6.1 Let
\[
\ell_\infty := \{ f : \mathbb{N} \to \mathbb{R} : \| f \| := \sup_{x \in \mathbb{N}} |f(x)| < \infty \},
\]
\[
c_0 := \{ f \in \ell_\infty : \lim_{x \to \infty} |f(x)| = 0, \| f \| := \sup_{x \in \mathbb{N}} |f(x)| \}.\]

Let \( t \mapsto \eta_t \in \mathbb{N} \) be a time-homogeneous continuous time Markov chain on \( \mathbb{N} \). Its transition operators are
\[
P_t : \ell_\infty \to \ell_\infty, \quad P_t f(x) := \mathbb{E}(f(\eta_t) \mid \eta_0 = x).
\]

(a) Show that the one parameter family of operators \( t \mapsto P_t \) form a semigroup of contractions on \( \ell_\infty \).

(b) Give examples when \( P_t : c_0 \to c_0 \), and when \( P_t : c_0 \not\to c_0 \).

(c) Prove that if \( P_t : c_0 \to c_0 \) then by force the semigroup \( t \mapsto P_t : c_0 \to c_0 \) is strongly continuous.

(d) Give an example when \( P_t : c_0 \not\to c_0 \) and the semigroup \( t \mapsto P_t : \ell_\infty \to \ell_\infty \) is strongly continuous.

(e) Give an example when the semigroup \( t \mapsto P_t : \ell_\infty \to \ell_\infty \) is not strongly continuous.

6.2 Let \( \mathcal{B} \) be a Banach space and \( \mathcal{C} \subset \mathcal{B} \) a dense subspace. Recall that we call the densely defined operator \( A : \mathcal{C} \to \mathcal{B} \) to be dissipative (or \( -A \) to be accretive) if \( \forall \varphi \in \mathcal{C} \) there exists a normalized tangent functional \( \ell_\varphi \in \mathcal{B}^* \) to the vector \( \varphi \), such that \( \ell_\varphi(-A\varphi) \geq 0 \). We showed in class that this implies that
\[
\| (\lambda I - A)\varphi \| \geq \lambda \| \varphi \|, \quad \text{for all } \varphi \in \mathcal{C}, \text{ and } \lambda > 0. \tag{1}
\]

Conversely, if \( A \) is the infinitesimal generator of a strongly continuous contraction semigroup, then it is dissipative.

(a) Show that (1) implies that \( A : \mathcal{C} \to \mathcal{B} \) is closable.
(b) Let

\[ B = C_0[0, \infty) = \{ f : [0, \infty) \to \mathbb{R} : f \text{ continuous, } \lim_{x \to \infty} |f(x)| = 0, \text{ with } \|f\| := \sup_{0 \leq x < \infty} |f(x)|. \}\]

Consider \( A \) defined on \( \tilde{C} := C_0[0, \infty) \cap C_2^0[0, \infty) \).

Show that \( A \) defined on \( \tilde{C} \) does not satisfy (1).

(c) Show that, on the other hand, \( A \) defined on \( C := C_0[0, \infty) \cap C_2^0[0, \infty) \cap \{ f'(0) = 0 \} \)

does satisfy (1). The closure of this operator is the infinitesimal generator of Brownian motion on \([0, \infty)\) reflecting at 0.

6.3 Young’s inequality for convolutions says that if \( 1 \leq p, q, r \leq \infty \) satisfy \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \), then

\[ \| f * g \|_r \leq \| f \|_p \| g \|_q. \]

Using this, show that \( t \mapsto e^{\frac{1}{2} \Delta t} \) is a strongly continuous contraction semigroup on \( L^p, 1 \leq p < \infty \).

\textbf{Hint:} Use the explicit form of the heat-kernel:

\[ e^{\frac{1}{2} \Delta t} f(x) = (2\pi t)^{d/2} \int_{\mathbb{R}^d} e^{-|x-y|^2/2t} f(y) dy. \]

6.4 In this problem we consider the infinitesimal generator of Brownian motion in \( \mathbb{R}^d \), that is: the Laplacian \( \Delta \) on the Banach space

\[ B = C_0(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{R} : f \text{ continuous, } |x| \to \infty \} \]

In \( d = 1 \) we have seen that the domain \( \text{Dom}(\Delta) = C_0(\mathbb{R}) \cap C_2^0(\mathbb{R}) \), i.e., vanishing value and vanishing 2nd derivative at infinity. We have also seen that on \( \mathbb{R}^d \), \( d \geq 2 \), the Schwarz space \( \mathcal{S}(\mathbb{R}^d) \) is a good core: the operator \(-\Delta\) defined on \( \mathcal{S}(\mathbb{R}^d) \) is dissipative, hence closable, and \( \{ \varphi - \Delta \varphi : \varphi \in \mathcal{S}(\mathbb{R}^d) \} = C_0(\mathbb{R}^d) \), and thus \( \Delta \) is indeed an infinitesimal generator, as we already knew. But what is \( \text{Dom}(\Delta) \) obtained this way, in \( \mathbb{R}^1 \)? I.e., what domain do we get when we close the operator from \( \mathcal{S}(\mathbb{R}^1) \)? It certainly contains \( C_0(\mathbb{R}) \cap C_2^0(\mathbb{R}) \), but isn’t it larger?
6.5 (a) Let $\psi$ be a bounded continuous function on $\mathbb{R}^d$, and $\lambda > 0$. Find a bounded solution $u$ of the equation
\[
\lambda u - \frac{1}{2} \Delta u = \psi \quad \text{on } \mathbb{R}^d.
\]
Prove that the solution is unique.

(b) Let $B(t)$ be $d$-dimensional Brownian motion ($d \geq 1$) and let $F$ be a Borel set in $\mathbb{R}^d$. Let
\[
T_F := | \{ t \leq 1 : B(t) \in F \} |,
\]
where $| \ldots |$ denotes Lebesgue measure. Prove that $\mathbb{E}(T_F) = 0$ if and only if $|F| = 0$.

Hint: Consider the resolvent $R_\lambda$ for $\lambda > 0$ and then let $\lambda \to 0$.

6.6 In connection with the derivation of the Black-Scholes formula for the price of an option, the following partial differential equation appears for $u = u(t,x)$, $t \in [0, \infty)$, $x \in \mathbb{R}$:
\[
\frac{\partial u}{\partial t}(a,x) = -\rho u(t,x) + \alpha x \frac{\partial u}{\partial x}(t,x) + \frac{1}{2} \beta^2 x^2 \frac{\partial^2 u}{\partial x^2}(t,x) \quad t > 0, \ x \in \mathbb{R}
\]
\[
u(0,x) = (x - K)_+ \quad x \in \mathbb{R},
\]
where $\rho > 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $K > 0$ are constants.

Use the Feynman-Kac formula to prove that the solution $u(t,x)$ of this initial value problem is given by
\[
u(t,x) = \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} \left( x e^{(\alpha - \beta^2/2)t + \beta y - K} + e^{-y^2/(2t)} dy, \quad t > 0.\right.
\]

6.7 The elliptic Feynman-Kac formula, with Dirichlet boundary conditions.
Let $D \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary, $c, f : \mathbb{R}^d \to \mathbb{R}$ smooth functions and $c \geq 0$. Prove the following statement:

The unique solution of the elliptic boundary value problem
\[
\frac{1}{2} \Delta u - cu = f \quad \text{in } D
\]
\[
u = 0 \quad \text{on } \partial D,
\]
is given by
\[
u(x) = \mathbb{E}(\int_0^\tau f(B(t)) \exp\left\{- \int_0^t c(B(s)) ds\right\} - B(t) \in F) \bigg| B(0) = x, \quad x \in D,
\]
where $B(t)$ is Brownian motion starting from $x \in D$ and $\tau$ is the first hitting time of $\partial D$. 

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