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A Note on Percolation

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Summary. An improvement of Harris' theorem on percolation is obtained; it implies relations between critical points of matching graphs of the type of the one stated by Essam and Sykes. As another consequence, it is proved that the percolation probability, as a function of the probability of occupation of a given site, is infinitely differentiable, except at most in the critical point.

1. Introduction

Percolation problems have been considerably studied in the literature (for an extensive bibliography see references quoted in [1–5]). Nevertheless, even in the case of Bernoulli measures, there are very few rigorous results; among them Harris' theorem ([6, 7]) on non coexistence of infinite clusters of "opposite type" in the two-dimensional lattices plays a central role; it implies the relation $p_c + p_c^* \geqq 1$ between the critical probabilities of two "matching graphs". The same relation was stated as an equality by Essam and Sykes ([8]) on the grounds of very plausible arguments based on an analogy with the case of Bethe lattices where exact results are known.

Harris proved his theorem by showing that if in a two-dimensional graph there is a.s. an infinite cluster, say, of $+$, one can find an increasing sequence $\{\Lambda_n\}$ of squares centered in a given site of the graph such that in $\Lambda_{n+1} \setminus \Lambda_n$ there is a closed chain of $+$ with probability greater than some constant not depending on n. This implies that a.s. some of these chains actually occur, forcing the clusters of opposite type to be finite.

In this note by using essentially the same techniques as in [6] we complement Harris' proof and show that one can control the size of the squares Λ_n by proving that the finite clusters are forced to have also finite mean size. In this way one gets a statement which can be inverted so that the inequality $p_c + p_c^* \geqq 1$ can be replaced by some equalities. We have not been able, however, to prove the relation $p_c + p_c^* = 1$.

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In Section 2 we define the quantities of interest. The main statement is proved in Section 3. In Section 4 we draw some consequences and prove that the percolation probability is infinitely many times derivable except at most in the critical point. Section 5 contains some final remarks.

2. Definitions

Two points in $\mathbb{Z}^2$ which differ only by one unit in one coordinate are called adjacent; they are called *adjacent if they are adjacent or such that both their coordinates differ by one unit[1]. A finite sequence $(x_1, \ldots, x_n)$ of distinct points in $\mathbb{Z}^2$ is called a (self-avoiding) chain if x_i and x_j are adjacent if and only if $|i-j|=1$ and a circuit if $(x_2, \ldots, x_n)$ is a chain and x_1 is adjacent to x_2 and to x_n; *chains and *circuits are defined in an analogous way. A subset $Y \subset \mathbb{Z}^2$ is connected [*connected] if, for all pairs x, y of points in Y, there is a chain [*chain] made up of points in Y, having x, y as terminal points. The boundary [*boundary] of a given subset $Y \subset \mathbb{Z}^2$ is the set $\partial Y [\partial^* Y]$ of all points in $\mathbb{Z}^2 \setminus Y$ that are adjacent [*adjacent] to at least one point in Y.

Remark. The external boundary of a connected set is *connected and the external *boundary of a *connected set is connected[2].

We consider the configuration space $\Omega = \{-1, 1\}^{\mathbb{Z}^2}$. If $\omega \in \Omega$, the (+)clusters [(+)*clusters] of ω are the maximal connected [*connected] components of $\omega^{-1}(1)$; (−)clusters and (−)*clusters are defined in the same way. A (+)chain in ω is a chain included in $\omega^{-1}(1)$[3]. Two sets A, $B \subset \mathbb{Z}^2$ are (+)connected in ω if there is in ω a (+)chain starting in A and ending in B.

If μ is a translation invariant probability measure on Ω, the μ-probability that a given site of the lattice belongs to a (+)cluster of size k is

$$p_k(\mu) = \sum_{|\gamma| = k} \mu(C_\gamma^+) \tag{2.1}$$

where for any finite connected subset γ of $\mathbb{Z}^2$ containing the origin, C_γ^+ is the set of all configurations ω such that γ is a (+)cluster in ω, and $|\gamma|$ means number of sites in γ.

We put:

$$P_k(\mu) = \sum_{n=k+1}^{\infty} p_n(\mu). \tag{2.2}$$

The percolation probability, $P_\infty(\mu) [P_\infty^*(\mu)]$, is defined as the μ-probability that the origin belongs to an infinite (+)cluster [(+)*cluster].

The mean cluster size, $S(\mu)$, is the expected value for the measure μ of the function $n(\omega)$ which is defined as the size of the finite (+)cluster to which the origin belongs if it belongs to any (+)cluster at all and zero otherwise.

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[1] In [8] a definition of matching graphs is given, which, in particular, applies to the graphs $G = \{\mathbb{Z}^2, L\}$, $G^* = \{\mathbb{Z}^2, L^*\}$, where $L[L^*]$ is the set of bonds between adjacent [*adjacent] points in $\mathbb{Z}^2$

[2] A statement equivalent to this remark is proven in [8] for any pair of matching graphs. All the following can be restated in this more general case

[3] Here and in the following we will not distinguish between chains and their images

We have

$$S(\mu)=\sum_{k=1}^{\infty} k p_k(\mu)=\sum_{k=0}^{\infty} P_k(\mu). \tag{2.3}$$

We denote with a bar, –, the functions analogous to $P_k(\mu)$, $P_\infty(\mu)$, $S(\mu)$ referred to as $(-)^*$clusters.

We are particularly interested in the case of the measures:

$$\mu_x=\prod_{i\in\mathbb{Z}^2} v_x \tag{2.4}$$

where $x\in[0,1]$ and v_x is the measure on $\{-1,1\}$ which assigns weights x and $1-x$ respectively to 1 and -1. In the case of the measures (2.4) we shall write simply $p_k(x)$, $P_k(x)$, ... instead of $p_k(\mu_x)$, $P_k(\mu_x)$, ...

We have

$$p_k(x)=\sum_{|\gamma|=k} x^{|\gamma|}(1-x)^{|\partial\gamma|}, \tag{2.5}$$

$$P_0(x)+P_\infty(x)=x. \tag{2.6}$$

If Λ is a square in $\mathbb{Z}^2$ and we call $P_\Lambda(x)$ the μ_x-probability that the origin is $(+)$connected with $\partial\Lambda$, we have:

$$P_\infty(x)=\operatorname*{Inf}_{\Lambda\ 0} P_\Lambda(x). \tag{2.7}$$

(2.7) implies that $P_\infty(x)$ is an upper semicontinuous function; this and the obvious non-decreasing property of $P_\infty(x)$ imply that $P_\infty(x)$ is a right-continuous function.

The critical points are defined by:

$$p_c=\operatorname{Sup}\{x\,|\,P_\infty(x)=0\},\qquad p_c^*=\operatorname{Sup}\{x\,|\,P_\infty^*(x)=0\}.$$

Note that if we call E_∞ the event that there exists in ω an infinite $(+)$cluster, the zero-one law implies that $P_\infty(x)>0$ if and only if $\mu_x(E_\infty)=1$, so that p_c can also be defined as the separation element between the x's such that $\mu_x(E_\infty)=0$ and those such that $\mu_x(E_\infty)=1$.

It is easy to check that $0<p_c<1$: one can get crude bounds on p_c by applying the Peierls argument.

3. An Improvement of Harris' Theorem

Our starting point is the following

Theorem 1. *If* $\bar{P}_\infty(x)>0$, *then* $P_\infty(x)=0$.

This theorem was first proved by Harris ([6]) in the particular case of the bond problem on the square lattice, but, as Fisher has remarked ([7]), the proof (for two-dimensional graphs) works quite in general. Recently it has been extended, in a slightly weaker form, to the case of equilibrium measures for the two-dimensional Ising model ([9, 10]).

Theorem 1 is equivalent to the relation $p_c + p_c^* \geqq 1$. In order to get exact relations between the critical points one should require, besides Theorem 1, some existence theorem for infinite clusters. On the other hand it is clear that Harris' theorem cannot be inverted, because it implies that in those cases (bond problem in the square lattice or site problem in the triangular lattice) in which connection and *connection coincide both P_∞ and $\bar{P}_\infty$ must be zero in $x=1/2$.

We observe, however, that the following proposition holds:

Proposition 1. *If μ is a translation-invariant measure on Ω, $P_\infty(\mu)=0$ and $S(\mu)<\infty$, then $\bar{P}_\infty(\mu)>0$.*

Proof. We consider the event

$$E_k = \{\omega \in \Omega \mid \text{in } \omega\ (k,0) \text{ is } (+)\text{connected with the 2-axis}\}.$$

We have

$$\sum_{k=0}^{\infty} \mu(E_k) \leqq \sum_{k=0}^{\infty} P_k(\mu) = S(\mu) < \infty.$$

Hence the Borel-Cantelli lemma implies that μ-a.e. at most a finite number of the events E_k can occur. For a given configuration ω, if none of the E_k's occurs the origin $(0,0)\in\mathbb{Z}^2$ belongs to an infinite $(-)$*cluster. Also, if a finite, non-zero, number of E_k's occur and n is the largest integer such that $\omega\in E_n$, it is clear that $\omega((n+1,0))=-1$, and the point $(n+1,0)$ cannot belong to a finite $(-)$*cluster γ, because in this case $\partial^*\gamma$ should intersect the 1-axis in some point $(N,0)$ such that $N>n+1$, $\omega\in E_N$. Hence μ-a.e. ω contains an infinite $(-)$*cluster.

Proposition 1 means that if there are neither infinite $(+)$clusters, nor infinite $(-)$*clusters, both $S(x)$ and $\bar{S}(x)$ must diverge. This is intuitively clear, because in this case each site in $\mathbb{Z}^2$ is "surrounded" by infinitely many finite clusters of both types. Actually, for the measures μ_x, this is the only case in which the mean cluster size can diverge. In order to prove this we need some additional lemmas.

Lemma 1 (FKG inequalities). *If we consider in Ω the partial ordering $\leqq$ defined setting $\omega_1 \leqq \omega_2$ if and only if $\omega_1(x)\leqq\omega_2(x)$ for all $x\in\mathbb{Z}^2$ and we call an event A positive [negative] if its characteristic function is increasing [decreasing], then if A and B are both positive (or both negative)*

$$\forall x\in[0,1] \qquad \mu_x(A\cap B)\geqq\mu_x(A)\mu_x(B). \tag{3.1}$$

The inequalities (3.1), known as FKG inequalities in the general case of ferromagnetic measures, were first proved in the present case by Harris in [6].

If L is a positive integer, we call $R_{L,1}(x)$ the μ_x-probability that two given opposite sides of the square

$$\Lambda_L = \{(x_1,x_2)\in\mathbb{Z}^2 \mid |x_1|\leqq L, |x_2|\leqq L\}$$

are connected in Λ_L by a $(+)$chain.

Lemma 2. *If $P_\infty(x)>0$, then $\lim_{L\to\infty} R_{L,1}(x)=1$.*

Proof. If $P_\infty(x)>0$, Theorem 1 implies that μ_x-a.e. there are infinitely many $(+)$circuits surrounding the origin (otherwise, if n is the largest integer such that $(n,0)$ belongs to such a circuit, the site $(n+1,0)$ should belong to an infinite $(-)^*$cluster). μ_x-a.e. all but at most a finite number of these circuits must belong to the same infinite $(+)$cluster (otherwise no infinite $(+)$cluster should exist). Hence for any $\varepsilon>0$ there is L_0 such that for all $L\geqq L_0$ the event A_L "there is in Λ_L a $(+)$circuit surrounding the origin and included in an infinite $(+)$cluster" has μ_x-measure bigger than $1-\varepsilon$. We call A_L^i $(i=1,\dots 4)$ the event that there is in Λ_L a $(+)$circuit surrounding the origin and $(+)$connected in Λ_L with the i-th side of Λ_L. The symmetry properties of μ_x and the inequalities (3.1) imply:

$$\mu_x\left(\bigcap_{i=1}^4 \tilde{A}_L^i\right)\geqq[\mu_x(\tilde{A}_L^i)]^4$$

(here and in the following " $\sim$ " means complementation). On the other hand it is obvious that

$$\bigcap_{i=1}^4 \tilde{A}_L^i\subset\tilde{A}_L;$$

hence $\mu_x(\tilde{A}_L^i)\leqq\varepsilon^{1/4}$ and by using again inequalities (3.1)

$$\mu_x(A_L^i\cap A_L^j)\geqq(1-\varepsilon^{1/4})^2.$$

If i and j are referred to two opposite sides we get

$$R_{L,1}(x)\geqq\mu_x(A_L^i\cap A_L^j)\geqq(1-\varepsilon^{1/4})^2,$$

and this ends the proof.

The proof of the following lemma is based on arguments similar to the ones used in [6] in the proof of Theorem 1.

We call $R_{L,2}(x)$ the μ_x-probability that the two opposite smaller sides of the rectangle

$$\Lambda_{L,2}=\{(x_1,x_2)\in\mathbb{Z}^2\,|\,|x_1|\leqq 2L,|x_2|\leqq L\}$$

are connected in $\Lambda_{L,2}$ by a $(+)$chain.

Lemma 3. *If* $P_\infty(x)>0$, *then* $\lim_{L\to\infty} R_{L,2}(x)=1$.

Proof. Given $\varepsilon>0$, we choose two integers n, L_0 such that, for all $L\geqq L_0$, with μ_x-probability bigger than $1-\varepsilon$:

a) there is in Λ_n a $(+)$circuit surrounding the origin and belonging to the infinite $(+)$cluster.

b) there is in Λ_L a $(+)$chain connecting two given opposite sides of Λ_L.

c) there is a $(+)$circuit surrounding the origin and contained in $\Lambda_L\backslash\Lambda_n$.

The existence of n, L_0 follows from Lemma 2 and the arguments used in its proof. Let $L\geqq L_0$. Besides Λ_n and Λ_L, we consider the two translates of Λ_L

$$B_1=\{(x_1,x_2)\in\mathbb{Z}^2\,|\,-2L\leqq x_1\leqq 0;|x_2|\leqq L\};$$
$$B_2=\{(x_1,x_2)\in\mathbb{Z}^2\,|\,0\leqq x_1\leqq 2L;|x_2|\leqq L\}.$$

We call $s_i(\Lambda)$ ($i=1, \ldots, 4$) the four sides of a square Λ, where $s_1(\Lambda)$ is the "left side" and the others are numbered clockwise. Using b) and the inequalities (3.1) in the same way as in the proof of Lemma 2 we get that with μ_x-probability bigger than $1-\varepsilon^{1/2}$ there is in B_1 a $(+)$chain connecting $s_1(B_1)$ with $s_3(B_1)$ and having as endpoint a site of the 2-positive-half-axis; using (3.1) and c) we get that with μ_x-probability bigger than $(1-\varepsilon^{1/2})(1-\varepsilon)$ such a chain can be found in $B_1 \setminus \Lambda_n$. We call C the set of the chains of this type. If $c \in C$ we call c' the chain obtained by reflecting c with respect to the 2-axis and we call $S_c[S_{c'}]$ the set of sites in $B_1[B_2]$ "below $c[c']$". We order C by putting $c_1 \leqq c_2$ if $S_{c_1} \subseteq S_{c_2}$. It is easy to see that if in a given configuration ω there is a non-empty set I of $(+)$chains belonging to C, then there is in I only one maximal element, namely the chain $c_I = (\partial^*(\bigcup_{c \in I} S_c)) \cap B_1$.

We define

$$D_c = \{\omega \in \Omega \mid \text{in } \omega \ c \text{ is the maximal } (+)\text{chain connecting } s_1(B_1) \text{ with } s_3(B_1) \text{ and it is } (+)\text{connected in } B_1 \text{ with } s_2(B_1)\}.$$

We have:

$$\mu_x\left(\bigcup_{c \in C} D_c\right) = \sum_{c \in C} \mu_x(D_c) \geqq (1-\varepsilon)^2 (1-\varepsilon^{1/2}) \tag{3.2}$$

where we have used the fact that if there is a $(+)$chain in B_1 connecting $s_2(B_1)$ with $s_4(B_1)$, then c is certainly $(+)$connected with $s_2(B_1)$.

We consider now the following events:

$$F = \{\omega \in \Omega \mid \text{in } \omega \text{ there is in } \Lambda_n \text{ a } (+)\text{circuit surrounding the origin and } (+)\text{connected in } \Lambda_L \text{ with } s_2(\Lambda_L)\}.$$

$$E_c[E_{c'}] = \{\omega \in \Omega \mid \text{in } \omega \ c[c'] \text{ is } (+)\text{connected in } S_c \cup S_{c'} \text{ with a } (+)\text{circuit surrounding the origin}\}.$$

a) and the inequalities (3.1) imply that $\mu_x(F) \geqq 1-\varepsilon^{1/4}$. On the other hand, for any $c \in C$, $\tilde{E}_c \cap \tilde{E}_{c'} \subset \tilde{F}$ and $\mu_x(E_c) = \mu_x(E_{c'})$; hence we have

$$\mu_x(\tilde{E}_c)^2 = \mu_x(\tilde{E}_c)\, \mu_x(\tilde{E}_{c'}) \leqq \mu_x(\tilde{E}_c \cap \tilde{E}_{c'}) \leqq \mu_x(\tilde{F}) \leqq \varepsilon^{1/4}$$

so that

$$\forall c \in C \qquad \mu_x(E_c) \geqq 1-\varepsilon^{1/8}. \tag{3.3}$$

We consider now the event $A = \bigcup_{c \in C} (D_c \cap E_c)$. (3.2), (3.3) and the remark that for any $c \in C$ D_c and E_c are independent events imply:

$$\mu_x(A) = \sum_{c \in C} \mu_x(D_c)\, \mu_x(E_c) \geqq (1-\varepsilon)^2 (1-\varepsilon^{1/2})(1-\varepsilon^{1/8}).$$

We call A' the event obtained from A by reflection with respect to the 2-axis. A and A' are positive events; hence

$$\mu_x(A \cap A') \geqq (1-\varepsilon)^4 (1-\varepsilon^{1/2})^2 (1-\varepsilon^{1/8})^2.$$

The lemma is proved by the remark that if $A \cap A'$ occurs, then there is a $(+)$chain in $B_1 \cup B_2$ connecting $s_1(B_1)$ with $s_3(B_2)$.

Lemma 4. *If* $P_\infty(x)>0$, *then* $\lim_{L\to\infty} R_{L,3}(x)=1$ *(where* $R_{L,3}(x)$ *is defined in an obvious way).*

Proof. This lemma is an easy corollary of Lemma 3; it suffices to observe that if we consider a rectangle R of sides L and $3L$ as the union of three adjacent squares Q_1, Q_2, Q_3 and if there are $(+)$chains connecting the opposite smaller sides of the rectangles $Q_1 \cup Q_2$ and $Q_2 \cup Q_3$ and there is a $(+)$chain connecting the two sides of Q_2 adjacent to $\tilde{R}$ (note that all these are positive events, so that (3.1) can be applied), then there is also in R a $(+)$chain connecting the smaller sides of R.

Theorem 2. $P_\infty(x)>0$ *if and only if* $\bar{P}_\infty(x)=0$ *and* $\bar{S}(x)<\infty$.

Proof. The "if" part of the theorem is proved by Proposition 1.

Suppose $P_\infty(x)>0$, given $\varepsilon>0$, Lemma 4 implies that we can choose an integer $L_0(x)$ such that for any $L \geqq L_0(x)$, $R_{L,3}(x)>1-\varepsilon$. We consider the sequence of squares Λ_{L_n}, $L_n=3^n L_0(x)$. By applying once more the inequalities (3.1) it is easy to check that for any $n \geqq 1$, a $(+)$circuit surrounding the origin is contained in $\Lambda_{L_n} \setminus \Lambda_{L_{n-1}}$ with μ_x-probability bigger than $(1-\varepsilon)^4$. The origin can belong to a $(-)^*$cluster of size greater than $a_n = 4 \cdot 3^{2n}(L_0(x))^2$ only if for all $i \in \{1, \ldots, n\}$ such a circuit does not exist in $\Lambda_{L_i} \setminus \Lambda_{L_{i-1}}$.

Hence we have:

$$\bar{P}_{a_n}(x) \leqq (1-(1-\varepsilon)^4)^n \tag{3.4}$$

(3.4) and the remark that $\bar{P}_k(x)$ is a non-increasing sequence implies:

$$\begin{aligned}
\bar{S}(x) &= \sum_{k=0}^{\infty} \bar{P}_k(x) = \sum_{k=0}^{a_0} \bar{P}_k(x) + \sum_{n=0}^{\infty} \sum_{k=a_n+1}^{a_{n+1}} \bar{P}_k(x) \\
&\leqq 4(L_0(x))^2 + \sum_{n=0}^{\infty} (a_{n+1}-a_n) \bar{P}_{a_n}(x) \\
&\leqq 4(L_0(x))^2 \left[1 + 8 \sum_{n=0}^{\infty} 9^n (1-(1-\varepsilon)^4)^n\right].
\end{aligned} \tag{3.5}$$

The last series is convergent if ε is small enough. This, together with Theorem 1, proves Theorem 2.

4. The Percolation Probability

In this section we consider the consequences of Theorem 2 on the regularity properties of the function $P_\infty(x)$. For this we need a slight improvement of Theorem 2.

If in the configuration ω there is an infinite $(+)$cluster C, we call "hole" of ω a maximal *connected component of $\mathbb{Z}^2 \setminus C$. It is easy to see that, if H is a hole, the set $B(H) = H \cap \partial C$ is included in a $(-)$*cluster and $|B(H)|^2 \geqq |H|$ [4].

We consider the function $h(\omega)$ defined as the size of the hole to which the origin belongs if it belongs to any hole at all and zero otherwise, and we put

$$q_k(x) = \mu_x(h^{-1}(k)); \qquad Q_k(x) = \sum_{n=k+1}^{\infty} q_n(x). \tag{4.1}$$

The expected value for the measure μ_x of the function $h(\omega)$ is

$$H(x) = \sum_{k=0}^{\infty} k q_k(x) = \sum_{k=0}^{\infty} Q_k(x). \tag{4.2}$$

The following proposition holds as a corollary of Theorem 2:

Proposition 2. *If* $P_\infty(x) > 0$, *then* $H(x) < \infty$.

Proof. We use the same notations as in the proof of Theorem 2 and rewrite (3.4) in the form:

$$\bar{P}_{a_n}(x) \leqq \theta^n. \tag{4.3}$$

We consider the square $\Lambda_{a_n^2}$. (4.3) and the inequalities (3.1) imply that the event "no site in $\Lambda_{a_n^2}$ belongs to a $(-)$*cluster of size greater than a_n" has a μ_x-probability greater than

$$(1-\theta^n)^{(2a_n^2+1)^2} \geqq (1-\theta^n)^{5a_n^4}.$$

On the other hand if the origin belongs to a hole H of size between a_n^2 and a_{n+1}^2, then $B(H)$ must intersect $\Lambda_{a_n^2}$ and all sites in $B(H)$ belong to a $(-)$*cluster greater than a_n. Hence we get:

$$\sum_{k=a_n^2+1}^{a_{n+1}^2} q_k(x) \leqq 1-(1-\theta^n)^{5a_n^4} = 1-(1-\theta^n)^{5(2L_0(x))^8 3^{8n}}$$
$$\leqq 5(2L_0(x))^8 (3^8\theta)^n,$$

$$Q_{a_n^2}(x) = \sum_{k=a_n^2+1}^{\infty} q_k(x) = \sum_{h=n}^{\infty} \sum_{k=a_h^2+1}^{a_{h+1}^2} q_k(x) \leqq 10(2L_0(x))^8 (3^8\theta)^n \tag{4.4}$$

(where we have supposed $3^8\theta < 1/2$, as it is certainly true if $L_0(x)$ is big enough). The bound (4.4) together with the remark that $Q_k(x)$ is a non-increasing sequence imply the convergence of (4.2).

Proposition 3. *If* $P_\infty(x) > 0$, *then* $S(x) < \infty$.

Proof. It is an immediate corollary of Proposition 2.

Proposition 4. *If* $0 < x < 1$, $x \neq p_c$, *then* P_∞ *is infinitely many times derivable in* x.

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[4] This inequality can be proven by observing that if we put

$L = \sup_{i=1,2} \sup_{x,y \in H} |x_i - y_i|$, then $|H| \leqq L^2$, $|B(H)| \geqq L$

Proof. If $x<p_c$ the statement is trivially true. Suppose $x>p_c$. We observe that

$$P_\infty(x)=x-\sum_{k=1}^{\infty} p_k(x)=x-\sum_{\gamma} x^{|\gamma|}(1-x)^{|\partial\gamma|} \tag{4.5}$$

where γ runs over the family of all finite connected subsets of $\mathbb{Z}^2$ containing the origin. Differentiating term by term we get the series:

$$1-(1/x)\sum_{k=1}^{\infty} kp_k(x)+(1/(1-x))\sum_{h=1}^{\infty} hr_h(x) \tag{4.6}$$

where $r_h(x)$ is the μ_x-probability that the boundary of the $(+)$cluster to which the origin belongs has size h. Using the bound (4.4) and taking into account that $Q_{a_n^2}$ is a non-increasing function of x, it is easy to check the uniformity of the convergence of the series (4.6) in any interval (x_1, x_2), $x_1>p_c$, $x_2<1$. In the same way one can check the existence of the higher order derivatives.

5. Concluding Remarks

If one defines, besides p_c and p_c^*, the other two "critical points"

$$\pi_c=\mathrm{Sup}\{x\,|\,P_\infty(x)=0,\, S(x)<\infty\},$$
$$\pi_c^*=\mathrm{Sup}\{x\,|\,P_\infty^*(x)=0,\, S^*(x)<\infty\}.$$

Theorem 2 can be written

$$p_c+\pi_c^*=\pi_c+p_c^*=1. \tag{5.1}$$

If one can prove, as one expects, that the relation $p_c+p_c^*=1$ holds, then (5.1) should imply that $p_c=\pi_c$, so that the mean cluster size should diverge in at most one point. We remark that conversely if one is able to prove that $p_c=\pi_c$ [5], then the relation $p_c+p_c^*=1$ should follow directly from Proposition 1.

The problem above is related to the other open problem of the continuity properties of P_∞ in p_c. In fact the right-continuity of P_∞ implies that it is continuous in p_c if and only if $P_\infty(p_c)=0$; at least for the triangular lattice by Theorem 1 this is certainly true if $p_c=\pi_c$ (so that both are equal to 1/2) because in this case $P_\infty(p_c)=\bar{P}_\infty(p_c)$.

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[5] This should be an assumption very similar to the one made in [8] that the function "mean number of clusters per site" has p_c as unique singular point

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