Balázs T. H.: Stochastic integration
the Itô integral

Motivation:

A typical SDE:

\[ X(t) = b(t) + \sigma(t) w(t) \]

where:

- \( b(t) = b(t, \omega) \), \( \sigma(t) = \sigma(t, \omega) \)
- \( w(t) = w(t, \omega) \), \( X(t) = X(t, \omega) \)

are random/stochastic processes defined on the same filtered probab. space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, P)\) and are adapted to the filtration

- \( t \mapsto b(t) \), \( t \mapsto \sigma(t) \) are
Continuous (slightly less regularity would be sufficient)

- \( b(t), \xi(t) \) may also depend on
  \( X(s) : 0 < s < t \)

- \( \omega(t) = \) random driving
  assumed: Stationary + "White"
  "White" = uncorrelated, zero mean
  \( E(\omega(t)) = 0, E(\omega(t)\omega(s)) = \delta(t-s) \)

- \( b^*(t) = \) instantaneous drift
- \( \sigma(t) = \) instantaneous dispersion
What is the "white noise" $\hat{w}(t)$?

$E(\hat{w}(t)) = 0, \ E(\hat{w}(s)\hat{w}(t)) = \delta(t-s)$

+ Gaussian

looks like: $\hat{w}(t) = \frac{dB(t)}{dt}$

doesn't exist as a decent process.

Better rewrite: $\hat{w}(t) dt = dB(t)$

The SDE re-written:

$dt X(t) = b(t) dt + \tilde{w}(t) dB(t)$

Integrated:

$X(t) = X(0) + \int b(s) ds + \int \tilde{w}(s) dB(s)$

Riemann integral
Examples of SDE's

1. Linear
   \[ \frac{dX(t)}{dt} = \gamma X(t) \] (interest rate)
   \[ \delta(t) = a \cdot X(t) \]
   \[ dX(t) = \gamma X(t) dt + a X(t) dB(t) \]
   in finance
   Slu: "geometric BM"

2. Langevin's eq: \( b(t) = -\sigma V(t) \)
   \[ \delta(t) = \delta \]
   \[ dV(t) = -\gamma V(t) dt + \sigma dB(t) \]
   velocity of a particle
   friction/damping
   random forcing
   in physics
   Slu: "Ornstein-Uhlenbeck process"

3. Population dynamics
   \( dX(t) = X(t)(1-X(t)) dB(t) \)
   \( X(t) \in (0,1) \)
   Slu: "Fisher-Wright process"
4. An example from electric circuits:

\[ L \ddot{Q}(t) + R \dot{Q}(t) + C^2 Q(t) = U(t) \]

\[ \dot{Q}(t) = \text{charge}; \quad \dot{Q}(t) = \text{current}; \quad \ddot{Q}(t) = \text{voltage} \]

\[ L = \text{inductance}; \quad R = \text{resistance}; \quad C = \text{capacity} \]

\[ U(t) = \text{tension / voltage / potential difference} \]

Randomized voltage:

\[ U(t) = F(t) \cdot W(t) \]

Call \( X_1 = Q \), \( X_2 = \dot{Q} \): \[
\dot{X}_1 = X_2; \quad \dot{X}_2 = -\frac{R}{L} X_2 - \frac{1}{CL} X_1 + \frac{1}{L} F(t) + \frac{\dot{W}}{L}
\]

\[
\text{d}X(t) = -G X(t) \text{d}t + H(t) \text{d}t + K \text{d}B(t)
\]

\[
X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}; \quad H(t) = \begin{pmatrix} 0 \\ L^2 F(t) \end{pmatrix}; \quad K = \begin{pmatrix} 0 \\ \alpha L \end{pmatrix}; \quad G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
What is the "stochastic integral" $\int_0^t \nu(s) \, dB(s)$?

Try Riemann sums

$$\int_0^t \nu(s) \, dB(s) \overset{?}{=} \lim_{\Sigma} \sum_{i} \nu(s_i)(B(s_{i+1}) - B(s_i))$$

The limit may not exist since $B(\cdot)$ has no bounded variation.

(However: if $\nu(\cdot)$ has bounded variation, then partial summation helps. But typically this is not the case!)

An instructive (counter)example:

$$\int_0^t B(s) \, dB(s) \overset{?}{=} \frac{1}{2} B(t)^2$$
do it in two ways:

1. \(\sum_{n}^{1} \equiv \sum_{i} B(ti)(B(t_{i+n})-B(ti))\)

2. \(\sum_{n}^{2} \equiv \sum_{i} B(ti) (B(t_{i+n})-B(ti))\)

\[\sum_{n}^{2} - \sum_{n}^{1} = \sum_{i} \left( B(t_{i+n}) - B(ti) \right)^{2} \rightarrow t\]

see: quadratic variation of BM

TheIts choice: \(t_{i}^{*} = ti\)

The Stratonovich choice: \(t_{i}^{*} = (ti + t_{i+n})/2\)

The Its integral - definition/construction:

Ingredients:

\((\Omega, (\mathcal{F}_{t}), \mathbb{P})\) filtered prob. space,

\(t \rightarrow B(t)\) BM on it

- adapted: \(\mathcal{F}_{t} B \subseteq \mathcal{F}_{t}\)
- martingale w.r.t. \(\mathcal{F}_{t}\)
Definition: "progressively measurable process"

\( y(t) \) stochastic process on \((\Omega, (\mathcal{F}_t), \mathbb{P})\)

such that

\((\forall t) \ (s, w) \mapsto y(s, w) \quad 0 \leq s \leq t\)

is jointly \((B \times \mathcal{F}_t)\) - measurable

(Borel \(\sigma\)-alg on \(\mathbb{R}_t\))

(slightly more than just adapted)

\(0 < T < \infty\) fixed.

The class \( V_T : \phi : [0, T] \times \Omega \to \mathbb{R} \)

1. \(\phi\) progressively measurable

2. \(\|\phi\|^2 : = E \left( \int_0^T \phi(t, w)^2 dt \right) < \infty\).
$L^2(\Omega \times [0,T], \mathcal{F}_T \text{meas.}, d\mathbb{P} \times ds) \quad \mathbb{7}$

$(V_T, \| \cdot \|)$ is a Hilbert space.

$V_T^s := \{ \varphi \in V_T : \varphi(t_i | w) = \sum_{i=1}^{n} \varphi_i(w) x_{i-1, t_i} \}$

$0 = t_0 < t_1 < \cdots < t_n = T, \quad \varphi_i \ell_{t_i} \text{-measurable}$

"Simple" processes, piecewise constant.

The partition is deterministic / doesn't depend on $\omega \in \Omega$!

For $\varphi \in V_T^s$ define

$$\int_0^T \varphi^2(s) dB(s) := \sum_{i=1}^{n} \varphi_{i-1}(B(t_i) - B(t_{i-1}))$$

Let

$I : (V_T^s, \| \cdot \|)^2 \rightarrow L^2(\Omega \times [0,T], \mathbb{P})$

$I(\varphi) := \int_0^T \varphi(s) dB(s)$
Lemma 1: I is an isometry:

\[ E \left( (\int_0^T \varphi(s) \, dB(s))^2 \right) = \]

\[ E \left( \int_0^T \varphi(s)^2 \, ds \right) = \| \varphi \|^2 \]

Proof:

\[ E \left( (\int_0^T \varphi(s) \, dB(s))^2 \right) = \]

\[ E \left( \sum_i \varphi_i (B(t_i) - B(t_{i-1})) \right)^2 = \]

\[ \sum_i E \left( \varphi_i^2 (B(t_i) - B(t_{i-1}))^2 \right) + \]

\[ 2 \sum_{i<j} E \left( \varphi_i \varphi_j (B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1})) \right) = 0 \text{ (indep. increments of B)} \]

\[ = \sum_i (t_i - t_{i-1}) \, E(\varphi_i^2) = \| \varphi \|^2. \]
Lemma 2: $V_T^s$ is dense in $(V_T, \| \cdot \|_1)$

Proof: Standard (but tedious...)

Theorem

$\int_0^T \! \Phi(t) dB(s) \text{ extends continuously from } (V_T, \| \cdot \|_1) \text{ to } (V_T, \| \cdot \|_1)$.

That is, for $\Phi \in V_T$

$$\int_0^T \! \Phi(t) dB(s) := \lim_{n \to \infty} \int_0^T \! \Phi_n(s) dB(s)$$

where $\Phi_n \in V_T^s$, $\| \Phi_n - \Phi \|_1 \to 0$.

The limit (in $L^2(\mathbb{R}, \mathcal{F}_T, \mathbb{P})$) doesn't depend on the approximating sequence $\Phi_n$.

$$E\left( \int_0^T \! \Phi(t) dB(s) \right)^2 = E\left( \int_0^T \! \Phi(s)^2 dB(s) \right)$$
Some basic properties of Ito integral:

Given \( 0 \leq S \leq T \leq U < \infty \):

1. \( \int_{T}^{U} \phi(s) \, dB(s) \) is \( \mathcal{F}_T \)-measurable

2. \( \int_{S}^{T} \phi(s) \, dB(s) = \int_{S}^{U} \phi(s) \, dB(s) + \int_{T}^{U} \phi(s) \, dB(s) \)

3. Given deterministic numbers \( a, b \):

   \[
   \int_{S}^{T} (a \phi(s) + b \psi(s)) \, dB(s) = \]
   \[
   a \int_{S}^{T} \phi(s) \, dB(s) + b \int_{S}^{T} \psi(s) \, dB(s) \]

4. \( \mathbb{E}(\int_{S}^{T} \phi(s) \, dB(s)) = 0 \)

   \[
   \mathbb{E} \left( \left( \int_{S}^{T} \phi(s) \, dB(s) \right)^2 \right) = \]
   \[
   \mathbb{E} \left( \int_{S}^{T} \phi(s)^2 \, ds \right) .
   \]
Examples: \[ \int_0^T B(s) dB(s) = \frac{1}{2} (B(T)^2 - T) \]

HW: Hint: \[ \Phi(t) = B(t) \]
\[ t_i = \frac{i T}{n}, \quad 0 \leq i \leq n \]

\[ \Phi_n(t) = \sum_{i=1}^n B(t_{i-1}) \int_{t_{i-1}}^{t_i} 1 \]

Step 1: \[ \| \Phi - \Phi_n \|^2 \to 0 \]

Step 2: Compute \[ \lim_{n \to \infty} \int_0^T \Phi_n(s) dB(s) \]

2. \[ \int_0^T B(s)^2 dB(s) = \frac{1}{3} B(t)^3 - \int_0^T B(s) ds \]

HW: Similar
Question: does a (sufficiently regular) 

\[ t \mapsto \int_0^t \varphi(s) \, dB(s) \]

exist on the probability space \( (\Omega, (\mathcal{F}_t)_t, \mathbb{P}) \)?

Mind: \( \int_0^t \varphi(s) \, dB(s) \) was defined as \( L^2 \)-limit, for \( t \) fixed. It's a.s. defined for \( t \) fixed or for countably many \( (t_j)_{j=0}^{\infty} \). They are a priori not a.s. simultaneously defined for all \( t \in [0, T] \).
Theorem: Let $\varphi \in V_T$

There exists an adapted process
\[ t \mapsto \mathcal{F}(t) \] with a.s. continuous sample paths such that
\[ \forall t \in [0,T]: \quad P\left( \int_0^t \varphi(s) \, dB(s) = \mathcal{F}(t) \right) = 1. \]

The process
\[ t \mapsto \mathcal{F}(t) \]

is actually a martingale and \( \forall \lambda > 0: \)
\[ P\left( \sup_{0 \leq t \leq T} |\mathcal{F}(t)| \geq \lambda \right) \leq \frac{\lambda^2}{\mathbb{E} \left( \int_0^T \varphi(s)^2 \, ds \right)}. \]
Proof: Choose \( \varphi_n \in \mathcal{V}_T \) so that
\[
\| \varphi - \varphi_n \|^2 = E \left( \int_0^T (\varphi(s) - \varphi_n(s))^2 \, ds \right) < 2^{-3n}
\]
\[
F_n(t) := \int_0^t \varphi_n(s) \, dB(s) \quad \text{continuous weakly} \quad (\text{HW})
\]

By Doob's inequality + Schwarz's inequality,
\[
P \left( \sup_{0 \leq t \leq T} |F_n(t) - F(t)| > 2^{-n} \right) \leq 2^{2n} E \left( \int_0^T (\varphi_n(s) - \varphi(s))^2 \, ds \right) \leq C \cdot 2^{-n}
\]

By Borel-Cantelli a.s. \( E \, N_0 = N_0 / (\ln 2) \) a.s. for \( N \geq N_0 \):
\[
\sup_{0 \leq t \leq T} |F_n(t) - F_{n+1}(t)| \leq 2^{-n}
\]
\[
\Rightarrow \text{a.s. } F_n(t) \xrightarrow{\text{uniformly}} F(t)
\]

\( F(t) \) continuous weakly
\[
F(t) = \int_0^t \varphi(s) \, dB(s)
\]
The multidimensional joyful integral

\( (\Omega, (\mathcal{F}_t)_{t \geq 0}, P) \)

\[ B(t) = (B_1(t), \ldots, B_k(t)) \in \mathbb{R}^k \]

\( k \)-dimensional BM, adapted to \( (\mathcal{F}_t) \)

\[ \varphi = (\varphi_1, \ldots, \varphi_k) \quad \text{and} \quad \varphi_j \in \mathcal{V}_t \]

\[
\int_0^t \varphi(s) \cdot dB(s) = \sum_{j=1}^k \int_0^t \varphi_j(s) \, dB_j(s)
\]