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INEQUALITIES WITH APPLICATIONS TO PERCOLATION AND RELIABILITY

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Abstract

A probability measure μ on $\mathbb{R}_+^n$ is defined to be strongly new better than used (SNBU) if $\mu(A+B) \leqq \mu(A)\mu(B)$ for all increasing subsets $A, B \subset \mathbb{R}_+^n$. For $n = 1$ this is equivalent to being new better than used (NBU distributions play an important role in reliability theory). We derive an inequality concerning products of NBU probability measures, which has as a consequence that if $\mu_1, \mu_2, \cdots, \mu_n$ are NBU probability measures on $\mathbb{R}_+$, then the product-measure $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ on $\mathbb{R}_+^n$ is SNBU. A discrete analog (i.e., with $\mathbb{N}$ instead of $\mathbb{R}_+$) also holds.

Applications are given to reliability and percolation. The latter are based on a new inequality for Bernoulli sequences, going in the opposite direction to the FKG–Harris inequality. The main application (3.15) gives a lower bound for the tail of the cluster size distribution for bond-percolation at the critical probability. Further applications are simplified proofs of some known results in percolation. A more general inequality (which contains the above as well as the FKG–Harris inequality) is conjectured, and connections with an inequality of Hammersley [12] and others ([17], [19] and [7]) are indicated.

CORRELATION INEQUALITIES; FKG INEQUALITY; NEW BETTER THAN USED

1. Definitions and main results

Because our main theorem holds for $\mathbb{R}_+ = [0, \infty)$ as well as $\mathbb{N} = \{0, 1, 2, \cdots\}$ we shall use the symbol R to denote either of these sets.

If $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$, then $x \geqq y$ means $x_i \geqq y_i$, $i = 1, \cdots, n$. A function f on R^n is called *increasing* if $x \geqq y$ implies $f(x) \geqq f(y)$. A subset A of R^n is called increasing if its indicator function (denoted by I_A) is increasing. If A and B are two subsets of R^n, then $A + B \equiv \{a + b \mid a \in A, b \in B\}$. It follows from Dellacherie and Meyer [7], Theorem III.18 and Section III.33a, that $A + B$ is universally measurable when A and B are Borel sets of $\mathbb{R}_+^n$. In particular, $A + B$ belongs to the completion of the Borel σ-field of $\mathbb{R}_+^n$ with respect to each probability measure.

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A probability measure μ on R^n is *strongly new better than used* (SNBU) if

$$\mu(A+B) \leqq \mu(A)\mu(B), \tag{1.1}$$

for all increasing Borel sets $A, B \subseteq R^n$.

For $n = 1$ and $R = \mathbb{R}_+$ this is equivalent to the usual definition of a *new better than used* (NBU) distribution. Therefore, in the one-dimensional case, we can say NBU instead of SNBU (see also Section 2).

Let $n \geqq 2$. For an increasing set $A \subset R^n$ and $i, j \leqq n$, $i \neq j$, we define *the image of A under (i,j)-identification* as the set of all $x \in R^n$ for which there exists an $a \in A$ such that $x_i \geqq a_i + a_j$ and $x_k \geqq a_k$, $k \neq i, j$.

This definition is illustrated by the following example.

(1.2) *Example.* Suppose someone receives a certain amount n_a of apples, n_p of pears and n_b of bananas. He is satisfied if, for a certain increasing set $A \subset \mathbb{N}^3$, $(n_a, n_p, n_b) \in A$. However, if he changes his mind, and wants each pear to be replaced by an apple, then he is satisfied if $(n_a, n_p, n_b) \in A^*$ where A^* is the above-defined image of A under $(1,2)$-identification.

The above definition has the following natural extension. Let A be an increasing subset of R^n and let $\mathcal{F}$ be a partition of the set $\{1, 2, \cdots, n\}$. Choose for each class $F \in \mathcal{F}$ a representative $i_F \in F$. Now *the image of A under identification acccording to the pair* $(\mathcal{F}, \{i_F : F \in \mathcal{F}\})$ is defined as the set of all $x \in R^n$ for which there exists an $a \in A$ such that for each class F: $x_{i_F} \geqq \Sigma_{j \in F}\, a_j$. Again, Theorem III.18 and Section III.33a of Dellacherie and Meyer [7] show that for a Borel set A of $\mathbb{R}^n_+$ its image under identification belongs to the completion of the Borel sets with respect to any probability measure.

(1.3) *Lemma. Let $\mu_1, \cdots, \mu_n$ be* NBU *probability measures on R^n and let $i, j \leqq n$, $i \neq j$ be such that $\mu_i = \mu_j$. Then for all increasing Borel sets $A \subset R^n$*

$$\mu(A) \geqq \mu(A^*), \tag{1.4}$$

where A^* denotes the image of A under (i,j)-identification, and μ is the product-measure $\mu_1 \times \mu_2 \times \cdots \times \mu_n$ on R^n.

Proof. Without loss of generality we may assume $i = 1$, $j = 2$. In terms of random variables (1.4) is equivalent to saying that if $X_1, X_2, \cdots, X_n$ are independent random variables whose distribution on R is NBU, and X_1 and X_2 are identically distributed, then

$$\Pr[(X_1, X_2, \cdots, X_n) \in A] \geqq \Pr[(X_1, X_3, X_4, \cdots, X_n) \in A'], \tag{1.5}$$

where $A' = \{(x_1 + x_2, x_3, x_4, \cdots, x_n) : (x_1, x_2, \cdots, x_n) \in A\} \subset R^{n-1}$. This inequality can now be proved as follows. Given $X_3 = x_3, X_4 = x_4, \cdots, X_n = x_n$, the conditional probability of the event in the left-hand side of (1.5) is, for each x_1, x_2 with $(x_1, x_2, x_3, \cdots, x_n) \in A$, larger than or equal to $\Pr[X_1 \geqq x_1, X_2 \geqq x_2]$. Since X_1 and X_2 are i.i.d. this probability equals $\Pr[X_1 \geqq x_1]\Pr[X_1 \geqq x_2]$. Hence the above-mentioned conditional probability is at least

$$\sup\{\Pr[X_1 \geqq x_1]\Pr[X_1 \geqq x_2]: (x_1, x_2, x_3, \cdots, x_n) \in A\}.$$

On the other hand, the conditional probability of the event in the right-hand side of (1.5) is exactly $\Pr[X_1 \in \{x_1 + x_2: (x_1, x_2, x_3, \cdots, x_n) \in A\}]$ which, because X_1 is a one-dimensional random variable, equals

$$\sup\{\Pr[X_1 \geqq x_1 + x_2]: (x_1, x_2, x_3, \cdots, x_n) \in A]\}$$

and this is, by the NBU property, at most

$$\sup\{\Pr[X_1 \geqq x_1]\Pr[X_1 \geqq x_2]: (x_1, x_2, x_3, \cdots, x_n) \in A\}.$$

(1.6) *Theorem.* (i) *Let $\mu_1, \mu_2, \cdots, \mu_n$ be* NBU *probability measures on R and let $\mathcal{F}$ be a partition of the index set $\{1, \cdots, n\}$, with the property that μ_i's with indices in the same class are identical. Further, choose for each class $F \in \mathcal{F}$ a representative $i_F \in F$ and let, for an increasing Borel set $A \subset R^n$, A^* denote the image of A under identification according to $(\mathcal{F}, \{i_F: F \in \mathcal{F}\})$. Then*

$$\mu(A^*) \leqq \mu(A), \tag{1.7}$$

where μ is the product-measure $\mu_1 \times \mu_2 \times \cdots \times \mu_n$ on R^n.

(ii) *Let $\nu_1, \nu_2, \cdots, \nu_n$ be* NBU *probability measures on R. Denote by ν the product-measure $\nu_1 \times \nu_2 \times \cdots \times \nu_n$ on R^n, and let $A_1, A_2, \cdots, A_k$ and $B_1, B_2, \cdots, B_k$ be increasing Borel sets of R^m. (Hence $\bigcup_{1 \leqq i \leqq k} (A_i \times B_i)$ is a subset of R^{2m} and $\nu \times \nu$ is a probability measure on R^{2m}.) Then*

$$\nu\Big(\bigcup_{1 \leqq i \leqq k} (A_i + B_i)\Big) \leqq (\nu \times \nu)\Big(\bigcup_{1 \leqq i \leqq k} (A_i \times B_i)\Big). \tag{1.8}$$

(iii) *Let $\mu_1, \mu_2, \cdots, \mu_n$ be* NBU *probability measures on R and $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ the product measure. Then, for all increasing Borel sets $A, B \subset R^n$,*

$$\mu(A + B) \leqq \mu(A)\mu(B), \tag{1.9}$$

i.e., μ is SNBU.

Proof. (i) follows by applying Lemma 1.3 successively to all pairs (i, j) with, for some class $F \in \mathcal{F}$, $i = i_F$ and $j \in F$, $j \neq i$.

(ii) If we take, in (i), $n = 2m$, $\mu_1 = \nu_1, \mu_2 = \nu_2, \cdots, \mu_m = \nu_m$, $\mu_{m+1} = \nu_1, \mu_{m+2} = \nu_2, \cdots, \mu_{2m} = \nu_m$ (hence $\mu = \nu \times \nu$), $\mathcal{F}$ the partition with classes $\{1, m+1\}$, $\{2, m+2\}, \cdots, \{m, 2m\}$, and set of representatives $\{1, 2, \cdots, m\}$ and $A = \bigcup_{1 \leqq i \leqq k} (A_i \times B_i)$, then according to (1.7) we get

$$(\nu \times \nu)(A^*) \leqq (\nu \times \nu)(A).$$

This reduces to (1.8) because, as is easily seen,

$$A^* = \Big(\bigcup_{1 \leqq i \leqq k} (A_i + B_i) \Big) \times R^m, \quad \text{so that } (\nu \times \nu)(A^*) = \nu \Big(\bigcup_{1 \leqq i \leqq k} (A_i + B_i) \Big).$$

(iii) follows immediately from (ii) by taking $k = 1$.

(1.10) *Remarks.* (a) Originally we had a different proof, of part (iii) of the above theorem only. We later noticed that the special case of (iii) with all μ_i concentrated on $\{0, 1\}$ can also be derived from results in [12], [17] (or [18]), [19] or [7] (see also Remark 3.5(b)).

(b) We have also proved that if μ is an SNBU probability measure on R^n and ν is an NBU probability measure on R, then the product measure $\mu \times \nu$ on R^{n+1} is SNBU (the proof of this involves some more technicalities than that of (iii)). However, the following problem, which arises naturally in the context of the above results, is still unsolved.

(1.11) *Problem.* Let μ and ν be SNBU probability measures on R^n and R^m respectively. Is the product measure on R^{n+m} always SNBU?

(c) Note that (1.8) proves a stronger property than SNBU for the product measure ν. The following simple argument shows that if ν is a probability measure on R^n then ν has property (1.8) if and only if ν is the product of its one-dimensional marginals and the marginals are NBU. The 'if' part is Theorem 1.6(iii). For the 'only if' part, consider the following events:

$$A_1 = \{X_1 \geqq a_1, \cdots, X_{n-1} \geqq a_{n-1}\}, \quad B_1 = R^n, \quad A_2 = R^n, \quad B_2 = \{X_n \geqq a_n\}.$$

Then $(A_1 + B_1) \cup (A_2 + B_2) = A_1 \cup B_2$ and therefore (1.8) implies

$$\begin{aligned}
&\nu(A_1) + \nu(B_2) - \nu(A_1 \cap B_2) \\
&\quad = \nu(A_1 \cup B_2) \\
&\quad \leqq (\nu \times \nu)((A_1 \times B_1) \cup (A_2 \times B_2)) \\
&\quad = (\nu \times \nu)(A_1 \times B_1) + (\nu \times \nu)(A_2 \times B_2) - (\nu \times \nu)((A_1 \times B_1) \cap (A_2 \times B_2)) \\
&\quad = \nu(A_1) + \nu(B_2) - \nu(A_1)\nu(B_2),
\end{aligned}$$

hence

$$\nu(A_1 \cap B_2) \geqq \nu(A_1)\nu(B_2).$$

On the other hand $A_1 \cap B_2 = A_1 + B_2$, and hence, by (1.8),

$$\nu(A_1 \cap B_2) \leqq \nu(A_1)\nu(B_2).$$

Consequently

$$\nu(X_i \geqq a_i, 1 \leqq i \leqq n) = \nu(X_i \geqq a_i, 1 \leqq i \leqq n-1)\nu(X_n \geqq a_n) = \cdots = \prod_{i=1}^{n} \nu(X_i \geqq a_i).$$

2. Applications to reliability

In reliability theory (for a description of the subject see e.g., Barlow and Proschan [3]) a non-negative random variable T is called NBU if its corresponding probability measure on $\mathbb{R}_+$ is NBU which means that (see Section 1), for all $t_1, t_2 \geqq 0$,

$$P[T > t_1 + t_2 \mid T > t_1] \leqq P[T > t_2], \tag{2.1}$$

or equivalently,

$$P[T > t_1 + t_2] \leqq P[T > t_1]P[T > t_2]. \tag{2.2}$$

Marshall and Shaked [14] introduced a multivariate extension of (2.2) by defining a random vector $T = (T_1, \cdots, T_n)$ to be *multivariate new better than used* (MNBU) if, for all increasing Borel sets $A \subset \mathbb{R}_+^n$ and all $\lambda, \mu \geqq 0$,

$$P[T \in (\lambda + \mu)A] \leqq P[T \in \lambda A]P[T \in \mu A], \tag{2.3}$$

where $\lambda A \equiv \{\lambda a \colon a \in A\}$. The main result in their paper was that if S and T are MNBU and if T and S are independent, then (S, T) is also MNBU (compare with Problem (1.11)). This yielded the following corollary.

Corollary. If $T_1, \cdots, T_n$ are independent NBU *random variables, then*

(i) $T = (T_1, \cdots, T_n)$ *is* MNBU,

(ii) $g(T_1, \cdots, T_n)$ *is* NBU, *whenever g is a non-negative measurable subhomogeneous increasing function.*

(*A* function g on $\mathbb{R}_+^n$ is called *subhomogeneous* if $g(\alpha x) \leqq \alpha g(x)$ for all $x \in \mathbb{R}_+^n$ and all $\alpha \geqq 1$.) This corollary is improved by the following corollary of Theorem 1.6(iii).

(2.4) *Corollary. If $T_1, \cdots, T_n$ are independent one-dimensional* NBU *random variables, then*

(a) *For all increasing Borel sets $A, B \subset \mathbb{R}_+^n$,*

$$P[(T_1, \cdots, T_n) \in A + B] \leqq P[(T_1, \cdots, T_n) \in A]P[(T_1, \cdots, T_n) \in B].$$

(b) $g(T_1, \cdots, T_n)$ *is* NBU *whenever $g \colon \mathbb{R}_+^n \to \mathbb{R}_+$ is a measurable increasing function with the property*

$$g^{-1}(a + b, \infty) \subset g^{-1}(a, \infty) + g^{-1}(b, \infty), \quad \forall a, b > 0, \tag{2.5}$$

where $g^{-1}A \equiv \{x \mid g(x) \in A\}$.

Proof. (a) follows immediately from Theorem 1.6(iii).

(b) Suppose $T_1, \cdots, T_n$ and g fulfill the conditions. Then:

$$\begin{aligned}
&P[g(T_1,\cdots,T_n) > s+t] \\
&\quad = P[(T_1,\cdots,T_n) \in g^{-1}(s+t,\infty)] \\
&\quad \leqq P[(T_1,\cdots,T_n) \in g^{-1}(s,\infty) + g^{-1}(t,\infty)] \\
&\quad \leqq P[(T_1,\cdots,T_n) \in g^{-1}(s,\infty)]P[(T_1,\cdots,T_n) \in g^{-1}(t,\infty)] \\
&\quad = P[g(T_1,\cdots,T_n) > s]P[g(T_1,\cdots,T_n) > t].
\end{aligned}$$

(2.6) *Remarks.* (a) Part (a) of the corollary implies (i) because $(\lambda+\mu)A \subset \lambda A + \mu A$. Part (b) implies (ii) because each increasing non-negative subhomogeneous function has the property (2.5), which can be seen as follows. Let $g: \mathbb{R}_+^n \to \mathbb{R}_+$ be increasing and subhomogeneous and let, for certain $a, b > 0$, $x \in g^{-1}(a+b,\infty)$, i.e., $g(x) > a+b$. Then $g((a+b)^{-1}ax) \geqq (a+b)^{-1}ag(x) > a$ and, analogously, $g((a+b)^{-1}bx) > b$. Hence $x = (a+b)^{-1}ax + (a+b)^{-1}bx$ is the sum of an element of $g^{-1}(a,\infty)$ and an element of $g^{-1}(b,\infty)$.

(b) In studies of NBU random variables these variables usually represent life lengths. However, the following interpretation of Corollary (3.1)(a), in which the variables represent amounts of certain products, might also be interesting. Suppose two people, say A and B, have to share the random output of a certain producer. A wants at least an amount a, B at least an amount b. If the output has an NBU distribution, then, by the definition of NBU, the following statement holds: the probability that the output can be shared such that A and B are both satisfied is not larger than the product of the probability that A would be satisfied if he had the total output for himself and the analogous probability for B. Now consider the case of n producers with independent random outputs, each having an NBU distribution. If A (B) wants at least an amount a_1 (b_1) of the first product, a_2 (b_2) of the second product, etc., then by the independence of the variables, it is obvious that the probability that A and B are both satisfied is still no larger than the product of the probability that A (respectively B), is satisfied. However, Corollary (2.4)(a) says that the statement still holds in the case that A and B are, within certain limits, willing to obtain somewhat less of one product in exchange for somewhat more of some of the other products.

3. Applications to Bernoulli sequences and percolation

Let $\Omega = \{0,1\}^n$. An event in Ω is called *increasing* or *positive* if its indicator function is an increasing function on Ω (i.e., increasing in each coordinate separately). An event is called *decreasing* or *negative* if its complement is increasing.

If A and B are positive events we denote by $A \circ B$ the event that A and B 'occur disjointly'. More precisely, $A \circ B$ is defined as follows. Each $\omega = (\omega_1, \cdots, \omega_n) \in \Omega$ is uniquely determined by the set $K(\omega) \subset \{1, \cdots, n\}$ of all indices i for which $\omega_i = 1$. Now $\omega \in A \circ B$ if and only if there exists a $K' \subset K(\omega)$ such that ω', determined by $K(\omega') = K'$, belongs to A, and ω'', determined by $K(\omega'') = K(\omega) \setminus K'$ belongs to B.

Example. If A is the event {at least k_1 of the ω_i's are equal to 1} and B the event {at least k_2 of the ω_i's are equal to 1}, then $A \circ B$ is the event {at least $k_1 + k_2$ of the ω_i's are equal to 1} (see below for further examples).

It is clear that $A \circ B$ is contained in $A \cap B$. Further, notice that $A \circ B = B \circ A$ and $A \circ (B \circ C) = (A \circ B) \circ C$.

Now let P be the probability measure on Ω under which $\omega_1, \cdots, \omega_n$ are independent and $P[\omega_i = 1] = 1 - P[\omega_i = 0]$. Harris [11] proved that

(3.1) $P[A \cap B] \geqq P[A]P[B]$, if A and B are both positive events,

or, equivalently,

(3.2) $P[A \cap B] \leqq P[A]P[B]$, if A is positive and B is negative.

This inequality, which is one of the basic tools in percolation theory, is now usually considered as a special case of the FKG inequality first proven in [9]. We now show that the inequality (3.1) is reversed if $A \cap B$ is replaced by $A \circ B$. This new inequality turns out to be a special case of Theorem 1.6(iii).

(3.3) *Theorem. If A and B are positive events, then*

$$P[A \circ B] \leqq P[A]P[B]. \tag{3.4}$$

Proof. In order to use Theorem 1.6(iii) we imbed the state space Ω in $\mathbf{N}^n = \{0, 1, \cdots\}^n$. We still use P to denote the image measure under this imbedding. Thus $P[\mathbf{N}^n \setminus \Omega] = 0$ and $P[\{x\}]$ is unchanged if $x \in \Omega$. Further, we replace each positive event $A \subset \Omega$ by the smallest increasing subset $\tilde{A}$ of $\mathbf{N}^n$ containing A. Thus A is replaced by

$$\tilde{A} = \{y \in \mathbf{N}^n : \exists x \in A \text{ such that } x \leqq y\}.$$

This operation does not change the probability of A because only a set of probability 0 is added. One now easily sees that $A \circ B$ differs from $\tilde{A} + \tilde{B}$ by a set of probability 0. In fact $z = (z_1, \cdots, z_n) \in \tilde{A} + \tilde{B}$ can have positive mass only if each z_i equals 0 or 1. Thus, if $z = x + y$, $x \in \tilde{A}$, $y \in \tilde{B}$, then one must actually have $x \in A$, $y \in B$ and the ones among the coordinates of x and y cannot occur at the same place (since $x_i = y_i = 1$ implies $z_i = 2$). Finally, noting that a probability measure on $\mathbf{N}$ with all mass concentrated on $\{0, 1\}$ is always NBU, the theorem follows directly from Theorem 1.6(iii).

(3.5) *Remarks.* (a) Analogously, a special case of Theorem 1.6(ii) is that for positive events $A_1, B_1, A_2, B_2, \cdots, A_k, B_k \subset \Omega$,

$$\begin{aligned} &P[A_1 \circ B_1 \cup A_2 \circ B_2 \cup \cdots \cup A_k \circ B_k] \\ &\leqq (P \times P)[A_1 \times B_1 \cup A_2 \times B_2 \cup \cdots \cup A_k \times B_k]. \end{aligned} \tag{3.6}$$

Roughly speaking, this means that the probability that, for at least one i, A_i and B_i occur disjointly, is smaller than the probability that, for at least one i, A_i and B_i occur on independent copies of the probability space.

(b) In the same way the following result, appearing in various forms in [12], [17], [19] and [7] can be derived as a special case of Theorem 1.6(i). Let $\mathscr{F}$ be a partition of $\{1, \cdots, n\}$ and let $\mathscr{C}$ be a family of subsets of $\{1, \cdots, n\}$ such that for each $C \in \mathscr{C}$ and $F \in \mathscr{F}$, $C \cap F$ contains at most one element. Consider, for a given $p \in [0,1]$, two probability measures P_p and $P_{p,\mathscr{F}}$ on Ω under both of which each ω_i is equal to 1 with probability p and equal to 0 with probability $1-p$ $(i = 1, \cdots, n)$. Under P_p the ω_i, $i = 1, \cdots, n$ are independent. Under $P_{p,\mathscr{F}}$, all ω_i's with indices in the same class are equal with probability 1, while the families $V_F := \{\omega_i : i \in F\}$, $F \in \mathscr{F}$ are independent. Now let A be the event that, for at least one $C \in \mathscr{C}$, $\omega_i = 1$ for all $i \in C$. Then

$$P_{p,\mathscr{F}}[A] \leqq P_p[A]. \tag{3.7}$$

In order to show that this follows from Theorem 1.6(i), imbed Ω again in $\mathbf{N}^n$, and replace A by $\tilde{A}$, exactly as in the proof of Theorem 3.3. Denote the image of P_p under the imbedding of Ω in $\mathbf{N}^n$ by $\tilde{P}_p$. Choose a representative i_F for each class $F \in \mathscr{F}$, and form $(\tilde{A})^*$ from $\tilde{A}$ by identification according to $(\mathscr{F}, \{i_F\})$. One can verify that

$$\begin{aligned} \tilde{P}_p[(\tilde{A})^*] &= P_p\left[\left(\bigcup_{C \in \mathscr{C}} \{x : x_i = 1 \text{ for each } i \in C\}\right)^*\right] \\ &= \tilde{P}_p\left[\bigcup_{C \in \mathscr{C}} \{x : x_{i_F} = 1 \text{ for each } F \text{ with } F \cap C \neq \varnothing\}\right] \\ &= P_{p,\mathscr{F}}\left[\bigcup_{C \in \mathscr{C}} \{x : x_i = 1 \text{ for each } i \in C\}\right] = P_{p,\mathscr{F}}[A]. \end{aligned}$$

(In the second equality we use the fact that $C \cap F$ is either empty or consists of a single element only.) Thus by (1.7)

$$P_{p,\mathscr{F}}[A] = \tilde{P}_p[(\tilde{A})^*] \leqq \tilde{P}_p[\tilde{A}] = P_p[A],$$

which is just (3.7).

Conversely, it is possible to derive (3.4) and (3.6) from (3.7) by applying (3.7) in the space Ω^2 with suitable choices of $\mathscr{F}$ and $\mathscr{C}$.

(c) Ahlswede and Daykin [1] have presented a rather general theory of correlation inequalities, including the FKG inequality. However, it seems that (3.4) does not fit in this framework and it might be the first step in a new direction (see also (d)).

(d) The operation $\circ$ has been defined for positive events only. However, define for arbitrary events A and $B \subset \Omega$ the event $A \square B$ as follows. First, for $\omega = (\omega_1, \cdots, \omega_n) \in \Omega$ and $K \subset \{1, \cdots, n\}$, let $C(K, \omega)$ denote the cylinder event $\{\omega' : \omega' \in \Omega \text{ and } \omega'_i = \omega_i \text{ for all } i \in K\}$. Let $\bar{K}$ denote $\{1, \cdots, n\} \setminus K$. Now define

(3.8) $$A \square B = \{\omega : \exists K \subset \{1, \cdots, n\} \text{ such that } C(K, \omega) \subset A \text{ and } C(\bar{K}, \omega) \subset B\}.$$

Clearly $A \square B \subset A \cap B$. We have the following conjecture:

(3.9) $$P[A \square B] \leqq P[A]P[B] \quad \text{for all events } A \text{ and } B.$$

It is easily seen that if A is positive and B negative, $A \square B$ is exactly $A \cap B$, and if A and B are both positive it equals $A \circ B$, so that (3.9) includes the FKG–Harris inequality as well as our inequality (3.9). Moreover, if the answer to problem (1.11) is affirmative for the case that μ or ν is a probability measure on $\mathbb{N}^2$, concentrated on the elements $(1, 0)$ and $(0, 1)$, then (3.9) follows in a way comparable with the derivation of Theorem 3.3 from Theorem 1.6(iii).

(e) Several special cases of (3.9) are proven in [5].

Examples and applications in percolation theory. Let $\mathcal{G}$ be a finite or countably infinite graph. A *path* from site s to site s' is a finite sequence of the form $(s_1 = s, e_1, s_2, e_2, \cdots, s_{n-1}, s_n = s')$, where each e_i is an edge connecting the sites s_i and s_{i+1}. There is no loss of generality for our purposes if we restrict ourselves to paths which are *self-avoiding* (which means that all s_i's in the above sequence are different). The *length* of a path is the number of edges it contains. Now suppose that the edge is *open* (or passable) with probability p_e and *closed* with probability $1 - p_e$, and that all these events for different edges are independent. A path or, more generally, a subgraph, is said to be open if all its edges are open. An *open cluster* is a maximal connected open subgraph of $\mathcal{G}$. Percolation theory (introduced by Broadbent and Hammersley [6]) studies questions like: what is the probability of the existence of an open path between two specified sites, and (in the case where $\mathcal{G}$ is infinite) do there exist, with positive probability, infinite open clusters? The above case is called *bond-percolation.* If, instead of the edges, the sites of $\mathcal{G}$ are randomly open or closed, one speaks of *site-percolation.* For a recent introduction to these problems see, for example, [14], Chapter 1. Also models have been studied in which the edges are only passable in one direction (see for example [9]).

The following special case of Theorem (3.3) is useful in percolation theory (see also (3.12) below).

(3.10) *Corollary. Let, for some $k \geqq 2$, $V_1, V_2, \cdots, V_k$ be sets of paths of a* *graph $\mathscr{G}$. Assume that all the edges (sites) of $\mathscr{G}$ are independently open or closed. Call two paths disjoint if they have no edge (site) in common. Let E_i, $i = 1, \cdots, k$, be the event that at least one of the paths in V_i is open. Then:*

$$\begin{aligned} &P[\text{there exist pairwise-disjoint open paths } \pi_1 \in V_1, \pi_2 \in V_2, \cdots, \pi_k \in V_k] \\ (3.11) \qquad &\leqq P[E_1]P[E_2]\cdots P[E_k]. \end{aligned}$$

Proof. We may restrict ourselves to the case where $\mathscr{G}$ is finite (by obvious limit arguments). Now if we take $\Omega = \{0, 1\}^E$, where E is the set of edges of $\mathscr{G}$ ($\Omega = \{0, 1\}^S$, where S is the set of sites of $\mathscr{G}$) and take $\omega_e = 1$ or 0 ($\omega_s = 1$ or 0) according as the edge e (site s) is open or closed, then it is not difficult to see that the event in the left-hand side of (3.11) corresponds with $E_1 \circ E_2 \circ \cdots \circ E_k$ and the result follows by repeated application of Theorem (3.3).

(3.12) *Remark.* By using (3.6) or (3.7) one can also derive a similar result in first-passage percolation (see [13], Section 4).

The following result is a simple proof of the first tree graph bound of Aizenman and Newman ([2], Proposition 4.1). Their bounds for higher connectivity functions can be derived in the same way. Let $t(v, w) = P$ [v is connected to w by an open path].

(3.13) *Corollary. Consider bond-percolation on a graph $\mathscr{G}$. Let s_1, s_2 and s_3 be sites of $\mathscr{G}$. Then*

$$\begin{aligned} &P[s_1, s_2 \text{ and } s_3 \text{ belong to the same open cluster}] \\ (3.14) \qquad &\leqq \sum_{\substack{s \text{ a site} \\ \text{of } \mathscr{G}}} t(s_1, s)t(s_2, s)t(s_3, s). \end{aligned}$$

Proof. The result follows by using Corollary 3.10 and the observation that s_1, s_2 and s_3 belong to the same open cluster if and only if there exists a site s (which may be equal to one of the s_i's such that there are disjoint open paths from s_1 to s, from s_2 to s and from s_3 to s, respectively.

The nicest application is an improvement of a result for critical percolation in two dimensions. As an example we consider bond percolation on the square lattice, which is the graph with sites $\{(n, m) \mid n, m \in \mathbb{Z}\}$. (It is easy to derive analogous results for other two-dimensional lattices.) On this graph each site (n, m) has exactly four edges incident to it, namely those between (n, m) and the sites $(n \pm 1, m \pm 1)$. Suppose all edges are independently open with probability p, and denote the corresponding probability measure by P_p. Let B_n be the event that there exists an open path from the origin to some site at distance $\geqq n$ from the origin. (The distance from (n_1, n_2) to (m_1, m_2) is defined as $|n_1 - m_1| + |n_2 - m_2|$.) Clearly $P_p[B_n]$ is decreasing in n. It is known ([12], p. 54 and Theorem 5.1) that for $p<\frac{1}{2}$ there exists a $\lambda(P)<1$ such that $P_p[B_n]<\lambda^n(p)$, while for $p>\frac{1}{2}$ $\lim_{n\to\infty} P_p[B_n]>0$. When p is equal to the critical probability $\frac{1}{2}$ then $P_p[B_n]$ tends to 0, but not exponentially. Smythe and Wierman ([16], p. 61) gave an easy proof of $P_{\frac{1}{2}}[B_n]\geqq 1/2n$. Later Kesten ([12], Theorem 8.2) showed that there exist $C, \gamma>0$ such that $P_{\frac{1}{2}}[B_n]>Cn^{-1+\gamma}$. However, the value of γ which follows from his calculations appears to be very small. It is believed that $P_{\frac{1}{2}}[B_n]\sim Cn^{-\delta}$ for some $C>0$, $0<\delta<1$ (see [17]). Even though we cannot prove such a power law, the following result greatly improves the estimates for γ obtainable from [12]. The proof uses a refinement of Smythe and Wierman's idea and Corollary (3.10). (Another proof can be based on the (known) inequality (3.17).)

(3.15) *Corollary.*

$$P_{\frac{1}{2}}[B_n]\geqq\frac{1}{2\sqrt{n}}.$$

Proof. Consider the subgraph $S(n)$ of S which consists of the part of S situated in the rectangle $0\leqq x\leqq 2n, 0\leqq y\leqq 2n-1$. It is well known from duality arguments (see [15], or [16], p. 31) that the $P_{\frac{1}{2}}$-probability that there exists an open path which lies in $S(n)$ and which connects the left-hand edge of $S(n)$ with its right-hand edge equals $\frac{1}{2}$. Further, it is clear that such a path passes through at least one of the sites $\{n\}\times[0,2n-1]$. Hence at least one of the $2n$ sites in the above set has two disjoint open connections with the left- and right-hand edge of $S(n)$, respectively. Also, the distance between a site in $\{n\}\times[0,2n-1]$ and a site in the left- or right-hand edge of $S(n)$ is always $\geqq n$. Consequently, by Corollary (3.10)

$$\tfrac{1}{2}\leqq\sum_{i=0}^{2n-1} P_{\frac{1}{2}}[(n,i) \text{ is connected by two disjoint open paths}$$

$$\text{to the left and right edge of } S(n)]\leqq 2n\{P_{\frac{1}{2}}[B_n]\}^2.$$

Lastly we give a new and simplified proof of a result of Hammersley [10]. First consider bond-percolation on a graph $\mathscr{G}$. By the *distance* between two sites of $\mathscr{G}$ we mean the minimal number of edges in any path which connects these sites. For any site s of $\mathscr{G}$ define

$$N_n(s) = \text{collection of sites at distance} \leqq n \text{ from } s,$$

$$B_n(s) = \text{collection of sites at distance exactly } n \text{ from } s,$$

$$P_n(s) = P[\exists \text{ open path from } s \text{ to a site in } B_n(s)]$$

$$\text{if } n\geqq 1, \text{ and } P_0(s)=1.$$

We say that a path belongs to $N_n(s)$ if all sites of the path, except for its endpoint, lie in $N_n(s)$, and we define, for $n\geqq 1$,

$E_n(s)$ = expected number of sites $s' \in B_n(s)$ for which there exists an open path from s to s' belonging to $N_{n-1}(s)$.

We take $E_0(s) = 1$. Finally, for $n \geqq 0$ we set

$$P_n = \sup_s P_n(s), \quad E_n = \sup E_n(s). \tag{3.16}$$

Hammersley [10] has proved that

$$P_n \leqq (E_m)^{\lfloor n/m \rfloor}, \tag{3.17}$$

where $\lfloor n/m \rfloor$ is the integer part of n/m. A direct consequence of this result is that if the expected size of the open cluster is finite, then the radius of the open cluster has a distribution with an exponentially bounded tail (see also [12], Section 5.1 and [2], Section 5 for a stronger result). Here we give an easy proof of the following inequality which is somewhat stronger than (3.17) (since by induction (3.18) will imply $P_{nm} \leqq (E_m)^n$).

(3.18) *Corollary.*

$$P_{n+m} \leqq E_m P_n, \qquad n, m \geqq 0.$$

Proof. If n or m equals 0 the result is trivial. Assume $n, m > 0$ and fix s. Suppose there exists an open path from s to $B^{n+m}(s)$. Denote by s' the first site on the path (starting from s) which lies in $B_m(s)$. Then, clearly, there exist two disjoint open paths, the first from s to s' and belonging to $N_{m-1}(s)$, and the second from s' to $B_{n+m}(s)$. Furthermore it is clear that $B_{n+m}(s)$ has distance at least n from s', so that the second path passes through $B_n(s')$. Thus

$$P_{n+m}(s) \leqq \sum_{s' \in B_m(s)} P[\exists \text{ two disjoint open paths, one from } s \text{ to } s' \text{ and belonging to } N_{m-1}(s), \text{ and the other from } s' \text{ to some site in } B_n(s')].$$

By Corollary 3.10 this expression is at most

$$\sum_{s' \in B_m(s)} P[\exists \text{ open path from } s \text{ to } s' \text{ which belongs to } N_{m-1}(s)] P_n(s')$$
$$\leqq E_m(s) P_n.$$

This holds for all s, so that (3.18) follows.

If one considers site percolation then (3.18) remains valid (and the proof goes through practically unchanged) provided one redefines P_n and E_n as follows: $N_n(s)$, $B_n(s)$ and (3.16) remain as before, but

$$P_n(s) = P[\exists \text{ open path from a neighbor of } s \text{ to a site of } B_n(s)], \quad n \geqq 1, \quad P_0(s) = 1,$$

$$E_n(s) = \text{expected number of sites } s' \in B_n(s) \text{ for which there exists an open path from a neighbor of } s \text{ to } s' \text{ and belonging to } N_{n-1}(s), \quad n \geqq 1, \quad E_0(s) = 1.$$

Another application of Corollary 3.10 is to be found in van den Berg [4], where it is used to prove that for one-parameter bond-percolation on $\mathbb{Z}^2$ the site (0, 0) always has at least as high a probability of being connected by an open path to (1,0) as to (2, 0).

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