The supercritical phase of percolation is well behaved

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We prove a general result concerning the critical probabilities of subsets of a lattice $\mathscr{L}$. It is a consequence of this result that the critical probability of a percolation process on $\mathscr{L}$ equals the limit of the critical probability of a slice of $\mathscr{L}$ as the thickness of the slice tends to infinity. This verification of one of the standard hypotheses of the subject settles many questions concerning supercritical percolation.

1. Introduction

Let d be a positive integer, let $0 < p < 1$, and declare each vertex of the lattice $\mathbb{Z}^d$ to be *open* with probability p and *closed* otherwise, independently of all other vertices. Write C for the set of vertices connected to the origin by paths of open vertices; that is, C is the (open) cluster containing the origin, and let

$$\theta(p) = P_p(|C| = \infty),$$

where P_p is the relevant probability measure on subsets of $\mathbb{Z}^d$.

It is well known that if $d > 1$ then there exists a critical value p_c of p satisfying $0 < p_c < 1$ and

$$\theta(p) = 0 \quad \text{if} \quad p < p_c,$$
$$\theta(p) > 0 \quad \text{if} \quad p > p_c.$$

Let A be an infinite connected subset of $\mathbb{Z}^d$; it is also well known (see Grimmett 1989, p. 120) that there is a critical value $p_c(A)$ such that there exists a.s. an infinite open cluster in A if $p > p_c(A)$ and a.s. no such cluster if $p < p_c(A)$, where a.s. denotes 'almost surely with respect to the measure P_p'. In particular $p_c(\mathbb{Z}^d) = p_c$. It is of special interest when A takes one of the following two forms:

(*a*) a 'slice with thickness k' given by

$$S(k) = \{x \in \mathbb{Z}^d : 0 \leqslant x_j \leqslant k, j > 2\}, \tag{1.1}$$

where $x = (x_1, x_2, \ldots, x_d)$ and k is a positive integer;

(*b*) a 'half-space' given by

$$\mathbb{H} = \{x \in \mathbb{Z}^d : x_d \geqslant 0\}. \tag{1.2}$$

Now $S(k) \subseteq \mathbb{H} \subseteq \mathbb{Z}^d$ for all d, k, and it easily follows that

$$p_c \leqslant p_c(\mathbb{H}) \leqslant p_c(S(k)). \tag{1.3}$$

Moreover for a similar reason $p_c(S(k))$ is a decreasing function of k, so we can write

$$p_c(S) = \lim_{k\to\infty} p_c(S(k)). \tag{1.4}$$

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The problem of proving that the mean cluster size is finite throughout the subcritical phase (when $p < p_c$) was settled by Menshikov (1986) and Aizenman & Barsky (1987). As a consequence of their work, the subcritical régime is now well understood. The supercritical régime (when $p > p_c$) has remained something of a challenge for $d \geqslant 3$. Although many results of interest have been established for $p > p_c(S)$, few have been obtained under the hypothesis that $p > p_c$. However, the corresponding full results will follow by an application of the following theorem, which is one of the main results of this paper.

Theorem. *If* $d \geqslant 3$, *then* $p_c = p_c(S)$.

By (1.3) and (1.4) it follows at once that $p_c = p_c(\mathbb{H})$. In fact we shall deduce this theorem from the more general Theorem A which is stated and proved in §4. Finally in §5 we shall outline some of the consequences.

Our result that $p_c = p_c(S)$ extends a recent result of Barsky *et al.* (1990*a*, *b*) who have proved the weaker equality $p_c(\mathbb{H}) = p_c(S)$. On the other hand Barsky *et al.* have also shown that $\theta_{\mathbb{H}}(p_c(\mathbb{H})) = 0$, where $\theta_{\mathbb{H}}(p)$ is the P_p-probability that the origin belongs to an infinite open cluster in $\mathbb{H}$. Although our method is based in part on ideas in Barsky *et al.* we are unable to settle the question as to whether or not $\theta(p_c) = 0$. Our difficulty lies in our use of 'sprinkling'; that is, the technique of adding a small density of extra open vertices to create open paths. We shall show that, if $\theta(p) > 0$ and $\eta > 0$, then $p+\eta > p_c(S(k))$ for some $k = k(\eta)$.

In earlier work, Kesten (unpublished work, 1988) has proved by quite different techniques that $p_c(S)$ is equal to the limit as $k \to \infty$ of the critical probability of the slab $\mathbb{Z}^{d-1} \times \{1, 2, \ldots, k\}$.

We shall work throughout this paper with percolation as briefly described in the first paragraph of this section; that is, site percolation on the d-dimensional hypercubic lattice $\mathbb{Z}^d$. However, our methods and conclusions are valid for a large category of processes including many bond and site processes on crystalline lattices in dimensions $d \geqslant 3$. See Grimmett (1989) for a general account of the mathematical theory of percolation.

2. Definitions and notation

This section contains the principal definitions and notation used in the paper, and the reader can use it subsequently for reference. For special purposes some further definitions and notation will be given in the later sections. On the other hand for the expressions $\theta(p)$, P_p, p_c, $p_c(A)$, $S(k)$, $\mathbb{H}$, S, the reader is referred back to the first part of §1.

Writing $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, then for each $x, y \in \mathbb{Z}^d$ the *distance* from x to y is given by

$$|x-y| = \max_j |x_j - y_j|,$$

where $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$. If $\sum_{i=1}^d |x_i - y_i| = 1$, $\{x, y\}$ is an *edge*; we then write $x \sim y$ and say that x is *adjacent* to y. We denote by $\mathbb{E}^d$ the set of all edges in $\mathbb{Z}^d$ and by $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ the corresponding 'hypercubic' lattice, although we normally identify $\mathbb{L}^d$ with $\mathbb{Z}^d$. The elements $x \in \mathbb{Z}^d$ are called *vertices*. In particular, for $j = 1, 2, \ldots, d$ the *unit vertices* are given by

$$i_j = (\delta_1^{(j)}, \delta_2^{(j)}, \ldots, \delta_d^{(j)}),$$

where $\delta_r^{(j)} = 1$ if $r = j$ and $\delta_r^{(j)} = 0$ otherwise.

For $A \subseteq \mathbb{Z}^d$, the *internal boundary* of A is given by

$$\partial A = \{x \in A : \exists y \in \mathbb{Z}^d - A \text{ such that } x \sim y\},$$

and the *external boundary* is given by

$$\Delta A = \{x \in \mathbb{Z}^d - A : \exists y \in A \text{ such that } x \sim y\} = \partial(\mathbb{Z}^d - A).$$

Let $0 < p < 1$ and declare each vertex of $\mathbb{Z}^d$ to be *open* with probability p and *closed* otherwise independently of all other vertices in the lattice. We identify the set of realizations (or configurations) with

$$\Omega = \{0, 1\}^{\mathbb{Z}^d},$$

where

$$\omega = (\omega(x) : x \in \mathbb{Z}^d) \in \Omega$$

represents the realization in which x is open if $\omega(x) = 1$ and closed if $\omega(x) = 0$. With $\mathscr{F}$ the usual σ-field we denote by P_p the relevant probability measure on $(\Omega, \mathscr{F})$.

Let $(f(x) : x \in \mathbb{Z}^d)$ be a family of independent random variables having the uniform distribution on $[0, 1]$, and write

$$\Omega^* = [0, 1]^{\mathbb{Z}^d}.$$

With $\mathscr{F}^*$ the usual σ-field of Ω^*, we denote by P the relevant product probability measure on $(\Omega^*, \mathscr{F}^*)$. For each $\rho \in [0, 1]$ and $x \in \mathbb{Z}^d$ we say that x is *ρ-open* if $f(x) < \rho$ and *ρ-closed* if $f(x) \geqslant \rho$. Thus $P(x \text{ is } \rho\text{-open}) = \rho$ and $P(x \text{ is } \rho\text{-closed}) = 1 - \rho$. Later we shall use the measure space $(\Omega^*, \mathscr{F}^*, P)$ in order to realize different percolation processes on the same sample space. When $\rho = p$, we shall identify 'ρ-open' with 'open' and 'ρ-closed' with 'closed'. We say that a set $A \subseteq \mathbb{Z}^d$ is *open* if x is open for all $x \in A$, and similarly we define *closed*, *ρ-open*, *ρ-closed* subsets of $\mathbb{Z}^d$.

A *path* in $\mathbb{Z}^d$ is a finite sequence $x(0), x(1), \ldots, x(n)$, or an infinite sequence $x(0), x(1), \ldots$, of distinct vertices in $\mathbb{Z}^d$ such that $x(i) \sim x(i+1)$ for all relevant i. A set $A \subseteq \mathbb{Z}^d$ is *connected* if each pair $x, y \in A$ belong to some path contained in A. We call a path *open* if all its vertices are open. We write '$A \leftrightarrow B$' if there exist $a \in A$ and $b \in B$ such that a and b belong to some open path, and write '$A \leftrightarrow B$ in D' if there exists such an open path contained in D. We write '$A \leftrightarrow \infty$' if there is an infinite open path containing some $a \in A$. The *open cluster* at the vertex y is given by

$$C(y) = \{x \in \mathbb{Z}^d : x \leftrightarrow y\}.$$

We set $C(y) = \varnothing$ if y is closed, and we recall that $C(0) = C$, where 0 denotes the origin.

For each positive integer n, the *box* $B(n)$ is the set

$$B(n) = \{x \in \mathbb{Z}^d : |x| \leqslant n\},$$

and we write

$$T(n) = \{x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d : x_1 = n,\ 0 \leqslant x_j \leqslant n \text{ for } j = 2, 3, \ldots, d\},$$

a special subset of a face of $B(n)$. For positive integers m, n we write

$$T(m, n) = \bigcup_{j=1}^{2m+1} \{j i_1 + T(n)\}.$$

For each $x \in \mathbb{Z}^d$, we call an open box $x + B(m)$ a (*launching*) *m-pad*, and if $2m < n$, we write

$$K(m, n) = \{x \in T(n) : x + i_1 \text{ belongs to some } m\text{-pad in } T(m, n)\}.$$

Thus we may regard $K(m,n)$ as the subset of $T(n)$ 'adjacent' to the union of all m-pads which 'rest' on $T(n)$.

3. The lemmas

In this section we state and prove six lemmas. Lemma 1 is really based on a certain technique, which we describe beforehand. Lemmas 2, 3, 4 and 5 are preliminary to Lemma 6 which together with Lemma 1 we shall need in §4.

The proof of our theorem in §4 makes use of a 'block' argument, in which we shall consider a 'lattice' of large boxes in $\mathbb{Z}^d$ and define any one of these to be open if there exist certain long open paths within and near it. We aim to show that with positive probability there exists an infinite 'path' of open boxes, each one 'adjacent' to the next. Thus that part of our proof will use a renormalization of the original site percolation on $\mathbb{Z}^d$; that is, percolation on $k\mathbb{Z}^d$ for some large integer k, in fact more precisely on a connected subset of $k\mathbb{Z}^d$. In the meantime, leading up to and including Lemma 1, we regard percolation on a connected set $F \subseteq \mathbb{Z}^d$ from the standpoint of a cluster generated from a single vertex, say the origin, by a certain stochastic process. When we eventually apply this in §4, however, this $\mathbb{Z}^d$ will be renormalized to $k\mathbb{Z}^d$ for a suitable k. Of special importance to us will be the case when $F = \mathbb{Z}^2$, regarded as a subset of $\mathbb{Z}^d$.

Let F be an infinite connected subset of $\mathbb{Z}^d$ containing the origin. Following a standard procedure, we let $e(1), e(2), \dots$ be a fixed ordering of the edges of the graph induced by F. Let $g: \mathbb{Z}^d \to \{0,1\}$ be given, where for each $x \in \mathbb{Z}^d$, x is open if $g(x) = 1$ and closed if $g(x) = 0$. Then there arises a sequence (S_n) of ordered pairs of subsets of $\mathbb{Z}^d$ defined inductively:

$$S_0 = (\varnothing, \varnothing),$$

$$S_1 = \begin{cases} (\{x_1\}, \varnothing) & \text{if} \quad g(x_1) = 1, \\ (\varnothing, \{x_1\}) & \text{if} \quad g(x_1) = 0, \end{cases}$$

where $x_1 = 0$, the origin.

Having defined $S_r = (A_r, B_r)$ for $r = 0, 1, \dots, t$, we then define S_{t+1} as follows.

Case 1. If possible let e_t denote the earliest edge in the fixed ordering with one endvertex belonging to A_t and the other endvertex, x_{t+1} say, belonging to $(A_t \cup B_t)^c$. We then write

$$S_{t+1} = \begin{cases} (A_t \cup \{x_{t+1}\}, B_t) & \text{if} \quad g(x_{t+1}) = 1, \\ (A_t, B_t \cup \{x_{t+1}\}) & \text{if} \quad g(x_{t+1}) = 0. \end{cases}$$

Case 2. If x_{t+1} under Case 1 does not exist then we set $S_{t+1} = S_t$.

In the limit we define $S = (A, B)$ where $A = \bigcup_{t=1}^{\infty} A_t$ and $B = \bigcup_{t=1}^{\infty} B_t$.

Thus we examine the vertices of F one by one, and it is not difficult to see that, if the origin 0 is open, we eventually build up the open cluster A containing 0 and contained in F. We note that B is the external boundary of A in F.

The above routine results in an infinite sequence $\mathscr{S} = (S_0, S_1, \dots)$, and in doing so it examines the values $g(x_1), g(x_2), \dots$ in turn. The same routine may be used if the $g(x)$ are random variables taking values in $\{0,1\}$, and the outcome $\mathscr{S}$ is then a random sequence. Let us write, for $t = 0, 1, \dots,$

$$\rho(\mathscr{S}, t) = \begin{cases} \mu(g(x_{t+1}) = 1 \mid S_0, S_1, \dots, S_t) & \text{if } x_{t+1} \text{ exists}, \\ 1 & \text{otherwise}, \end{cases} \tag{3.1}$$

where μ is the probability measure associated with the sequence of random variables $g(x_1), g(x_2), \ldots$.

Lemma 1. *Using the notation introduced above, suppose that there exists γ such that $\gamma > p_c(F)$ and*

$$\rho(\mathscr{S}, t) \geqslant \gamma \quad \textit{for all } \mathscr{S} \textit{ and } t. \tag{3.2}$$

Then

$$\mu(|A| = \infty) > 0. \tag{3.3}$$

Proof. Let $(f(x): x \in \mathbb{Z}^d)$ be independent random variables with the uniform distribution on $[0, 1]$, and consider the following cluster-growth process $\mathscr{J}$ with associated product measure P as defined in §2. Declare the origin of $\mathbb{Z}^d$ to be *green* if $f(0) \leqslant \mu(g(0) = 1)$, and *red* otherwise. If the origin is red, we terminate the process. If the origin is green, find the earliest edge e_1 incident with the origin, and declare its other endvertex x_2 green if $f(x_2) \leqslant \mu(g(x_2) = 1 \mid S_1 = \sigma_1)$, where $\sigma_1 = (A_1, B_1)$, and $A_1 = \{0\}$, $B_1 = \varnothing$. We iterate this process in the step by step manner described before Lemma 1. The general step is to declare x_{t+1} (if it exists) to be green if

$$f(x_{t+1}) \leqslant \mu(g(x_{t+1}) = 1 \mid S_i = \sigma_i \text{ for } 0 \leqslant i \leqslant t),$$

and red otherwise, where $\sigma_i = (A_i, B_i)$ is the vector comprising the set A_i of green vertices and the set B_i of red vertices after the consideration of x_i.

Let A be the (limiting) cluster of green vertices, with external boundary B in F. It follows from (3.2) that every vertex in B is γ-closed. The cluster C_γ of γ-open vertices at the origin cannot intersect B, and hence C_γ is a subset of A. On the other hand $\gamma > p_c(F)$ so that there is strictly positive probability that C_γ is infinite, and therefore $P(|C_\gamma| = \infty) > 0$. It is clear that the process $\mathscr{J}$ has the same distribution as the process $\mathscr{S}$, and (3.3) follows as required. □

The next lemma is based on the idea that if two sets are joined to one another by an open path with large probability, then they are likely to be joined by many such paths. It is related to the 'sprinkling' lemma of Aizenman *et al.* (1983), but it does not use sprinkling.

Lemma 2. *Let $R, K \subseteq B \subseteq \mathbb{Z}^d$ with $(R \cup \Delta R) \cap K = \varnothing$ and let*

$$U = \{x \in \Delta R \cap B : \exists y \in B \textit{ such that } x \sim y \textit{ and } y \leftrightarrow K \textit{ in } B - (R \cup \Delta R)\}.$$

Then for each positive integer t,

$$P_p(|U| \leqslant t) \leqslant (1-p)^{-t} P_p(\Delta R \nleftrightarrow K \textit{ in } B - R).$$

Proof.

$$\begin{aligned} P_p(\Delta R \nleftrightarrow K \text{ in } B - R) &= P_p(U \text{ is closed (or empty)}) \\ &\geqslant P_p(|U| \leqslant t,\ U \text{ is closed}) \\ &= \sum_{\substack{A \subseteq \Delta R \cap B \\ |A| \leqslant t}} P_p(U \text{ is closed} \mid U = A)\, P_p(U = A) \\ &= \sum_{\substack{A \subseteq \Delta R \cap B \\ |A| \leqslant t}} (1-p)^{|A|} P_p(U = A), \end{aligned}$$

since the event $\{U = A\}$ is independent of the states of the vertices in A. Hence

$$P_p(\Delta R \nleftrightarrow K \text{ in } B - R) \geqslant (1-p)^t P_p(|U| \leqslant t),$$

as required. □

Now it is obvious that if for a given m we have $B(m) \leftrightarrow \infty$, then for all $n > m$ there exists $x \in \partial B(n)$ such that $B(m) \leftrightarrow x$ in $B(n)$. It follows from our next lemma that, roughly, in this case for large n there will probably exist many x with this property.

Lemma 3. *For any positive integers k and m, we have*

$$\sum_{n=m}^{\infty} P_p(|U_n| < k, B(m) \leftrightarrow \infty) < (1-p)^{-dk}, \tag{3.4}$$

where
$$U_n = \{x \in \partial B(n) : B(m) \leftrightarrow x \text{ in } B(n)\}. \tag{3.5}$$

Proof. For $n \geqslant m$, we have

$$\begin{aligned}(1-p)^{dk} P_p(|U_n| < k, B(m) \leftrightarrow \infty) &\leqslant (1-p)^{dk} P_p(1 \leqslant |U_n| < k) \\ &< P_p(U_{n+1} = \varnothing \mid 1 \leqslant |U_n| < k)\, P_p(1 \leqslant |U_n| < k) \\ &= P_p(U_{n+1} = \varnothing, 1 \leqslant |U_n| < k) = P_p(E_n),\end{aligned}$$

say, where the events E_n are disjoint, and the lemma follows by summation. □

We deduce the next lemma from Lemma 3 by using a certain idea of symmetry together with the FKG inequality (Harris 1960).

Lemma 4. *For any positive integers l and m, we have*

$$\liminf_{n \to \infty} P_p(|V(n)| \geqslant l) \geqslant 1 - P_p(B(m) \nleftrightarrow \infty)^{1/w} \tag{3.6}$$

where $w = d2^d$ and
$$V(n) = \{x \in T(n) : B(m) \leftrightarrow x \text{ in } B(n)\}. \tag{3.7}$$

Proof. By the definition of $T(n)$ in §2, we note that there exists a group of symmetries of the cube, of order $w = (2d)\,2^{d-1} = d2^d$, and with the following property: if the elements of the group transform $T(n)$ respectively into $T_1(n)$, $T_2(n), \ldots, T_w(n)$, then

$$\bigcup_{i=1}^{w} T_i(n) = \partial B(n).$$

It follows from the definition of U_n in (3.5) that

$$\{|U_n| < wl\} \supseteq \bigcap_{i=1}^{w} \{|V_i(n)| < l\},$$

where
$$V_i(n) = \{x \in T_i(n) : B(m) \leftrightarrow x \text{ in } B(n)\}$$

for $i = 1, 2, \ldots, w$. This intersection is of decreasing events, whence by the FKG inequality

$$\begin{aligned}P_p(|U_n| < wl) &\geqslant \prod_{i=1}^{w} P_p(|V_i(n)| < l) \\ &= P_p(|V(n)| < l)^w.\end{aligned}$$

It follows that
$$P_p(|V(n)| \geqslant l) \geqslant 1 - P_p(|U_n| < wl)^{1/w}.$$

However, writing $k = wl$ in (3.4) of Lemma 3, the series on the left is convergent, and hence

$$\begin{aligned}P_p(|U_n| < wl) &\leqslant P_p(|U_n| < wl, B(m) \leftrightarrow \infty) + P_p(B(m) \nleftrightarrow \infty) \\ &\to P_p(B(m) \nleftrightarrow \infty) \quad \text{as} \quad n \to \infty,\end{aligned}$$

by Lemma 3. □

We shall now deduce from Lemma 4 that, if $\theta(p) > 0$, then, for sufficiently large m, n, it is very likely that $B(m)$ can be joined to some m-pad 'resting' on $T(n)$; the reader is referred back to §2 for the relevant definitions.

Lemma 5. *If $\theta(p) > 0$, then for each $\eta > 0$ there exist positive integers $m = m(d,p,\eta)$ and $n = n(d,p,\eta)$ such that $2m < n$ and*

$$P_p(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) > 1-\eta.$$

Proof. Since $\theta(p) > 0$, there exists $m = m(d,p,\eta)$ such that

$$P_p(B(m) \leftrightarrow \infty) > 1-(\tfrac{1}{2}\eta)^w, \tag{3.8}$$

where $w = d2^d$.

Then in turn there exists $M = M(d,p,\eta)$ such that

$$\mu\left(\bigcup_{i=1}^{M} \mathscr{E}_i\right) > 1-\tfrac{1}{2}\eta, \tag{3.9}$$

where $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_M$ are independent events in some probability space, each having μ-probability given by

$$\mu(\mathscr{E}_i) = P_p(B(m) \text{ is open}), \quad i = 1,2,\ldots,M. \tag{3.10}$$

In fact it is sufficient that

$$(1-p^{(2m+1)^d})^M < \tfrac{1}{2}\eta,$$

but we do not require the precise condition.

Writing

$$l = (2m+1)^{d-1}M, \tag{3.11}$$

by Lemma 4 and by (3.8) there exists $n = n(d,p,\eta)$ such that

$$P_p(|V(n)| \geqslant l) > 1-\tfrac{1}{2}\eta, \tag{3.12}$$

where $V(n)$ is given by (3.7).

Supposing, as we may, that $2m+1$ divides $n+1$, we can partition $T(n)$ into disjoint $(d-1)$-dimensional boxes with sides of length $2m$. Then by (3.7) and (3.11), $|V(n)| \geqslant l$ implies that $B(m)$ is joined in $B(n)$ to at least M of these boxes, and by (3.9) and (3.10), with probability greater than $1-\frac{1}{2}\eta$, at least one of these boxes corresponds to a 'parallel' m-pad in $T(m,n)$. The lemma now follows by (3.12). □

In the next lemma and in §4 to follow, we use the probability measure P on Ω^* and the notions of 'ρ-open' and 'ρ-closed', as defined in §2.

Lemma 6. *If $\theta(p) > 0$, then for each $\epsilon, \delta > 0$ there exist positive integers $m = m(d,p,\epsilon,\delta)$ and $n = n(d,p,\epsilon,\delta)$ such that $2m < n$ and with the following property: for each set R such that $B(m) \subseteq R \subseteq B(n)$ and $(R \cup \Delta R) \cap T(n) = \varnothing$, and each function $\beta: \Delta R \cap B(n) \to [0, 1-\delta]$ we have*

$$P(G \mid H) > 1-\epsilon, \tag{3.13}$$

where

$$G = \{\text{there exists a path in } B(n)-R \text{ from } \Delta R \cap B(n) \text{ to } K(m,n) \text{, this path being } p\text{-open outside } \Delta R \cap B(n) \text{ and } (\beta(u)+\delta)\text{-open at its only vertex } u \in \Delta R \cap B(n)\} \tag{3.14}$$

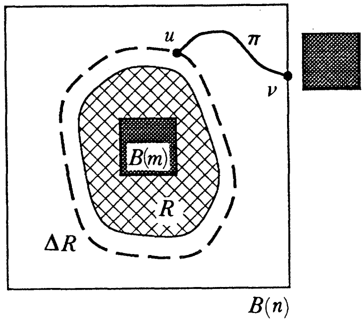

Figure 1. A path π in $B(n)-R$. The endpoint u of π lies in ΔR, and the endpoint v belongs to $\partial B(n)$ and is adjacent to an m-pad lying next to $\partial B(n)$.

and

$$H = \{x \text{ is } \beta(x)\text{-closed for all } x \in \Delta R \cap B(n)\}. \tag{3.15}$$

The event G of (3.14) is illustrated in figure 1.

In the proof of our theorem in §4, we shall show that with positive probability we can construct a suitable sequence of m-pads $B_1, B_2, \ldots$, all connected to B_1 by open paths. Essentially this is carried out step by step by a sequence of applications of Lemma 5, but we cannot use Lemma 5 directly because certain relevant events are not independent.

The purpose of Lemma 6 above is to overcome this difficulty, and enable us to extend an open connected set C' to a larger open connected set C'' containing the next m-pad. After translation of coordinates in Lemma 6 we can think of R as $C' \cap B(n)$. Then to extend C' in $B(n)$ we have to cross over ΔR, each of whose vertices x is already determined as $\beta(x)$-closed. However, there is a good chance that some of these x are $(\beta(x)+\delta)$-open, and this is part of the idea of the proof which follows.

Proof of Lemma 6. Assume that $\theta(p) > 0$ and let $\epsilon, \delta > 0$. Choose a positive integer t so large that

$$(1-\delta)^t < \tfrac{1}{2}\epsilon, \tag{3.16}$$

and then choose a positive number η so that

$$\eta < \tfrac{1}{2}\epsilon(1-p)^t. \tag{3.17}$$

Then we can, by Lemma 5, choose positive integers $m = m(d,p,\epsilon,\delta)$ and $n = n(d,p,\epsilon,\delta)$ such that $2m < n$ and

$$P_p(B(m) \leftrightarrow K(m,n) \text{ in } B(n)) > 1-\eta. \tag{3.18}$$

Suppose that $B(m) \subseteq R \subseteq B(n)$, $(R \cup \Delta R) \cap T(n) = \varnothing$ and $\beta: \Delta R \cap B(n) \to [0, 1-\delta]$. Since $K(m,n) \subseteq T(n)$, it is easy to see that

$$\{B(m) \leftrightarrow K(m,n) \text{ in } B(n)\} \subseteq \{\Delta R \leftrightarrow K(m,n) \text{ in } B(n)-R\},$$

whence by (3.18),

$$P_p(\Delta R \leftrightarrow K(m,n) \text{ in } B(n)-R) > 1-\eta. \tag{3.19}$$

For each $K \subseteq T(n)$, writing

$$U(K) = \{x \in \Delta R \cap B(n) : \exists y \in B(n) \text{ such that } x \sim y \text{ and } y \leftrightarrow K \text{ in } B(n)-(R \cup \Delta R)\}, \tag{3.20}$$

the events $\{|U(K)| \leqslant t\}$ and $\{K(m,n) = K\}$ are independent. It follows by Lemma 2 that

$$\begin{aligned} P_p(|U(K(m,n))| \leqslant t) &= \sum_{K \subseteq T(n)} P_p(|U(K)| \leqslant t)\, P_p(K(m,n) = K) \\ &\leqslant \sum_{K \subseteq T(n)} (1-p)^{-t} P_p(\Delta R \nleftrightarrow K \text{ in } B(n) - R)\, P_p(K(m,n) = K) \\ &= (1-p)^{-t} P_p(\Delta R \nleftrightarrow K(m,n) \text{ in } B(n) - R), \end{aligned}$$

again by the independence of the relevant events. We now have, by (3.19),

$$P_p(|U(K(m,n))| > t) \geqslant 1 - (1-p)^{-t}\eta. \tag{3.21}$$

For each $U \subseteq \Delta R \cap B(n)$, the event $\{U(K(m,n)) = U\}$ is independent of the states of the vertices in U, and these states are independent of each other. Consequently, writing P^H for the probability measure P conditioned on the event H given in (3.15), we have

$$\begin{aligned} &P^H(x \text{ is } (\beta(x)+\delta)\text{-closed for all } x \in U(K(m,n)), |U(K(m,n))| > t) \\ &\quad = \sum_{\substack{U \subseteq \Delta R \cap B(n) \\ |U| > t}} P^H(x \text{ is } (\beta(x)+\delta)\text{-closed for all } x \in U, U(K(m,n)) = U) \\ &\quad \leqslant \sum_{\substack{U \subseteq \Delta R \cap B(n) \\ |U| > t}} (1-\delta)^{|U|} P^H(U(K(m,n)) = U) \leqslant (1-\delta)^t. \end{aligned}$$

It follows by (3.16), (3.17) and (3.21) that

$$\begin{aligned} &P^H(\text{there exists a } (\beta(u)+\delta)\text{-open vertex } u \in U(K(m,n))) \\ &\quad \geqslant P^H(|U(K(m,n))| > t) - (1-\delta)^t \\ &\quad = P_p(|U(K(m,n))| > t) - (1-\delta)^t \\ &\quad \geqslant 1 - (1-p)^{-t}\eta - (1-\delta)^t \geqslant 1 - \epsilon. \end{aligned}$$

Thus $P^H(G) > 1-\epsilon$, which is (3.13) as required. □

4. The main theorem

Theorem A. *If F is an infinite connected subset of $\mathbb{Z}^d$, and $p_c(F) < 1$, then for each $\eta > 0$ there exists an integer $k > 0$ such that*

$$p_c(2kF + B(k)) \leqslant p_c(\mathbb{Z}^d) + \eta.$$

Our theorem of §1 follows at once by the choice $F = \mathbb{Z}^2$, for in this case $2kF + B(k)$ is a translation of the slice $S(2k)$ having thickness $2k$. Other consequences will be discussed in §5.

Proof. We shall work with the family $(f(x): x \in \mathbb{Z}^d)$ of independent random variables having the uniform distribution on $[0,1]$ and with the associated ideas as discussed in §2. We shall show that, if $\eta > 0$ and $p = p_c + \frac{1}{2}\eta$ (where $p_c = p_c(\mathbb{Z}^d)$), then there is a strictly positive probability that there exists an infinite $(p+\eta)$-open cluster in $2kF + B(k)$, for an appropriate choice of k. This we achieve by describing a procedure for building such a cluster, and appealing to Lemma 1. We shall verify the principal hypothesis of Lemma 1, that 'success' at each step has a sufficiently large probability, by using Lemma 6.

Let $0 < \eta < p_c$ and write

$$p = p_c + \tfrac{1}{2}\eta, \tag{4.1}$$

$$\delta = \eta/(4d), \tag{4.2}$$

$$\epsilon = (1-p_c(F))/(8d). \tag{4.3}$$

Since $p > p_c$, we have $\theta(p) > 0$. Moreover, since $p_c(F) < 1$, we have $\epsilon, \delta > 0$. Consequently there exist $m = m(d, \eta, \epsilon, \delta)$, $n = n(d, \eta, \epsilon, \delta)$ such that $2m < n$ and with the property in the statement of Lemma 6. We write

$$N = m+n+1, \tag{4.4}$$

and we shall show that the statement of Theorem A is valid with $k = 2N$.

We 'renormalize' $\mathbb{Z}^d$ by considering the set of vertices $\{4Nx : x \in \mathbb{Z}^d\}$; of special importance to us are the corresponding boxes $\{4Nx + B(N) : x \in \mathbb{Z}^d\}$, and we call such boxes *site-boxes*. A pair of site-boxes are called *adjacent* if they are centred at adjacent vertices of the renormalized lattice. Adjacent site-boxes are linked up by partly overlapping *bond-boxes*, that is, boxes of the form $Ny + B(N)$ with $y \in \mathbb{Z}^d$ such that exactly one component of y is not divisible by 4; such a box is called a *half-way* box if the exceptional component of y is even.

We shall examine site-boxes one by one, declaring them to be either 'occupied' or 'unoccupied' according to certain rules. We do this in turn in the manner described before Lemma 1. Let $e(1), e(2), \ldots$ be a fixed ordering of the edges of F; we may suppose without loss of generality that the origin 0 belongs to F. We begin by examining the site-box $B(N)$, corresponding to the origin. If $B(N)$ is unoccupied then we terminate the process. If it is occupied, we find the earliest edge e_1 incident with the origin, and consider the site-box $4Nx_2 + B(N)$ corresponding to the other endpoint x_2 of e_1. If this site-box is occupied, then we add x_2 to the 'occupied cluster' at the origin; in any case we shall never return to it. This process is continued, at each stage finding the earliest edge one of whose endpoints lies in the occupied cluster at the origin and the other of which corresponds to a site-box which has not yet been examined. We shall present a suitable definition of the occupied state, such that (*a*) Lemma 1 will imply the existence of an infinite occupied cluster with positive probability, and (*b*) the existence of such a cluster necessarily entails the existence of an infinite open cluster in the original subset $2kF + B(k)$ of $\mathbb{Z}^d$.

Once we have specified what is meant by saying that the origin is (or is not) occupied, then much of the work will have been done; the event in question is illustrated in figure 2. Consider then the site-box $B(N)$, noting that $B(m) \subset B(N)$. We say that 'the first step is successful' if $B(m)$ is p-open; if the first step is not successful then we terminate the process. Note that

$$P(B(m) \text{ is } p\text{-open}) > 0, \tag{4.5}$$

and assume henceforth that $B(m)$ is p-open (this will turn out to be more a matter of convenience than necessity). We recall that an 'm-pad' is a translation of $B(m)$ every vertex of which is p-open.

We write $C_1 = B(m)$ and define, for $x \in \mathbb{Z}^d$,

$$\gamma_1(x) = \begin{cases} p & \text{if} \quad x \in C_1, \\ 1 & \text{otherwise}, \end{cases}$$

$$\beta_1(x) = 0 \quad \text{for all} \quad x \in \mathbb{Z}^d,$$

so that every vertex x of $\mathbb{Z}^d$ is $\gamma_1(x)$-open and $\beta_1(x)$-closed. Let L_1 denote the identity

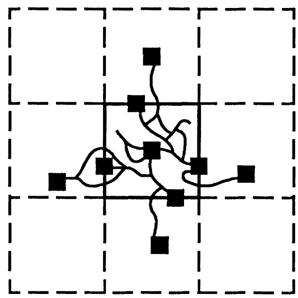

Figure 2. An illustration of the event that the origin of the renormalized lattice is occupied. The black boxes are m-pads; that is, they are p-open translates of $B(m)$, which is the central black box of the figure. The larger boxes are translates of $B(N)$. As indicated, $B(m)$ is joined through a network of paths and black boxes to m-pads contained in each of the copies of $B(N)$ having a face common with $B(N)$.

and, for $j = 2, 3, \ldots, 2d$, denote by L_j the 'earliest' isometry of $\mathbb{Z}^d$ which preserves the origin and which maps the first coordinate direction onto the jth; the word 'earliest' refers to some natural fixed ordering of the set of such isometries. Define C_2 by

$$C_2 = C_1 \cup E_1 \cup F_1, \tag{4.6}$$

where E_1 is the set of $x \in \Delta C_1$ which are $(\beta_1(x)+\delta)$-open, and F_1 is the set of (p-open) vertices $y \in B_1' = B(n) \cup \{\bigcup_{j=1}^{2d} L_j(T(m,n))\}$ such that y can be joined to ΔC_1 by a path of B_1' which is p-open outside ΔC_1 and whose unique vertex $u \in \Delta C_1$ is $(\beta_1(u)+\delta)$-open. We call this step successful if C_2 contains at least one vertex in $K_j(m,n)$ for each $j = 1, 2, \ldots, 2d$, where $K_j(m,n)$ is the set of vertices z in $L_j(T(n))$ such that $z + L_j(i_1)$ lies in an m-pad of $L_j(T(m,n))$. If this step is unsuccessful, we terminate the process.

Before continuing, we estimate the probability that the step is successful, conditional on the success of the previous step. Let G be the event that there exists a path in $B(n)-B(m)$ from $\Delta B(m)$ to $K(m,n)$, this path being p-open outside $\Delta B(m)$ and $(\beta_1(u)+\delta)$-open at its unique vertex u in $\Delta B(m)$. We denote by G_j the corresponding event with $K(m,n)$ replaced by $K_j(m,n)$. Then, by an application of Lemma 6 with $R = B(m)$, $\beta = \beta_1$ restricted to $\Delta R \cap B(n)$, we have

$$P(G_j \mid B(m) \text{ is } p\text{-open}) > 1-\epsilon, \quad j = 1, 2, \ldots, 2d,$$

so that

$$P\left(\bigcap_{j=1}^{2d} G_j \,\middle|\, B(m) \text{ is } p\text{-open}\right) > 1-2d\epsilon. \tag{4.7}$$

Thus this step is successful with probability at least $1-2d\epsilon$.

Let us assume that the last step was successful. We define, for $x \in \mathbb{Z}^d$,

$$\gamma_2(x) = \begin{cases} \gamma_1(x) & \text{if } x \in C_1, \\ \beta_1(x)+\delta & \text{if } x \in \Delta C_1 \cap C_2, \\ p & \text{if } x \in C_2 - (C_1 \cup \Delta C_1), \\ 1 & \text{otherwise,} \end{cases} \tag{4.8}$$

$$\beta_2(x) = \begin{cases} \beta_1(x) & \text{if } x \notin B_1', \\ \beta_1(x)+\delta & \text{if } x \in (\Delta C_1 - C_2) \cap B_1', \\ p & \text{if } x \in (\Delta C_2 - \Delta C_1) \cap B_1', \\ 0 & \text{otherwise,} \end{cases} \tag{4.9}$$

so that every vertex x of $\mathbb{Z}^d$ is $\gamma_2(x)$-open and $\beta_2(x)$-closed. In general, for each positive integer t in the process we shall define real functions β_t, γ_t on $\mathbb{Z}^d$ satisfying

$$0 \leqslant \beta_t(x) \leqslant \beta_{t+1}(x) \leqslant f(x) < \gamma_{t+1}(x) \leqslant \gamma_t(x) \leqslant 1 \quad \text{for all} \quad x \in \mathbb{Z}^d, \quad t = 1, 2, \ldots,$$

and the resulting pair of sequences of functions (β_t), (γ_t) will give a complete description of the history of the construction. We shall in fact form sequences of connected sets $C_1 \subseteq C_2 \subseteq \ldots$ and $B'_1, B'_2, \ldots$, in such a way that

$$\beta_t(x) \leqslant f(x) < \gamma_t(x) \quad \text{for all} \quad x \in \mathbb{Z}^d \quad \text{and} \quad t \geqslant 1$$

and

$$\beta_t(x) = 0, \quad \gamma_t(x) = 1 \quad \text{for all} \quad x \notin C_t \cup \Delta C_t.$$

Now it is clear that $T(m,n)$ is contained in the bond-box $Ni_1 + B(N)$. Let B_2 denote the earliest m-pad (earliest in some fixed ordering of $\{u + B(m) : u \in \mathbb{Z}^d\}$) in $C_2 \cap T(m,n)$. We aim to extend C_2 to join up with a third m-pad, B_3 say, lying in the half-way box $2Ni_1 + B(N)$, and accordingly we use a 'steering action'. It happens that, writing $B_2 = b + B(m)$, all the coordinates of b are positive, and consequently $b + T(m,n)$ is not a subset of $2Ni_1 + B(N)$. We therefore seek a suitable target point not in $b + T(m,n)$ but rather in $b + T^*(m,n)$, where U^* is the image of the set U under the symmetry $x_1 \mapsto \xi_1$, $x_j \mapsto -\xi_j$ for $j = 2, 3, \ldots, d$. We write

$$K_*(m,n) = \{x \in b + T^*(n) : x + i_1 \text{ belongs to some } m\text{-pad in } b + T^*(m,n)\}.$$

We then define

$$C_3 = C_2 \cup E_2 \cup F_2, \tag{4.10}$$

where E_2 is the set of $x \in \Delta C_2 \cap (b + B(n))$ which are $(\beta_2(x) + \delta)$-open, and F_2 is the set of (p-open) vertices $y \in B'_2 = b + (B(n) \cup T^*(m,n))$ such that y can be joined to ΔC_2 by a path of B'_2 which is p-open outside ΔC_2 and whose unique vertex $u \in \Delta C_2$ is $(\beta_2(u) + \delta)$-open. We call this step successful if C_3 contains at least one vertex in $K_*(m,n)$. If it is not successful, we terminate the process.

Let us estimate the probability that this step is successful, conditional on the success of all earlier steps. We apply Lemma 6 centred at b, with $K(m,n)$ replaced by $K_*(m,n)$, $R = C_2 \cap (b + B(n))$, $\beta = \beta_2$ restricted to $\Delta R \cap (b + B(n))$, to find that this conditional probability exceeds $1 - \epsilon$. We assume henceforth that this step is successful.

Similar to (4.8) and (4.9), we define for $x \in \mathbb{Z}^d$

$$\gamma_3(x) = \begin{cases} \gamma_2(x) & \text{if } x \in C_2, \\ \beta_2(x) + \delta & \text{if } x \in \Delta C_2 \cap C_3, \\ p & \text{if } x \in C_3 - (C_2 \cup \Delta C_2), \\ 1 & \text{otherwise,} \end{cases}$$

$$\beta_3(x) = \begin{cases} \beta_2(x) & \text{if } x \notin B'_2, \\ \beta_2(x) + \delta & \text{if } x \in (\Delta C_2 - C_3) \cap B'_2, \\ p & \text{if } x \in (\Delta C_3 - \Delta C_2) \cap B'_2, \\ 0 & \text{otherwise.} \end{cases}$$

Having completed a 'link-up' from the first site-box $B(N)$ to the half-way box $2Ni_1 + B(N)$, we now return to the first site-box and attempt to 'link-up' with the half-way box $2Ni_2+B(N)$ in a manner similar to that just described. In fact, by virtue of the very first step we have already 'linked-up' with the bond-box $Ni_2+B(N)$, and so just one more step is needed.

Let $B_2^{(2)} = b^{(2)}+B(m)$ be the earliest m-pad in $C_3 \cap L_2(T(m,n))$. We define

$$C_4 = C_3 \cup E_3 \cup F_3, \tag{4.11}$$

where E_3 is the set of $x \in \Delta C_3 \cap (b^{(2)}+B(n))$ which are $(\beta_3(x)+\delta)$-open, and F_3 is the set of (p-open) vertices $y \in B_3' = b^{(2)}+\{B(n) \cup L_2(T^*(m,n))\}$ such that y can be joined to ΔC_3 by a path of B_3' which is p-open outside ΔC_3 and whose unique vertex $u \in \Delta C_3$ is $(\beta_3(u)+\delta)$-open. We call this step successful if C_4 contains at least one vertex in

$$K_*^{(2)}(m,n) = \{x \in b^{(2)}+L_2(T^*(n)) : x+i_2 \text{ belongs to some } m\text{-pad in } b^{(2)}+L_2(T^*(m,n))\}.$$

If it is not successful we terminate the process.

Now

$$(C_3 \cup \Delta C_3) \cap \{b^{(2)}+L_2(T^*(m,n))\} = \varnothing,$$

so that the joint distribution of $f(x)$, for vertices x belonging to potential new m-pads, conditional on C_3, is the original product measure. Therefore, we may apply Lemma 6 as usual to deduce that the last step succeeds with (conditional) probability exceeding $1-\epsilon$. Assuming success, we define the functions γ_4 and β_4 in the usual manner.

In a similar way we attempt to 'link-up' in each of the remaining coordinate directions $j = 3, 4, \ldots, 2d$, and we declare the origin 0 of the renormalized lattice to be *occupied* if the entire construction up to that point is successful. This can only occur at the moment of definition of C_{2d+2}, and therefore the construction is successful with conditional probability satisfying

$$\begin{aligned} P(0 \text{ is occupied} \mid B(m) \text{ is open}) &> (1-2d\epsilon)(1-\epsilon)^{2d} \\ &> 1-4d\epsilon = \tfrac{1}{2}(1+p_c(F)) \end{aligned} \tag{4.12}$$

by (4.3). If 0 is not occupied, we terminate the process.

Before continuing, we pause to estimate the functions γ_q and β_q. We recall that formulae with the same form as (4.8) and (4.9) hold for all $q \geqslant 2$. Following the history of the construction, it is not difficult to see that, for $q = 1, 2, \ldots, 2d+2$,

$$\beta_q(x) \leqslant \gamma_q(x) \leqslant p+(w_q(x)-1)\,\delta \quad \text{for all} \quad x \in C_q, \tag{4.13}$$

where $w_q(x)$ is the number of integers $r \leqslant q$ for which $x \in B_r(n)$, where $B_r(n)$ is the box which replaces $B(n)$ in Lemma 6 when we apply that lemma in extending C_{r-1} to C_r. For $q = 1, 2, \ldots, 2d+2$, we have the crude bound

$$w_q(x) \leqslant 2d+1 \quad \text{for all} \quad x \in \mathbb{Z}^d, \tag{4.14}$$

and it will be clear later that both (4.13) and (4.14) are valid for all q. Consequently, by (4.1), (4.2), (4.13) and (4.14),

$$\beta_q(x) \leqslant \gamma_q(x) \leqslant p_c+\eta \quad \text{for all} \quad x \in C_q, \quad q = 1, 2, \ldots. \tag{4.15}$$

Much of the work is now done. Assume that 0 is occupied, and find the earliest edge e_1 of F incident to the origin. We may assume without loss of generality that the other endvertex of e_1 is i_1. In this case we move our attention to the site-box $4Ni_1+B(N)$ corresponding to i_1, and to determine whether or not it is occupied requires a procedure similar to that for the origin. A significant difference, however, is that the first stage is to attempt in two steps a 'link-up' to it from the half-way box $2Ni_1 + B(N)$ which we recall is already 'linked-up' with $B(N)$. The rest of the procedure consists of attempting to 'branch out' from $4Ni_1 + B(N)$, and to 'link-up' with the $2d-1$ adjacent half-way boxes not considered so far.

We recall from the second step of our construction that $b + T^*(m, n) \subset 2Ni_1 + B(N)$. We also recall that $C_3 \cap (b + T^*(m, n))$ and hence $C_{2d+2} \cap (b + T^*(m, n))$ contains at least one m-pad; let B_3 denote the earliest such m-pad. Thus $B_3 \subset 2Ni_1 + B(N)$, and we attempt to link C_{2d+2} with an m-pad B_4 in the next bond-box $3Ni_1 + B(N)$ in a manner similar to the 'link-up' from $Ni_1 + B(N)$ to $2Ni_1 + B(N)$ in the second step. We recall that we then needed a 'steering action' and we need this again but now in its general form. Accordingly, in place of $T^*(m, n)$ used in the second step we use the image of $T(m, n)$ under the appropriate symmetry of the form $x_1 \mapsto \xi_1$, $x_j \mapsto \pm \xi_j$ for $j = 2, 3, \ldots, d$, where, for each such j, we take the plus sign when the jth coordinate of the centre of B_3 is negative, and the minus sign when it is non-negative. Performing a step similar to the much earlier second step, we may extend C_{2d+2} to a set C_{2d+3} containing a suitable m-pad B_4 in $3Ni_1 + B(N)$, and the (conditional) probability of success exceeds $1-\epsilon$ as usual.

One more step similar to the last completes the 'link-up' with the site-box $4Ni_1 + B(N)$, and here we extend C_{2d+3} to C_{2d+4} containing an m-pad $B_5 = b^* + B(m)$, say, in this site-box. As anticipated, we next attempt to 'branch out' in the relevant $2d-1$ directions, by forming $2d-1$ m-pads 'resting' on the faces of $b^* + B(n)$, in a manner similar to the 'branching-out' from the origin in our very first step in $2d$ directions. Since b^* will generally differ from $4Ni_1$, we need to use the usual steering technique to ensure that the new m-pads lie in their respective bond-boxes. There is an additional proviso. It happens that the first coordinate b_1^* of b^* equals $4N$ exactly. In this 'ambiguous state' we override the usual convention and ensure, as we may, that all the new m-pads rest on the 'upper half' of $b^* + B(n)$ in the sense that all vertices in these m-pads have first coordinates not less than $4N$. This in turn ensures that these m-pads contain no vertices x for which $f(x)$ has already been inspected. The procedure of 'steering away from the inlet branch' will continue throughout all future steps of the construction corresponding to this one.

The remaining steps consist of the $2d-1$ 'link-ups' with the relevant half-way boxes. Since b^* is generally not 'centralized', it is the case that in some of the $2d-1$ directions we may already have a connecting m-pad in the target half-way box, and for such directions the 'link-up' is deemed complete. For the remaining directions we proceed as we did originally when 'branching out' from the origin, and if the entire construction is successful, then i_1 is declared to be occupied.

The number of new m-pads required in deciding that i_1 is occupied is no greater than

$$2 + (2d-1) + (2d-1) = 4d,$$

and therefore, as in (4.12),

$$P(i_1 \text{ is occupied} \mid 0 \text{ is occupied}) > \tfrac{1}{2}(1 + p_c(F)). \tag{4.16}$$

The procedure does not necessarily terminate even if i_1 is unoccupied, as there may well be other site-boxes adjacent to $B(N)$ which we need to test. If i_1 is unoccupied, then there was some step in the construction which was unsuccessful; on encountering this step, we define the new γ and β functions, and then abandon this site-box to move on to another.

Following the procedure explained before Lemma 1, we examine the relevant site-boxes one by one as described for the second site-box. Thus we first 'link-up' from

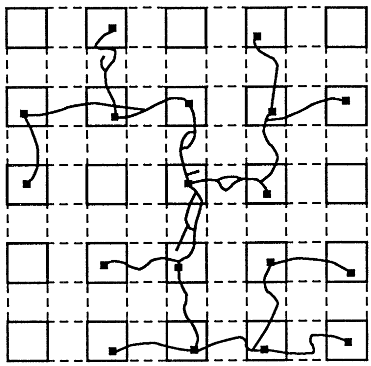

Figure 3. Successful connections in the renormalized growth process. The black central box is $B(m)$, which is contained in $B(N)$. The box $B(m)$ is joined by a network of paths and open boxes to open boxes contained in copies of $B(N)$. These paths are contained in the 'growing cluster', a few of whose other connections are displayed.

the relevant half-way box to the new site-box B, say, and then 'branch out' to the half-way boxes which are 'half-way' to those adjacent site-boxes not yet examined. In general there may be fewer than $2d-1$ adjacent site-boxes not yet examined; it is even possible that they have all been examined in which case we declare the renormalized site to be unoccupied and move on to the next site-box.

Proceeding thus with the step by step approach described before Lemma 1 and illustrated in figure 3, we have in the notation of that lemma that $\mu = P$, that $g(x_{t+1}) = 1$ if and only if the $(t+1)$th vertex x_{t+1} considered in the renormalized lattice is occupied, and that

$$\rho(\mathscr{S}, t) > \tfrac{1}{2}(1+p_c(F)) \quad \text{for} \quad t \geqslant 1.$$

Applying Lemma 1, we find that, conditional on the event $\{B(m) \text{ is open}\}$, there is strictly positive probability that the cluster A of occupied vertices in the renormalized lattice is infinite, so that

$$P(|A| = \infty) = P(|A| = \infty \,|\, B(m) \text{ is open})\, P(B(m) \text{ is open}) > 0.$$

If A is infinite, then there is a connected cluster of $4NF + B(2N)$ containing the origin, and such that (by (4.15)) every vertex therein is $(p_c+\eta)$-open; the claim of the theorem follows. □

5. The supercritical phase

We note some consequences of Theorem A and its method of proof.

(*a*) We have seen already that the theorem implies that, for each $\epsilon > 0$, there exists k such that $p_c(S(k)) < p_c + \epsilon$. Using results of Grimmett (1981, 1983), it may be deduced also that, if $h: [0, \infty) \rightarrow [0, \infty)$ satisfies

$$h(u)/\log u \rightarrow \infty \quad \text{as} \quad u \rightarrow \infty,$$

then, for each $\epsilon > 0$, there exists a positive integer k such that the critical probability of the subset

$$\{x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d : 0 \leqslant x_2 \leqslant h(x_1), x_1 \geqslant 0, |x_j| \leqslant k \text{ for } j > 2\}$$

is less than $p_c + \epsilon$.

(*b*) The second of our consequences is that, if $p_c < p < 1$, then

$$P_p(0 \leftrightarrow ni_1, |C| < \infty) < e^{-n/\xi(p)}$$

for some $\xi(p)$ satisfying $0 < \xi(p) < \infty$. This was proved by Chayes *et al.* (1987) for sufficiently large p, and the full result now follows from their work together with the formula $p_c = p_c(S)$. In the language of mathematical physics, the correlation length is finite throughout the supercritical phase. See Grimmett (1989, §§6.5, 6.6) and Kesten (unpublished work, 1988).

(*c*) A similar application of our theorem to a result of Kesten & Zhang (1989) gives us that

$$P_p(n \leqslant |C| < \infty) \leqslant \exp(-\alpha(p)\, n^{(d-1)/d})$$

for some $\alpha(p)$ satisfying $\alpha(p) > 0$ if $p > p_c$.

(*d*) Let Π_n be the maximal number of vertex-disjoint open paths crossing an n-cube from one face to the opposite face. If $p > p_c$ then there exist $\beta(p), \gamma(p) > 0$ such that

$$P_p(\Pi_n \geqslant \beta(p)\, n^{d-1}) \geqslant 1 - \exp(-\gamma(p)\, n^{d-1}).$$

This now follows from Chayes & Chayes (1986) who assumed that $p > p_c(S)$. In fact their argument makes use of Lemma 4.9 in Aizenman *et al.* (1983).

An easier proof of this result may be obtained by direct use of the argument in the proof of Theorem A, and we sketch this. Let N, λ be positive integers and write $n = \lambda N$. Divide the '$(n-1)$-cube' $D(n-1) = \{0, 1, \ldots, n-1\}^d$ into λ^{d-2} 'slices' of the form

$$D_k(n-1) = \{x \in D(n-1): k_i N \leqslant x_i < (k_i+1)N, 1 \leqslant i \leqslant d-2\}$$

for

$$k = (k_1, k_2, \ldots, k_{d-2}) \in \{0, 1, \ldots, \lambda-1\}^{d-2},$$

where x_i denotes the ith component of x. We partition each $D_k(n-1)$ into $(N-1)$-cubes of side-length $N-1$ in the usual way. If $p > p_c$ and $N = N(p)$ is a sufficiently large function of p, then we may think of the $(N-1)$-cubes within $D_k(n-1)$ as being the vertices of a supercritical site-percolation process. Such a process contains $\delta\lambda$ disjoint crossings of the slice with probability at least $1 - e^{-\eta n}$ for some $\delta(p)$, $\eta(p) > 0$; this may be proved as done by Grimmett & Kesten (1984) or Chayes & Chayes (1986) for 'independent percolation'. These crossings correspond in the original process to at least $\delta\lambda$ disjoint open crossings of the slice. Different slices do not overlap, and hence Π_n is at least the sum of λ^{d-2} independent contributions, each of which exceeds $\delta\lambda$ with probability at least $1 - e^{-\eta n}$. Since there is no essential loss of generality under the assumption that $n = \lambda N$ is an integer multiple of N, the conclusion now follows by a simple estimate for the binomial distribution.

(*e*) In a disordered electrical network containing a proportion p of conductors and $1-p$ of insulators, the effective resistance R_n between opposite faces of an n-cube satisfies

$$P_p(\limsup_{n\to\infty} \{n^{d-2} R_n\} < \infty) = 1, \text{ if } p > p_c.$$

This is a consequence of (*d*); see Grimmett & Kesten (1984), and Chayes & Chayes (1986).

(*f*) If $p > p_c$, there is only a small probability that two vertices x and y are in the infinite open cluster but joined by no open path of 'reasonable' length. We make this statement more precise as follows. Let $x \in \mathbb{Z}^d$ where $d \geqslant 3$, and $M \geqslant 1$, and write $D(x, M)$ for the smallest cube of the form $\prod_{i=1}^{d} [a_i, b_i]$ containing both $B(M)$ and

$x+B(M)$; if, for example, $x = (x_1, x_2, \ldots, x_d)$ and $x_i \geqslant 0$ for all i, we should take $a_i = -M$, $b_i = x_i + M$ for $i = 1, 2, \ldots, d$. We claim that there exists $\alpha(p)$, satisfying $\alpha(p) > 0$ if $p > p_c$, such that

$$P_p(0 \leftrightarrow \infty, x \leftrightarrow \infty, 0 \nleftrightarrow x \text{ in } D(x, M)) \leqslant e^{-\alpha(p) M} \quad (5.1)$$

for all M.

Here is a sketch of a direct proof of (5.1). Let $p > p_c$, and let N be a positive integer sufficiently large that the renormalized percolation process of the proof of Theorem A, with $F = \mathbb{Z}^2$ say, is supercritical; write $R = 2N+1$. Let

$$A_M = \{0 \leftrightarrow \infty, x \leftrightarrow \infty, 0 \nleftrightarrow x \text{ in } D(x, M)\},$$

noting that $A_{M+1} \subseteq A_M$. Now

$$P_p(A_{R(j+1)} \mid A_{Rj}) \leqslant \max\{P(u \nleftrightarrow v \text{ in } D(x, R(j+1)) - D(x, Rj)\}, \quad (5.2)$$

where the maximum is taken over all pairs u, v of vertices contained in $\Delta D(x, Rj)$. This holds since, if A_{Rj} occurs, then there exists some 'earliest' pair $u, v \in \Delta D(x, Rj)$ such that u and v are joined (respectively) to 0 and x by open paths contained (apart from their endpoints) in $D(x, Rj)$; if, in addition, $A_{R(j+1)}$ occurs, then $u \nleftrightarrow v$ in $D(x, R(j+1)) - D(x, Rj)$. The latter d-dimensional 'annulus', when unwrapped, resembles part of $\mathbb{Z}^{d-1} \times \{0, 1, \ldots, 2N\}$. It may be seen, using the block construction in the proof of Theorem A, that there exists $\gamma(p) > 0$ such that

$$P_p(u \leftrightarrow v \text{ in } D(x, R(j+1)) - D(x, Rj)) \geqslant \gamma(p)$$

uniformly in u, v, j and x. Hence

$$P_p(A_{R(j+1)} \mid A_{Rj}) \leqslant 1 - \gamma(p) \quad \text{for} \quad j = 0, 1, \ldots,$$

implying that
$$P_p(A_{Rj}) \leqslant (1-\gamma(p))^j \quad \text{for} \quad j = 0, 1, \ldots,$$

from which (5.1) follows.

Similar arguments appear in Aizenman *et al.* (1983), under the hypothesis $p > p_c(S)$.

(*g*) Our final observation, (5.5) to follow, concerns the length of the shortest open path between two distant vertices. Let $L(x, y)$ be the number of edges in the shortest open path joining vertex x to vertex y, with the convention that $L(x, y) = \infty$ if $x \nleftrightarrow y$. The random variable $L(x, y)$ has an atom at ∞, and it is therefore more convenient to work instead with the random variable $L'(x, y)$ defined as follows. For each $x \in \mathbb{Z}^d$, let $i(x)$ denote 'the vertex nearest to x' which lies in an infinite open cluster; that is

$$|x - i(x)| = \min\{|x - z| : z \leftrightarrow \infty\}$$

and $i(x)$ is chosen to be the earliest vertex which achieves this minimum, according to some predetermined ordering of the vertices of $\mathbb{Z}^d$. We define $L'(x, y) = L(i(x), i(y))$, noting that $L'(x, y) < \infty$ a.s. (almost surely), by the almost sure uniqueness of the infinite cluster.

Suppose that $p > p_c$ and $d \geqslant 3$. It is clear that $\{L'(x, y) : x, y \in \mathbb{Z}^d\}$ and $\{L'(x+t, y+t) : x, y \in \mathbb{Z}^d\}$ are identically distributed families of random variables, for any fixed $t \in \mathbb{Z}^d$. Furthermore, it is easily checked that

$$L'(x, y) \leqslant L'(x, z) + L'(z, y) \quad \text{for} \quad x, y, z \in \mathbb{Z}^d,$$

so that L' satisfies the subadditive inequality. Once we have proved that

$$E(L'(x, y)) < \infty \quad \text{for all} \quad x, y \in \mathbb{Z}^d, \quad (5.3)$$

where E denotes expectation, then it will follow immediately by the subadditive ergodic theorem (see Kingman 1976) that, as $n \to \infty$,

$$\frac{1}{n} L'(0, nx) \to \lambda(x) \text{ a.s. and in } L^1, \tag{5.4}$$

for some real $\lambda(x)$ satisfying

$$\sum_{i=1}^{d} |x_i| \leqslant \lambda(x) < \infty \quad \text{for} \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d.$$

It is a consequence of (5.4) that, as $n \to \infty$,

$$\{(1/n) L(0, nx) - \lambda(x)\} I_{\{0 \leftrightarrow \infty\}} I_{\{x \leftrightarrow \infty\}} \to 0 \text{ a.s.}, \tag{5.5}$$

where I_A is the indicator function of the event A. This extends a conclusion of Zhang & Zhang (1984) valid in the case of two dimensions.

We need only prove that $E(L'(x, y)) < \infty$. Suppose that $M, N > 0$. Then

$$\begin{aligned} P_p(L'(x, y) \geqslant M) &= \sum_{u, v \in \mathbb{Z}^d} P_p(L(u, v) \geqslant M,\ i(x) = u, i(y) = v) \\ &\leqslant P_p(|i(x) - x| \geqslant N) + P_p(|i(y) - y| \geqslant N) \\ &\quad + \sum_{\substack{u, v: \\ |u-x| < N \\ |v-y| < N}} P_p(L(u, v) \geqslant M, u \leftrightarrow \infty, v \leftrightarrow \infty). \end{aligned} \tag{5.6}$$

We claim first that there exists $a(p) > 0$ such that

$$P_p(|i(z) - z| \geqslant N) \leqslant e^{-a(p) N} \quad \text{for} \quad N > 0, z \in \mathbb{Z}^d. \tag{5.7}$$

Certainly $p > p_c$ entails $p > p_c(S(k))$ for some k; with this choice of k, the probability that no vertex in $\{z + j i_1 : 0 \leqslant j \leqslant N\}$ lies in an infinite cluster is at most $\exp(-A(p) \lfloor N/(k+1) \rfloor)$ for some $A(p) > 0$, and (5.7) follows. Secondly, we have from (5.1) that there exists $\alpha(p) > 0$ such that

$$P_p(u \leftrightarrow \infty, v \leftrightarrow \infty, u \not\leftrightarrow v \text{ in } u + D(v-u, N)) \leqslant e^{-\alpha(p) N}.$$

The number of edges in $D(v-u, N)$ is at most $d\{1 + |v-u| + 2N\}^d$, which is no greater than $d\{4N + |x-y|\}^d$ if $|u-x| < N$ and $|v-y| < N$. Therefore

$$P_p(L(u, v) \geqslant M, u \leftrightarrow \infty, v \leftrightarrow \infty) \leqslant e^{-\alpha(p) N} \tag{5.8}$$

whenever

$$M \geqslant d\{4N + |x-y|\}^d \tag{5.9}$$

and $|u-x| < N$, $|v-y| < N$. We set $N = \lfloor 5^{-1} d^{-1/d} M^{1/d} \rfloor$ and then (5.9) holds for all sufficiently large M. It follows from (5.6), (5.7) and (5.8) that for some $c < \infty$ and $\beta(p) > 0$ we have

$$P_p(L'(x, y) \geqslant M) \leqslant cM \exp(-\beta(p) M^{1/d}) \text{ for all large } M.$$

Summing over integer values of M, we obtain $E(L'(x, y)) < \infty$ as required.

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