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DISJOINT OCCURRENCES OF EVENTS:RESULTS AND CONJECTURES

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ABSTRACT.In a recent joint article with H.Kesten the notion "disjoint occurrences of events" has been intoduced. In the present note we give a brief outline of some results and problems presented in the above article and in a joint article with U.Fiebig.

1. INTUITIVE APPROACH. To explain the notion "disjoint occurrences of events" we start with the following example from percolation theory.

(1.1) EXAMPLE. Consider a finite or countably infinite graph G of which each bond b, independent of the other bonds, is open with probability p_b and closed with probability $1-p_b$. Let Q,R,S and T be sites of G. It follows from a correlation inequality of Harris [9] (which is a special case of the well-known FKG inequality [5]) that the event that there exists an open path from Q to R and the event that there exists an open path from S to T are positively correlated, i.e.

(1.2) $$P\{Q \to R,\ S \to T\} \geq P\{Q \to R\}\, P\{S \to T\},$$

or, equivalently,

(1.3) $$P\{Q \to R\} \mid S \to T\} \geq P\{Q \to R\}.$$

This is intuitively plausible because, roughly speaking, the information that there exists an open path from S to T makes the bonds of G more likely to be open, which, in turn, makes it more likely that there exists an open path from Q to R. One would also intuitively expect that, on the other hand, the probability that there are open paths from Q to R and from S to T which are <u>disjoint</u> (i.e. have no bonds in common) is <u>smaller</u> than the product of the individual probabilities, i.e.

(1.4) $$P\{\text{ there exist disjoint open paths from Q to R and from S to T }\} \leq P\{Q \to R\}\, P\{S \to T\}.$$

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This is indeed true and follows from more general results in [3] concerning products of so-called NBU distributions. The following proof, related to that of the clutter theorem (see e.g, [4],[8] and [10]) is more direct. First note that, by obvious limit arguments, we may restrict ourselves to the case that G is finite. Let b be a bond of G.Now replace b by two parallel bonds b' and b" which are both, independent of each other and of all the other bonds, open or closed with the same probabilities as the original bond. Suppose that, in the new graph thus obtained, we allow only those paths from Q to R which do not contain b" and those paths from S to T which do not contain b'. Moreover, we apply this splitting operation step by step to all the bonds of G and at each step we allow only those paths from Q to R which contain only unsplit and '-bonds and those paths from S to T which contain only unsplit and "-bonds. One can check that at each step of this splitting process the l.h.s. of (1.4) increases (i.e. strictly increases or remains unchanged) while both factors in the r.h.s. remain unchanged. The inequality now follows from the above argument and from the following facts: When all bonds have been split, we have obtained two separate copies G' and G" of G so that the event in the l.h.s. of (1.4) is replaced by { there exists an open path from Q to R in G' and an open path from S to T in G"}. The probability of this event is, by independence and symmetry, equal to the r.h.s. of (1.4).

REMARK. It is also easy to check that at each step in the above splitting process the l.h.s. of (1.2) <u>decreases</u> so that this method yields an alternative proof of (1.2).

(1.5) EXAMPLE. Consider again the situation in the first example. However, this time we are not interested in open paths but in alternating paths, i.e. paths which, if followed from one endpoint to the other, exhibit an alternating sequence of open and closed bonds.(These and related models have recently been studied in, e.g. [6]). Simple examples show that the analog of (1.2) fails in this new situation, but we believe that the analog of (1.4) holds. This belief is part of a more general conjecture stated in [3] which will be discussed in the next section.

2. FORMAL STATEMENT OF RESULTS AND CONJECTURES.

Let $\Omega = S_1 \times \ldots \times S_n$ and $\mu = \mu_1 \times \ldots \times \mu_n$, where S_i is a finite subset of $\mathbb{N}$ and μ_i a probability measure on S_i, $i=1,\ldots,n$. Elements of Ω are denoted by $\omega = (\omega_1,\ldots,\omega_n)$.
A set $A \subset \Omega$ is called <u>increasing</u> (<u>decreasing</u>) if $\omega \in A$, $\omega' \in \Omega$, $\omega_i' \geq \omega_i$, $i=1,\ldots,n$ implies $\omega' \in A$.

Harris' inequality, mentioned in section 1, is the following:

(2.1) THEOREM. If $A,B \subset \Omega$ are increasing, then

(2.2) $\mu(A \cap B) \geq \mu(A)\, \mu(B)$.

If $\omega \in \Omega$ and $K \subset \{1,\ldots,n\}$ then the cylinder $[\omega]_K$ is defined as the set $\{\omega' \mid \omega'_i = \omega_i, i \in K\}$. K is called the support of the cylinder.

The set $A \,\square\, B$ of disjoint realisations of A and B is defined as

(2.3) $A \,\square\, B = \{\omega \mid \exists K,L \subset \{1,\ldots,n\} \text{ s.t. } K \cap L = \emptyset, [\omega]_K \subset A, [\omega]_L \subset B\}$.

It has been shown in [3] (as a corollary of results concerning products of NBU distributions) that the probability that two increasing events occur disjointly is smaller than the product of their individual probabilities. In fact [3] only gives a proof for the case that $\Omega = \{0,1\}^n$ but that can easily be extended. So we have:

(2.4) THEOREM. If $A,B \subset \Omega$ are increasing, then

(2.5) $\mu(A \,\square\, B) \leq \mu(A)\, \mu(B)$.

In [3] also the following stronger result has been shown:

(2.6) $\mu\left(\bigcup_{1\leq i\leq k} A_i \,\square\, B_i\right) \leq (\mu\times\mu)\left(\bigcup_{1\leq i\leq k} A_i \times B_i\right)$,

where A_i, B_i are increasing subsets of Ω, $i = 1,\ldots,k$.

Coming back to the example in section 1, if we take $\Omega = \{0,1\}^E$, where E is the set of bonds of G and take $\omega_b = 1$ or 0 according as the bond b is open or closed, then it is clear that (1.2) is a special case of Harris' inequality(2.1) and that (1.4) is a special case of theorem (2.4).

An interesting conjecture in [3] is that the monotonicity condition in theorem 2.4 is immaterial, i.e. that (2.5) holds for all events:

(2.7) CONJECTURE. For all $A,B \subset \Omega$

(2.8) $\mu(A \,\square\, B) \leq \mu(A)\, \mu(B)$.

REMARK. Simple examples show that (2.6) does not hold for all events.

In [2] the above conjecture has been investigated and the following theorem has been proved. First we need some definitions: An event A is called convex if ($\omega,\omega'' \in A$, $\omega' \in \Omega$, $\omega_i \leq \omega'_i \leq \omega''_i$, $i = 1,\ldots,n$) implies $\omega' \in A$. It is not difficult to show that an event is convex if and only if it is the inter - section of an increasing and a decreasing event. An event A is called permutation invariant if permutation of the coordinates of an element of A always yields again an element of A.

Two cylinders C and C' are said to be perpendicular to each other, denoted by $C \perp C'$, if their supports are disjoint.

(2.9) THEOREM. In the following cases inequality (2.8) holds:

a) A and B are both convex.

b) $\Omega = \{0,1\}^n$ and A and B are both permutation invariant.

c) $A = \bigcup_{i\in I} C_i$, $B = \bigcup_{j\in J} C'_j$, where for each $i \in I$ and $j \in J$, C_i and C'_j are cylinders s.t. $C_i \perp C'_j$ or $C_i \cap C'_j = \emptyset$.

d) $A = \bigcup_{i\in I} C_i$, where each C_i is a maximal cylinder of A and for all $i,j \in I$ $\mathrm{supp}(C_i) = \mathrm{supp}(C_j)$ or $\mathrm{supp}(C_i) \cap \mathrm{supp}(C_j) = \emptyset$.

Note that case a) above yields an extension of theorem 2.4 since each increasing event is convex. The proofs of case a) and b) in [2] are based on the splitting method mentioned in section 1. Ref. [2] also shows corollaries and examples of theorem 2.9.

3. APPLICATIONS. Theorem 2.4 appears to be useful in percolation theory. Several inequalities in [1] and [7] follow easily from our theorem, as has been shown in [3]. Some of these inequalities can be strengthened. Our theorem also yields that, for critical bond percolation on the square lattice,

$$P(B_n) \geq 1/(2\sqrt{n}),$$

where B_n is the event that there exist sites at distance $\geq n$ from (0,0) which can be reached from (0,0) by an open path. Applications of inequality 2.6 to first-passage percolation are shown in [11]. Conjecture 2.7 is interesting in itself, but, if it appears to be true, it will probably have applications, e.g. to alternative percolation models (see example 1.5). Example 1.1 in [2] shows an interpretation outside percolation theory.

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