4.1 Check that the following processes solve the corresponding SDE’s, where $B(t)$ is 1-dimensional standard Brownian motion:

(a) $X(t) = e^{B(t)}$, with $B(0) = b$ solves

$$dX(t) = \frac{1}{2} X(t)dt + X(t)dB(t), \quad X(0) = e^b.$$ 

(b) $X(t) = \frac{B(t)}{1+t}$, with $B_0 = b$, solves

$$dX(t) = -\frac{X(t)}{1+t}dt + \frac{1}{1+t}dB_t, \quad X(0) = b.$$ 

(c) $X(t) = \sin B(t)$, with $B(0) = b \in (-\pi/2, \pi/2)$, and $t < \min\{ t : |B(t)| = \pi/2 \}$, solves

$$dX(t) = -\frac{1}{2} X(t)dt + \sqrt{1 - X(t)^2}dB_t, \quad X(0) = \sin b.$$ 

(d) $(X_1(t), X_2(t)) = (\cosh B(t), \sinh B(t))$, with $B(0) = b$, solves

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} dt + \begin{pmatrix} X_2(t) \\ X_1(t) \end{pmatrix} dB(t).$$

4.2 Let $B(t)$ be a standard 1-dimensional Brownian motion with $B(0) = b$, and $(U(t), V(t)) := (\cos B(t), \sin B(t))$. Write down in vectorial notation the SDE driving the 2-dimensional process $(U(t), V(t))$.

4.3 Solve the following SDE’s, where $B(t)$ is 1-dimensional standard Brownian motion starting from $B(0) = 0$:

(a)

$$dX(t) = -X(t)dt + e^{-t}dB(t).$$

(b)

$$dX(t) = rdt + \alpha X(t)dB(t),$$

with $r, \alpha \in \mathbb{R}$ constants.

**Hint:** Multiply by $\exp\left(-\alpha B(t) + \frac{\alpha^2}{2} t\right)$. 


(c) Now, $X(t) = (X_1(t), X_2(t)) \in \mathbb{R}^2$, and $B(t) = (B_1(t), B_2(t))$ is standard 2-dimensional Brownian motion.

$$
\begin{align*}
dX_1(t) & = X_2(t)dt + \alpha dB_1(t) \\
dX_2(t) & = -X_1(t)dt + \beta dB_2(t),
\end{align*}
$$
or in vector notation,

$$
dX(t) = JX(t)dt + AD(t),
$$
where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$.

*Hint:* Multiply by left by $e^{-Jt}$.

4.4 The Ornstein-Uhlenbeck process:

(a) Solve explicitly the stochastic differential equation

$$
dX(t) = -\gamma X(t)dt + dB(t), \quad X(0) = x_0,
$$
and show that the process $X(t)$ is Gaussian.

*Hint:* Multiply by $e^{\gamma t}$.

(b) Compute $E(X(t))$ and $\text{Cov}(X(s), X(t))$.

(c) Let $Y_k^{(n)}$ be the Markov chain on the state space $S^{(n)} := \{0, 1, \ldots, n\}$ with transition matrix

$$
P_{i,j}^{(n)} = \frac{i}{n}\delta_{i-1,j} + \frac{n-i}{n}\delta_{i+1,j}, \quad i, j \in S^{(n)}.
$$

The Markov chain $Y_k^{(n)}$ is called Ehrenfest’s Urn Model (or Dogs and Fleas). Define the sequence of continuous time processes

$$
X^{(n)}(t) := \frac{Y_{\lfloor nt \rfloor}^{(n)} - (n/2)}{\sqrt{n}}, \quad t \geq 0.
$$

Write down an approximate stochastic differential equation for $X^{(n)}(t)$, with time increments $dt = \frac{1}{n}$ and conclude (non-rigorously) that the Ornstein-Uhlenbeck process is – in some sense – the limit of the processes $X^{(n)}(t)$ (that is: the scaling limit of Ehrenfest’s Urn Model.)

4.5 Recall that a continuous Gaussian process $X(t)$ is uniquely determined by its expectations $m(t) := E(X(t))$ and pairwise covariances $c(s, t) := \text{Cov}(X(s), X(t)) = E(X(s)X(t)) - E(X(s))E(X(t))$. The one-dimensional Brownian bridge (from 0 to 0) is such a Gaussian process defined on the time interval $[0, 1]$, with $m(t) = 0$ and $c(s, t) = \min(s, t)(1 - \max(s, t))$. Prove that the law of this process is given by any of the following three representations. In all expressions $t \in [0, 1]$ and $t \mapsto B(t)$ is standard 1-dimensional Brownian motion.
(a) \( X(t) = B(t) - tB(1). \)

(b) \( Y(t) = (1 - t)B\left(\frac{1}{1-t}\right), \) for \( t \in (0, 1), \) and \( Y(1) = 0. \) Note that continuity at \( t = 1 \) needs an argument. See the hint at the end of the exercise.

(c) \( Z(t) = \int_0^t (1-t)/(1-s)dB(s), \) for \( t \in (0, 1), \) and \( Z(1) = 0. \) Note again that continuity at \( t = 1 \) needs an argument. See the hint at the end of the exercise.

(d) \( t \mapsto Z(t) \) in the previous expression is in fact the strong solution of the SDE

\[
dZ(t) = -\frac{Z(t)}{1-t} dt + dB(t), \quad t \in [0, 1), \quad X_0 = 0.
\]

**Hint:** In order to prove continuity at \( t = 0 \) note that \( t \mapsto (1-t)^{-1}Y(t) \) and \( t \mapsto (1-t)^{-1}Z(t) \) are continuous martingales on \([0, 1)\). Use Doob’s maximal inequality to estimate \( P(\sup_{0 < t < t_1} |Z(t) - Z(t_0)| > \varepsilon) \), where \( 0 \leq t_0 < t_1 < 1, \varepsilon > 0 \). Then proceed via a Borel-Cantelli argument.

**Remarks:**

1. Yet another alternative definition of the Brownian bridge is \( X(t) := (B(t) \mid B(1) = 0) \). That is: Brownian motion conditioned to be at 0 at the terminal time \( t = 1 \).
2. The Brownian bridge from \( a \) to \( b \) (where \( a, b \in \mathbb{R} \)) is \( X_{a,b}(t) := bt + a(1-t) + X_{0,0}(t) \), where \( X_{0,0} \) is a Brownina bridge from 0 to 0, as defined above.
3. Note that \( X(t), Y(t), \) and \( Z(t) \) are genuinely different representations. They have the same law but they are different path-wise.

4.6 Let \( t \mapsto B(t) = (B_k(t) : 1 \leq k \leq m) \in \mathbb{R}^m \) be an \( m \)-dimensional standard Brownian motion, \( t \mapsto v(t) = (v_{ik}(t) : 1 \leq i \leq n, 1 \leq k \leq m) \in \mathbb{R}^{n \times m} \) progressively measurable (with the usual conditions) and \( Y(t) = (Y_i(t) : 1 \leq i \leq n) \in \mathbb{R}^n \) defined by the Itô integral

\[
Y_i(t) := \sum_{k=1}^m \int_0^t v_{ik}(s) dB_k(s).
\]

Prove the following theorem due to Paul Lévy:

If

\[
\sum_{k=1}^m v(t)_{ik}v(t)_{jk} \equiv \delta_{i,j}, \quad 1 \leq i, j \leq n, \quad (\ast)
\]

then \( t \mapsto Y(t) \) is an \( n \)-dimensional standard Brownian motion. (Note that condition \((\ast)\) forcibly implies \( n \leq m \).)

**Hint:** Using the exponential martingales of Problem 3.7 prove that for any deterministic continuous function \( h : [0, \infty) \to \mathbb{R}^n \) of compact support, \( E(\exp\{\int_0^\infty h(s) \cdot dY(s)\}) = \exp\{\frac{1}{2} \int_0^\infty |h(s)|^2 ds\} \).
4.7 In this problem \( t \mapsto B(t) \) be a standard 1-dimensional Brownian motion,
\[
t \mapsto L(t) := \mathcal{L}^2 \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|B(s)| \leq \varepsilon\}} ds,
\]
its local time at \( x = 0 \) and
\[
t \mapsto M(t) := \max_{0 \leq s \leq t} B(s)
\]
its maximum before time \( t \).

Recall Tanaka’s formula (proved in class):
\[
|B(t)| - |B(0)| = \int_0^t \text{sgn}(B(s)) dB(s) + L(t). \quad \text{(T)}
\]

(a) Let \( S_n \) be simple symmetric random walk on \( \mathbb{Z} \), and
\[
\ell_n := \sum_{m=0}^{n-1} \mathbb{1}_{\{|S_m| = 0\}}
\]
denote the number of visits of 0 by \( S \) before time \( n \). (This is the discrete analogue of local time.) Prove the following discrete version of Tanaka’s formula (T):
\[
|S_n| - |S_0| = \sum_{m=0}^{n-1} \text{sgn}(S_m)(S_{m+1} - S_m) + \ell_n.
\]

(b) Using Tanaka’s formula (T) prove the following identity in law:
\[
\left( |B(t)|, L(t) \right)_{t \geq 0} \overset{d}{=} \left( M(t) - B(t), M(t) \right)_{t \geq 0}.
\]
This is a theorem due to Paul Lévy.