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Coexistence of the Infinite (*) Clusters: – A Remark on the Square Lattice Site Percolation

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Summary. We show that the critical probability p_c is strictly greater than 1/2 for the square lattice site percolation.

§ 1. Introduction

The bond percolation problem on the square lattice was solved by Kesten [1], that the critical probability equals 1/2. On the other hand, for the site percolation on the square lattice, no one doubts that $p_c > 1/2$, though it has never been rigorously proved. The essential idea of finding p_c was given by Sykes and Essam [5], but unfortunately the argument was not sufficiently rigorous. The best rigorous result for this problem is that $p_c + p_c^* = 1$ which was proved by Russo [4]. In this note, we prove that $p_c > 1/2$ by using arguments of Kesten [1] and Russo [3, 4].

Hereafter we consider the square lattice $\mathbf{Z}^2$ and the configuration space $\Omega = \{+1, -1\}^{\mathbf{Z}^2}$. For $0 \leqq p \leqq 1$, we denote by $P^{(p)}$ the Bernoulli probability measure on Ω, taking probability p of finding + spin at $\underline{x} \in \mathbf{Z}^2$. We say that $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$ are nearest neighbours (and denote it by $\langle \underline{x}, \underline{y} \rangle$) iff $|x_1 - y_1| + |x_2 - y_2| = 1$. $\underline{x}$ and $\underline{y}$ are (*) nearest neighbours (we denote it by $\langle \underline{x}, \underline{y} \rangle^*$) iff $\max(|x_1 - y_1|, |x_2 - y_2|) = 1$. Let $\mathbf{L}$ be the sublattice of $\mathbf{Z}^2$ such that $\mathbf{L} = \{\underline{x} \in \mathbf{Z}^2;\ x_1 + x_2 \text{ is even}\}$. $\mathbf{L}$ is isomorphic to $\mathbf{Z}^2$. We say that $\underline{x} \in \mathbf{L}$ and $\underline{y} \in \mathbf{L}$ are $\mathbf{L}$-nearest neighbours [(*)$\mathbf{L}$-nearest neighbours] iff $|x_i - y_i| = 1$, $i = 1, 2$ [$|x_1 - y_1| + |x_2 - y_2| = 2$] and denote it by $\langle \underline{x}, \underline{y} \rangle_{\mathbf{L}}$ [$\langle \underline{x}, \underline{y} \rangle_{\mathbf{L}}^*$].

A sequence $\{\underline{x}_1, \ldots, \underline{x}_n\}$ of mutually distinct points in $\mathbf{Z}^2$ is called a (self avoiding) chain [(*)chain] iff $\langle \underline{x}_i, \underline{x}_j \rangle \Leftrightarrow |i-j| = 1$ [$\langle \underline{x}_i, \underline{x}_j \rangle^* \Leftrightarrow |i-j| = 1$], and is called a circuit [(*)circuit] iff $\{\underline{x}_1, \ldots, \underline{x}_{n-1}\}$ and $\{\underline{x}_2, \ldots, \underline{x}_n\}$ are chains [(*)chains] and $\langle \underline{x}_n, \underline{x}_1 \rangle$ [$\langle \underline{x}_n, \underline{x}_1 \rangle^*$]. A subset Λ of $\mathbf{Z}^2$ is said to be connected [(*)connected] iff for any $\underline{x}, \underline{y} \in \Lambda$, there is a chain [(*)chain] $\{\underline{x}_1, \ldots, \underline{x}_n\}$ in Λ with $\underline{x} = \underline{x}_1$, $\underline{y} = \underline{x}_n$. $\mathbf{L}$-chain, $\mathbf{L}$-connectedness, (*)$\mathbf{L}$-chain, and (*)$\mathbf{L}$-connectedness are defined in the same way.

Note that $\hat{\Lambda} = \Lambda \cap \mathbf{L}$ is (*)$\mathbf{L}$-connected if Λ is connected.

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The percolation probability $\Pi(p)$ is defined by

$$\Pi(p)=P^{(p)}\left\{\begin{matrix}\text{there exists an infinite } (+)\text{chain} \\ \text{including the origin}\end{matrix}\right\}.$$

The problem is to find the critical probability p_c;

$$p_c=\inf\{p;\ \Pi(p)>0\}.$$

Putting $p_c^*=\inf\{p;\ \Pi^*(p)>0\}$, where

$$\Pi^*(p)=P^{(p)}\left\{\begin{matrix}\text{there exists an infinite } (+{}^*)\text{chain} \\ \text{including the origin}\end{matrix}\right\},$$

we can easily see that $p_c^*\leqq p_c$. Moreover, Russo proved the following;

Theorem (Russo [4]).

(i) *$\Pi(p)$, $\Pi^*(p)$ are continuous in $p\in[0,1]$,*

(ii) $p_c+p_c^*=1$.

The estimate $p_c\geqq 1/2$ is the direct consequence of the above theorem. Here, we give a little sharper result;

Theorem 1. $p_c>1/2$.

In §2, we prove an essential lemma whose statement looks rather trivial, and in §3 we prove Theorem 1.

§2. Sponge Percolation Problem

For any positive integers m and n, put

$$\Lambda^+(m,n)\equiv\{\underline{x}\in\mathbf{Z}^2;\ 0\leqq x_1\leqq m,\ 0\leqq x_2\leqq n\},$$
$$\Lambda^-(m,n)\equiv\{\underline{x}\in\mathbf{Z}^2;\ 0\leqq x_1\leqq m,\ -n\leqq x_2\leqq 0\},$$
$$\Lambda(m,n)\equiv\{\underline{x}\in\mathbf{Z}^2;\ 0\leqq |x_1|\leqq m,\ 0\leqq |x_2|\leqq n\}.$$

A chain [(*)chain] in $\Lambda^{(\pm)}(m,n)$ is called a vertical cut [vertical (*)cut] in $\Lambda^{(\pm)}(m,n)$ if it connects the upper side of $\Lambda^{(\pm)}(m,n)$ with the lower side of $\Lambda^{(\pm)}(m,n)$, and if this chain [(*)chain] intersects with each horizontal side of $\Lambda^{(\pm)}(m,n)$ at only one point. A chain [(*)chain] in $\Lambda^{(\pm)}(m,n)$ is called a horizontal cut [horizontal (*)cut] if it connects the left side of $\Lambda^{(\pm)}(m,n)$ with the right side of $\Lambda^{(\pm)}(m,n)$, and if this chain [(*)chain] intersects with each vertical side of $\Lambda^{(\pm)}(m,n)$ at only one point. We can define a vertical [horizontal] **L**-cut [(*)**L**-cut] in $\hat{\Lambda}^{(\pm)}(m,n)\equiv\mathbf{L}\cap\Lambda^{(\pm)}(m,n)$ in the same way.

Finally, a (*)**L**-chain $\gamma\equiv\{\underline{x}_1,\dots,\underline{x}_k\}$ in $\hat{\Lambda}^+(m,n)$ is called a weak vertical [horizontal] (*)**L**-cut if it connects $\{x_2=0 \text{ or } 1\}$ with $\{x_2=n-1 \text{ or } n\}$ [$\{x_1=0$ or $1\}$ with $\{x_1=m-1$ or $m\}$], and both $\gamma\cap\{x_2=0 \text{ or } 1\}$ and $\gamma\cap\{x_2=n-1$ or $n\}$ are single points. [$\gamma\cap\{x_1=0 \text{ or } 1\}$ and $\gamma\cap\{x_1=m-1 \text{ or } m\}$ are single points.]

Now let us define the sponge percolation probabilities as in the following;

$$a_p^{[*]}(m,n) \equiv P^{(p)}\{\text{there exists a horizontal } (+[*])\text{cut in } \Lambda(m,n)\},$$
$$a_p^{\pm[*]}(m,n) \equiv P^{(p)}\{\text{there exists a horizontal } (+[*])\text{cut in } \Lambda^{\pm}(m,n)\},$$
$$\hat{a}_p^{[*]}(m,n) \equiv P^{(p)}\{\text{there exists a horizontal } (+[*])\mathbf{L}\text{-cut in } \hat{\Lambda}(m,n)\},$$
$$\hat{a}_p^{\pm[*]}(m,n) \equiv P^{(p)}\{\text{there exists a horizontal } (+[*])\mathbf{L}\text{-cut in } \hat{\Lambda}^{\pm}(m,n)\}.$$

For the weak (*)**L**-cut, we use the notation "$w-$" in front of $\hat{a}_p^{(\pm)*}(m,n)$, e.g. $w-\hat{a}_p^*(m,n)$.

Lemma 1. *Let $\alpha(m,n)$ be $a_p^{[*]}(m,n)$, $\hat{a}_p^{[*]}(m,n)$ or $w-\hat{a}_p^*(m,n)$. If $\alpha(3N,N) > 1-5^{-4}$ for some integer $N \geqq 1$, then $\Pi^{[*]}(p) > 0$.*

Proof. This was already proved in [4] except for $\alpha(m,n) = w-\hat{a}_p^*(m,n)$. In this case, by applying the argument as in [4], we have

(1) $P_{\mathbf{L}}^{(p)}\{$there exists an infinite $(+*)\mathbf{L}$-chain in $\mathbf{L}\} = 1$, where $P_{\mathbf{Z}}^{(p)}$ is the restriction of $P^{(p)}$ to $\Omega_{\mathbf{L}} = \{+1, -1\}^{\mathbf{L}}$. Since $(\Omega_{\mathbf{L}}, P_{\mathbf{L}}^{(p)})$ is isomorphic to $(\Omega, P^{(p)})$, (1) implies that $\Pi^*(p) > 0$. (Q.E.D.)

Lemma 2. *Let $\alpha(m,n)$ be the same as in Lemma 1. Then there exists an increasing function f: $[0,1] \to [0,1]$ with $f(0)=0$, $f(1)=1$ such that*

$$\alpha(2N,N) \geqq f(\alpha(N,N)),$$
$$\alpha(3N,N) \geqq \alpha(N,N) \cdot f^2(\alpha(N,N)) \equiv g(\alpha(N,N)).$$

Proof. This is also in [4] except for $\alpha(m,n) = w-\hat{a}_p^*(m,n)$. If N is even, the same proof as in [4] works. If N is odd, the same argument makes the estimate little worse;

$$\alpha(2N-1,N) \geqq \alpha(N,N)[1-\sqrt{1-\alpha(N,N)}]^6.$$

Therefore for $f(x) = x^3(1-\sqrt{1-x})^{12}$, we have the statement of Lemma 2. (Q.E.D.)

Lemma 3. *If $\limsup_{N\to\infty} a_p^+(N,N) < 1$, then for any positive integer $k > 0$,*

$$\lim_{N\to\infty} [a_p^+(N,N) - a_p^+(N,N-k)] = 0.$$

Proof. It is enough to prove that

$$\lim_{N\to\infty} [a_p^+(N,N-k+1) - a_p^+(N,N-k)] = 0$$

for any positive integer k. Since $a_p^+(m,n) = a_p^-(m,n)$,

(2) $a_p^+(N,N-k+1) - a_p^+(N,N-k)$

$$= P^{(p)}\left\{\begin{matrix}\text{there exists a vertical } (-*)\text{cut in } \Lambda^-(N,N-k), \\ \text{but there are no vertical } (-*)\text{cuts in } \Lambda^-(N,N-k+1)\end{matrix}\right\}.$$

Put

$$C_{N,k} \equiv \{\omega \in \Omega;\ \text{there exists a vertical } (-*)\text{cut in } \Lambda^-(N,N-k)\},$$

$$C_{N,k}(r) \equiv \{\omega \in \Omega;\ \text{there exists a vertical } (-*)\text{cut in } \Lambda^-(N, N-k) \text{ which connects } \{x_1 \geqq N/2,\ x_2 = -N+k\} \text{ with } \{x_2 = 0\}\}.$$

$$C_{N,k}(l) \equiv \{\omega \in \Omega;\ \text{there exists a vertical } (-*)\text{cut in } \Lambda^-(N, N-k) \text{ which connects } \{x_1 \leqq N/2, x_2 = -N+k\} \text{ with } \{x_2 = 0\}\}.$$

For $\omega \in C_{N,k}(r)$, we denote by $R(\omega)$ the right-most vertical $(-*)$cut in $\Lambda^-(N, N-k)$. Let $\Delta_R(\omega)$ be the intersection point of $R(\omega)$ with $\{x_2 = -N+k\}$. Then $\Delta_R(\omega) \subset \{x_1 \geqq N/2\}$. For any $j \geqq 1$, let

$$G_{2j}(\omega) \equiv \text{square centered at } \Delta_R(\omega) \text{ with the length of its side equal to } 2(3^{2j}-1),$$

$$G_{2j+1}(\omega) \equiv \text{square centered at } \Delta_R(\omega) \text{ with the length of its side equal to } 2 \cdot 3^{2j+1},$$

and $\Gamma_j(\omega) \equiv G_{2j+1}(\omega) \setminus G_{2j}(\omega)$. Putting

$$J_{N,k}(\omega) \equiv \max\{j;\ \text{the left upper corner of } \Gamma_j(\omega) \text{ is in } \Lambda^-(N, N-k)\},$$

we have

$$J_{N,k}(\omega) \geqq (2\log 3)^{-1}[\log(\min\{N/2, N-k\}) - \log 3].$$

We denote the right hand side of the above inequality by $\delta_{N,k}$.

Since $\limsup_{N\to\infty} a_p^+(N, N) < 1$, there exists $n_0 > 0$ such that

$$P^{(p)}\{\text{there exists a vertical } (-*)\text{cut in } \Lambda^-(n, n)\} > [1 - \limsup_{N\to\infty} a_p^+(N, N)]/2$$

for $n > n_0$. Choosing J_0 sufficiently large such that $3^{2J_0} > n_0$, we obtain for $j > J_0$,

$$P^{(p)}\{\text{there exists a } (-*)\text{circuit surrounding the origin in } \Gamma_j^0\} \geqq g^4([1 - \limsup_{N\to\infty} a_p^+(N, N)]/2) \equiv \beta > 0,$$

where Γ_j^0 is the same square as $\Gamma_j(\omega)$ but centered at the origin.

By applying Kesten's argument in Proposition 1 of [1], we obtain that

$$P^{(p)}\{\omega \in C_{N,k}(r);\ \text{there are at most } \nu\ (-*)\text{chains in } \Lambda^-(N, N-k) \text{ which connect } R(\omega) \text{ with the lower side of } \Lambda^-(N, N-k)\} \leqq A_{N,k}(\nu, \beta),$$

$$A_{N,k}(\nu, \beta) \equiv \sum_{j=0}^{\nu} \binom{[\delta_{N,k}]^1 - J_0}{\nu} \beta^j (1-\beta)^{\delta_{N,k} - J_0 - j},$$

which goes to 0 as $N \to \infty$ for fixed k, ν and β. The same estimate holds for $C_{N,k}(l)$. Hence we obtain that

[1] This denotes the integer part of $\delta_{N,k}$

(3) $P^{(p)}\{\omega\in C_{N,k};$ there are at most ν vertical $(-*)$cuts in $\Lambda^-(N,N-k)$ with distinct end points on the lower side of $\Lambda^-(N,N-k)\}\leqq 2A_{N,k}(\nu,\beta)$.

Let

$$D_{N,k}(\omega)\equiv\left\{\begin{matrix}\underline{x}\in\Lambda^-(N,N-k)\cap\{x_2=-N+k\}; \\ \underline{x} \text{ is } (-*)\text{connected with } \{x_2=0\} \text{ in } \Lambda^-(N,N-k)\end{matrix}\right\}.$$

Then for any subset D of the lower side of $\Lambda^-(N,N-k)$, (i.e. $D\subset\Lambda^-(N,N-k)\cap\{x_2=-N+k\}$) we have

$$P^{(p)}\left\{\begin{matrix}D \text{ is not } (-*)\text{connected with} \\ \{x_2=-N+k-1\} \text{ in } \Lambda^-(N,N-k+1)\end{matrix}\middle| D_{N,k}(\omega)=D\right\}\leqq p^{-|D|}.$$

From this and (1), (2), we have for any $\nu>0$,

$$\begin{aligned}(4)\quad & a_p^+(N,N-k+1)-a_p^+(N,N-k) \\ & \leqq 2A_{N,k}(\nu,\beta)+\sum_{i=\nu+1}^{\infty} p^{-i}\sum_{|D|=i} P^{(p)}\{D_{N,k}(\omega)=D\}.\end{aligned}$$

(4) proves the assertion of the lemma. (Q.E.D.)

§ 3. Coexistence of Infinite (*)Chains

Let σ be a weak horizontal (*)**L**-cut in $\Lambda^+(m,n)$. There corresponds a horizontal (*)cut $\bar{\sigma}$ in $\Lambda^+(m,n)$ to σ in the following way. Let $\sigma=\{\underline{x}_1,\dots,\underline{x}_k\}$ with $\langle\underline{x}_i,\underline{x}_{i+1}\rangle^*_{\mathbf{L}}$, $i=1,2,\dots,k-1$, $\underline{x}_1\in\{x_1=0 \text{ or } 1\}$ and $\underline{x}_k\in\{x_1=m-1 \text{ or } m\}$. The corresponding $\bar{\sigma}=\{\underline{y}_1,\underline{y}_2,\dots,\underline{y}_l\}$ $(l\leqq k)$ is defined as follows;

1°) If $\underline{x}_1\in\{x_1=0\}$, then $\underline{y}_1=\underline{x}_1$. Otherwise $\underline{y}_1=\underline{x}_1-e_1$ and $\underline{y}_2=\underline{x}_1$, where $e_1=(0,1)\in\mathbf{Z}^2$.

2°) If $\underline{y}_i=\underline{x}_j$, and $\langle\underline{x}_j,\underline{x}_{j+1}\rangle^*$, then $\underline{y}_{i+1}=\underline{x}_{j+1}$.

3°) If $\underline{y}_i=\underline{x}_j$, but $\underline{x}_j$ and $\underline{x}_{j+1}$ are not (*)nearest neighbours, then there is a point $\underline{x}^*$ such that $\langle\underline{x}_j,\underline{x}^*\rangle$ and $\langle\underline{x}^*,\underline{x}_{j+1}\rangle$. (This point $\underline{x}^*$ is unique!) In this case, we put $\underline{y}_{i+1}=\underline{x}^*$ and $\underline{y}_{i+2}=\underline{x}_{j+1}$.

4°) If $\underline{x}_k\in\{x_1=m\}$, then $\underline{y}_l=\underline{x}_k$. Otherwise we put $\underline{y}_l=\underline{x}_k+e_1$.

It is easy to check that $\bar{\sigma}=\{\underline{y}_1,\dots,\underline{y}_l\}$ is a horizontal (*)cut in $\Lambda^+(m,n)$.

Theorem 1′. *If* $\limsup_{N\to\infty} a_p^+(N,N)<1$, *then either*

$$\lim_{N\to\infty} a_p^+(N,N)=0 \quad or \quad \lim_{N\to\infty} a_{1-p}^+(N,N)=0.$$

Proof. For any $\nu>0$, we take k, N sufficiently large so that $k>3^{2(J_0+\nu)+1}$, and $N>2k$, where J_0 is defined in the proof of Lemma 3. First, note that if there exists a horizontal $(+)$cut in $\Lambda^+(N,N-k)$, then there exists a weak horizontal $(+*)$**L**-cut in $\hat{\Lambda}^+(N,N-k)$. Therefore we have

$$a_p^+(N,N-k)=\{w-\hat{a}_p^{+*}(N,N-k)\}\cdot b_p^+(N,N-k),$$

where $b_p^+(N,N-k)$ is defined by

(5) $b_p^+(N, N-k)$

$$=P^{(p)}\left\{\begin{matrix}\text{there exists a horizontal} \\ (+)\text{cut in } \Lambda^+(N,N-k)\end{matrix}\middle|\begin{matrix}\text{there exists a weak horizontal} \\ (+*)\mathbf{L}\text{-cut in } \Lambda^+(N,N-k)\end{matrix}\right\}.$$

Let $\sigma(\omega)$ be the lowest weak horizontal (*)**L**-cut in $\hat{\Lambda}^+(N, N-k)$, and $\bar{\sigma}(\omega)$ be the corresponding horizontal (*)cut in $\Lambda^+(N, N-k)$.

For any weak horizontal (*)**L**-cut σ in $\hat{\Lambda}^+(N, N-k)$, let

$$H(\sigma)\equiv\{\underline{x}\in\Lambda^+(N,N-k);\ \underline{x} \text{ is above } \bar{\sigma}\}.$$

Then we have

$$1-P^{(p)}\{\text{there exists a horizontal } (+)\text{cut in } \Lambda^+(N,N-k)|\sigma(\omega)=\sigma\}$$

$$\geqq P^{(p)}\left[\left\{\begin{matrix}\text{there exist at least } v \text{ vertical } (-*)\text{cuts in } H(\sigma) \text{ with} \\ \text{distinct endpoints on the lower side of } H(\sigma)\text{, and for} \\ \text{one of these endpoints } \underline{x},\ \omega(\underline{y})=-1 \text{ for any } \underline{y} \text{ from} \\ \partial(\partial^*(\underline{x})\cap\sigma)\backslash\sigma\end{matrix}\right\}\middle|\sigma(\omega)=\sigma\right]$$

$$\geqq a_{1-p}^{+*}(N,N)(1-A_{2k,0}(v,\beta))(1-(1-(1-p)^{21})^v),$$

since $\#[\partial(\partial^*(\underline{x}))]=21$, where $\partial^{[*]}(\Lambda)=\{\underline{y}\in\mathbf{Z}^2;\ \langle\underline{x},\underline{y}\rangle^{[*]} \text{ for some } \underline{x}\in\Lambda\}$. Hence we have

$$b_p^+(N,N-k)\leqq 1-a_{1-p}^{+*}(N,N)(1-A_{2k,0}(v,\beta))(1-(1-(1-p^{21})^v).$$

Now we assume that

$$\limsup_{N\to\infty} w-\hat{a}_p^{+*}(N,N-k)=1.$$

Then from Lemmas 1 and 2, we obtain

$$P^{(p)}\{\text{there exists an infinite } (+*)\text{chain}\}=1,$$

which implies that $\lim_{N\to\infty} a_{1-p}^+(N,N)=0$. (See Lemma 2 of [3].)

Next, assume that $\limsup_{N\to\infty} w-a_p^{+*}(N,N)=\eta<1$. We take a subsequence $\{N_j\}$ such that $\lim_{j\to\infty} a_p^+(N_j,N_j)$ exists. From Lemma 3, we obtain

$$\lim_{j\to\infty} a_p^+(N_j,N_j)\leqq\eta\cdot[1-\lim_{j\to\infty} a_{1-p}^{+*}(N_j,N_j)(1-A_{2k,0}(v,\beta))(1-(1-(1-p)^{21})^v)].$$

Letting first $k\to\infty$, and then $v\to\infty$, we obtain

$$\lim_{j\to\infty} a_p^+(N_j,N_j)=0,$$

which implies that $\lim_{N\to\infty} a_p^+(N,N)=0$. (Q.E.D.)

Proof of Theorem 1. Since $a_{1/2}^+(N,N)\leqq 1/2$, from Theorem 1′ we obtain that $\lim_{N\to\infty} a_{1/2}^*(N,N)=1-\lim_{N\to\infty} a_{1/2}^+(2N,2N)=1$.

Therefore from Lemmas 1 and 2, we have $\Pi^*(1/2)>0$. This, combined with Russo's theorem (i), and (ii), implies that $p_c>1/2$. (Q.E.D.)

Corollary. *For* $p_c^* < p < p_c$,

$$P^{(p)}\{\text{there exist both infinite } (+*) \text{ and } (-*)\text{chains}\} = 1.$$

In particular, this holds for $p = 1/2$.

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