}(B_2)\cap\mathcal{N}(B_3)\big)}. \tag{7.3}$$

Since the distance between $\cup B_3$ and $\cup(B_1\cup Q)$ is greater than $2.2L$, the events $\mathcal{N}(B_3)\cap\mathcal{A}$ and $\mathcal{F}(Q)\cap\mathcal{N}(B_1)$ are independent; that is,

$$P_{\lambda,p}\big(\mathcal{F}(Q)\cap\mathcal{N}(B_1)\cap\mathcal{N}(B_3)\cap\mathcal{A}\big) = P_\lambda\big(\mathcal{F}(Q)\cap\mathcal{N}(B_1)\big)P_{\lambda,p}\big(\mathcal{N}(B_3)\cap\mathcal{A}\big). \tag{7.4}$$

Let A_1 be the set of points in M with distance at most L from $(\cup B_1)\cap(\cup B_2)$, let A_0 be the points in connected components of $M - A_1$ that intersect $\cup Q$, and let $A_2 = M - A_0 - A_1$. We want to show that the events $\mathcal{F}(Q)\cap\mathcal{N}(B_1)$ and $\mathcal{N}(B_2)\cap\mathcal{N}(B_3)$ are positively correlated. For this, the FKG inequality can be used, but not immediately. Any $\omega \in \hat\Omega$ can be decomposed into $(\omega_0, \omega_1, \omega_2)$, where $\omega_j = A_j \cap \omega$. This induces a decomposition $\Omega = \Omega_0 \times \Omega_1 \times \Omega_2$ of Ω. Note that $\mathcal{F}(Q)\cap\mathcal{N}(B_1)$ is an event that's independent of ω_2 and is monotone decreasing in $\pi\omega_1$. Similarly, $\mathcal{N}(B_2)\cap\mathcal{N}(B_2)$ is independent of ω_0 and is monotone decreasing in $\pi\omega_1$. Given any $\omega_1 \in \Omega_1$, let $f(\omega_1)$ be the probability that $(\omega_0, \omega_1, \omega_2) \in \mathcal{F}(Q)\cap\mathcal{N}(B_1)$, and let $g(\omega_1)$ be the probability that $(\omega_0, \omega_1, \omega_2) \in \mathcal{N}(B_2)\cap\mathcal{N}(B_2)$, where $\omega_0 \in \Omega_0$ and $\omega_2 \in \Omega_2$ are random. Then f and g are monotone decreasing random variables on Ω_1. Hence, the FKG inequality for random variables, $E(fg) \geq Ef\,Eg$, gives,

$$P_\lambda\big(\mathcal{F}(Q)\cap\mathcal{N}(B_1)\cap\mathcal{N}(B_2)\cap\mathcal{N}(B_3)\big) \geq P_\lambda\big(\mathcal{F}(Q)\cap\mathcal{N}(B_1)\big)P_\lambda\big(\mathcal{N}(B_2)\cap\mathcal{N}(B_3)\big). \tag{7.5}$$

A similar argument shows that

$$P_{\lambda,p}\big(\mathcal{D}(\emptyset)\cap\mathcal{A}\big) \geq P_{\lambda,p}\big(\mathcal{N}(B_3)\cap\mathcal{A}\big)P_\lambda\big(\mathcal{N}(Q)\cap\mathcal{N}(B_1)\cap\mathcal{N}(B_2)\big)$$
$$= P_{\lambda,p}\big(\mathcal{N}(B_3)\cap\mathcal{A}\big)P_\lambda\big(\mathcal{N}(Q_{6L})\big). \tag{7.6}$$

Now combine (7.3), (7.4), (7.5) and (7.6), to obtain,

$$
\begin{aligned}
P_{\lambda,p}\big(\mathcal{A}|\mathcal{D}(Q)\big) &\le \frac{P_{\lambda,p}\big(\mathcal{N}(B_3)\cap\mathcal{A}\big)}{P_\lambda\big(\mathcal{N}(B_2)\cap\mathcal{N}(B_3)\big)} \\
&\le \frac{P_{\lambda,p}\big(\mathcal{A}\cap\mathcal{D}(\emptyset)\big)}{P_\lambda\big(\mathcal{N}(Q_{6L})\big)P_\lambda\big(\mathcal{N}(B_2)\cap\mathcal{N}(B_3)\big)} \le \frac{P_{\lambda,p}\big(\mathcal{A}|\mathcal{D}(\emptyset)\big)}{P_\lambda\big(\mathcal{N}(Q_{6L})\big)}.
\end{aligned}
\tag{7.7}
$$

This proves the lemma. □

7.3. Lemma (clean configurations have no giant tiles). *Let K be a compact subset of M, and let $\mathcal{S}$ be the event that all tiles in $\mathbf{T}(\omega, ds)$ which meet K have diameter smaller than L. Then*

$$\lim_{\lambda\to\infty} P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big) = 1.$$

Proof. Let U be an open set in M whose closure is compact and which contains K. Let X be a maximal subset of U such that the distance between any two elements of X is at least $L/9$. For $x \in X$ let $\mathcal{E}_x$ be the event that the ball $B_0(x, L/9)$ is disjoint from ω. Let $Q(x)$ be the set of tiles in $\mathbf{T}_B$ whose distance from x is at most $3L$. Since $\mathcal{E}_x$ depends only on the intersection of ω with $B_0(x, L/9)$, the First Lemma of Clean Configurations 7 gives,

$$P_\lambda\big(\mathcal{E}_x|\mathcal{D}(\emptyset)\big) \le \frac{P_\lambda(\mathcal{E}_x)}{P_\lambda\Big(\mathcal{N}\big(Q(x)\big)\Big)} = \frac{e^{-L^d\lambda/O(1)}}{P_\lambda\Big(\mathcal{N}\big(Q(x)\big)\Big)}.$$

Since

$$\operatorname{vol}\big(Q(x)\big) = O(1)L^d = O(1)\lambda^{-1}(\log\lambda)^{O(1)},$$

the inequality (7.1) implies that $P_\lambda\Big(\mathcal{N}\big(Q(x)\big)\Big) \underset{\lambda\to\infty}{\longrightarrow} 1$. Therefore,

$$P_\lambda\Big(\cup_{x\in X}\,\mathcal{E}_x\Big|\mathcal{D}(\emptyset)\Big) \le O(1)|X|e^{-L^d\lambda/O(1)} \le O(1)\lambda\exp\Big(-(\log\lambda)^{1+\frac{1}{d}}/O(1)\Big) \underset{\lambda\to\infty}{\longrightarrow} 0$$

The proof is now completed as the proof of Lemma 5.5. □

8. Insensitivity

This section can be avoided if one is only interested in the case $M = \mathbb{R}^2$.

Let X be some finite set. We denote by 2^X the set of funtions from X to $\{0,1\}$, and make the usual identification of 2^X with the collection subsets of X. Given an element $a \in 2^X$, we denote by $|a|$ the cardinality of a, thought of as a set, which is the same as the L^1 norm of a, thought of as a function. If ν_1, ν_2 are two measures on 2^X, we let $\nu_1 \cup \nu_2$ denote the image of the measure $\nu_1 \times \nu_2$ under the map $\cup : 2^X \times 2^X \to 2^X$. (In other words, $\nu_1 \cup \nu_2$ is the distribution of $a \cup b$, if a and b are independent random elements of $\big(2^X, \nu_1\big)$ and $\big(2^X, \nu_2\big)$.) Similarly, the measure $\nu_1 \cap \nu_2$ is defined.

Fix some $p \in [0,1]$, and let η denote the product measure on 2^X with $\eta\{a : x \in a\} = p$ for each $x \in X$.

8.1. Insensitivity Lemma. *Let ν be a measure on 2^X. Then the following estimate holds for the measure norm of the difference $\eta \cup \nu - \eta$,*

$$\|\eta\cup\nu - \eta\| \le \sqrt{E_{\nu\cap\nu}\left(p^{-|a|}\right) - 1}.$$

The expression $E_{\nu\cap\nu}\left(p^{-|a|}\right)$ means the expectation of $p^{-|a|}$ when a is distributed according to $\nu \cap \nu$.

The lemma was partly motivated by the concept of influence of a boolean variable on a function, introduced by Ben-Or and Linial [3].

Proof. What can one say? Cauchy–Schwarz!

$$\begin{aligned} \|\eta\cup\nu-\eta\|^2 &= \Big(\sum_{a\in 2^X} |\eta\cup\nu(a)-\eta(a)|\Big)^2 \\ &\le \sum_{a\in 2^X}\eta(a)\sum_{a\in 2^X}\eta(a)^{-1}\big(\eta\cup\nu(a)-\eta(a)\big)^2 \\ &= \sum_{a\in 2^X}\eta(a)^{-1}\Big(\eta\cup\nu(a)^2-2\eta\cup\nu(a)\,\eta(a)+\eta(a)^2\Big) \\ &= \sum_{a\in 2^X}\eta(a)^{-1}\,\eta\cup\nu(a)^2-1. \end{aligned} \tag{8.1}$$

Observe that

$$\eta(a) = p^{|a|}(1-p)^{n-|a|},$$

where $n = |X|$. We may write an equality of the form $b\cup c = a$ as $a-c \subset b \subset a$. Hence,

$$\eta\cup\nu(a) = \sum_{c\subset a}\nu(c)p^{|a|-|c|}(1-p)^{n-|a|} = \eta(a)\sum_{c\subset a}\nu(c)p^{-|c|}.$$

We use these expressions to simplify (8.1),

$$\begin{aligned} \|\eta\cup\nu-\eta\|^2 &\le \sum_{a\in 2^X}\eta(a)\left(\sum_{c\subset a}\nu(c)p^{-|c|}\right)^2-1 \\ &= \sum_{a\in 2^X}\sum_{b\subset a}\sum_{c\subset a}\eta(a)\nu(b)\nu(c)p^{-|b|-|c|}-1 \\ &= \sum_{b\in 2^X}\sum_{c\in 2^X}\nu(b)\nu(c)p^{-|b|-|c|}\sum_{a\supset b\cup c}\eta(a)-1 \\ &= \sum_{b\in 2^X}\sum_{c\in 2^X}\nu(b)\nu(c)p^{-|b|-|c|}p^{|b\cup c|}-1 \\ &= \sum_{b\in 2^X}\sum_{c\in 2^X}\nu(b)\nu(c)p^{-|b\cap c|}-1 = E_{\nu\cap\nu}\left(p^{-|a|}\right)-1. \end{aligned}$$

□

8.2 Corollary. *Let ν^c denote the image of ν under the map $a \to X-a$ from 2^X to 2^X. Then*

$$\|\eta\cap\nu^c\| \le \sqrt{E_{\nu\cap\nu}\left((1-p)^{-|a|}\right)-1}.$$

Proof. Use $\eta\cap\nu^c = \left(\eta^c\cup\nu\right)^c$, and apply the lemma. □

9. Assembly

Proof of Theorems 2.1 and 2.2. In the proof, we shall assume that M is compact. This is basically for convenience of notation, and it is easy to modify the arguments to apply in general.

Let $\mathcal{C}$ denote the event of crossing, that is $\mathcal{C} = \mathcal{C}(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds) \subset \hat{\Omega}$, in the situation of Theorem 2.1 and $\mathcal{C} = \mathcal{C}(M, M', \alpha, ds)$, in the situation of Theorem 2.2. Similarly, let $\mathcal{C}_u$ denote the crossing, but with respect to the conformal metric $e^u ds$. Given any set Q of tiles in $\mathbf{T}_B$, let $\mathcal{P}(Q)$ be the event that $\cup Q$ is pivotal for $\mathcal{C}$; that is, $\mathcal{P}(Q)$ is the set of all ω such that there is an ω' which equals ω outside of Q, and one of them is in $\mathcal{C}$ while the other is not. Let $\Delta\mathcal{C}$ be the event that there is a crossing with respect to the metric ds, but not with respect to the metric $e^u\, ds$. In other words, $\Delta\mathcal{C} = \mathcal{C} - \mathcal{C}_u$. We need to estimate $P_{\lambda,p}(\Delta\mathcal{C})$ when λ is large.

We shall continue to use the book-keeping tiling $\mathbf{T}_B$ from Sect. 6. For each set Q of tiles in $\mathbf{T}_B$, and for $a > 0$, let Q_a denote the set of tiles in $\mathbf{T}_B$ with distance at most a to $\cup Q$. Recall the definition 5 of L. We assume that λ is so large that in the scale of L the sets $M', \mathfrak{S}_1, \mathfrak{S}_2$ are 'very smooth'. Let $\mathcal{S}$ be the event all tiles in $\mathbf{T}(\omega, ds)$ have diameter at most L.

Recall that for any set Q of tiles in $\mathbf{T}_B$, $\mathcal{D}(Q)$ denotes the event that Q is the set of tiles containing navels of local potential defects. Since when $\omega \in \mathcal{S}$, defects can occur only at local potential defects, and because defects effect the connectivity only for the tiles close by, we have,

$$\Delta\mathcal{C} \cap \mathcal{S} \cap \mathcal{D}(Q) \subset \mathcal{P}(Q_{6L}). \tag{9.1}$$

We shall now estimate $P_{\lambda,p}(\Delta\mathcal{C})$. Our first goal is to have an estimate on $P_{\lambda,p}(\Delta\mathcal{C})$ in terms of a random clean configuration with defects and an independent collection of defects added on top of it. (While this is not a precise mathematical statement, we hope it aids the intuition of the reader.) First write,

$$P_{\lambda,p}(\Delta\mathcal{C}) \leq 1 - P_\lambda(\mathcal{S}) + P_{\lambda,p}(\Delta\mathcal{C} \cap \mathcal{S}). \tag{9.2}$$

Now estimate the last summand, using (9.1) and Lemma 7,

$$\begin{aligned} P_{\lambda,p}(\Delta\mathcal{C} \cap \mathcal{S}) &= \sum_Q P_{\lambda,p}\big(\Delta\mathcal{C} \cap \mathcal{S} | \mathcal{D}(Q)\big) P_\lambda\big(D(Q)\big) \\ &\leq \sum_Q P_{\lambda,p}\big(\mathcal{P}(Q_{6L}) | \mathcal{D}(Q)\big) P_\lambda\big(D(Q)\big) \\ &\leq \sum_Q \min\Big\{1, P_{\lambda,p}\big(\mathcal{P}(Q_{6L})|\mathcal{D}(\emptyset)\big) P_\lambda\big(\mathcal{N}(Q_{6L})\big)^{-1}\Big\} P_\lambda\big(D(Q)\big) \\ &\leq 2\sum_Q P_{\lambda,p}\big(\mathcal{P}(Q_{6L})|\mathcal{D}(\emptyset)\big) P_\lambda\big(D(Q)\big) + \\ &\qquad + \sum_{P_\lambda(\mathcal{N}(Q_{6L}))<1/2} P_\lambda\big(D(Q)\big). \end{aligned} \tag{9.3}$$

Our first goal of reducing to the situation where there is a clean configuration with defects added on top can now be considered as accomplished. (This is the meaning of the left summand, which is the more significant one.) We now estimate the left summand.

Let X be a maximal set of points in M with the property that the distance between any two points in X is at least L, and for each $x \in X$ let $S(x)$ denote the sphere of radius $15L$ about x. For each set Q of tiles of $\mathbf{T}_B$, we let $X(Q)$ denote the intersection of X with $\cup Q_L$. It follows that the balls of radius L and centers in $X(Q)$ cover $\cup Q$.

Fix for a moment some $\omega = (\omega_o, \omega_c) \in \hat{\Omega}$ and some Q. Let $W(\omega, Q)$ denote the nuclei of tiles in $\mathbf{T}(\omega, ds)$ that intersect $\cup_{x \in X(Q)} S(x)$. Set

$$\omega_Q^+ = \big(\omega_o \cup W(\omega, Q), \omega_c - W(\omega, Q)\big),$$
$$\omega_Q^- = \big(\omega_o - W(\omega, Q), \omega_c \cup W(\omega, Q)\big).$$

In other words, ω_Q^+ is obtained from ω by opening all the nuclei of tiles which intersect $\cup_{x \in X(Q)} S(x)$, and ω_Q^- is obtained from ω by closing them. Let $\mathcal{K}(Q)$ denote the event that there is a crossing for ω_Q^+, but not for ω_Q^-. Observe that $\mathcal{S} \cap \mathcal{P}(Q_{6L}) \subset \mathcal{K}(Q)$, which gives,

$$P_{\lambda,p}\big(\mathcal{P}(Q_{6L}) | \mathcal{D}(\emptyset)\big) \le P_{\lambda,p}\big(\mathcal{K}(Q) | \mathcal{D}(\emptyset)\big) + 1 - P_\lambda\big(\mathcal{S} | \mathcal{D}(\emptyset)\big).$$

Now, (9.3) implies,

$$P_{\lambda,p}(\Delta\mathcal{C} \cap \mathcal{S}) \le 2 \sum_Q P_{\lambda,p}\big(\mathcal{K}(Q)|\mathcal{D}(\emptyset)\big) P_\lambda\big(D(Q)\big) + 2 - 2P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big) + \\ + \sum_{P_\lambda(\mathcal{N}(Q_{6L})) < 1/2} P_\lambda\big(D(Q)\big). \tag{9.4}$$

We now estimate the sum

$$\sum_Q P_{\lambda,p}\big(\mathcal{K}(Q)|\mathcal{D}(\emptyset)\big) P_\lambda\big(\mathcal{D}(Q)\big). \tag{9.5}$$

Fix some clean $\omega \in \mathcal{D}(\emptyset)$, and let $\omega' \in \Omega$ be arbitrary. Recall that $Z(\omega')$ denotes the set of tiles in $\mathbf{T}_B$ that contain navels of local potential defects of ω'. This means, $P_\lambda\big(Z(\omega') = Q\big) = P_\lambda\big(\mathcal{D}(Q)\big)$. So (9.5) can be written as

$$P\Big(\omega \in \mathcal{K}\big(Z(\omega')\big) \Big| \omega \in \mathcal{D}(\emptyset)\Big), \tag{9.6}$$

where the probability is with respect to the joint distribution of ω and ω'. Set $\tau = \pi\omega$ (recall that this means that τ is the same as ω, except that it is not specified which nuclei of τ are open and which are closed). We may think of ω as a random coloring of τ, and rewrite (9.6) as,

$$E_\tau\Big(P\big(\mathcal{K}\big(Z(\omega')\big)\big) \Big| \tau \in \mathcal{D}(\emptyset)\Big). \tag{9.7}$$

Here the probability is with respect to the coloring of τ and with respect to the choice of ω'. Let us fix τ for a moment, and consider ω' and the coloring of τ as random. On 2^τ, the collection of subsets of τ, let η be the $(p, 1-p)$ product measure. In other words, η is the distribution of ω_o. Let ν be the measure on subsets of τ given by $\nu(A) = P_{\omega'}\big(W\big(\tau, Z(\omega')\big) \in A\big)$; that is, ν is the image of the measure P_λ under the map $\omega' \to W\big(\tau, Z(\omega')\big)$. Note that with the notations of the Insensitivity Lemma 8.1 and its Corollary 8.2, the open nuclei in $\omega^+_{Z(\omega')}$ are distributed according to $\eta \cup \nu$, and the open nuclei in $\omega^-_{Z(\omega')}$ are distributed according to $\eta \cap \nu^c$. So,

$$
\begin{aligned}
P\big(\mathcal{K}\big(Z(\omega')\big)\big) &= P_{\eta\cup\nu}(\mathcal{C}) - P_{\eta\cap\nu^c}(\mathcal{C}) \\
&\le \|\eta\cup\nu - \eta\cap\nu^c\| \\
&\le \|\eta\cup\nu - \eta\| + \|\eta\cap\nu^c - \eta\|.
\end{aligned} \tag{9.8}
$$

Let a be distributed according to the measure $\nu \cap \nu$, and set

$$\beta = \max\{-\log p, -\log(1-p)\}.$$

With the help of the Insensitivity Lemma and its corollary, (9.8) gives the following estimate,

$$P\big(\mathcal{K}\big(Z(\omega')\big)\big) \le 2\sqrt{E_{\nu\cap\nu} e^{\beta|a|} - 1}, \tag{9.9}$$

Let ω'' be another random element of Ω, and set

$$m = m(\tau, \omega', \omega'') = \Big|W\big(\tau, Z(\omega')\big) \cap W\big(\tau, Z(\omega'')\big)\Big|.$$

Since (9.5) is equal to (9.7), the inequality (9.9) allows us to make the following estimate,

$$
\begin{aligned}
&\sum_Q P_{\lambda,p}\big(\mathcal{K}(Q)|\mathcal{D}(\emptyset)\big) P_\lambda\big(\mathcal{D}(Q)\big) \\
&\le E_\tau\left(\min\left\{1, 2\sqrt{E_{\omega',\omega''} e^{\beta m} - 1}\right\}\middle|\, \tau \in \mathcal{D}(\emptyset)\right) \\
&\le 1 - P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big) + \\
&\qquad + 2E_\tau\left(\sqrt{E_{\omega',\omega''} e^{\beta m} - 1}\,\middle|\, \tau \in \mathcal{D}(\emptyset)\cap\mathcal{S}\right) P_\lambda\big(\mathcal{S}|D(\emptyset)\big) \\
&\le 1 - P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big) + 2\sqrt{E_{\tau,\omega',\omega''}\left(e^{\beta m} - 1\,\middle|\, \tau \in \mathcal{D}(\emptyset)\cap\mathcal{S}\right)} \\
&= 1 - P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big) + 2\sqrt{E_{\tau,\omega',\omega''}\big(e^{\beta m}|\tau \in \mathcal{D}(\emptyset)\cap\mathcal{S}\big) - 1}.
\end{aligned} \tag{9.10}
$$

Let F be the (random) set of points $x \in X$ such that $Z(\omega')$ and $Z(\omega'')$ both intersect the ball of radius $50L$ about x, and let n_x be the number of tiles in $\mathbf{T}(\tau, ds)$ that intersect $S(x)$ and have diameter at most L. Set $n = n(F,\tau) = \sum_{x\in F} n_x$. Then for $\tau \in \mathcal{S}$ we have $n \ge m$. Consequently,

$$
\begin{aligned}
&E_{\tau,\omega',\omega''}\big(e^{\beta m}|\tau \in \mathcal{D}(\emptyset)\cap\mathcal{S}\big) \le E_{\tau,\omega',\omega''}\big(e^{\beta n(F,\tau)}|\tau \in \mathcal{D}(\emptyset)\cap\mathcal{S}\big) \\
&\le \frac{E_{\tau,\omega',\omega''}\big(e^{\beta n(F,\tau)}|\tau \in \mathcal{D}(\emptyset)\big)}{P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big)} \\
&= P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big)^{-1} \sum_{K\subset X} P(F = K) E_\tau\big(e^{\beta n(K,\tau)}|\tau \in \mathcal{D}(\emptyset)\big).
\end{aligned} \tag{9.11}
$$

Let $Q_s(K)$ denote the set of tiles in $\mathbf{T}_B$ that are within distance s of K, and note that $n(K,\tau)$ depends only on the intersection of τ with $Q_{20L}(K)$. We use the First Lemma of Clean Configurations (7.1) to estimate the above conditional expectation by an unconditional expectation, and then apply Proposition 6.1, as follows,

$$
\begin{aligned}
E_\tau\big(e^{\beta n(K,\tau)}|\tau \in \mathcal{D}(\emptyset)\big) &\le \frac{E_\tau\big(e^{\beta n(K,\tau)}\big)}{P_\lambda\Big(\mathcal{N}\big(Q_{30L}(K)\big)\Big)} \\
&\le \frac{\exp\big(O(1)|K|L^{d-1}\lambda^{(d-1)/d}\big)}{P_\lambda\Big(\mathcal{N}\big(Q_{30L}(K)\big)\Big)} = \frac{\exp\Big(O(1)|K|(\log\lambda)^{1-d^{-2}}\Big)}{P_\lambda\Big(\mathcal{N}\big(Q_{30L}(K)\big)\Big)} .
\end{aligned}
\tag{9.12}
$$

The number of tiles of $\mathbf{T}_B$ in $Q_{30L}(K)$ is $O(1)|K|\,(\log\lambda)^{O(1)}$. Consequently, by (7.2),

$$
P_\lambda\Big(\mathcal{N}\big(Q_{30L}(K)\big)\Big) \ge \exp\Big(-O\,(1)\,|K|\lambda^{-2/d}(\log\lambda)^{O(1)}\Big) .
$$

Hence, (9.12) may be improved to

$$
E_\tau\big(e^{\beta n(K,\tau)}|\tau \in \mathcal{D}(\emptyset)\big) \le \exp\Big(O(1)|K|(\log\lambda)^{1-d^{-2}}\Big) . \tag{9.13}
$$

In order to get a good estimate for the right hand side of (9.11), we now study the distribution of F. For any $x \in X$, the inequality (7.1) provides the following estimate for the probability that $x \in F$.

$$
P_\lambda(x \in F) \le O\,(1)\,\Big(L^d\lambda^{(d-2)/d}(\log\lambda)^{O(1)}\Big)^2 = \lambda^{-4/d}(\log\lambda)^{O(1)}. \tag{9.14}
$$

Let $X = X_1 \cup \cdots \cup X_N$ be a partition of X into disjoint sets X_j with the property that for each j the distance between any two elements of X_j is at least $150L$. We take N to be bounded by a constant, which depends only on d. This is possible, since there is a bound on the number of points of X in a ball of radius $150L$. Note that if $x, x' \in X_j$, then the events $x \in F$ and $x' \in F$ are independent. Using (9.14) and $|X| = o(1)\lambda$, this gives,

$$
P\big(|F \cap X_j| = k\big) \le \lambda^k \lambda^{-4k/d}(\log\lambda)^{O(1)k} = \lambda^{(d-4)k/d}(\log\lambda)^{O(1)k}.
$$

If $|F| = k$, we must have $k \ge |F \cap X_j| \ge k/N$, for some j. Consequently,

$$
P\big(|F| = k\big) \le N(k+1)\lambda^{(d-4)k/(Nd)}(\log\lambda)^{O(1)k}.
$$

Together with (9.13) and (9.11), this gives,

$$
\begin{aligned}
&E_{\tau,\omega',\omega''}\big(e^{\beta m}|\tau \in \mathcal{D}(\emptyset) \cap \mathcal{S}\big) \\
&\quad \le P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big)^{-1} \sum_{K \subset X} P(F = K) E_\tau\big(e^{\beta n(K,\tau)}|\tau \in \mathcal{D}(\emptyset)\big) \\
&\le P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big)^{-1} + \\
&+ P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big)^{-1} \sum_{k=1}^{\infty} N(k+1)\lambda^{(d-4)k/(Nd)}(\log\lambda)^{O(1)k} \exp\Big(O(1)k(\log\lambda)^{1-d^{-2}}\Big) \\
&= P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big)^{-1}\big(1 + o(1)\big),
\end{aligned}
\tag{9.15}
$$

as $\lambda \to \infty$, because $d < 4$. Recall that Lemma 7 says that

$$
\lim_{\lambda \to \infty} P_\lambda\big(\mathcal{S}|\mathcal{D}(\emptyset)\big) = 1. \tag{9.16}
$$

With (9.10) and (9.15), this gives,

$$\sum_Q P_{\lambda,p}\big(\mathcal{K}(Q)|\mathcal{D}(\emptyset)\big)\,P_\lambda\big(\mathcal{D}(Q)\big) \underset{\lambda\to\infty}{\longrightarrow} 0.$$

From this, (9.16) and (9.4), we get,

$$P_{\lambda,p}(\Delta\mathcal{C}\cap\mathcal{S}) = o(1) + \sum_{P_\lambda\left(\mathcal{N}(Q_{6L})\right)<1/2} P_\lambda\big(D(Q)\big). \tag{9.17}$$

For any given tile in $\mathbf{T}_B$ the probability that it is in $Z(\omega)$ is bounded by $O(1)\lambda^{-2/d}$ $(\log\lambda)^{O(1)}$, by (7.1). Because the total number of tiles in $\mathbf{T}_B$ is $O(\lambda)$ the expected number of tiles in $Z(\omega)$ satisfies,

$$E\Big(|Z(\omega)|\Big) \le O(1)\lambda^{(d-2)/d}\,(\log\lambda)^{O(1)}. \tag{9.18}$$

On the other hand, (7.1) also implies that the number of tiles in Q must be at least $\lambda^{2/d}(\log\lambda)^{-O(1)}$, if $P_\lambda\big(\mathcal{N}(Q_{6L})\big) < 1/2$. This gives the inequality,

$$E\Big(|Z(\omega)|\Big) \ge \lambda^{2/d}(\log\lambda)^{-O(1)} \sum_{P_\lambda\left(\mathcal{N}(Q_{6L})\right)<1/2} P_\lambda\big(D(Q)\big). \tag{9.19}$$

The combination of (9.18) and (9.19) implies,

$$\sum_{P_\lambda\left(\mathcal{N}(Q_{6L})\right)<1/2} P_\lambda\big(D(Q)\big) \le O(1)\lambda^{(d-4)/d}\,(\log\lambda)^{O(1)} \underset{\lambda\to\infty}{\longrightarrow} 0,$$

because $d < 4$. Now from (9.17), (9.2) and Lemma 5.5, it follows that $P_{\lambda,p}(\Delta\mathcal{C}) \underset{\lambda\to\infty}{\longrightarrow} 0$, which completes the proof of the theorem. □

10. The Density Invariance Conjecture

The following conjecture is probably true only in dimension $d = 2$.

10.1. Density invariance conjecture. *Let $\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right)$ be the crossings event, as in Theorem 2.1, let μ be a measure on M, comparable to* vol, *and let $P^\mu_{\lambda,p}$ denote the resulting measure on $\hat{\Omega}$, where we have stressed the dependence on μ. Then the limit crossing probability*

$$PC(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds, \mu, p) = \lim_{\lambda\to\infty} P^\mu_{\lambda,p}\big(\mathcal{C}\left(M, M', \mathfrak{S}_1, \mathfrak{S}_2, ds\right)\big)$$

exists, and does not depend on μ. A similar statement holds for the percolation in homotopy classes of Theorem 2.2.

One may consider a weaker version of the conjecture, where the claim is only that the difference in the probabilities corresponding to two measures μ, μ^* tends to zero as $\lambda \to \infty$, instead of claiming that the limit exists. A stronger version of the conjecture would state that the convergence is uniform in μ, as long as the constant $c > 0$ such that $c^{-1}\,\text{vol} \le \mu \le c\,\text{vol}$ is held fixed. At least in the plane, the numerical evidence below also suggests that the limiting crossing probabilities are the same as for the bond percolation model.

The requirement that μ be comparable to vol is probably stronger than needed. On the other hand, assuming only that its support is M would not be sufficient. Consider the following example. Let $\{A_j\}$ be a sequence of vertical lines whose union is dense in the plane, and let μ_j be the length measure on A_j. Let $\{a_j\}$ be a sequence of positive numbers that tends to zero very fast, and let $\mu = \sum_j a_j \mu_j$. Then it is not hard to see that when $\lambda \to \infty$ the probability for crossing a horizontal rectangle from left to right tends to 1.

Numerical evidence. Following is some numerical evidence which supports the conjecture in the plane. We have tested five different measures $\mu_1, \ldots, \mu_5$. Their densities $f_1(x, y), \ldots, f_5(x, y)$, respectively, all depend only on the x variable, and are given in Table 1. Figure 2 shows a Voronoi tiling for a configuration obtained with the measure μ_4.

Table 1. The densities of the measures tested

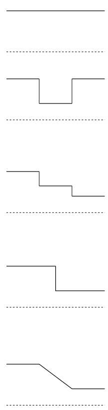
$$f_1(x,y) = 1,$$

$$f_2(x,y) = \begin{cases} 2/5, & 1/3 < x < 2/3, \\ 1, & \text{otherwise}, \end{cases}$$

$$f_3(x,y) = \begin{cases} 1, & x < 1/3, \\ \sqrt{2/5}, & 1/3 \le x \le 2/3, \\ 2/5, & 2/3 < x, \end{cases}$$

$$f_4(x,y) = \begin{cases} 1, & x < 1/2, \\ 2/5, & 1/2 \le x, \end{cases}$$

$$f_5(x,y) = \begin{cases} 1, & x < 1/3, \\ (8-9x)/5, & 1/3 \le x \le 2/3, \\ 2/5, & 2/3 < x. \end{cases}$$



With each of these measures, we ran the following experiment 200 times. Set $R = [a, b] \times [c, d] = [0, 1.2] \times [0, 1]$ and $R' = [a', b'] \times [c', d'] = [.08, 1.12] \times [.08, .92]$. Then the rectangle, R' fits in R with a margin of 0.08. In the rectangle R, 100,000 points were distributed independently, according to the given measure. The Voronoi tiling was then computed. Following that, 1,000 times, random colorings of the resulting tilings were computed, in each coloring the probability for a tile to be open was taken to be $1/2$, independently[1]. Then the algorithm determined the largest $r_0 \in \big[0, (b' - a')/(d' - c')\big]$ such that some connected component of the intersection of the union of open tiles with

[1] Actually, with the objective of saving computing time, the complete coloring was not computed, only the colors of the tiles that the algorithm queried were determined, but the result is the same.

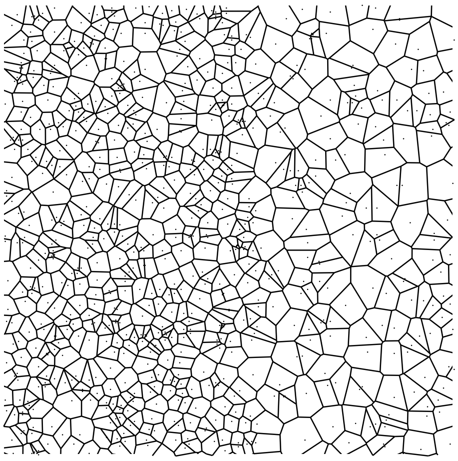


Fig. 2. A Voronoi tiling for a random configuration obtained with $\mu = \mu_4$

the rectangle R' intersects both lines $\{(a', y) : y \in \mathbb{R}\}$, $\Big\{\big(a' + r_0(d' - c'), y\big) : y \in \mathbb{R}\Big\}$. After all these runs, for any r in the left hand column of Table 2, the proportion of the runs for which $r \geq r_0$ was computed, which is a statistical estimate for the probability for left to right crossing of the rectangle $[0, r] \times [0, 1]$. The resulting figures, denoted $P_{\mu_j}(r)$ are listed in Table 2, together with the values obtained from Cardy's formula [6], and the numerical values given in Langlands et. al. [12], for the $\mathbb{Z}^2$ percolation model. We wish to stress that different entries in the column corresponding to any μ_j were obtained using the same trials, and are therefore dependent. On the other hand, entries in different columns may be considered independent.

Note that the largest deviation between an entry in the middle columns to Cardy's value is in the order of 0.005. This is roughly comparable to an error in r that is equal to the typical size of a Voronoi cell in these Voronoi tilings.

Qhull, a program created at the Geometry Center in Minnesota, was used to compute the Voronoi tilings. We wish to thank the authors of qhull, C. Bradford Barber and Hannu Huhdanpaa, and the Geometry Center, for making it available.

Invariance under conformal mappings. Following is the conjecture from Langlands et. al. [12], adapted to the Voronoi model.

Conjecture 10.2. *Let J be a closed topological disk in the plane $\mathbb{R}^2 = \mathbb{C}$, and let $\gamma_1, \gamma_2 \subset \partial J$ be two disjoint arcs. Let $\omega \in \hat{\Omega}$ be a random colored Poisson point process in the plane, with respect to ordinary area measure, with density λ and $p = 1/2$, and consider the resulting Voronoi tiling* **T**. *Let $PC_\lambda(J, \gamma_1, \gamma_2)$ be the probability that there is some path in J that connects γ_1 and γ_2, and is contained in the union of open tiles of* **T**. *Suppose that $f : J \to \mathbb{R}^2$ is a continuous injective mapping, which is conformal in the interior of J. Then*

$$\lim_{\lambda\to\infty} PC_\lambda(J, \gamma_1, \gamma_2) = \lim_{\lambda\to\infty} PC_\lambda\big(f(J), f(\gamma_1), f(\gamma_2)\big).$$

Let's talk about duality in the plane. Observe that the probability that there is some point that belongs to more than 3 Voronoi tiles is zero. A configuration in which 4

Table 2.

r	$P_{\mu_1}(r)$	$P_{\mu_2}(r)$	$P_{\mu_3}(r)$	$P_{\mu_4}(r)$	$P_{\mu_5}(r)$	Lang. et. al.	Cardy's value
.5000	.8214	.8229	.8234	.8229	.8200	–	.8244
.5235	.8037	.8063	.8070	.8046	.8028	.8065	.8070
.5481	.7854	.7883	.7888	.7867	.7847	.7783	.7889
.5779	.7636	.7673	.7673	.7646	.7626	.7666	.7671
.6070	.7428	.7462	.7456	.7438	.7418	.7453	.7459
.6400	.7197	.7220	.7222	.7204	.7188	.7217	.7223
.6667	.7011	.7028	.7036	.7015	.7001	–	.7035
.6721	.6974	.6992	.6998	.6978	.6966	.6994	.6997
.7059	.6742	.6770	.6762	.6747	.6737	.6762	.6765
.7414	.6508	.6532	.6525	.6510	.6498	.6522	.6527
.7500	.6451	.6475	.6469	.6456	.6441	–	.6470
.7753	.6287	.6313	.6303	.6298	.6279	.6301	.6306
.8190	.6011	.6037	.6029	.6022	.6007	.6026	.6030
.8611	.5758	.5782	.5772	.5775	.5755	.5768	.5774
.9048	.5508	.5534	.5516	.5522	.5507	.5516	.5519
.9512	.5254	.5271	.5263	.5271	.5251	.5257	.5260
1.000	.4994	.5010	.5004	.5012	.4997	.4999	.5000
1.051	.4730	.4750	.4750	.4757	.4742	.4743	.4741
1.105	.4475	.4492	.4495	.4506	.4490	.4484	.4482
1.161	.4222	.4238	.4244	.4254	.4238	.4230	.4227
1.221	.3965	.3974	.3989	.3997	.3978	.3974	.3970

Voronoi tiles have a nonempty intersection will be called *degenerate*. It follows that the boundary of any union of tiles of a nondegenerate configuration is a disjoint collection of paths in the plane. In the situation of the conjecture, let $\hat\gamma_1$ and $\hat\gamma_2$ be the two arcs in $\partial J - (\gamma_1 \cup \gamma_2)$, and let $\omega \in \hat\Omega$ be a nondegenerate configuration. Let A be the set of all points in J that are either on γ_1 or may be joined to γ_1 by a path in J contained in open tiles. Then either A intersects γ_2, or there is a boundary component of $A \cap J$ that connects $\hat\gamma_1$ and $\hat\gamma_2$. In the latter case, it follows that there is a path in J from $\hat\gamma_1$ to $\hat\gamma_2$ that is contained entirely in closed tiles. On the other hand, if there is such a path connecting $\hat\gamma_1$ and $\hat\gamma_2$, then there cannot be an open path in J connecting γ_1 and γ_2. We conclude that either there is in J an open crossing from γ_1 to γ_2, or there is a closed crossing from $\hat\gamma_1$ to $\hat\gamma_2$, and these cases are mutually exclusive. Since the probability for an open crossing is the same as the probability for a closed crossing, we get,

$$PC_\lambda(J, \gamma_1, \gamma_2) + PC_\lambda\,(J, \hat\gamma_1, \hat\gamma_2) = 1. \tag{10.1}$$

Proposition 10.3. *Conjecture 10.1 implies Conjecture 10.2.*

The proof uses Theorem 2.1, and monotonicity and continuity properties of crossing probabilities.

If one assumes that Conjecture 10.1 is valid also for intersections of crossing events, then the proof below can be used to show that 10.2 is valid for intersections of crossing events, as discussed in [12].

Proof. Let the situation be as in Conjecture 10.2. Since for any such J there is a continuous injective mapping taking J to the unit disk, which is conformal in J, we assume, without loss of generality, that $f(J)$ is the closed unit disk $\overline{U}$.

We start with a one-sided estimate. Let α_1 be a closed arc on the unit circle which is contained in the relative interior of the arc $f(\gamma_1)$, and let α_2 be a closed arc on the unit circle which is contained in the relative interior of $f(\gamma_2)$. We shall show that

$$\liminf_{\lambda\to\infty} \Big(PC_\lambda(J,\gamma_1,\gamma_2) - PC_\lambda\left(\overline{U},\alpha_1,\alpha_2\right) \Big) \geq 0. \tag{10.2}$$

Let β be an analytic simple closed curve which approximates ∂J, and has the pattern of intersection with ∂J as indicated in Figure 3, and let J' be the closed disk bounded by β. Let $\mathfrak{S}_1$ be a smooth open topological disk in $J' - J$, such that $\partial\mathfrak{S}_1 \cap \beta$ is an arc approximating γ_1, and let $\mathfrak{S}_2$ be a smooth open topological disk in $J' - J$, such that $\partial\mathfrak{S}_2 \cap \beta$ is an arc approximating γ_2. Let g be the Riemann map from J' to the unit disk, and assume that g is normalized so that $g(f^{-1}(0)) = 0$ and the derivative of $g \circ f^{-1}$ at 0 is real. Because J' is an approximation of J, $g^{-1} : \overline{U} \to J'$ is an approximation of $f^{-1} : \overline{U} \to J$. We assume that β has been chosen sufficiently close to ∂J so that the arc $g(\partial\mathfrak{S}_1 \cap \beta)$ contains α_1 in its interior, and the arc $g(\partial\mathfrak{S}_2 \cap \beta)$ contains α_2 in its interior. Hence for some $r > 1$, the images of α_1 and α_2 under the map $z \to r^{-1}z$ are contained in $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively. Fix such an r, and let $G(z) = rg(z)$.

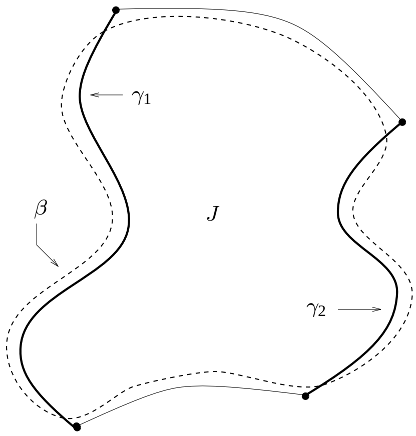


Fig. 3. The approximation β of ∂J

Since β is analytic, g extends to a conformal homeomorphism from a neighborhood W of J to a neighborhood of the closed unit disk. Let $M \supset J$ be a bounded open set whose closure is contained in W. By Theorem 2.1, when λ is large, the probability of $\mathcal{C}\big(M, J', \mathfrak{S}_1, \mathfrak{S}_2, |dz|\big)$ is approximately the same as the probability of $\mathcal{C}\big(M, J', \mathfrak{S}_1, \mathfrak{S}_2, |G'(z)dz|\big)$. By Conjecture 10, we may also change the measure from ordinary volume measure to the measure induced by the map G. But M with metric $|G'(z)dz|$ and measure induced by G is isomorphic to $G(M)$ with ordinary Euclidean metric and measure. Thus, as $\lambda \to \infty$, the probability of $\mathcal{C}\big(M, J', \mathfrak{S}_1, \mathfrak{S}_2, |dz|\big)$ tends to the probability of $\mathcal{C}\big(G(M), G(J'), G(\mathfrak{S}_1), G(\mathfrak{S}_2), |dz|\big)$. When λ is large, we may assume, with high probability, that all tiles near M and near $\overline{U}$ are very small. For such configurations, a crossing from α_1 to α_2 in $\overline{U}$ implies $\omega \in \mathcal{C}\big(G(M), G(J'), G(\mathfrak{S}_1), G(\mathfrak{S}_2), |dz|\big)$, and $\omega \in \mathcal{C}\big(M, J', \mathfrak{S}_1, \mathfrak{S}_2, |dz|\big)$ implies a crossing from γ_1 to γ_2 in J. This proves (10.2).

On the other hand, if α_1' and α_2' are arcs on ∂U which contain $f(\gamma_1)$ and $f(\gamma_2)$ in their interiors, respectively, then

$$\limsup_{\lambda\to\infty} \Big(PC_\lambda(J,\gamma_1,\gamma_2) - PC_\lambda\left(\overline{U},\alpha_1',\alpha_2'\right) \Big) \le 0. \tag{10.3}$$

This can be proved in the same way as (10.2), or deduced from (10.2), using duality. Conjecture 10.2 will follow from (10.2) and (10.3), once we prove that $PC_\lambda(\overline{U},\alpha_1,\alpha_2)$ is continuous in α_1 and α_2, with a modulus of continuity that's independent of λ. Therefore, the next lemma completes the proof. □

10.4 Continuity Lemma. *Let α_1 and α_2' be two disjoint arcs on ∂U, and let $\alpha_2 \subset \alpha_2'$ be an arc which has an endpoint a in common with α_2'. Let b be the other endpoint of α_2, let c be the other endpoint of α_2' and let d be the endpoint of α_1 that is separated in ∂U from a by the relative interior of $\alpha_1 \cup \alpha_2$. Set*

$$\rho = \frac{(a-c)(b-d)}{(a-d)(b-c)},$$

the cross ratio of a,b,c,d. Assuming Conjecture 10.1, for all λ sufficiently large,

$$PC_\lambda\left(\overline{U},\alpha_1,\alpha_2'\right) - PC_\lambda\left(\overline{U},\alpha_1,\alpha_2\right) \le \frac{O(1)}{\sqrt{\rho}}.$$

Proof. Let β be the component of $\partial U - \alpha_1 \cup \alpha_2'$ that has a as an endpoint, and let $h : \overline{U} \to \overline{U}$ be a conformal homeomorphism of the unit disk that takes $\alpha_2' - \alpha_2$ and $\alpha_1 \cup \beta_1$ into arcs of the same length, with centers on the real axis. Set $\gamma_1 = h(\alpha_1)$, $\delta = h(\beta)$, let γ_2 be an arc that is slightly shorter than $h(\alpha_2)$, is contained in $h(\alpha_2)$, and has $h(a)$ as one of its endpoints, and let γ_2' be an arc that is slightly longer than $h(\alpha_2')$, contains $h(\alpha_2')$ and has a as one of its endpoints. Note that there is a conformal automorphism h_1 of U, close to the identity, that takes γ_1 into an arc that contains γ_1 in its interior, and takes γ_2' into an arc that contains $h(\alpha_2')$ in its interior. (Recall that conformal automorphisms of U are determined by the images of three points on ∂U. One only needs to appropriately choose the images of the endpoints of γ_1 and the endpoint of γ_2' distinct from a.) By (10.3) with $\overline{U}$ replacing J, $h_1 \circ h$ replacing f, and arcs appropriately chosen, we have

$$PC_\lambda\left(\overline{U},\alpha_1,\alpha_2'\right) \le PC_\lambda\left(\overline{U},\gamma_1,\gamma_2'\right) + o(1), \tag{10.4}$$

as $\lambda \to \infty$. Similarly, by (10.3),

$$PC_\lambda\left(\overline{U},\alpha_1,\alpha_2\right) \ge PC_\lambda\left(\overline{U},\gamma_1,\gamma_2\right) + o(1), \tag{10.5}$$

Let $\mathcal{A} \subset \hat{\Omega}$ be the event that there is a crossing in $\overline{U}$ from γ_1 to γ_2' in open tiles, but there isn't such a crossing from γ_1 to γ_2, and consider some nondegenerate configuration $\omega \in \mathcal{A}$. There must be an open crossing from $\gamma_2' - \gamma_2$ to γ_1. Because γ_2 does not connect in open tiles to this crossing, by duality, there must be a crossing in closed tiles from $\gamma_2' - \gamma_2$ to $\gamma_1 \cup \delta$. Let $\mathcal{B}$ be the event that there is an open crossing from $\gamma_2' - \gamma_2$ to $\gamma_1 \cup \delta$, and there is also a closed crossing between these arcs. Then,

$$PC_\lambda\left(\overline{U},\gamma_1,\gamma_2'\right) - PC_\lambda\left(\overline{U},\gamma_1,\gamma_2\right) = P_\lambda(\mathcal{A}) \le P_\lambda(\mathcal{B}). \tag{10.6}$$

Let n be the largest integer such that the length of the arc $\gamma_1 \cup \delta = h(\alpha_1 \cup \beta)$ is less than π/n. Since the cross ratio is invariant under conformal automorphisms of U, it is easy to verify, using the definition of ρ, that

$$\rho = O\left(n^2\right). \tag{10.7}$$

Recall that by the choice of h, the arc $h(\alpha_2' - \alpha_2)$ has the same length as $\gamma_1 \cup \delta$. We also assume, with no loss of generality, that the length of $\gamma_2' - \gamma_2$ is less than π/n. For any integer k, let $\mathcal{B}_k$ be the rotation of $\mathcal{B}$ by $k\pi/n$; that is, the set of all $\omega \in \hat{\Omega}$ such that the rotation of ω about 0 by $k\pi/n$ is in $\mathcal{B}$. Observe that if $\omega \in \mathcal{B}_j$ is nondegenerate and k is not divisible by n, then $\omega \notin \mathcal{B}_{j+k}$, because any crossing from $\gamma_2' - \gamma_2$ to $\gamma_1 \cup \delta$ separates the rotation by $k\pi/n$ of $\gamma_2' - \gamma_2$ and the rotation by $k\pi/n$ of $\gamma_1 \cup \delta$.

The events $\mathcal{B}_j$, $j = 0, \ldots, n-1$ are n events with the same probability, and the intersection of any two of them has zero probability. Therefore,

$$P_\lambda(\mathcal{B}) \le 1/n. \tag{10.8}$$

From (10.4), (10.5), (10.6), and (10.8), we get

$$PC_\lambda\left(\overline{U}, \alpha_1, \alpha_2'\right) - PC_\lambda\left(\overline{U}, \alpha_1, \alpha_2\right) \le 1/n + o(1). \tag{10.9}$$

Therefore, (10.7) completes the proof. □

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