Problem Set 2
Filtrations, Stopping Times, Markov Property, Martingales, ...

2.1 Let \( t \mapsto X(t) \) be a stochastic process in a complete separable metric space \( S \). Prove that the following two formulations of the Markov property are actually equivalent. (Note that formulation (b) is a priori stronger than (a).)

(a) For any \( 0 \leq t, 0 \leq u \) and \( F : S \rightarrow \mathbb{R} \) bounded and measurable
\[
E(F(X(t+u)) \mid \mathcal{F}_t^X) = E(F(X(t+u)) \mid \sigma(X_t)).
\]

(b) For any \( 0 \leq t, n \in \mathbb{N}, 0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \) and \( F : S^n \rightarrow \mathbb{R} \) bounded and measurable
\[
E(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \mid \mathcal{F}_t^X) =
E(F(X(t+u_1), X(t+u_2), \ldots, X(t+u_n)) \mid \sigma(X_t)).
\]

Hint: Apply successive conditioning (i.e. the "tower rule") of conditional probabilities.

2.2 (a) Prove that \( t \mapsto B(t) \) is a martingale and \( t \mapsto B(t)^2 \) is a submartingale (with respect to the filtration \( (\mathcal{F}_t^B)_{t \geq 0} \)).

(b) Let \( t \mapsto M(t) \) be a martingale (w.r.t. a filtration \( (\mathcal{F}_t)_{t \geq 0} \)) and \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) a convex function. Let
\[
Y(t) := \psi(M(t)).
\]
Assuming that \( E(|\psi(M(t))|) < \infty \) for all \( t \geq 0 \), prove that \( t \mapsto Y(t) \) is a submartingale.

Hint: Use Jensen’s inequality.
2.3 Show that the processes $t \mapsto B(t)$, $t \mapsto B(t)^2 - t$ and $t \mapsto B(t)^3 - 3tB(t)$ are martingales adapted to the filtration $\{F^B_t\}_{t \geq 0}$.

2.4 Check whether the following processes are martingales with respect to the filtration $\{F^B_t\}_{t \geq 0}$:

(a) $X(t) = B(t) + 4t$,

(b) $X(t) = B(t)^2$,

(c) $X(t) = t^2 B(t) - 2 \int_0^t sB(s)ds$,

(d) $X(t) = B_1(t)B_2(t)$,

where $B_1$ and $B_2$ are two independent Brownian motions.

2.5 Let $-a < 0 < b$ and denote

$$\tau_{\text{left}} := \inf\{s > 0 : B(s) = -a\}, \quad \tau_{\text{right}} := \inf\{s > 0 : B(s) = b\}, \quad \tau := \min\{\tau_{\text{left}}, \tau_{\text{right}}\}.$$ 

(a) By applying the Optional Stopping Theorem compute $P(\tau_{\text{left}} < \tau_{\text{right}})$ and $E(\tau)$.

(b) By "applying" the Optional Stopping Theorem it would "follow" that $E(B(\tau_a)) = 0$. However, clearly $B(\tau_a) = a$ by definition (and continuity of the Brownian motion). What is wrong with the argument?

2.6 (a) Let $\theta \in \mathbb{R}$ be a fixed parameter. Show that the processes $t \mapsto \exp\{\theta B(t) - \theta^2 t/2\}$ is a martingale with respect to the filtration $\{F^B_t\}_{t \geq 0}$.

(b) By differentiating with respect to $\theta$ and letting then $\theta = 0$ derive a martingale which is a fourth order polynomial expression of $B(t)$.

(c) For any $n \in \mathbb{N}$ let

$$H_n(x) := e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}. $$

Show that $H_n(x)$ is a polynomial of order $n$ in the variable $x$. (It is called the Hermite polynomial of order $n$.) Compute $H_n(x)$ for $n = 1, 2, 3, 4$.

(d) Show that for any $n \in \mathbb{N}$ the process $t \mapsto t^{n/2}H_n(B(t)/\sqrt{t})$ is a martingale.
2.7 Let \( t \mapsto B(t) \) be standard 1d Brownian motion and \( \tau := \inf\{t > 0 : |B(t)| = 1\} \).

Prove that
\[
E (e^{-\lambda \tau}) = \cosh(\sqrt{2\lambda})^{-1}, \quad \lambda \geq 0.
\]

*Hint:* Apply the Optional Stopping Theorem to the exponential martingale defined in problem 2.6.

2.8 Denote
\[
J : \mathbb{R} \to \mathbb{R}, \quad J(\lambda) := \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{\lambda \cos \theta} d\theta.
\]

Let \( B(t) = (B_1(t), B_2(t)) \) be a two-dimensional Brownian motion and \( \tau := \inf\{t : |B(t)| = 1\} \).

That is: \( \tau \) is the first hitting time of the circle centred at the origin, with radius 1.

Prove that
\[
E (e^{-\lambda \tau}) = J(\sqrt{2\lambda})^{-1}, \quad \lambda \geq 0.
\]

*Hint:* Apply the Optional Stopping Theorem to the martingale \( t \mapsto \exp\{\theta \cdot B(t) - |\theta|^2 t/2\} \), where \( \theta \in \mathbb{R}^2 \), with the stopping time \( \tau \).