Brownian Motion/2

Construction of Brownian Motion:

Nevertheless, in a miraculous way, BM exists, as a decent mathematical object!!!
Soft remarks:

1) Sufficient to prove for $t \in [0, 1]$:
   let $B_k(t)$, $k=1, 2, \ldots$ be independent BM-s on $t \in [0, 1]$ and
   
   $x \mapsto B(x)$, $x \in [0, \infty)$

defined as:

   $B_n(t) = \sum_{k=1}^{\lfloor t \rfloor} B_k(1) + B_{\lfloor t \rfloor + 1}(t - \lfloor t \rfloor)$

BM-s glued together at endpoints.

2) Sufficient to consider the $\delta = 1$ case

   $B_n(\cdot) = \delta B_n(\cdot)$ variance $\delta^2$

   variance 1
Sketch of N. Wiener's proof: \( t \in (0,1) \)

Idea: Try expansion with respect to an orthonormal basis in \( L^2(0,1) \), with independent Gaussian coefficients.

Let \( \{ \psi_n(t) \}_{n=1}^{\infty} \) be orthonormal basis in \( L^2(0,1) \)

\[
\int_0^1 \psi_n(t) \psi_m(t) \, dt = \delta_{n,m}
\]

(to be specified later)

Let \( \{ \xi_n \}_{n=1}^{\infty} \) be i.i.d. \( N(0,1) \) random variables.

Let \( \{ C_n \}_{n=1}^{\infty} \) be real constants

(to be specified later)

And write (formally)

\[
B(t) = \sum_{n=1}^{\infty} C_n \xi_n \psi_n(t)
\]
Hope that \((\Psi_n)_{n=1}^\infty\) and \((C_n)_{n=1}^\infty\) can be chosen so that the r.h.s. converges in suff. strong sense the result has the desired distribution (Gaussian with corr. min \((t, s)\)).

\[ a) \]
\[
E(\mathcal{B}(t)\mathcal{B}(s)) = E\left(\sum_{n=1}^{\infty} C_n \frac{3}{n} \Psi_n(t) \left(\sum_{m=1}^{\infty} C_m \frac{3}{m} \Psi_m(s)\right)\right)
\]
\[
= \sum_{n,m=1}^{\infty} C_n C_m \Psi_n(t) \Psi_m(s) \quad E(\frac{3}{n}, \frac{3}{m})
\]
\[
= \sum_{n=1}^{\infty} C_n \Psi_n(t) \Psi_n(s)
\]
\[
= \min(t, s)
\]

the two must be equal
Define on $L^2(0,1)$ the bounded operator $K \in B(L^2(0,1))$

$$Kf(t) := \int_0^1 K(t,s)f(s) \, ds$$

with kernel $K(t,s) = \min(t,s)$

$K = K^*$, compact op. (actually: Hilbert-Schmidt)

has same properties as self-adj. matrices

Full set of eigenvectors/eigenvalues:

$$K \psi_n = \lambda_n \psi_n, \quad n = 1, 2, \ldots$$

and

$$K(t,s) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t) \psi_n(s)$$

For this particular kernel

$$\lambda_n = \left(\frac{n-\frac{1}{2}}{\frac{1}{4}}\right)^2, \quad \psi_n(t) = \sqrt{2} \sin\left(n-\frac{1}{2}\right) t \cos(n-\frac{1}{2}) t; \quad n = 1, 2, \ldots$$

HW: check these
good candidate:

\[ B(t) = \sum_{n=1}^{\infty} \left( n - \frac{1}{2} \right)^{-1} \frac{3}{2^n} \sqrt{2} \sin\left( n - \frac{1}{2} \right) t^n \]

\[ \text{i.i.d. } N(0,1) \]

uniform in \( t \)

difficult

Remark: Convergence in \( L^2([0,1]) \) is easy:

also easy: a.s.

\[ \sum_{n=1}^{\infty} \left( n - \frac{1}{2} \right)^{-2} \left| \frac{3}{2^n} \right|^2 < \infty \quad \text{almost surely!} \]

\[ B_n(t) = \sum_{n=1}^{\infty} c_n \frac{3}{2^n} \varphi_n(t) \quad \text{continuous} \]

Need: (almost sure) uniform convergence

Wiener's proof: uniform a.s. along a well chosen subsequence of \( N \), hard.
Paul Lévy's proof (in full detail)

Idea: Sample at diadic rationals $(k2^{-n})^{2^n}_{k=0}$ and interpolate linearly.

Ingredients

$$c_0 = \frac{1}{2^n}, \quad \frac{3}{2^{k+1}}$$

$$n = 1, 2, \ldots$$

$$k = 0, 1, \ldots, 2^{n-1} - 1$$

i.i.d. $\mathcal{N}(0,1)$ random variables indexed by diadic rationals in [0,1].
\[ c_n = 2^{\frac{-n+1}{2}} \]

Successive approximation:

\[ B(0) = 0 \quad B(1) = \frac{3}{4} \]

\[ B\left(\frac{1}{2}\right) = \frac{1}{2} (B(0) + B(1)) + C_1 \frac{3}{4} \]

\[ B\left(\frac{1}{4}\right) = \frac{1}{2} (B(0) + B\left(\frac{1}{2}\right)) + C_2 \frac{3}{4} \]

\[ B\left(\frac{3}{4}\right) = \frac{1}{2} (B\left(\frac{1}{2}\right) + B(1)) + C_2 \frac{3}{4} \]

\[ \vdots \]

\[ B\left(\frac{2k+1}{2^n}\right) = \frac{1}{2} (B\left(\frac{k}{2^n}\right) + B\left(\frac{k+1}{2^n}\right)) + C_{n+1} \frac{3}{4^{2k+1}} \]

\[ k = 0, \ldots, 2^n - 1 \]

\[ B^{(m)}(t) \text{ obtained by linear interpolation between } B\left(\frac{k}{2^n}\right) : k = 0, 1, \ldots, 2^n \]
Proof of almost sure uniform convergence of the sequence $B^{(n)}(t)$:

Note that

$$\sup_{0 \leq t \leq 1} |B^{(n)}(t) - B^{(n)}(t)| = C_{nH} \max_{0 \leq k \leq 2^n - 1} \left| \frac{\xi_{2kH}}{2^{nH}} \right|$$

$$P(\sup_{0 \leq t \leq 1} |B^{(n)}(t) - B^{(n)}(t)| > 2^{-\frac{n}{4}}) =$$

$$P\left( \max_{0 \leq k \leq 2^n - 1} \left| \frac{\xi_{2kH}}{2^{nH}} \right| > 2^{\frac{n+2}{4}} \right) \leq 2^n P\left( \left| \frac{\xi}{3} \right| > 2^{\frac{n+2}{4}} \right) \leq 3 \kappa M_{\xi}$$

$$\sqrt{\frac{2}{\pi}} \cdot 2^n \exp\left(-\frac{n^2}{2}\right) \leftarrow \text{this is summable}$$

By Borel-Cantelli: almost surely

$$\exists N = N(\omega) \text{ (random)} \text{ such that}$$
for $n \geq N$

$$\sup_{0 \leq t \leq 1} \left| B^{(n)}(t) - B^{(n)}(t) \right| \leq \frac{n}{4}$$

Hence, the sequence of functions

$$t \mapsto B(t)$$

is uniformly (in $t$ and $a$) convergent.

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**Standard BM in $\mathbb{R}^d$:**

$$t \mapsto B(t) \in \mathbb{R}^d$$

$$B(t) = (B_1(t), B_2(t), \ldots, B_d(t))$$

where $(B_j(t))^{d}_{j=1}$ are independent $1d$ Brownian Motions.

The distribution of $B(t)$ is invariant under orthogonal transformations (rotations) of $\mathbb{R}^d$. 