Dynamically defined subsets of generic self-affine sets

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Abstract

In dynamical systems, shrinking target sets and pointwise recurrent sets are two important classes of dynamically defined subsets. In this article we introduce a mild condition on the linear parts of the affine mappings that allow us to bound the Hausdorff dimension of cylindrical shrinking target and recurrence sets. For generic self-affine sets in the sense of Falconer, that is by randomising the translation part of the affine maps, we prove that these bounds are sharp. These mild assumptions mean that our results significantly extend and complement the existing literature for recurrence on self-affine sets.

1 Introduction

The shrinking target problem in dynamical systems investigates the "size" of the set of points that recur to a collection of (shrinking) targets infinitely many times. Letting (X, T, μ) be a dynamical system with invariant measure μ and a collection of (measurable) subsets $(B_k)_{k\in\mathbb{N}}$, $B_k \subseteq X$ one investigates the shrinking target set

$$S((B_k)_k) = \{x \in X : T^k(x) \in B_k \text{ for infinitely many } k \in \mathbb{N}\}.$$

Similarly, given a function $\psi : \mathbb{N} \to \mathbb{R}^+$, the pointwise recurrent set is defined as

$$R(\psi) = \{x \in X : T^k(x) \in B(x, \psi(k)) \text{ for infinitely many } k \in \mathbb{N}\}.$$

Often these sets are dense in the original space X, as well as G_δ, and so dimension theory is used to classify the sizes of such sets. The Hausdorff dimension is the most appropriate choice here, as dense G_δ sets have full dimension for, e.g. the packing-, Minkowski-, and Assouad-type dimensions.

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The shrinking target problem was first investigated by Hill and Velani for Julia sets who analysed their Hausdorff dimension [12] and found a zero-one law for its Hausdorff measure [13]. The shrinking target problem has intricate links to number theory when using naturally arising sets in Diophantine approximation as the shrinking targets. This has received a lot of attention over recent years, see for instance [1, 5, 18, 23, 24] for shrinking target sets and [2, 6, 8, 9, 16, 19, 20, 21] for related research.

The literature of recurrence sets so far has focussed mostly on zero-one laws for conformal and one dimensional dynamics, such as β-transformations, see Tan and Wang [26], and Zheng and Wu [28]. For self-similar and self-conformal dynamics these questions were explored by Seuret and Wang [27], who also gave a pressure formula for the Hausdorff dimension, as well as Baker and Farmer [3] who stated a zero-one law dependent on a convergence condition of the size of the neighbourhoods. Finally, and most recently, Kirsebom, Kude, and Persson [17] studied linear maps on the d-dimensional torus.

The above works mostly concern dynamical systems in $\mathbb{R}^1$ or conformal dynamics and transitioning to higher dimensional non-conformal dynamics presents severe challenges. To circumvent the extreme challenges that affinities pose, a common strategy is to "randomise" the affine maps by considering typical translation parameter. This approach was first considered by Falconer in his seminal article [7], whose conditions were significantly relaxed by Solomyak [25] and generalised by Jordan, Pollicott and Simon [14].

This typicality with respect to the translation parameter allows one to say more about the regularity of the attractors and is a commonly employed strategy, see for example [14]. Using such randomisation, Koivusalo and Ramírez [18] gave an expression for the Hausdorff dimension of a self-affine shrinking target problem. They show that for a fixed symbolic target with exponentially shrinking diameter and well-behaved affine maps, the Hausdorff dimension is typically given by the zero of an appropriate pressure function. Strong assumptions are made on the affine system, as well as the fixed target and in this article we significantly improve upon their results.

We will show that for a large family of self-affine systems and dynamical targets with non-fixed centres the Hausdorff dimension is given by the intersection of two pressures: one being the standard self-affine pressure function, the other being an inverse lower pressure related to the target. Crucially, we do not expect the target to be fixed and the inverse pressure to exist.

Our condition also allows us to investigate the dimensions of sets with a pointwise recurrence, a quantitative version of recurrence for self-affine dynamics. As far as we are aware, this is the first time this was attempted for non-conformal dynamics in higher dimensions.

2 Results

2.1 Self-affine sets and symbolic space

Let $\mathbf{A} = \{A_1, A_2, \cdots, A_N\}$ be a collection of non-singular $d \times d$ contracting matrices. Let $\mathbf{t} = \{t_1, t_2, \cdots, t_N\}$ be a collection of N vectors in $\mathbb{R}^d$.

Let $\{1, \cdots, N\}$ be a finite alphabet and write $\Sigma_n, \Sigma_*, \Sigma$ for the union of words of length n, the union of all finite length words, and all infinite words, respectively. For words $\mathtt{i} \in \Sigma_n$ and $\mathtt{j} \in \Sigma$ we write $\mathtt{i} = i_1 i_2 \cdots i_n$ and $\mathtt{j} = j_1 j_2 \cdots$ to denote the individual letters of $\mathtt{i}$ and $\mathtt{j}$. For a word $\mathtt{i} \in \Sigma_*$, let $|\,\mathtt{i}\,|$ denote the length of $\mathtt{i}$. For any two words $\mathtt{i}, \mathtt{j} \in \Sigma$, let us denote the common prefix by $\mathtt{i} \wedge \mathtt{j}$, that is, $\mathtt{i} \wedge \mathtt{j} := i_1 \cdots i_{|\,\mathtt{i} \wedge \mathtt{j}\,|}$ and $|\,\mathtt{i} \wedge \mathtt{j}\,| := \min\{k \geq 1 : i_k \neq j_k\} - 1$. We adapt the notation that if $|\,\mathtt{i} \wedge \mathtt{j}\,| = 0$ then $\mathtt{i} \wedge \mathtt{j} := \emptyset$. For two $\mathtt{i}, \mathtt{j} \in \Sigma_*$, denote by $\mathtt{i} \prec \mathtt{j}$ if $\mathtt{j}$ is a prefix of $\mathtt{i}$, that is, $|\,\mathtt{i} \wedge \mathtt{j}\,| = |\,\mathtt{j}\,| \leq |\,\mathtt{i}\,|$. Let $\sigma : \Sigma \to \Sigma$ be the left-shift operator on Σ, i.e. $\sigma(\mathtt{i}) = \sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 i_4 \cdots$. Let $\Phi_{\mathbf{t}} = \{f_i(x) = A_i x + \mathbf{t}_i\}_{i=1}^N$ be an iterated function system formed by affine maps on $\mathbb{R}^d$. For a finite word $\mathtt{i} \in \Sigma_*$, let $A_{\mathtt{i}} = A_{i_1} \cdots A_{i_n}$ and $f_{\mathtt{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$. It is a classical result that there exists a unique non-empty compact set $\Lambda \subset \mathbb{R}^d$ such that

$$\Lambda = \bigcup_{i=1}^{N} f_i(\Lambda).$$

To avoid singleton sets we assume that $N \geq 2$ throughout. Let us denote by $\pi = \pi_{\mathbf{t}}$ the natural projection from Σ to the attractor of $\Phi_{\mathbf{t}}$, that is,

$$\pi_{\mathbf{t}}(\mathtt{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = \sum_{k=1}^{\infty} A_{i_1} \cdots A_{i_{k-1}} \mathbf{t}_{i_k} .$$

Clearly, $\pi_{\mathbf{t}}(\mathtt{i}) = f_{i_1}(\pi_{\mathbf{t}}(\sigma\, \mathtt{i}))$ and so

$$\pi_{\mathbf{t}}(\mathtt{i}) - \pi_{\mathbf{t}}(\mathtt{j}) = A_{\mathtt{i}\wedge\mathtt{j}}\big(\pi_{\mathbf{t}}(\sigma^{|\mathtt{i}\wedge\mathtt{j}|}\,\mathtt{i}) - \pi_{\mathbf{t}}(\sigma^{|\mathtt{i}\wedge\mathtt{j}|}\,\mathtt{j})\big).$$

For a $d \times d$ matrix $A \in \mathrm{GL}_d(\mathbb{R})$ let $\varphi^s(A)$ be the usual singular value function defined by

$$\varphi^s(A) = \begin{cases} \alpha_1(A)\alpha_2(A)\ldots\alpha_{\lfloor s\rfloor}(A)\alpha_{\lfloor s\rfloor+1}(A)^{s-\lfloor s\rfloor} & \text{for } 0 \leq s < d, \\ (\alpha_1(A)\ldots\alpha_{d-1}(A)\alpha_d(A))^{s/d} & \text{for } s \geq d, \end{cases}$$

where $\alpha_1(A) \geq \alpha_2(A) \geq \cdots \geq \alpha_n(A)$ are the singular values of an $n \times n$ matrix A. Recall that φ^t is submultiplicative, i.e. $\varphi(A_1A_2) \leq \varphi(A_1)\varphi(A_2)$. For any ball B, clearly, $A(B)$ is an ellipsoid and, as it was shown in [7, Proof of Proposition 5.1], it can be covered by at most $(4|B|)^d \frac{\alpha_1(A)\cdots\alpha_{\lfloor s\rfloor}(A)}{\alpha_{\lceil s\rceil}(A)^{\lfloor s\rfloor}}$-many cubes with side length $\alpha_{\lceil s\rceil}(A)$.

The pressure of the self-affine system is defined as

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \varphi^s(A_{\mathtt{i}}),$$

where we note that this limit exists because of the submultiplicativity of $\varphi^s(A)$. Further, the pressure is continuous in s, strictly decreasing, and satisfies $P(0) = \log N$ and $P(s) \to -\infty$ as $s \to \infty$.

Throughout the paper we will use the following extra condition:

Condition 2.1. *Assume that* $\mathbf{A}$ *is such that for every* $s > 0$ *there exists* $C > 0$ *and* $K \in \mathbb{N}$ *such that for every* $\mathtt{i}, \mathtt{j} \in \Sigma_*$ *there exists* $\mathtt{k} \in \Sigma_K$ *with*

$$\varphi^s(A_{\mathtt{i}\,\mathtt{k}\,\mathtt{j}}) \geq C\varphi^s(A_{\mathtt{i}})\varphi^s(A_{\mathtt{j}}).$$

Similar conditions has been introduced earlier by Feng [10] and Käenmäki and Morris [15]. Feng [10, Proposition 2.8] showed that under a mild irreducibility condition there exists $C > 0$ and $K > 0$ such that for every $\mathtt{i}, \mathtt{j} \in \Sigma_*$ there exists $\mathtt{k}$ with $|\,\mathtt{k}\,| \leq K$ such that $\|A_{\mathtt{i}\,\mathtt{k}\,\mathtt{j}}\| \geq C\|A_{\mathtt{i}}\|\|A_{\mathtt{j}}\|$. Later, this inequality was generalised by Käenmäki and Morris [15, Lemma 3.5] for the singular value function under more restrictive but natural irreducibility conditions. Unfortunately, the uncertainty of the length of the "buffer" word $\mathtt{k}$ in the previous conditions does not allow us to study shrinking target and recurrence sets effectively. We will show in Section 2.4 and Section 5 that under some irreducibility and proximality assumptions, Condition 2.1 holds.

2.2 Shrinking targets

Let $(\lambda_k)_{k\in\mathbb{N}} \in (\Sigma_*)^{\mathbb{N}}$ be a sequence of target cylinders. We are interested in the shrinking target set

$$S_{\mathbf{t}}((\lambda_k)_{k\in\mathbb{N}}) = \pi_{\mathbf{t}} \left\{ \mathtt{i} \in \Sigma : \sigma^k \mathtt{i} \in [\lambda_k] \text{ for infinitely many } k \in \mathbb{N} \right\}.$$

For our sequence of target cylinders, we define the following inverse lower pressure:

$$\alpha(t) = \liminf_{k\to\infty} \frac{-1}{k} \log \varphi^t(A_{\lambda_k}) \tag{2.1}$$

Let

$$s_0 := \inf\{t > 0 : P(t) \leq \alpha(t)\}. \tag{2.2}$$

If $\liminf_{n\to\infty} \frac{|\lambda_k|}{k} < \infty$ then there exists a unique solution s_0 to the equation $P(s_0) = \alpha(s_0) \geq 0$, see Lemma 3.4. Otherwise $s_0 = 0$. We prove that this value gives the Hausdorff dimension of the shrinking target set under some assumptions on the matrices $\mathbf{A}$. Throughout the paper, we denote the Hausdorff dimension of a set $X \subseteq \mathbb{R}^d$ by $\dim_H X$ and its d-dimensional Lebesgue measure by $\mathcal{L}_d(X)$.

Theorem 2.2. *Let $\mathbf{A}$ be a collection of $d \times d$ matrices and let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of target cylinders. Suppose that $\mathbf{A}$ satisfies Condition 2.1 and $\|A\| < 1/2$ for all $A \in \mathbf{A}$. Then*

$$\dim_H S_{\mathbf{t}}((\lambda_k)_k) = \min\{d, s_0\} \textit{ for Lebesgue-almost every } \mathbf{t}.$$

Moreover, $\mathcal{L}_d(R_{\mathbf{t}}((\lambda_k)_k)) > 0$ for Lebesgue-almost every $\mathbf{t}$ if $s_0 > d$.

Remark 2.3. The upper dimension bounds do not just hold for almost every translation $\mathbf{t}$, but hold for all translations.

Similar result has been obtained by Koivusalo and Ramírez [18] for shrinking targets on self-affine sets. Firstly, they assume that there exists a constant $C > 0$ such that for every $\mathtt{i}, \mathtt{j} \in \Sigma_*$, $\varphi^s(A_{\mathtt{i}}A_{\mathtt{j}}) \geq C\varphi^s(A_{\mathtt{i}})\varphi^s(A_{\mathtt{j}})$, secondly, they assume that $\alpha(t)$ is taken as a limit. The first condition holds only for a restrictive family of matrices, see Remark 2.6. By using a more detailed analysis on the pressure function, we were able to relax the condition on the limit as well.

2.3 Recurrence sets

Now, we turn our attention to the recurrence sets. Let $\psi\colon \mathbb{N} \mapsto \mathbb{N}$, and let $\beta = \liminf_{n\to\infty} \frac{\psi(n)}{n}$. Consider the set

$$R_{\mathbf{t}}(\psi) := \pi_{\mathbf{t}} \left\{ \mathtt{i} \in \Sigma : \sigma^k \mathtt{i} \in [\mathtt{i}|_{\psi(k)}] \text{ for infinitely many } k \in \mathbb{N} \right\}.$$

Let us define the square-pressure function

$$P_2(t) = \lim_{n\to\infty} \frac{-1}{n} \log \sum_{\mathtt{i}\in\Sigma_n} \left(\varphi^t(A_{\mathtt{i}})\right)^2.$$

Note that the limit exists again because of the subadditivity of $\varphi^t(A)$. Further, the pressure is continuous in t, strictly increasing, and satisfies $P_2(0) = -\log N$ and $P_2(t) \to \infty$ as $t \to \infty$.

Theorem 2.4. *Let* $\mathbf{A}$ *be a collection of* $d \times d$ *matrices. Suppose that* $\mathbf{A}$ *satisfies Condition 2.1 and* $\|A\| < 1/2$ *for all* $A \in \mathbf{A}$*. Let* $\psi : \mathbb{N} \to \mathbb{N}$ *with* $\beta := \liminf_{n\to\infty} \psi(n)/n < 1$ *then*

$$\dim_H R_{\mathbf{t}}(\psi) = \min\{d, r_0\} \text{ for Lebesgue-almost every } \mathbf{t},$$

where r_0 *is the unique solution of the equation*

$$(1 - \beta)P(r_0) = \beta P_2(r_0). \tag{2.3}$$

Moreover, $\mathcal{L}_d(R_{\mathbf{t}}(\psi)) > 0$ *for Lebesgue-almost every* $\mathbf{t}$ *if* $r_0 > d$*.*

The equation (2.3) applies specifically only to the case when $\beta \leq 1$, for other values of β it needs to be modified accordingly. The condition $\beta < 1$ is purely technical and relies on the fact that the buffer word in Condition 2.1 depends on both of the words before and after it. Hence, for recurrence rates greater than 1 it might cause "self-dependence" in the buffer word, which then may not exist. We note that under the stronger assumption on the matrices by Koivusalo and Ramírez [18], Theorem 2.4 can be generalized for any value $\beta \in [0, \infty]$ with a straightforward modification of (2.3) and the proof of Theorem 2.4.

2.4 Irreducibility of matrices

Let us denote by $\wedge^k \mathbb{R}^d$ the *k-th exterior product of* $\mathbb{R}^d$. For $A \in \mathrm{GL}_d(\mathbb{R})$, we can define an invertible linear map $A^{\wedge k}\colon \wedge^k \mathbb{R}^d \mapsto \wedge^k \mathbb{R}^d$ by setting

$$A^{\wedge k}(u_1 \wedge \cdots \wedge u_k) = (Au_1) \wedge \cdots \wedge (Au_k).$$

Let us consider the following tensor product of the exterior algebras

$$\widehat{W} = \wedge^1 \mathbb{R}^d \otimes \cdots \otimes \wedge^{d-1} \mathbb{R}^d.$$

Again, for $A \in \mathrm{GL}_d(\mathbb{R})$, we can define an invertible linear map $\widehat{A}\colon \widehat{W} \mapsto \widehat{W}$ by setting for $u = u_1 \otimes \cdots \otimes u_{d-1}$,

$$\widehat{A}(u_1 \otimes \cdots \otimes u_{d-1}) = (A^{\wedge 1}u_1) \otimes \cdots \otimes (A^{\wedge(d-1)}u_{d-1}).$$

We define a linear subspace W of $\widehat{W}$, which is generated by the flags of $\mathbb{R}^d$ as follows:

$$W = \mathrm{span}\{u_1 \otimes (u_1 \wedge u_2) \otimes \cdots \otimes (u_1 \wedge \cdots \wedge u_{d-1}) : \{u_1, \ldots, u_{d-1}\} \text{ linearly independent in } \mathbb{R}^d\}.$$

We call W the *flag vector space.* Note that the flag space W is invariant with respect to the linear map $\widehat{A}$ for $A \in \mathrm{GL}_d(\mathbb{R})$.

We say that a linear map A is 1-*proximal* if it has a unique eigenvalue of maximal absolute value and this eigenvalue has algebraic (and hence geometric) multiplicity one. We say that $A \in \mathrm{GL}_d(\mathbb{R})$ is *fully proximal* if it has d distinct eigenvalues in absolute value. Simple calculations show that A is fully proximal then $A^{\wedge k}$ is 1-proximal for every k and $\widehat{A}$ is 1-proximal on W. We say that the tuple $\mathbf{A}$ is fully proximal if there exists a finite product $A_{i_1} \cdots A_{i_k}$ formed by the elements in $\mathbf{A}$, which is fully proximal.

We say that the tuple $\mathbf{A}$ is *fully strongly irreducible* or *strongly irreducible over* W if there are no finite collections $V_1, \ldots, V_n$ of proper subspaces of W such that

$$\bigcup_{A\in\mathbf{A}} \bigcup_{k=1}^{n} \widehat{A}V_k = \bigcup_{k=1}^{n} V_k.$$

Roughly speaking, the tuple $\mathbf{A}$ being fully proximal and fully irreducible means that it generates the most general geometric picture.

Proposition 2.5. *Let* $\mathbf{A}$ *be a tuple of matrices in* $\mathrm{GL}_d(\mathbb{R})$ *such that* $\mathbf{A}$ *is fully proximal and fully strongly irreducible. Then for every* $0 < s < d$ *there exists* $C > 0$ *and* $K \in \mathbb{N}$ *such that for every* $\mathtt{i}, \mathtt{j} \in \Sigma_*$ *there exists* $\mathtt{k} \in \Sigma_K$ *with*

$$\varphi^s(A_{\mathtt{i}\,\mathtt{k}\,\mathtt{j}}) \geq C\varphi^s(A_{\mathtt{i}})\varphi^s(A_{\mathtt{j}}).$$

Remark 2.6. Koivusalo and Ramírez [18] assumed that there exists a constant $C > 0$ such that for every $\mathtt{i}, \mathtt{j} \in \Sigma_*$

$$\varphi^s(A_{\mathtt{i}} A_{\mathtt{j}}) \geq C\varphi^s(A_{\mathtt{i}})\varphi^s(A_{\mathtt{j}}).$$

Bárány, Käenmäki and Morris [4, Corollary 2.5] showed that this condition for planar matrix tuples $\mathbf{A}$ is equivalent with the following: $\mathbf{A}$ can be decomposed into two sets $\mathbf{A}_e$ and $\mathbf{A}_h$ such that $\mathbf{A}_e$ is strongly conformal (i.e. can be transformed into orthonormal matrices with a common base transformation) and if $\mathbf{A}_h \neq \emptyset$, then $\mathbf{A}_h$ has a strongly invariant multicone $\mathcal{C}$ (i.e. $\bigcup_{A\in\mathbf{A}_h} A\overline{\mathcal{C}} \subset \mathcal{C}^o$) such that $A\mathcal{C} = \mathcal{C}$ for all $A \in \mathbf{A}_e$.

Assuming fully strong irreducibilty and fully proximality is clearly a less restrictive requirement. For instance, in case of planar matrices fully strong irreducibility and fully proximality is equivalent with strong irreducibility and proximality.

Using Proposition 2.5 we obtain the following immediate corollaries.

Corollary 2.7. *Let* $\mathbf{A}$ *be a collection of* $d \times d$ *matrices and let* $(\lambda_k)_{k\in\mathbb{N}}$ *be a sequence of target cylinders. Suppose that* $\mathbf{A}$ *is fully strongly irreducible and fully proximal and* $\|A\| < 1/2$ *for all* $A \in \mathbf{A}$*. Then*

$$\dim_H S_{\mathbf{t}}((\lambda_k)_k) = \min\{d, s_0\} \text{ for Lebesgue-almost every } \mathbf{t},$$

where s_0 *is defined in* (2.2)*. Moreover,* $\mathcal{L}_d(S_{\mathbf{t}}((\lambda_k)_k)) > 0$ *for Lebesgue-almost every* $\mathbf{t}$ *if* $s_0 > d$*.*

Corollary 2.8. *Let* $\mathbf{A}$ *be a collection of* $d\times d$ *matrices. Suppose that* $\mathbf{A}$ *is fully strongly irreducible and fully proximal and* $\|A\| < 1/2$ *for all* $A \in \mathbf{A}$*. Let* $\psi : \mathbb{N} \to \mathbb{N}$ *with* $\beta := \liminf \psi(n)/n < 1$ *then*

$$\dim_H R_{\mathbf{t}}(\psi) = \min\{d, r_0\} \text{ for Lebesgue-almost every } \mathbf{t},$$

where r_0 *is the unique solution of the equation* (2.3)*. Moreover,* $\mathcal{L}_d(R_{\mathbf{t}}(\psi)) > 0$ *for Lebesgue-almost every* $\mathbf{t}$ *if* $r_0 > d$*.*

Structure. We prove Theorem 2.2 in Section 3 and Theorem 2.4 in Section 4 using Condition 2.1. First, we derive elementary results on the inverse lower pressure α defined in (2.1) in Section 3.1. We will also recall results about the pressure P and prove the uniqueness of the solution of $P(s_0) = \alpha(s_0)$. We proceed in Section 3.2 by proving the upper bound to Theorem 2.2 and finish the lower bound proof in Section 3.3 with an energy estimate. Similarly, Section 4.1 is devoted to show the upper bound and Section 4.2 is to show the lower bound of Theorem 2.4. Section 5 contains the proof of Proposition 2.5, which shows that the assumptions in Corollary 2.7 and 2.8 are sufficient.

3 Dimension of shrinking targets

3.1 Basic properties and the inverse lower pressure function

Let $(\lambda_k)_{k\in\mathbb{N}} \in (\Sigma_*)^{\mathbb{N}}$ be a sequence and let α be the corresponding inverse lower pressure defined in (2.1).

Lemma 3.1. *If $\alpha(s) = 0$ for some $s > 0$, then $\liminf_{n\to\infty} |\lambda_n|/n = 0$. Conversely, if $\liminf_{n\to\infty} |\lambda_n|/n = 0$, then $\alpha(t) = 0$ for all $t > 0$.*

In particular, if there exists $s > 0$ such that $\alpha(s) = 0$, then $\alpha(t) = 0$ for all $t \geq 0$.

Proof. Let $\gamma = \max_{i\in\Sigma_1}\{\alpha_1(A_i)\}$ and $\underline{\gamma} = \min_{i\in\Sigma_1}\{\alpha_d(A_i)\}$. Observe that by definition

$$\underline{\gamma}^{s|\mathtt{i}|} \leq \varphi^s(A_{\mathtt{i}}) \leq \gamma^{s|\mathtt{i}|}. \tag{3.1}$$

Assume that $\alpha(s) = 0$. Since $-1/n \log \varphi^s(A_{\lambda_n}) \geq 0$, this implies that there is a subsequence n_k such that $1/n_k \log \varphi^s(A_{\lambda_{n_k}}) \nearrow 0$ and that the ratios between subsequent terms are greater than $\gamma/\underline{\gamma}$. But then $1/n_k \log \gamma^{s|\lambda_{n_k}|} = s|\lambda_{n_k}|/n_k \log \gamma \nearrow 0$ and so $|\lambda_{n_k}|/n_k \searrow 0$, as required.

For the other direction assume $|\lambda_{n_k}|/n_k \to 0$ for some subsequence n_k. Then, for any $t \geq 0$,

$$\alpha(t) = \liminf_{n\to\infty} -\frac{1}{n} \log \varphi^t(A_{\lambda_n}) \leq \liminf_{n\to\infty} -\frac{1}{n} \log \underline{\gamma}^{t|\lambda_n|} \leq \liminf_{k\to\infty} \frac{|\lambda_{n_k}|}{n_k} t(-\log \underline{\gamma}) \leq 0.$$

Combining this with the trivial inequality $\alpha(t) \geq 0$ we get the desired conclusion that $\alpha(t) = 0$ for all $t \geq 0$. □

Similarly, if the modified pressure function is extremal in the other direction it must be extremal everywhere.

Lemma 3.2. *If $\alpha(s) = \infty$ for some $s > 0$, then $\lim_{n\to\infty} |\lambda_n|/n = \infty$. Conversely, if $\lim_{n\to\infty} |\lambda_n|/n = \infty$, then $\alpha(t) = \infty$ for every $t > 0$.*

In particular, if there exists $s > 0$ such that $\alpha(s) = \infty$, then $\alpha(t) = \infty$ for all $t > 0$.

The proof is analogous to that of Lemma 3.1 and is left to the reader.

Lemma 3.3. *Assume that $0 < \liminf_{k\to\infty} |\lambda_k|/k < \infty$. Then the function $t \mapsto \alpha(t)$ is strictly monotone increasing, and continuous in t. Moreover, $\alpha(t) \to \infty$ as $t \to \infty$ and $\alpha(0) = 0$.*

Proof. Note that by Lemma 3.1 and Lemma 3.2 the inverse lower pressure satisfies $0 < \alpha(t) < \infty$ for all $t > 0$. Letting $t = 0$, we have

$$\alpha(0) = \liminf_{n\to\infty} -\frac{1}{n} \log \varphi^0(A_{\lambda_n}) = \log 1 = 0.$$

For every $k \in \mathbb{N}$,

$$\frac{-1}{k} \log \varphi^s(A_{\lambda_k}) \leq \frac{|\lambda_k|}{k} s(-\log \underline{\gamma}),$$

and so

$$s \liminf_{k\to\infty} \frac{|\lambda_k|}{k} \geq \frac{\alpha(s)}{-\log \underline{\gamma}} \text{ for every } s > 0. \tag{3.2}$$

This shows that $\alpha(t)$ is continuous at $t = 0$.

For any $t > 0$ and $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned}
\alpha(t-\varepsilon) &\leq \liminf_{k\to\infty} -\frac{1}{k} \log\left(\alpha_1(A_{\lambda_k})^{-\varepsilon} \varphi^t(A_{\lambda_k})\right) \\
&\leq \liminf_{k\to\infty} -\frac{1}{k} \log\left(\gamma^{-\varepsilon|\lambda_k|} \varphi^t(A_{\lambda_k})\right) \\
&= \liminf_{k\to\infty} \frac{1}{k}\left(\varepsilon|\lambda_k| \log \gamma - \log \varphi^t(A_{\lambda_k})\right) \\
&\leq \limsup_{k\to\infty} \frac{\varepsilon|\lambda_k| \log \gamma}{k} + \liminf_{k\to\infty}\left(-\frac{1}{k} \log \varphi^t(A_{\lambda_k})\right) \\
&= \alpha(t) - \varepsilon \frac{\log \gamma}{(t-\varepsilon)\log \underline{\gamma}} \alpha(t-\varepsilon),
\end{aligned}$$

where in the last inequality we applied (3.2) with $s = t - \varepsilon$. Hence,

$$\alpha(t-\varepsilon) \le \alpha(t)\left(1 + \frac{\varepsilon \log \gamma}{(t-\varepsilon)\log \underline{\gamma}}\right)^{-1} < \alpha(t), \tag{3.3}$$

which shows that $\alpha(t)$ is strictly monotone increasing on $(0, \infty)$.

For an $s > 0$, let $n_k(s)$ be a sequence for which the lower limit in $\alpha(s)$ is achieved. Then by (3.1),

$$\frac{-|\lambda_{n_k(s)}| s \log \gamma}{n_k(s)} \le \frac{-1}{n_k(s)} \log \varphi^s(A_{\lambda_{n_k(s)}}).$$

Hence for every $s > 0$,

$$\limsup_{k\to\infty} \frac{|\lambda_{n_k(s)}|}{n_k(s)} \le \frac{\alpha(s)}{-s\log \gamma}.$$

This implies that

$$\begin{aligned}
\alpha(t-\varepsilon) &= \lim_{k\to\infty} -\frac{1}{n_k(t-\varepsilon)} \log \varphi^{t-\varepsilon}(A_{\lambda_{n_k(t-\varepsilon)}}) \\
&\ge \liminf_{k\to\infty} -\frac{1}{n_k(t-\varepsilon)} \log \left(\alpha_d(A_{\lambda_{n_k(t-\varepsilon)}})^{-\varepsilon} \varphi^t(A_{\lambda_{n_k(t-\varepsilon)}})\right) \\
&\ge \liminf_{k\to\infty} -\frac{1}{n_k(t-\varepsilon)} \log \left(\underline{\gamma}^{-\varepsilon |\lambda_{n_k(t-\varepsilon)}|} \varphi^t(A_{\lambda_{n_k(t-\varepsilon)}})\right) \\
&= \liminf_{k\to\infty} \left(-\frac{1}{n_k(t-\varepsilon)} \log \varphi^t(A_{\lambda_{n_k(t-\varepsilon)}}) + \varepsilon \frac{|\lambda_{n_k(t-\varepsilon)}|}{n_k(t-\varepsilon)} \log \underline{\gamma}\right) \\
&\ge \liminf_{k\to\infty} \left(-\frac{1}{n_k(t-\varepsilon)} \log \varphi^t(A_{\lambda_{n_k(t-\varepsilon)}})\right) + \varepsilon \log \underline{\gamma} \limsup_{k\to\infty} \frac{|\lambda_{n_k(t-\varepsilon)}|}{n_k(t-\varepsilon)} \\
&\ge \alpha(t) - \varepsilon \frac{\alpha(t-\varepsilon)\log \underline{\gamma}}{(t-\varepsilon)\log \gamma}.
\end{aligned}$$

Thus,

$$\alpha(t-\varepsilon) \ge \alpha(t)\left(1 + \frac{\varepsilon \log \underline{\gamma}}{(t-\varepsilon)\log \gamma}\right)^{-1},$$

which together with (3.3) implies continuity.

To show the limit as $t \to \infty$, observe that

$$\alpha(t) = \liminf_{k\to\infty} -\frac{1}{k} \log \varphi^t(A_{\lambda_k}) \ge \liminf_{k\to\infty} -\frac{1}{k} \log \gamma^{t|\lambda_k|} \ge t(-\log \gamma) \liminf_{k\to\infty} \frac{|\lambda_k|}{k}. \qquad \square$$

Lemma 3.4. *Assume that* $\liminf_{k\to\infty} |\lambda_k|/k < \infty$. *Then there exists a unique* $s_0 > 0$ *such that* $P(s_0) = \alpha(s_0)$. *Further,* $P(s_0) \ge 0$.

Proof. If $\liminf_{k\to\infty} |\lambda_k|/k > 0$ then the first statement follows by Lemma 3.3 since $P(0) - \alpha(0) = \log N$, and $P(t) - \alpha(t) \to -\infty$ as $t \to \infty$ and $P(t) - \alpha(t)$ is strictly monotone decreasing. If $\liminf_{k\to\infty} |\lambda_k|/k = 0$ then by Lemma 3.1 $\alpha(t) \equiv 0$ for $t \ge 0$ and then the uniqueness of the solution follows by the uniqueness of the root of P.

The second conclusion follows from the observation that $\alpha(t) \ge 0$ for all $t \ge 0$. $\square$

The following lemma is standard, but we include it for completeness.

Lemma 3.5. *Let* $\mathtt{i} \in \Sigma_*$ *be a finite word and let* $0 < t < s$*. Then,*

$$\frac{\varphi^s(A_{\mathtt{i}})}{\varphi^t(A_{\mathtt{i}})} \le \gamma^{(s-t)|\,\mathtt{i}\,|} \tag{3.4}$$

for $0 < \gamma = \max_{i\in\Sigma_1} \{\alpha_1(A_i)\} < 1$ *as defined in Lemma 3.1.*

Proof. For $0 < t < s < d$,

$$\begin{aligned}\frac{\varphi^s(A_{\mathtt{i}})}{\varphi^t(A_{\mathtt{i}})} &= \frac{\alpha_1(A_{\mathtt{i}})\dots\alpha_{\lfloor s\rfloor}(A_{\mathtt{i}})\cdot\alpha_{\lfloor s\rfloor+1}(A_{\mathtt{i}})^{s-\lfloor s\rfloor}}{\alpha_1(A_{\mathtt{i}})\dots\alpha_{\lfloor t\rfloor}(A_{\mathtt{i}})\cdot\alpha_{\lfloor t\rfloor+1}(A_{\mathtt{i}})^{t-\lfloor t\rfloor}}\\
&\le \frac{\alpha_1(A_{\mathtt{i}})\dots\alpha_{\lfloor t\rfloor}(A_{\mathtt{i}})\cdot\alpha_{\lfloor t\rfloor+1}(A_{\mathtt{i}})^{\lfloor s\rfloor-\lfloor t\rfloor}\alpha_{\lfloor t\rfloor+1}(A_{\mathtt{i}})^{s-\lfloor s\rfloor}}{\alpha_1(A_{\mathtt{i}})\dots\alpha_{\lfloor t\rfloor}(A_{\mathtt{i}})\cdot\alpha_{\lfloor t\rfloor+1}(A_{\mathtt{i}})^{t-\lfloor t\rfloor}}\\
&= \alpha_{\lfloor t\rfloor+1}(A_{\mathtt{i}})^{s-t}.\end{aligned}$$

Similarly, for $0 < t < d \le s$,

$$\begin{aligned}\frac{\varphi^s(A_{\mathtt{i}})}{\varphi^t(A_{\mathtt{i}})} &= \frac{(\alpha_1(A_{\mathtt{i}})\dots\alpha_d(A_{\mathtt{i}}))^{s/d}}{\alpha_1(A_{\mathtt{i}})\dots\alpha_{\lfloor t\rfloor}(A_{\mathtt{i}})\cdot\alpha_{\lfloor t\rfloor+1}(A_{\mathtt{i}})^{t-\lfloor t\rfloor}}\\
&= \big(\alpha_1(A_{\mathtt{i}})\dots\alpha_{\lfloor t\rfloor}(A_{\mathtt{i}})\big)^{s/d-1}\,\alpha_{\lfloor t\rfloor+1}(A_{\mathtt{i}})^{s/d-t+\lfloor t\rfloor}\,\big(\alpha_{\lfloor t\rfloor+2}(A_{\mathtt{i}})\cdots\alpha_d(A_{\mathtt{i}})\big)^{s/d}\\
&\le \alpha_1(A_{\mathtt{i}})^{(s/d-1)\lfloor t\rfloor+s/d-t+\lfloor t\rfloor+s(d-\lfloor t\rfloor-1)/d} = \alpha_1(A_{\mathtt{i}})^{s-t}.\end{aligned}$$

Finally, for $d \le t < s$,

$$\frac{\varphi^s(A_{\mathtt{i}})}{\varphi^t(A_{\mathtt{i}})} = \frac{(\alpha_1(A_{\mathtt{i}})\dots\alpha_d(A_{\mathtt{i}}))^{s/d}}{(\alpha_1(A_{\mathtt{i}})\dots\alpha_d(A_{\mathtt{i}}))^{t/d}} = (\det(A_{\mathtt{i}}))^{(s-t)/d}.$$

We conclude that (3.4) holds for $\gamma := \max_{i\in\Sigma_1}\{\alpha_1(A_i)\}$ by submultiplicativity. □

3.2 Upper bound to Theorem 2.2

Note that $S_{\mathbf{t}}((\lambda_k)_k)$ is a lim sup set that can be written as

$$S_{\mathbf{t}}((\lambda_k)_k) = \bigcap_{k_0=1}^{\infty}\bigcup_{k=k_0}^{\infty}\bigcup_{\mathtt{i}\in\Sigma_k}\pi_{\mathbf{t}}([\mathtt{i}\,\lambda_k]).$$

Temporarily fix $t \ge 0$. By definition, for every $\delta > 0$ there exists k_0 large enough such that

$$-\frac{1}{k}\log\varphi^t(A_{\lambda_k}) \ge \alpha(t) - \delta$$

for all $k \ge k_0$. This can be rearranged to give

$$\varphi^t(A_{\lambda_k}) \le e^{-k(\alpha(t)-\delta)}. \tag{3.5}$$

Similarly, for every $\delta > 0$, we obtain

$$\sum_{\mathtt{i}\in\Sigma_k}\varphi^t(A_{\mathtt{i}}) \le e^{k(P(t)+\delta)} \tag{3.6}$$

for large enough k. For the lower bounds, we note that for all $\delta > 0$ there exists a subsequence k_n such that

$$\varphi^t(A_{\lambda_{k_n}}) \geq e^{-k_n(\alpha(t)+\delta)} \tag{3.7}$$

and for large enough k,

$$\sum_{\mathtt{i}\in\Sigma_k} \varphi^t(A_{\mathtt{i}}) \geq e^{k(P(t)-\delta)} \tag{3.8}$$

by submultiplicativity and existence of the limit.

Assume that $\liminf_k |\lambda_k|/k < \infty$. Let $s > s_0$ and note that $P(s) - \alpha(s) < 0$. We set $\delta > 0$ small enough such that $\eta := P(s) - \alpha(s) + 2\delta < 0$. We may now apply (3.5) and (3.6) with $t = s$ where $k \geq k_0$ with k_0 depending on s.

Let us repeat the construction of the cover given in [7, Proof of Proposition 5.1] for convenience. Let B be a ball with sufficiently large radius such that $f_i(B) \subset B$ for all $i = 1, \ldots, N$. Without loss of generality we may assume that $|B| \geq 1$. Clearly, the set $f_{\mathtt{i}}(B)$ is an ellipsoid with axis length $\alpha_1(A_{\mathtt{i}})|B|, \ldots, \alpha_d(A_{\mathtt{i}})|B|$. The ellipsoid $f_{\mathtt{i}}(B)$ can be covered by a d-dimensional rectangle $P_{\mathtt{i}}$ such that the edges of $P_{\mathtt{i}}$ are parallel to the corresponding axes of $f_{\mathtt{i}}(B)$ and of lengths $\alpha_1(A_{\mathtt{i}})|B|, \ldots, \alpha_d(A_{\mathtt{i}})|B|$. Furthermore, the rectangle $P_{\mathtt{i}}$ can be covered by $\prod_{j=1}^{\lfloor s\rfloor}\lceil \frac{\alpha_j(A_{\mathtt{i}})}{\alpha_{\lceil s\rceil}(A_{\mathtt{i}})}\rceil$-many cubes of side length $\alpha_{\lceil s\rceil}(A_{\mathtt{i}})|B|$. Since $S(\{\lambda_k\}_{k\in\mathbb{N}}) \subseteq \bigcup_{k=k_0}^{\infty}\bigcup_{\mathtt{i}\in\Sigma_k} f_{\mathtt{i}\,\lambda_k}(B)$ for every k_0, we obtain

$$\begin{aligned}
\mathcal{H}^s(S_{\mathbf{t}}((\lambda_k)_k)) &\leq \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty}\sum_{\mathtt{i}\in\Sigma_k}\prod_{j=1}^{\lfloor s\rfloor}\left\lceil \frac{\alpha_j(A_{\mathtt{i}\,\lambda_k})}{\alpha_{\lceil s\rceil}(A_{\mathtt{i}\,\lambda_k})}\right\rceil (\alpha_{\lceil s\rceil}(A_{\mathtt{i}\,\lambda_k})|B|\sqrt{d})^s \\
&\leq \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty}\sum_{\mathtt{i}\in\Sigma_k} \varphi^s(A_{\mathtt{i}\,\lambda_k})(4|B|\sqrt{d})^d \\
&\leq \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty}\sum_{\mathtt{i}\in\Sigma_k} \varphi^s(A_{\mathtt{i}})\varphi^s(A_{\lambda_k})(4|B|\sqrt{d})^d \\
&\leq \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} e^{-k(\alpha(s)-\delta)} \sum_{\mathtt{i}\in\Sigma_k} \varphi^s(A_{\mathtt{i}})(4|B|\sqrt{d})^d \\
&\leq \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} e^{k(P(s)+\delta-\alpha(s)+\delta)}(4|B|\sqrt{d})^d \\
&= \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} e^{\eta k}(4|B|\sqrt{d})^d \leq \liminf_{k_0\to\infty} \frac{e^{\eta k_0}}{1-e^{\eta}}(4|B|\sqrt{d})^d = 0.
\end{aligned}$$

Since $s > s_0$ was arbitrary, we conclude that $\dim_H S_{\mathbf{t}}((\lambda_k)_k) \leq s_0$ for all $\mathbf{t}$.

Finally, consider the case when $\liminf_k |\lambda_k|/k = \infty$. Let $s > 0$ be arbitrary and again write $\gamma = \max_{i\in\Sigma_1}\{\alpha_1(A_i)\}$. Recall that $\#\Sigma_1 = N$ and observe that there exists M such that $|\lambda_k| \geq 2k\log N/(s\log\gamma^{-1})$ for $k \geq M$. Therefore $\gamma^{s|\lambda_k|} \leq N^{-2k}$ for large enough k. The Hausdorff measure bound above becomes

$$
\begin{aligned}
\mathcal{H}^s(S_{\mathbf{t}}((\lambda_k)_k)) &\le \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} \sum_{\mathtt{i}\in\Sigma_k} \varphi^s(A_{\mathtt{i}\,\lambda_k})(4|B|\sqrt{d})^d \\
&\le \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} \sum_{\mathtt{i}\in\Sigma_k} \varphi^s(A_{\mathtt{i}})\varphi^s(A_{\lambda_k})(4|B|\sqrt{d})^d \\
&\le \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} \gamma^{|\lambda_k|s} \sum_{\mathtt{i}\in\Sigma_k} \varphi^s(A_{\mathtt{i}})(4|B|\sqrt{d})^d \\
&\le \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} N^{-2k}N^k(4|B|\sqrt{d})^d \\
&= \liminf_{k_0\to\infty} \sum_{k=k_0}^{\infty} N^{-k}(4|B|\sqrt{d})^d = 0.
\end{aligned}
$$

As $s > 0$ was arbitrary, this shows that $\dim_H S_{\mathbf{t}}((\lambda_k)_k) = 0$ for all $\mathbf{t}$. □

3.3 Lower bound to Theorem 2.2

To simplify the exposition we will abuse notation slightly and write $\varphi^s(\mathtt{i})$ instead of $\varphi^s(A_{\mathtt{i}})$ for $\mathtt{i} \in \Sigma_*$.

For every sufficiently large $p \in \mathbb{N}$ and $s < \min\{s_0, d\}$, we construct a measure ν_p^s on the symbolic space Σ and investigate its projection under the self-affine iterated function system. Let $(m_k)_{k\in\mathbb{N}}$ be a sequence on which the lower limit in $\alpha(s)$ is achieved and take a very sparse subsequence such that

$$\sum_{k=1}^{n} m_k \le (1+2^{-n})m_n \quad \text{and} \quad m_n \ge 2^n \sum_{i=1}^{n-1}(|\lambda_{m_i}| + K), \tag{3.9}$$

where K is the length of the buffer word defined in Condition 2.1. We may further assume, without loss of generality, that $m_1 \gg p$ and that $m_k \ge 2^k$ and $|\lambda_{m_k}| > p$ for all k. By the pigeonhole principle there exists $1 \le \widehat{p}_0 \le p+K$ such that $m_k = \widehat{p}_0 + (K+p)q$ for infinitely many q. Again, by taking subsequences, we may assume that m_k is always of the form $\widehat{p}_0 + (K+p)q$ for some q. If $\widehat{p}_0 > K$ then we define $p_0 := \widehat{p}_0 - K$ otherwise let $p_0 := \widehat{p}_0 + p$.

We will obtain ν_p^s as the weak limit of descending measures $\nu_{p,k}^s : \Sigma \to [0,1]$. The construction is fairly intricate and involves splitting the measure into blocks of length p with “buffers” of length K in-between that are given by Condition 2.1. However, at each position $m_\ell + 1$, we want to append λ_{m_ℓ}. To ensure consistency of lengths, we need to slightly modify λ_{m_ℓ} by extending the words to be of length $p + q(K+p)$ for some $q \ge 0$. To this end we define $\lambda'_{m_\ell} = \lambda_{m_\ell}11\ldots1$, where the number of symbol 1's is $p - |\lambda_{m_\ell}| \mod (K+p)$. Let

$$\Omega(k) = \begin{cases} \{\lambda'_{m_\ell}\} & \text{if } k = m_\ell \text{ for some } \ell \in \mathbb{N}, \\ \Sigma_p & \text{otherwise.} \end{cases}$$

For every $\mathtt{i}_1, \mathtt{i}_2 \in \Sigma_*$ denote the lexicographically smallest word in Condition 2.1 by $\mathtt{k}(\mathtt{i}_1, \mathtt{i}_2) \in \Sigma_K$. We define a collection of symbols $\mathfrak{K}_n$ by induction. Let $\mathfrak{K}_0 := \Sigma_{p_0}$ Suppose that $\mathfrak{K}_n$ is defined for some $n \ge 0$. Then let us define $\mathfrak{K}_{n+1}$ as

$$\mathfrak{K}_{n+1} = \{\mathtt{i}\,\mathtt{k}\,\mathtt{j} : \mathtt{i} \in \mathfrak{K}_n, \mathtt{j} \in \Omega(|\,\mathtt{i}\,\mathtt{k}\,|) \text{ and } \mathtt{k} = \mathtt{k}(\mathtt{i},\mathtt{j})\}.$$

To ease notation let ℓ_k denote the length of words in $\mathfrak{K}_k$. Observe that by construction, every $\mathtt{i} \in \mathfrak{K}_n$ can be written of the form

$$\mathtt{i} = \mathtt{i}_1\, \mathtt{k}_1\, \mathtt{i}_2\, \mathtt{k}_2 \ldots \mathtt{k}_n\, \mathtt{i}_{n+1},$$

where for every $k \in \{2, \ldots, n+1\}$, $\mathtt{i}_k \in \Omega(\ell_{k-1} + K)$ and $\mathtt{k}_k = \mathtt{k}(\mathtt{i}_1\, \mathtt{k}_1 \ldots \mathtt{k}_{k-1}\, \mathtt{i}_k, \mathtt{i}_{k+1})$. While the cylinders in $\mathfrak{K}_n$ consist of the same number of blocks $(n+1)$ and buffers (n), their lengths are not necessarily $p_0 + n(p+K)$ due to the different lengths of λ'_{m_i}. Their lengths are however, by construction, always of length $p_0 + q(p+K)$ for some integer $q \geq n$. Let us define a lim sup set of codings $\mathfrak{K}$ by

$$\mathfrak{K} = \bigcap_k \{[\mathtt{i}] : \mathtt{i} \in \mathfrak{K}_k\}.$$

Note that for every $\mathtt{i} \in \mathfrak{K}$, we have $\sigma^{m_\ell}\mathtt{i} \in [\lambda'_{m_\ell}] \subseteq [\lambda_{m_\ell}]$ for every $\ell = 1, 2, \ldots$. To show this, it is enough to check that $m_{\ell+1} \geq m_\ell + |\lambda'_{m_\ell}| + K$ for every $\ell = 1, 2, \ldots$. By (3.9), $m_{\ell+1} \geq 2^\ell(\sum_{i=1}^{\ell} |\lambda_{m_i}| + K + m_i) \geq m_\ell + |\lambda_{m_\ell}| + K$. Since $m_{\ell+1} = \widehat{p}_0 + q_{\ell+1}(p+K)$ for some $q_{\ell+1} \in \mathbb{N}$ and by definition $|\lambda'_{m_\ell}| - |\lambda_{m_\ell}| = \min\{k \geq 0 : |\lambda_{m_\ell}| + k \in \{p + q(K+p)\}_{q\in\mathbb{N}}\}$, the claim $m_{\ell+1} \geq m_\ell + |\lambda'_{m_\ell}| + K$ follows. Hence, the image $\pi_{\mathbf{t}}(\mathfrak{K})$ is a non-empty subset of the lim sup set $S_{\mathbf{t}}((\lambda_k)_k)$ by choice of p_0.

Let $\eta(n)$ denote the number of λ'_{m_ℓ} blocks in $\mathfrak{K}_n$. Then

$$\ell_n = \sum_{i=1}^{\eta(n)} |\lambda'_{m_i}| + (n - \eta(n))p + Kn + p_0. \tag{3.10}$$

We start by defining $\nu^s_{p,0}$ on cylinders of length no less than p_0 by

$$\nu^s_{p,0}([\mathtt{i}\,\mathtt{h}]) = \frac{\varphi^s(\mathtt{i})}{\sum_{\mathtt{j}\in\Sigma_{p_0}} \varphi^s(\mathtt{j})} N^{-|\mathtt{h}|}$$

for $\mathtt{i} \in \Sigma_{p_0} = \mathfrak{K}_0$ and $\mathtt{h} \in \Sigma_*$. This uniquely defines a probability measure on Σ, i.e. $\nu^s_{p,0}(\Sigma) = 1$.

We define $\nu^s_{p,n}$ on cylinders with prefix in $\mathfrak{K}_n$ by

$$\nu^s_{p,n+1}(\mathtt{i}) = \begin{cases} \dfrac{\varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1})}{\displaystyle\sum_{\mathtt{j}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{j}_{n+1}\in\mathfrak{K}_n} \varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})} N^{-|\mathtt{i}|+|\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1}|} & \text{if } \mathtt{i} \prec \mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1} \in \mathfrak{K}_n, \\ 0 & \text{otherwise.} \end{cases}$$

Now we show that $(\nu^s_{p,k})_{k\in\mathbb{N}}$ converges weakly to some unique measure ν^s_p. Since the probability measures of Σ are forming a compact set with respect to the weak*-topology, there exists an accumulation point ν^s_p of the sequence $(\nu^s_{p,k})_{k\in\mathbb{N}}$. Suppose for a contradiction that there exists another probability measure $(\nu^s_p)'$ which is an accumulation point of the sequence $(\nu^s_{p,k})_{k\in\mathbb{N}}$ in weak*-topology. Observe that for any $\mathtt{i} \in \Sigma_*$, the measures $\nu^s_{p,k}([\mathtt{i}])$ are eventually constant and hence, $\nu^s_p([\mathtt{i}]) = \nu^s_{p,k}([\mathtt{i}])$ for all sufficiently large $k \in \mathbb{N}$. Thus, for every cylinder set $[\mathtt{i}]$, $\nu^s_p([\mathtt{i}]) = (\nu^s_p)'([\mathtt{i}])$, which is a contradiction.

Lemma 3.6. *Let $k \in \mathbb{N}_0$. Then,*

$$C^k \sum_{\substack{\mathtt{i}_1 \in \Sigma_{p_0} \\ \mathtt{i}_j \in \Omega(\ell_{j-1}+K) \\ j \le k+1}} \varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\dots\varphi^s(\mathtt{i}_{k+1})$$

$$\le \sum_{\mathtt{i}_1\,\mathtt{k}_1\dots\mathtt{k}_k\,\mathtt{i}_{k+1} \in \mathfrak{K}_k} \varphi^s(\mathtt{i}_1\,\mathtt{k}_1\dots\mathtt{k}_k\,\mathtt{i}_{k+1})$$

$$\le \sum_{\substack{\mathtt{i}_1 \in \Sigma_{p_0} \\ \mathtt{i}_j \in \Omega(\ell_{j-1}+K) \\ j \le k+1}} \varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\dots\varphi^s(\mathtt{i}_{k+1}), \quad (3.11)$$

where C is the constant appearing in Condition 2.1.

Remark 3.7. Observe that the summations in (3.11) are all over the same set. We have changed the subscript to emphasise these two points of view of $\mathfrak{K}_k$ versus its constituent parts.

Proof. The last inequality follows from the submultiplicativity of φ^s and that $\varphi^s(\mathtt{k}_j) < 1$.

The first inequality follows inductively from repeated application of Condition 2.1 as follows: The base case $k = 0$ follows trivially, since $\mathfrak{K}_0 = \Sigma_{p_0}$. For the induction step assume that (3.11) holds for $k \ge 0$. Applying Condition 2.1 to words in $\mathfrak{K}_{k+1}$ gives

$$\sum_{\mathtt{i}_1\,\mathtt{k}_1\dots\mathtt{k}_{k+1}\,\mathtt{i}_{k+2} \in \mathfrak{K}_{k+1}} \varphi^s(\mathtt{i}_1\,\mathtt{k}_1\dots\mathtt{k}_{k+1}\,\mathtt{i}_{k+2}) \ge C \sum_{\substack{\mathtt{i}_1\,\mathtt{k}_1\dots\mathtt{k}_k\,\mathtt{i}_{k+1} \in \mathfrak{K}_k \\ \mathtt{i}_{k+2} \in \Omega(\ell_{k+1}+K)}} \varphi^s(\mathtt{i}_1\,\mathtt{k}_1\dots\mathtt{k}_k\,\mathtt{i}_{k+1})\varphi^s(\mathtt{i}_{k+2})$$

and the induction hypothesis immediately gives

$$\ge C^{k+1} \sum_{\substack{\mathtt{i}_1 \in \Sigma_{p_0} \\ \mathtt{i}_j \in \Omega(\ell_{j-1}+K) \\ j \le k+2}} \varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\dots\varphi^s(\mathtt{i}_{k+2}),$$

which completes the proof. □

The proof of Theorem 2.2 reduces mainly to the following technical lemma.

Lemma 3.8. *Let $s_0 > 0$ be such that $P(s_0) = \alpha(s_0)$. Then for all $0 < t < s < s_0$ and sufficiently large p,*

$$\iint_{\Sigma\times\Sigma} \frac{d\nu_p^s(\mathtt{i})d\nu_p^s(\mathtt{j})}{\varphi^t(A_{\mathtt{i}\wedge\mathtt{j}})} < \infty. \quad (3.12)$$

Proof. Let $p \in \mathbb{N}$ be large enough such that $\gamma^{(s-t)p} < C$, where $0 < \gamma < 1$ and $1 > C > 0$ are the constants appearing in Lemma 3.5 and Condition 2.1, respectively. Since $s < s_0$, we have $P(s) > \alpha(s)$ and we can pick $\delta > 0$ such that $P(s) - \alpha(s) > 4\delta$ and choose p (which so far only depends on the C and γ) large enough such that we may apply (3.7) and (3.8) with δ, moreover, we require that $p\delta > KP(s) - 2K\delta$.

Recall that ν_p^s is supported on $\mathfrak{K}$. First, we show that for all distinct $\mathtt{i}, \mathtt{j} \in \mathfrak{K}$, their longest common prefix $\mathtt{i} \wedge \mathtt{j}$ must be a word of the form $\mathtt{i}_1\,\mathtt{k}_1\dots\mathtt{i}_n\,\mathtt{i}'$ for some $\mathtt{i}' \in \Sigma_{\le(p+K)} = \bigcup_{k=0}^{p+K} \Sigma_k$ and maximal n such that $\ell_{n-1} \le |\,\mathtt{i}\wedge\mathtt{j}\,|$, where ℓ_{n-1} denotes the length of the finite words in $\mathfrak{K}_{n-1}$. To see this, assume $|\,\mathtt{i}'\,| > p + K$. By the construction of $\mathfrak{K}_n$, if $\ell_{n-1} + K = m_j$ for some j then $\sigma^{\ell_{n-1}}\,\mathtt{i}$ must have a prefix of the form $\mathtt{k}_n\,\lambda'_{m_j}$. But since all words $\mathtt{h}\in\mathfrak{K}$ satisfy $(\sigma^{m_j}\,\mathtt{h})|_{|\lambda'_{m_j}|}=\lambda'_{m_j}$, so must $\mathtt{j}$ and we obtain

$$\mathtt{i}\wedge\mathtt{j}=\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{i}_n\,\mathtt{k}_n\,\lambda'_{m_j}\,\mathtt{i}''$$

for some finite word $\mathtt{i}''$. This however, contradicts the maximality of n. On the other hand, if $\ell_{n-1}+K\neq m_j$ for all j then again by the construction of $\mathfrak{K}_n$, $\mathtt{i}'$ must have a prefix of the form $\mathtt{k}_n\,\mathtt{i}_{n+1}\,\mathtt{i}''$, with $\mathtt{i}_{n+1}=p$ and so, it again contradicts to the maximality of n and our claim follows.

Note further that by the boundedness of the length of $\mathtt{i}'$ by $p+K$ as well as the non-singularity of the matrices A_i, there exists a universal constant D for the IFS such that

$$1/D^{t(p+2K)}\varphi^t(\mathtt{j})\leq\varphi^t(\mathtt{j}\,\mathtt{i}')\leq D^{t(p+2K)}\varphi^t(\mathtt{j}). \tag{3.13}$$

The double integral (3.12), together with (3.13) simplifies to the following sum

$$\sum_{n=0}^{\infty}\sum_{\substack{\mathtt{i}\wedge\mathtt{j}=\mathtt{i}_1\ldots\mathtt{i}_n\,\mathtt{i}'\\ \mathtt{i},\mathtt{j}\in\mathfrak{K},\,\mathtt{i}'\in\Sigma_{\leq p+K}}}\frac{\nu_p^s([\mathtt{i}\,|_{|\,\mathtt{i}\wedge\mathtt{j}\,|}])\nu_p^s([\mathtt{j}\,|_{|\,\mathtt{i}\wedge\mathtt{j}\,|}])}{\varphi^t(\mathtt{i}\wedge\mathtt{j})}\leq D^{t(p+2K)}N^{p+K}\sum_{n=0}^{\infty}\sum_{\mathtt{i}\in\mathfrak{K}_n}\frac{\nu_p^s([\mathtt{i}])^2}{\varphi^t(\mathtt{i})}.$$

Thus,

$$\begin{aligned}
&\iint_{\Sigma\times\Sigma}\frac{d\nu_p^s(\mathtt{i})d\nu_p^s(\mathtt{j})}{\varphi^t(A_{\mathtt{i}\wedge\mathtt{j}})}\\
&\leq D^{t(p+2K)}N^{p+K}\sum_{n=0}^{\infty}\sum_{\mathtt{i}\in\mathfrak{K}_n}\frac{\nu_{p,n}^s([\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1}])^2}{\varphi^t(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})}\\
&=D^{t(p+2K)}N^{p+K}\sum_{n=0}^{\infty}\sum_{\mathtt{i}\in\mathfrak{K}_n}\left(\frac{\varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1})}{\sum_{\mathtt{j}\in\mathfrak{K}_n}\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})}\right)^2\varphi^t(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})^{-1}
\end{aligned}$$

and by definition of $\mathfrak{K}_n$,

$$\begin{aligned}
&=D^{t(p+2K)}N^{p+K}\sum_{n=0}^{\infty}\left(\frac{1}{\sum_{\mathtt{j}\in\mathfrak{K}_n}\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})}\right)^2\\
&\qquad\qquad\cdot\sum_{\mathtt{i}\in\mathfrak{K}_n}\frac{(\varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1}))^2}{\varphi^t(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})}\\
&\leq D^{t(p+2K)}N^{p+K}\sum_{n=0}^{\infty}\left(\frac{1}{\sum_{\mathtt{j}\in\mathfrak{K}_n}\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})}\right)^2\\
&\qquad\qquad\cdot\sum_{\mathtt{i}\in\mathfrak{K}_n}C^{-n}\cdot\frac{\varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1})\varphi^s(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})}{\varphi^t(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})}\\
&\leq D^{t(p+2K)}N^{p+K}\sum_{n=0}^{\infty}C^{-n}\gamma^{(s-t)\ell_n}\left(\sum_{\mathtt{j}\in\mathfrak{K}_n}\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})\right)^{-1}
\end{aligned}$$

by Condition 2.1 and Lemma 3.5 for some $0 < \gamma < 1$. Again, let $\eta(n)$ denote the number of λ_{m_l} blocks in $\mathfrak{K}_n$. Using (3.10), we can bound

$$= D^{t(p+2K)} N^{p+K} \sum_{n=0}^{\infty} C^{-n} \gamma^{(s-t)\ell_n} \left(\sum_{\substack{\mathtt{j}_1 \in \Sigma_{p_0} \\ \mathtt{j}_k \in \Omega(\ell_{k-1}+K) \\ k \leq n+1}} \varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\dots\varphi^s(\mathtt{j}_{n+1}) \right)^{-1}$$

$$\leq cD^{t(p+2K)} N^{p+K} \sum_{n=1}^{\infty} C^{-n} \gamma^{(s-t)\ell_n} \left(\left(\sum_{\mathtt{j}\in\Sigma_p} \varphi^s(\mathtt{j}) \right)^{n-\eta(n)} \cdot \prod_{i=1}^{\eta(n)} \varphi^s(\lambda'_{m_i}) \right)^{-1}$$

for some $c > 0$. Then by (3.7) and (3.8)

$$\leq cD^{t(p+2K)} N^{p+K} \sum_{n=1}^{\infty} C^{-n} \gamma^{(s-t)\ell_n} e^{-(n-\eta(n))p(P(s)-\delta)} \prod_{i=1}^{\eta(n)} e^{m_i(\alpha(s)+\delta)}.$$

Applying (3.10) and $p\delta > KP(s) - 2K\delta \Leftrightarrow \frac{p}{p+K}(P(s)-\delta) > P(s) - 2\delta$ we get

$$= cD^{t(p+2K)} N^{p+K} \sum_{n=1}^{\infty} C^{-n} \gamma^{(s-t)\ell_n} \exp\Bigg(- \left(\ell_n - \sum_{i=1}^{\eta(n)} |\lambda'_{m_i}| - K\eta(n) - p_0 \right) \cdot \frac{p}{p+K}(P(s)-\delta) + (\alpha(s)+\delta) \sum_{i=1}^{\eta(n)} m_i \Bigg)$$

$$\leq c' D^{t(p+2K)} N^{p+K} \sum_{n=1}^{\infty} C^{-n} \gamma^{(s-t)\ell_n} \exp\left(- \left(\ell_n - \sum_{i=1}^{\eta(n)} |\lambda'_{m_i}| - K\eta(n) \right) (P(s)-2\delta) + (\alpha(s)+\delta) \sum_{i=1}^{\eta(n)} m_i \right).$$

Clearly, $\ell_n \geq m_{\eta(n)} + |\lambda'_{m_{\eta(n)}}|$ and $\ell_n \geq \sum_{i=1}^{\eta(n)} |\lambda'_{m_i}| + (n-\eta(n))p \geq np$ so

$$\leq c' D^{t(p+2K)} N^{p+K} \sum_{n=1}^{\infty} C^{-n} \gamma^{(s-t)pn} \cdot \exp\left(- \left(m_{\eta(n)} - \sum_{i=1}^{\eta(n)-1} |\lambda'_{m_i}| - K\eta(n) \right) (P(s)-2\delta) + (\alpha(s)+\delta) \sum_{i=1}^{\eta(n)} m_i \right).$$

Now we can apply (3.9) to obtain,

$$\leq c'' D^{t(p+2K)} N^{p+K} \sum_{n=1}^{\infty} C^{-n} \gamma^{(s-t)pn} \cdot \exp\left(-(1-2^{-n}) m_{\eta(n)} (P(s)-2\delta) + (1+2^{-n}) m_{\eta(n)} (\alpha(s)+\delta) \right)$$

$$\leq c'' D^{t(p+2K)} N^{p+K} \sum_{n=1}^{\infty} C^{-n} \gamma^{(s-t)pn} \exp\left(m_{\eta(n)} ((1+2^{-n})\alpha(s) - (1-2^{-n})P(s) + 3\delta) \right).$$

Coupling this with the observation that $C^{-1}\gamma^{(s-t)p} < 1$ and $(1+2^{-n})\alpha(s)-(1-2^{-n})P(s))+3\delta < 0$ for sufficiently large n, the expression above is bounded by a geometric series with ratio less than one and hence is bounded. It immediately follows that (3.12) is bounded and the t energy of ν_p^s is finite, as required. $\square$

Before we prove Theorem 2.2, we state a well known result using the transversality method.

Lemma 3.9. *Let $\Phi_{\mathbf{t}} = \{f_i(x) = A_i x + t_i\}_{i=1}^N$ be an iterated function system of affine maps on $\mathbb{R}^d$ such that $\|A_i\| < 1/2$ for all $i = 1, \ldots, N$. Furthermore, let $\pi_{\mathbf{t}}$ be the natural projection. Then for every $R > 0$ there exists a constant $C > 0$ such that for every $\mathtt{i} \neq \mathtt{j} \in \Sigma$ and $s \in (0, d]$*

$$\int_{B(0,R)} \frac{d\mathbf{t}}{|\pi_{\mathbf{t}}(\mathtt{i}) - \pi_{\mathbf{t}}(\mathtt{j})|^s} \leq \frac{C}{\varphi^s(A_{\mathtt{i}\wedge\mathtt{j}})}.$$

The proof can be found in [7, Lemma 3.1] and [25, Proposition 3.1].

Lemma 3.10. *Let $\Phi_{\mathbf{t}} = \{f_i(x) = A_i x + t_i\}_{i=1}^N$ be an iterated function system of affine maps on $\mathbb{R}^d$ such that $\|A_i\| < 1/2$ for all $i = 1, \ldots, N$. Furthermore, let $\pi_{\mathbf{t}}$ be the natural projection. Then for every $R > 0$ there exists a constant $C > 0$ such that for every $\mathtt{i} \neq \mathtt{j} \in \Sigma$ with $i_1 \neq j_1$ and $r > 0$*

$$\mathcal{L}_{dN}\left(\{\mathbf{t} \in B(0,R) : |\pi_{\mathbf{t}}(\mathtt{i}) - \pi_{\mathbf{t}}(\mathtt{j})| \leq r\}\right) \leq Cr.$$

The proof can be found in [14, Lemma 7].

Proof of Theorem 2.2. To show that $\dim_H S_{\mathbf{t}}((\lambda_k)_k) \geq s_0$ for Lebesgue-almost every $\mathbf{t}$, it is enough to show that for every $t < s_0$ we have $\dim_H S_{\mathbf{t}}((\lambda_k)_k) \geq t$ for Lebesgue-almost every $\mathbf{t}$.

Let $t < s < s_0$ and p be as in Lemma 3.8. By Frostman's lemma (see for example [22, Chapter 8]), it is enough to show that

$$\iint \frac{d(\pi_{\mathbf{t}})_*\nu_p^s(x)d(\pi_{\mathbf{t}})_*\nu_p^s(y)}{|x-y|^t} < \infty$$

for almost every $\mathbf{t}$. To show that, it is enough to show that for every $R > 0$

$$\int_{B(0,R)} \iint \frac{d(\pi_{\mathbf{t}})_*\nu_p^s(x)d(\pi_{\mathbf{t}})_*\nu_p^s(y)}{|x-y|^t} < \infty.$$

By Lemma 3.9 and Fubini's Theorem, there exists a constant $C > 0$ such that

$$\int_{B(0,R)} \iint \frac{d(\pi_{\mathbf{t}})_*\nu_p^s(x)d(\pi_{\mathbf{t}})_*\nu_p^s(y)}{|x-y|^t} d\mathcal{L}_{dN}(\mathbf{t}) \leq C \iint \frac{d\nu_p^s(\mathtt{i})d\nu_p^s(\mathtt{j})}{\varphi^t(A_{\mathtt{i}\wedge\mathtt{j}})},$$

where the right-hand side is finite by Lemma 3.8.

Now, let us turn to the proof that $\mathcal{L}_d(S_{\mathbf{t}}((\lambda_k)_k)) > 0$ for Lebesgue-almost every $\mathbf{t}$ if $s_0 > d$. Let s be such that $s_0 > s > d$. It is enough to show that $(\pi_{\mathbf{t}})_*\nu_p^s \ll \mathcal{L}_d$ for almost every $\mathbf{t}$. By [22, Theorem 2.12], it is enough to show that $\liminf_{r\to 0} \frac{(\pi_{\mathbf{t}})_*\nu_p^s(B(y,r))}{r^d} < \infty$ for $(\pi_{\mathbf{t}})_*\nu_p^s$-almost every y, and hence, if we show that for every $R > 0$

$$\int_{B(0,R)} \int \liminf_{r\to 0} \frac{(\pi_{\mathbf{t}})_*\nu_p^s(B(y,r))}{r^d} d(\pi_{\mathbf{t}})_*\nu_p^s(y) d\mathcal{L}_{dN}(\mathbf{t}) < \infty.$$

the claim follows. Fatou's Lemma and Fubini's Theorem implies that

$$\begin{aligned}
&\int_{B(0,R)}\int \liminf_{r\to 0}\frac{(\pi_{\mathtt{t}})_*\nu_p^s(B(y,r))}{r^d}d(\pi_{\mathtt{t}})_*\nu_p^s(y)d\mathcal{L}_{dN}(\mathtt{t})\\
&\le \liminf_{r\to 0}\int_{B(0,R)}\int \frac{(\pi_{\mathtt{t}})_*\nu_p^s(B(y,r))}{r^d}d(\pi_{\mathtt{t}})_*\nu_p^s(y)d\mathcal{L}_{dN}(\mathtt{t})\\
&=\liminf_{r\to 0}\frac{1}{r^d}\iint \mathcal{L}_{dN}(\{\mathtt{t}\in B(0,R): |\pi_{\mathtt{t}}(\mathtt{i})-\pi_{\mathtt{t}}(\mathtt{j})|<r\})d\nu_p^s(\mathtt{i})d\nu_p^s(\mathtt{j})\\
&=\liminf_{r\to 0}\frac{1}{r^d}\iint \mathcal{L}_{dN}\left(\left\{\mathtt{t}\in B(0,R): |\pi_{\mathtt{t}}(\sigma^{|\mathtt{i}\wedge\mathtt{j}|}\mathtt{i})-\pi_{\mathtt{t}}(\sigma^{|\mathtt{i}\wedge\mathtt{j}|}\mathtt{j})|<\frac{r}{|\det(A_{\mathtt{i}\wedge\mathtt{j}})|}\right\}\right)d\nu_p^s(\mathtt{i})d\nu_p^s(\mathtt{j})\\
&\le C\iint\frac{d\nu_p^s(\mathtt{i})d\nu_p^s(\mathtt{j})}{\varphi^d(A_{\mathtt{i}\wedge\mathtt{j}})},
\end{aligned}$$

where the last inequality follows by Lemma 3.10 and the fact that $|\det(A)|=\varphi^d(A)$. The finiteness of the right-hand side follows again by Lemma 3.8. □

4 Pointwise recurrent sets

4.1 Proof of the upper bound

Let $\psi\colon \mathbb{N}\mapsto\mathbb{N}$ and let $\beta=\liminf_{n\to\infty}\frac{\psi(n)}{n}$. Suppose that $\beta<1$. For a finite word $\mathtt{i}\in\Sigma_*$, let $\bar{\mathtt{i}}\in\Sigma$ be the infinite word $\bar{\mathtt{i}}=\mathtt{i}\,\mathtt{i}\,\mathtt{i}\cdots$. Note that $R_{\mathtt{t}}(\psi)$ can be written as

$$R_{\mathtt{t}}(\psi)=\bigcap_{k_0=1}^{\infty}\bigcup_{k=k_0}^{\infty}\bigcup_{\mathtt{i}\in\Sigma_k}\pi_{\mathtt{t}}([\bar{\mathtt{i}}|_{k+\psi(k)}]).$$

Analogous to the properties of $\alpha(t)$, the map $t\mapsto(1-\beta)P(t)-\beta P_2(t)$ is continuous, strictly decreasing, diverges to $-\infty$ and $(1-\beta)P(0)-\beta P_2(0)=\log N$. Thus, there exists a unique solution $r_0>0$ of the equation $(1-\beta)P(r_0)=\beta P_2(r_0)$. Let $t>r_0$ be arbitrary.

Since $0<(1-\beta)P(t)<\beta P_2(t)$, one can choose $\delta>0$ sufficiently small such that

$$P(t)+P_2(t)-2\delta>0 \text{ and } (1-\beta+\delta)P(t)-(\beta-\delta)(P_2(t)-2\delta)+\delta<0.$$

There exists a constant $C>0$ such that for every $k\ge 1$

$$\sum_{\mathtt{i}\in\Sigma_k}\varphi^t(\mathtt{i})\le Ce^{k(P(t)+\delta)} \text{ and } \sum_{\mathtt{i}\in\Sigma_k}\left(\varphi^t(\mathtt{i})\right)^2\le Ce^{k(-P_2(t)+\delta)}.$$

Moreover, we may assume that $C>0$ is sufficiently large such that $k(\beta-\delta)-C\le\psi(k)$ for every $k\ge 1$. Hence, writing $\psi(k)_k:=\min\{k,\psi(k)\}$, we obtain

$$\begin{aligned}
\mathcal{H}^t(R_{\mathtt{t}}(\psi))&\le\liminf_{k_0\to\infty}\sum_{k=k_0}^{\infty}\sum_{\mathtt{i}\in\Sigma_k}\varphi^t(\bar{\mathtt{i}}|_{k+\psi(k)})\\
&\le\liminf_{k_0\to\infty}\sum_{k=k_0}^{\infty}\sum_{\mathtt{i}\in\Sigma_{\psi(k)_k}}\sum_{\mathtt{j}\in\Sigma_{k-\psi(k)_k}}\left(\varphi^t(\mathtt{i})\right)^2\varphi^t(\mathtt{j})\\
&\le C^2\liminf_{k_0\to\infty}\sum_{k=k_0}^{\infty}e^{k(P(t)+\delta)-\psi(k)_k(P(t)+P_2(t)-2\delta)}\\
&\le C^2e^{C(P(t)+P_2(t)-2\delta)}\liminf_{k_0\to\infty}\sum_{k=k_0}^{\infty}e^{k(P(t)+\delta-(\beta-\delta)(P(t)+P_2(t)-2\delta))}=0,
\end{aligned}$$

where we have used that $\Sigma_k = \Sigma_{\psi(k)_k} \times \Sigma_{k-\psi(k)_k}$. Since $t > r_0$ was arbitrary, the claim follows.

Remark 4.1. We note that if $\beta > 1$ then the argument above is not optimal.

4.2 Lower bound for Theorem 2.4

The proof is analogous to the lower bound of Theorem 2.2 with some necessary modifications. Let $p \in \mathbb{N}$ an integer which will be specified later. Let m_k be a sequence on which the lower limit $\beta = \liminf_{n\to\infty} \psi(n)/n$ is achieved and take a sparse subsequence such that

$$\sum_{k=1}^{n} m_k \le (1 + 2^{-n})m_n \text{ and } m_n \ge 2^n \sum_{k=1}^{n-1} (\psi(m_k) + K). \tag{4.1}$$

Let us choose p_0 as in Section 3.3, so $m_k = p_0 + (p + K)q$ for every $k \ge 1$ for some $q \in \mathbb{N}$. To ensure consistency of lengths again, we need to slightly modify $\psi(m_\ell)$ by extending the words to be of length $p + q(K + p)$ for some $q \ge 0$. To this end we define $\psi'(\ell) := \psi(m_\ell) + k$, where $k = p - \psi(m_\ell) \mod (K + p)$.

We construct a measure ν_p^s similarly to Section 3.3, except that the elements in $\Omega(k)$ depend on the previous elements. More precisely, let

$$\Omega(\mathtt{i}, k) = \begin{cases} \{\mathtt{i}\,|_{\psi'(\ell)}\} & \text{if } k = m_\ell \text{ for some } \ell \in \mathbb{N}, \\ \Sigma_p & \text{otherwise.} \end{cases}$$

For every $\mathtt{i}_1, \mathtt{i}_2 \in \Sigma_*$ denote the word in Condition 2.1 by $\mathtt{k}(\mathtt{i}_1, \mathtt{i}_2) \in \Sigma_K$, choosing the lexicographically first if multiple exist. We define a collection of symbols $\mathfrak{K}'_n$ by induction. Let $\mathfrak{K}'_0 := \Sigma_{p_0}$ Suppose that $\mathfrak{K}'_n$ is defined for some $n \ge 0$. Then let us define $\mathfrak{K}'_{n+1}$ as

$$\mathfrak{K}'_{n+1} = \{\mathtt{i}\,\mathtt{k}\,\mathtt{j} : \mathtt{i} \in \mathfrak{K}'_n, \mathtt{j} \in \Omega(\mathtt{i}, |\,\mathtt{i}\,\mathtt{k}\,|) \text{ and } \mathtt{k} = \mathtt{k}(\mathtt{i}, \mathtt{j})\}.$$

Denote by ℓ_k the length of words in $\mathfrak{K}'_k$. Observe that by construction, again every $\mathtt{i} \in \mathfrak{K}'_n$ can be written of the form

$$\mathtt{i} = \mathtt{i}_1\,\mathtt{k}_1\,\mathtt{i}_2\,\mathtt{k}_2 \ldots \mathtt{k}_n\,\mathtt{i}_{n+1},$$

where for every $k \in \{2, \ldots, n+1\}$, $\mathtt{i}_k \in \Omega(\mathtt{i}_1\,\mathtt{k}_1 \ldots \mathtt{i}_{k-1}, \ell_{k-1}+K)$ and $\mathtt{k}_k = \mathtt{k}(\mathtt{i}_1\,\mathtt{k}_1 \ldots \mathtt{k}_{k-1}\,\mathtt{i}_k, \mathtt{i}_{k+1})$.

Let $\eta'(n) = \max\{l \ge 0 : m_l \le \ell_n\}$, that is, $\eta'(n)$ is the number of returns $\sigma^{m_l}\bar{\mathtt{i}} \in [\mathtt{i}\,|_{\psi'(l)}]$ for every $\mathtt{i} \in \mathfrak{K}'_n$. Then

$$\ell_n = \sum_{i=1}^{\eta'(n)} \psi'(i) + (n - \eta'(n))p + Kn + p_0. \tag{4.2}$$

We start by defining $\nu_{p,0}^s$ on cylinders of length no less than p_0 by

$$\nu_{p,0}^s([\mathtt{i}\,\mathtt{h}]) = \frac{\varphi^s(\mathtt{i})}{\sum_{\mathtt{j}\in\Sigma_{p_0}} \varphi^s(\mathtt{j})} N^{-|\mathtt{h}|}$$

for $\mathtt{i} \in \Sigma_{p_0} = \mathfrak{K}'_0$ and $\mathtt{h} \in \Sigma_*$. This uniquely defines a probability measure on Σ, i.e. $\nu_{p,0}^s(\Sigma) = 1$.

We define $\nu_{p,n}^s$ on cylinders with prefix in $\mathfrak{K}'_n$ by

$$\nu_{p,n+1}^s(\mathtt{i}) = \begin{cases} \dfrac{\varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1})}{\displaystyle\sum_{\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1}\in\mathfrak{K}'_n} \varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1})} N^{-|\,\mathtt{i}\,|+|\,\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1}\,|} & \text{if } \mathtt{i} \prec \mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1} \in \mathfrak{K}'_n, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for any cylinder set the measure $\nu_{p,k}^s$ stabilises and thus it converges weakly to a unique measure ν_p^s in k in the same way as for the shrinking target set.

Lemma 4.2. *Let $r_0 > 0$ be such that $(1-\beta)P(r_0) = \beta P_2(r_0)$. Then for all $0 < t < s < r_0$ and sufficiently large p,*

$$\iint_{\Sigma\times\Sigma} \frac{d\nu_p^s(\mathtt{i})d\nu_p^s(\mathtt{j})}{\varphi^t(A_{\mathtt{i}\wedge\mathtt{j}})} < \infty.$$

Proof. By similar argument to the beginning of Lemma 3.8, it is enough to show that

$$\mathcal{I} = \sum_{n=0}^{\infty} \sum_{\mathtt{i}\in\mathfrak{K}'_n} \frac{\nu_p^s([\mathtt{i}])^2}{\varphi^t(\mathtt{i})} < \infty.$$

$$\begin{aligned}
\mathcal{I} &= \sum_{n=0}^{\infty} \sum_{\mathtt{i}\in\mathfrak{K}'_n} \frac{\nu_{p,n}^s([\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1}])^2}{\varphi^t(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})} \\
&= \sum_{n=0}^{\infty} \sum_{\mathtt{i}\in\mathfrak{K}'_n} \left(\frac{\varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1})}{\sum_{\mathtt{j}\in\mathfrak{K}'_n}\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})}\right)^2 \varphi^t(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})^{-1} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{\sum_{\mathtt{j}\in\mathfrak{K}'_n}\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})}\right)^2 \cdot \sum_{\mathtt{i}\in\mathfrak{K}'_n} \frac{(\varphi^s(\mathtt{i}_1)\varphi^s(\mathtt{i}_2)\ldots\varphi^s(\mathtt{i}_{n+1}))^2}{\varphi^t(\mathtt{i}_1\,\mathtt{k}_1\ldots\mathtt{k}_n\,\mathtt{i}_{n+1})} \\
&\le \sum_{n=0}^{\infty} C^{-n}\gamma^{(s-t)\ell_n} \left(\sum_{\mathtt{j}\in\mathfrak{K}'_n}\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{n+1})\right)^{-1}.
\end{aligned}$$

By definition, $m_{\eta'(n)}$ is the position of the last return, and it returns to $[\mathtt{j}\,|_{\psi(m_{\eta'(n)})}]$. Unfortunately, $\mathtt{j}\,|_{\psi(m_{\eta'(n)})}$ is not necessarily an element of $\mathfrak{K}'_k$ for all $k > 0$. Let k_n be the smallest integer such that $\psi(m_{\eta'(n)}) \le \ell_{k_n}$, where we recall that ℓ_n is the length of the elements of $\mathfrak{K}'_n$. Clearly, for every $\mathtt{j} = \mathtt{j}_1\,\mathtt{k}_1\ldots\mathtt{k}_{k_n}\,\mathtt{j}_{k_n+1} \in \mathfrak{K}'_{k_n}$

$$\varphi^s(\mathtt{j}_1)\varphi^s(\mathtt{j}_2)\ldots\varphi^s(\mathtt{j}_{k_n+1}) \ge \varphi^s(\mathtt{j}),$$

and for $\mathtt{j}\in\mathfrak{K}'_n$ $\varphi^s(\mathtt{j}\,|_{\psi(m_{\eta'(n)})})) \ge \varphi^s(\mathtt{j}')$, where $\mathtt{j}'$ is the unique element in $\mathfrak{K}'_{k_n}$ such that $\mathtt{j}\prec\mathtt{j}'$. Moreover, for every $\mathtt{j}\in\mathfrak{K}'_n$ there are $n-\eta'(n)-(k_n-\eta'(k_n))$-many Σ_p components in the sequence $\sigma^{\ell_{k_n}}\mathtt{j}$. Hence, we obtain that

$$\le \sum_{n=1}^{\infty} C^{-n}\gamma^{(s-t)\ell_n}\left(\left(\sum_{\mathtt{j}\in\Sigma_p}\varphi^s(\mathtt{j})\right)^{n-\eta'(n)-(k_n-\eta'(k_n))} \cdot \left(\sum_{\mathtt{j}\in\mathfrak{K}'_{k_n}}\varphi^s(\mathtt{j})^2\right)\cdot \underline{\gamma}^{s\sum_{i=1}^{\eta'(n)-1}\psi(m_i)}\right)^{-1}$$

$$\begin{aligned}
\le \sum_{n=1}^{\infty} C^{-n}\gamma^{(s-t)\ell_n}\Bigg(\left(\sum_{\mathtt{j}\in\Sigma_p}\varphi^s(\mathtt{j})\right)^{n-\eta'(n)-(k_n-\eta'(k_n))} &\cdot \left(\sum_{\mathtt{j}\in\Sigma_p}\varphi^s(\mathtt{j})^2\right)^{k_n-\eta'(k_n)} \\
&\cdot \underline{\gamma}^{s(\sum_{i=1}^{\eta'(n)-1}\psi(m_i)+2\sum_{i=1}^{\eta'(k_n)}\psi(m_i))}\Bigg)^{-1}
\end{aligned}$$

$$\begin{aligned}
\le \sum_{n=1}^{\infty} C^{-n}\gamma^{(s-t)\ell_n}\exp\Bigg(&-p(P(s)-\delta)(n-\eta'(n)-(k_n-\eta'(k_n)))+p(P_2(s)+\delta)(k_n-\eta'(k_n)) \\
&-\log\underline{\gamma}s\left(\sum_{i=1}^{\eta'(n)-1}\psi(m_i)+2\sum_{i=1}^{\eta'(k_n)}\psi(m_i)\right)\Bigg)
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} C^{-n}\gamma^{(s-t)\ell_n} \exp\Bigg(-p(P(s)-\delta)(n-\eta'(n)) + p(P(s)+P_2(s))(k_n-\eta'(k_n)) - \log \underline{\gamma} s \Bigg(\sum_{i=1}^{\eta'(n)-1} \psi(m_i) + 2\sum_{i=1}^{\eta'(k_n)} \psi(m_i) \Bigg) \Bigg).$$

Using (4.2), we get

$$\leq \sum_{n=1}^{\infty} C^{-n}\gamma^{(s-t)\ell_n} \exp\Bigg(-(P(s)-\delta)(\ell_n - \sum_{i=1}^{\eta'(n)} \psi(m_i) - K\eta'(n) - p_0) + (P(s)+P_2(s))(\ell_{k_n-1} + p(\eta'(k_n-1) - \eta'(k_n))) - \log \underline{\gamma} s \Bigg(\sum_{i=1}^{\eta'(n)-1} \psi(m_i) + 2\sum_{i=1}^{\eta'(k_n)} \psi(m_i) \Bigg) \Bigg)$$

$$\leq \sum_{n=1}^{\infty} C^{-n}\gamma^{(s-t)\ell_n} \exp\Bigg(-(P(s)-\delta)(m_{\eta'(n)} - \sum_{i=1}^{\eta'(n)-1} \psi(m_i) - K\eta'(n) - p_0) + (P(s)+P_2(s))(\psi(m_{\eta'(n)}) + p) - 3\log \underline{\gamma} s \Bigg(\sum_{i=1}^{\eta'(n)-1} \psi(m_i) \Bigg) \Bigg)$$

Using the defining properties (4.1) of the sequence m_n, we have

$$\leq c\sum_{n=1}^{\infty} C^{-n}\gamma^{(s-t)\ell_n} \exp\Big(-(P(s)-\delta)(1-2^{-n})m_{\eta'(n)} + (P(s)+P_2(s))(\beta+\delta)m_{\eta'(n)} - 3\log \underline{\gamma} s 2^{-n} m_{\eta'(n)} \Big)$$

Coupling this with the observation that $C^{-1}\gamma^{(s-t)p} < 1$ and $-(P(s)-\delta)(1-2^{-n}) + (P(s)+P_2(s))(\beta+\delta) - 2\log \underline{\gamma} s 2^{-n} < 0$ for sufficiently large n, the left hand side is finite and the proof is complete. □

Now, the proof of Theorem 2.4 is identical to the proof of Theorem 2.2 by replacing Lemma 3.8 with Lemma 4.2, so we omit it.

5 Justification of Condition 2.1

In this section, we give a sufficient condition under which Condition 2.1 holds. The proof is not only a modification of the proof but also an application of Käenmäki and Morris [15, Proposition 4.1]. First, let us recall some definitions and notations from algebraic geometry, following Goldsheid and Guivarc'h [11] and Käenmäki and Morris [15]. Some has already appeared in Section 2.4 but for the convenience of the reader, we repeat it here.

Let us denote by $\wedge^k \mathbb{R}^d$ the *kth exterior product of* $\mathbb{R}^d$. That is, let $\{e_1, \dots, e_d\}$ be the standard orthonormal basis of $\mathbb{R}^d$ and define

$$\wedge^k \mathbb{R}^d = \operatorname{span}\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq d\}.$$

for all $k = 1, \ldots, d$ and let $\wedge^0 \mathbb{R}^d = \mathbb{R}$ by convention. The *wedge product* $\wedge \colon \wedge^k \mathbb{R}^d \times \wedge^j \mathbb{R}^d \mapsto \wedge^{k+j} \mathbb{R}^d$ is an associative bilinear operator, which is anticommutative on the elements of $\mathbb{R}^d$, i.e. for $w \in \wedge^k \mathbb{R}^d$ and $v \in \wedge^j \mathbb{R}^d$

$$w \wedge v = (-1)^{kj} v \wedge w.$$

If $v \in \wedge^k \mathbb{R}^d$ can be expressed as a wedge product of k vectors of $\mathbb{R}^d$ then v is said to be *decomposable*. Let us define the *Hodge star operator* $* \colon \wedge^k \mathbb{R}^d \mapsto \wedge^{d-k} \mathbb{R}^d$ to be the bijective linear map satisfying

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \mathrm{sgn}(i_1, \ldots, i_d) e_{i_{k+1}} \wedge \cdots \wedge e_{i_d}$$

for all $1 \leq i_1 < \cdots < i_k \leq d$, where $1 \leq i_{k+1} < \cdots < i_d \leq d$ are such that $\{i_{k+1}, \ldots, i_d\} = \{1, \ldots, d\} \setminus \{i_1, \ldots, i_k\}$ and $\mathrm{sgn}(i_1, \ldots, i_d)$ is the signature of the permutation $(i_1, \ldots, i_d)$ of $(1, \ldots, d)$. Let us define the *inner product on* $\wedge^k \mathbb{R}^d$ by

$$\langle v, w \rangle_k = *(v \wedge (*w))$$

for all $v, w \in \wedge^k \mathbb{R}^d$. Moreover, we define the norm of $v \in \wedge^k \mathbb{R}^d$ by $\|v\|_k = \sqrt{\langle v, v \rangle_k}$. It can be shown that if $v, w \in \wedge^k \mathbb{R}^d$ are decomposable elements then

$$\langle v, w \rangle_k = \det(\langle v_i, w_j \rangle),$$

where $v = v_1 \wedge \cdots \wedge v_k$ and $w = w_1 \wedge \cdots \wedge w_k$. For $A \in \mathrm{GL}_d(\mathbb{R})$, we can define an invertible linear map $A^{\wedge k} \colon \wedge^k \mathbb{R}^d \mapsto \wedge^k \mathbb{R}^d$ by setting

$$A^{\wedge k}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (Ae_{i_1}) \wedge \cdots \wedge (Ae_{i_k})$$

and extending by linearity.

For every matrix $A \in \mathrm{GL}_d(\mathbb{R})$, there exists a basis of orthonormal vectors $\{u_1, \ldots, u_d\}$ such that $\|Au_i\| = \alpha_i(A)$ and $\{\alpha_1(A)^{-1} Au_1, \ldots, \alpha_d(A)^{-1} Au_d\}$ is orthonormal. Hence, the operator norm of $A^{\wedge k}$ is

$$\|A^{\wedge k}\|_k = \max\{\|A^{\wedge k} w\|_k : \|w\|_k = 1\} = \|A^{\wedge k}(u_1 \wedge \cdots \wedge u_k)\|_k = \alpha_1(A) \cdots \alpha_k(A).$$

Thus, for every $0 < s \leq d$, the singular value function can be written as

$$\varphi^s(A) = \left(\|A^{\wedge \lfloor s \rfloor}\|_{\lfloor s \rfloor}\right)^{1 + \lfloor s \rfloor - s} \left(\|A^{\wedge \lceil s \rceil}\|_{\lceil s \rceil}\right)^{s - \lfloor s \rfloor}.$$

The naïve intuition suggests that Condition 2.1 shall hold for general matrix tuples in some sense. For that, we introduce two families of definitions related to the action of the tuple, which describes the most general and most likely behaviour.

Let $\mathbf{A} = \{A_1, A_2, \cdots, A_N\}$ be a tuple of $\mathrm{GL}_d(\mathbb{R})^N$ matrices. We say that $\mathbf{A}$ is *k-irreducible* if there is no proper subspace V of $\wedge^k \mathbb{R}^d$ such that $A^{\wedge k} V = V$ for every $A \in \mathbf{A}$. Similarly, we say that $\mathbf{A}$ is *strongly k-irreducible* if there is no finite collection of proper subspaces $V_1, \ldots, V_n$ of $\wedge^k \mathbb{R}^d$ such that $\bigcup_{k=1}^n \bigcup_{A \in \mathbf{A}} A^{\wedge k} V_k = \bigcup_{k=1}^n V_k$. Denote by $\mathcal{S}(\mathbf{A})$ the semi-group induced by $\mathbf{A}$. The following lemma is due to Käenmäki and Morris [15, Proposition 4.1].

Lemma 5.1. *Let $\mathbf{A}$ be a tuple of matrices of $\mathrm{GL}_d(\mathbb{R})$ such that $\mathbf{A}$ is k- and $k+1$-irreducible. If there exist nonzero vectors $v_k, w_k \in \wedge^k \mathbb{R}^d$ and $v_{k+1}, w_{k+1} \in \wedge^{k+1} \mathbb{R}^d$ such that*

$$\langle v_k, A^{\wedge k} w_k \rangle_k \langle v_{k+1}, A^{\wedge k+1} w_{k+1} \rangle_{k+1} = 0$$

for every $A \in \mathcal{S}(\mathbf{A})$ then $\mathbf{A}$ is neither strongly k-irreducible nor strongly $(k+1)$-irreducible.

For two vector spaces V and W, let us define the *tensor product* $V \otimes W$ as follows

$$V \otimes W = \operatorname{span}\{v \otimes w : v \in V,\ w \in W\},$$

where for any $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}(v_1 + v_2) \otimes w_1 &= v_1 \otimes w_1 + v_2 \otimes w_1,\\ v_1 \otimes (w_1 + w_2) &= v_1 \otimes w_1 + v_1 \otimes w_2,\\ \alpha(v_1 \otimes w_1) &= (\alpha v_1) \otimes w_1 = v_1 \otimes (\alpha w_1).\end{aligned}$$

Let us consider the following tensor product of the exterior algebras

$$\widehat{W} = \wedge^1 \mathbb{R}^d \otimes \cdots \otimes \wedge^{d-1} \mathbb{R}^d .$$

We define the inner product of $\widehat{W}$ for $u = u_1 \otimes \cdots \otimes u_{d-1}, v = v_1 \otimes \cdots \otimes v_{d-1} \in \widehat{W}$

$$\langle u, v\rangle_\wedge = \prod_{i=1}^{d-1} \langle u_i, v_i\rangle_i,$$

and extend it in a bilinear, symmetric way. We define a linear subspace W of $\widehat{W}$, which is generated by the flags of $\mathbb{R}^d$ as follows:

$$W = \operatorname{span}\{u_1 \otimes (u_1 \wedge u_2) \otimes \cdots \otimes (u_1 \wedge \cdots \wedge u_{d-1}) : \{u_1, \ldots, u_{d-1}\} \text{ linearly independent in } \mathbb{R}^d\}.$$

We call W the *flag vector space.* Again, for an $A \in \mathrm{GL}_d(\mathbb{R})$, we can define an invertible linear mar $\widehat{A} \colon \widehat{W} \mapsto \widehat{W}$ by setting for $u = u_1 \otimes \cdots \otimes u_{d-1}$

$$\widehat{A}(u_1 \otimes \cdots \otimes u_{d-1}) = (A^{\wedge 1} u_1) \otimes \cdots \otimes (A^{\wedge (d-1)} u_{d-1})$$

and extending by linearity. It is easy to see that $\widehat{A} \colon W \mapsto W$ for $A \in GL_d(\mathbb{R})$. Let us denote the restriction of the inner product $\langle \cdot, \cdot\rangle_\wedge$ and norm $\| \cdot \|_\wedge$ to W by $\langle \cdot, \cdot\rangle_W$ and $\| \cdot \|_W$.

We say that $A \in GL_d(\mathbb{R})$ is 1-proximal if it has a unique simple eigenvalue of maximum modulus. We say that $A \in \mathrm{GL}_d(\mathbb{R})$ is *fully proximal* if it has d distinct eigenvalues in absolute value. Note that A is fully proximal then $A^{\wedge k}$ is 1-proximal for every k and $\widehat{A}$ is 1-proximal on W. We say that the tuple $\mathbf{A}$ is fully proximal if there exists an $A \in \mathcal{S}(\mathbf{A})$ which is fully proximal. We say that the tuple $\mathbf{A}$ is *fully strongly irreducible* or *strongly irreducible over* W if there are no finite collection $V_1, \ldots, V_n$ of proper subspaces of W such that

$$\bigcup_{A \in \mathbf{A}} \bigcup_{k=1}^{n} \widehat{A} V_k = \bigcup_{k=1}^{n} V_k.$$

Before we prove Proposition 2.5, we need to recall two important tools.

Lemma 5.2. *Suppose that* $\mathbf{A}$ *is fully proximal and fully strongly irreducible then its transpose* $\mathbf{A}^\top = \{A_1^\top, \ldots, A_N^\top\}$ *and* $\mathbf{A}^m = \{A_1 \cdots A_m\}_{A_1,\ldots,A_m \in \mathbf{A}}$ *are also fully proximal and fully strongly irreducible for* $m \geq 1$*.*

Proof. Let $A \in \mathrm{GL}_d(\mathbb{R})$ be a fully proximal matrix, and let $\lambda_1, \ldots, \lambda_d$ and $v_1, \ldots, v_d$ be the corresponding eigenvalues and eigenvectors. Then it is easy to see that any nonzero $w_i \in \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}^\perp$ is an eigenvector of $A^\top$ with eigenvalue λ_i. Indeed, $\langle A^\top w_i, v_j\rangle = \langle w_i, A v_j\rangle = \lambda_j \langle w_i, v_j\rangle = 0$, we get that $A^\top w_i \in \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}^\perp$ and since

$\dim \operatorname{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}^\perp = 1$ we have that $A^\top w_i = c w_i$ for some $c \in \mathbb{R}$. But since $c\langle w_i, v_i\rangle = \langle A^\top w_i, v_i\rangle = \langle w_i, A v_i\rangle = \lambda_i \langle w_i, v_i\rangle$ and $\langle w_i, v_i\rangle \neq 0$, we get that $A^\top w_i = \lambda_i w_i$.

Now, let us suppose that $\mathbf{A}$ is not fully strongly irreducible and we show that then $\mathbf{A}^\top$ is not fully strongly irreducible too. Let $V_1, \ldots, V_n$ be proper subspaces of W such that $\bigcup_{A\in\mathbf{A}} \bigcup_{i=1}^n \widehat{A} V_i = \bigcup_{i=1}^n V_i$. By the invertability, $\bigcup_{A\in\mathbf{A}} \bigcup_{i=1}^n \widehat{A^{-1}} V_i = \bigcup_{i=1}^n V_i$. Clearly, $\widehat{A^\top} V^\perp = (\widehat{A^{-1}} V)^\perp$ for every proper subspace V of W. So for any $i = 1, \ldots, n$ and $A \in \mathbf{A}$

$$\widehat{A^\top} V_i^\top = (\widehat{A^{-1}} V_i)^\perp \subset \bigcup_{i=1}^n V_i^\perp,$$

thus it follows that $\mathbf{A}^\top$ is not fully strongly irreducible.

Similarly, the fully proximality of $\mathbf{A}$ implies clearly the fully proximality of $\mathbf{A}^m$. Moreover, if $\mathbf{A}^m$ is not fully strongly irreducible then there exists a finite family of proper subspaces $V_1, \ldots, V_n$ of W such that $\bigcup_{A_1,\ldots,A_n\in\mathbf{A}} \bigcup_{i=1}^n \widehat{A_1 \cdots A_n} V_i = \bigcup_{i=1}^n V_i$. Thus, the tuple $\mathbf{A}$ is not fully strongly irreducible for the family $\bigcup_{i=1}^n \bigcup_{k=0}^{m-1} \bigcup_{A_1\cdots A_k\in\mathbf{A}} \{\widehat{A_1 \cdots A_k} V_i\}$. □

Denote by $\mathcal{P}(W)$ the projective space of W and $\mathrm{SL}(W)$ the space of linear maps of W to W. Let

$$\mathcal{R}(\mathbf{A}) = \left\{ B \in \mathrm{SL}(W) : \operatorname{rank}(B) = 1 \text{ and there exists } A_n \in \mathcal{S}(\mathbf{A}) \text{ such that } \frac{\widehat{A_n}}{\|\widehat{A_n}\|_W} \to B \right\}$$

and let

$$\mathcal{L}(\mathbf{A}) = \{BW : B \in \mathcal{R}(\mathbf{A})\} \subset \mathcal{P}(W).$$

For a linear map $B \in \mathrm{SL}(W)$, denote by $\mathrm{Im}(B)$ the image space of B and by $\mathrm{Ker}(B)$ the kernel of B. Thus, $\mathcal{L}(\mathbf{A}) = \{\mathrm{Im}(B) : B \in \mathcal{R}(\mathbf{A})\}$.

Finally, let us denote the set of the fully proximal elements of the semigroup $\mathcal{S}(\mathbf{A})$ by $\mathcal{S}_0(\mathbf{A})$. Then for every $A \in \mathcal{S}_0(\mathbf{A})$, the limit $\lim_{n\to\infty} \frac{\widehat{A}^n}{\|\widehat{A}^n\|_W}$ exists and belongs to $\mathcal{R}(\mathbf{A})$. This follows by a standard decomposition into eigenvectors for which only the one associated to the largest eigenvalue (which is unique by proximality) will remain upon normalisation. We denote this limit by $\widehat{G}(A)$. Similarly, the limit $G_k(A) := \lim_{n\to\infty} \frac{(A^{\wedge k})^n}{\|(A^{\wedge k})^n\|_k}$ exists and has rank 1. Moreover, $\mathrm{Im}(G_k(A)) = \operatorname{span}\{v_1 \wedge \ldots \wedge v_k\}$, where v_i is an eigenvector corresponding to the i-th largest eigenvalue in absolute value. Moreover, for a fully proximal matrix $A \in \mathcal{S}_0(\mathbf{A})$,

$$\mathrm{Im}(\widehat{G}(A)) = \mathrm{Im}(G_1(A)) \otimes \cdots \otimes \mathrm{Im}(G_{d-1}(A)). \tag{5.1}$$

The following lemma is a corollary of Goldsheid and Guivarc'h [11, Theorem 2.14].

Lemma 5.3. *If $\mathbf{A}$ is fully proximal and fully strongly irreducible then $\{\mathrm{Im}(\widehat{G}(A))\}_{A\in\mathcal{S}_0(\mathbf{A})}$ is dense in $\mathcal{L}(\mathbf{A})$ and $\bigcup_{A\in\mathbf{A}} \widehat{A}\mathcal{L}(\mathbf{A}) = \mathcal{L}(\mathbf{A})$.*

Lemma 5.4. *Let $\mathbf{A}$ be a fully strongly irreducible matrix tuple. Then $\mathbf{A}$ is strongly k-irreducible for every $k = 1, \ldots, d-1$ and there exists no finite collection $V_1, \ldots, V_n$ of proper subspaces of W such that $\mathcal{L}(\mathbf{A}) \subseteq \bigcup_{i=1}^n \mathcal{P}(V_i)$.*

Proof. Let us argue by contradiction. First, suppose that $\mathbf{A}$ is not strongly k-irreducible for some $k \in \{1, \ldots, d-1\}$. Let $V_1, \ldots, V_n$ be a finite collection of proper subspaces of $\wedge^k \mathbb{R}^d$ such that $\bigcup_{\ell=1}^n \bigcup_{A\in\mathbf{A}} A^{\wedge k} V_k = \bigcup_{\ell=1}^n V_\ell$. Let

$$\widehat{V}_\ell := \left(\wedge^1 \mathbb{R}^d \otimes \cdots \otimes \wedge^{k-1} \mathbb{R}^d \otimes V_\ell \otimes \wedge^{k+1} \mathbb{R}^d \otimes \cdots \otimes \wedge^{d-1} \mathbb{R}^d\right) \cap W.$$

It is easy to see that $\widehat{V}_\ell$ is a proper subspace of W for all $\ell = 1, \ldots, d-1$ and $\bigcup_{\ell=1}^n \bigcup_{A\in\mathbf{A}} \widehat{A}\widehat{V}_k = \bigcup_{\ell=1}^n \widehat{V}_\ell$, which is a contradiction.

Now, suppose that there exists a finite collection $V_1, \ldots, V_n$ of proper subspaces of W such that $\mathcal{L}(\mathbf{A}) \subseteq \bigcup_{i=1}^n \mathcal{P}(V_i)$. Without loss of generality, we may assume that $V_1, \ldots, V_n$ is minimal in the sense that $\mathcal{L}(\mathbf{A}) \cap \mathcal{P}(V_i)$ is not contained in a finite union of subspaces of V_i. Indeed, if $\mathcal{L}(\mathbf{A}) \cap \mathcal{P}(V_i) \bigcup \bigcup_{i=1}^{n'} \mathcal{P}(V_i')$ for a finite collection of proper subspaces $V_1', \ldots, V_{n'}'$ of V_i, then one can replace V_i with $V_1', \ldots, V_{n'}'$. Clearly, the procedure terminates in finitely many steps.

We will show that for every $A \in \mathbf{A}$ and every $j \in \{1, \ldots, n\}$ there exists $i \in \{1, \ldots, n\}$ such that $\widehat{A}V_j = V_i$. Clearly,

$$\widehat{A}\left(\mathcal{L}(\mathbf{A}) \cap \mathcal{P}(V_j)\right) \subseteq \bigcup_{i=1}^n \mathcal{P}(V_i),$$
$$\widehat{A}\left(\mathcal{L}(\mathbf{A}) \cap \mathcal{P}(V_j)\right) \subseteq \mathcal{P}(\widehat{A}V_j).$$

Since $\widehat{A}$ is invertible on W we get

$$\mathcal{L}(\mathbf{A}) \cap \mathcal{P}(V_j) \subseteq \mathcal{P}(V_j) \cap \bigcup_{i=1}^n \mathcal{P}(\widehat{A}^{-1}V_i) = \bigcup_{i=1}^n \mathcal{P}(\widehat{A}^{-1}V_i \cap V_j).$$

But by the minimality assumption of $V_1, \ldots, V_n$, the subspace $\widehat{A}^{-1}V_i \cap V_j$ must be equal to V_j for an $i \in \{1, \ldots, n\}$.

Thus, $\bigcup_{\ell=1}^n \bigcup_{A\in\mathbf{A}} \widehat{A}\widehat{V}_k = \bigcup_{\ell=1}^n \widehat{V}_\ell$, which is again a contradiction. □

Proof of Proposition 2.5. Let us argue by contradiction. Namely, there exists $s > 0$ such that for every $C > 0$ and $K \in \mathbb{N}$ there exist $\mathtt{i}_{C,K}, \mathtt{j}_{C,K} \in \Sigma_*$ such that for all $\mathtt{k} \in \Sigma_K$

$$\varphi^s(A_{\mathtt{i}_{C,K}\mathtt{k}\mathtt{j}_{C,K}}) < C\varphi^s(A_{\mathtt{i}_{C,K}})\varphi^s(A_{\mathtt{j}_{C,K}})$$

We may first assume that $s \notin \mathbb{N}$, the proof of the integer case is similar and even simpler. For short, let $\lfloor s \rfloor = k$ and $\lceil s \rceil = k+1$ By the singular value decomposition of $A_{\mathtt{i}_{C,K}}$ and $A_{\mathtt{j}_{C,K}}$, let $u_1^{(C,K)}, \ldots, u_d^{(C,K)}$ and $v_1^{(C,K)}, \ldots, v_d^{(C,K)}$ be the orthonormal bases such that $\left\|A_{\mathtt{j}_{C,K}} u_i^{(C,K)}\right\| = \alpha_i(A_{\mathtt{j}_{C,K}})$ and $\left\|(A_{\mathtt{i}_{C,K}})^\top v_i^{(C,K)}\right\| = \alpha_i(A_{\mathtt{i}_{C,K}})$. Hence,

$$\left\|A_{\mathtt{j}_{C,K}}^{\wedge j}\right\|_j = \left\|A_{\mathtt{j}_{C,K}}^{\wedge j} u_1^{(C,K)} \wedge \cdots \wedge u_j^{(C,K)}\right\|_j,$$
$$\left\|A_{\mathtt{i}_{C,K}}^{\wedge j}\right\|_j = \left\|\left(A_{\mathtt{i}_{C,K}}^{\wedge j}\right)^\top v_1^{(C,K)} \wedge \cdots \wedge v_j^{(C,K)}\right\|_j,$$

for all $j = 1, \ldots, d-1$. For short, let $u_{C,K}^{\wedge j} = u_1^{(C,K)} \wedge \cdots \wedge u_j^{(C,K)}$ and $v_{C,K}^{\wedge j} = v_1^{(C,K)} \wedge \cdots \wedge v_j^{(C,K)}$. So for every $C > 0$ and $K \in \mathbb{N}$ and for all $\mathtt{k} \in \Sigma_K$

$$\left\langle \left(A_{\mathtt{i}_{C,K}}^{\wedge k}\right)^\top v_{C,K}^{\wedge k}, A_{\mathtt{k}}^{\wedge k} A_{\mathtt{j}_{C,K}}^{\wedge k} u_{C,K}^{\wedge k} \right\rangle_k^{k+1-s} \left\langle \left(A_{\mathtt{i}_{C,K}}^{\wedge k+1}\right)^\top v_{C,K}^{\wedge k+1}, A_{\mathtt{k}}^{\wedge k+1} A_{\mathtt{j}_{C,K}}^{\wedge k+1} u_{C,K}^{\wedge k+1} \right\rangle_{k+1}^{s-k}$$
$$< C \left\|A_{\mathtt{j}_{C,K}}^{\wedge k} u_{C,K}^{\wedge k}\right\|_k^{k+1-s} \left\|A_{\mathtt{j}_{C,K}}^{\wedge k+1} u_{C,K}^{\wedge k+1}\right\|_{k+1}^{s-k} \left\|\left(A_{\mathtt{i}_{C,K}}^{\wedge k}\right)^\top v_{C,K}^{\wedge k}\right\|_k^{k+1-s} \left\|\left(A_{\mathtt{i}_{C,K}}^{\wedge k+1}\right)^\top v_{C,K}^{\wedge k+1}\right\|_{k+1}^{s-k}.$$

By compactness and possibly taking a subsequence, we may assume that $\frac{A_{\mathtt{j}_{C,K}} u_i^{(C,K)}}{\alpha_i(A_{\mathtt{j}_{C,K}})} \to u_i^{(K)}$ and $\frac{(A_{\mathtt{i}_{C,K}})^\top v_i^{(C,K)}}{\alpha_i(A_{\mathtt{i}_{C,K}})} \to v_i^{(K)}$ as $C \to 0$ and $\{u_1^{(K)}, \dots, u_d^{(K)}\}$ and $\{v_1^{(K)}, \dots, v_d^{(K)}\}$ are both orthonormal bases of $\mathbb{R}^d$. Hence, for every $K \in \mathbb{N}$ there exist orthonormal bases $\{u_1^{(K)}, \dots, u_d^{(K)}\}$ and $\{v_1^{(K)}, \dots, v_d^{(K)}\}$ of $\mathbb{R}^d$ such that for every $\mathtt{k} \in \Sigma_K$

$$\left\langle v_K^{\wedge k}, A_{\mathtt{k}}^{\wedge k} u_K^{\wedge k} \right\rangle_k \left\langle v_K^{\wedge k+1}, A_{\mathtt{k}}^{\wedge k+1} u_K^{\wedge k+1} \right\rangle_{k+1} = 0,$$

where $u_K^{\wedge j} = u_1^{(K)} \wedge \cdots \wedge u_j^{(K)}$ and $v_K^{\wedge j} = v_1^{(K)} \wedge \cdots \wedge v_j^{(K)}$ for all $j = 1, \dots, d-1$. Finally, again by compactness let K_n be the subsequence for which $\{u_1^{(K_n!)}, \dots, u_d^{(K_n!)}\} \to \{u_1^*, \dots, u_d^*\}$ and $\{v_1^{(K_n!)}, \dots, v_d^{(K_n!)} \to \{v_1^*, \dots, v_d^*\}$ and as $n \to \infty$, and $u_*^{\wedge j} = u_1^* \wedge \cdots \wedge u_j^*$ and $v_*^{\wedge j} = v_1^* \wedge \cdots \wedge v_j^*$.

Claim. *There exists a fully proximal matrix $A \in \mathcal{S}_0(\mathbf{A})$ such that $u_*^{\wedge k} \notin \mathrm{Ker}(G_k(A))$ and $u_*^{\wedge k+1} \notin \mathrm{Ker}(G_{k+1}(A))$.*

Proof of the Claim. Again, let us argue by contradiction, that is for every $A \in \mathcal{S}_0(\mathbf{A})$, $u_*^{\wedge k} \in \mathrm{Ker}(G_k(A))$ or $u_*^{\wedge k+1} \in \mathrm{Ker}(G_{k+1}(A))$. It is easy to see that $A^\top$ is also fully proximal and $\mathrm{Ker}(G_k(A))^\perp = \mathrm{Im}(G_k(A^\top))$. Hence,

$$\mathrm{Im}(G_k(A^\top)) \in \mathcal{P}(\mathrm{span}\{u_*^{\wedge k}\}^\perp) \text{ or } \mathrm{Im}(G_{k+1}(A^\top)) \in \mathcal{P}(\mathrm{span}\{u_*^{\wedge k+1}\}^\perp),$$

and so by (5.1)

$$\begin{aligned}\bigcup_{A^\top \in \mathcal{S}_0(\mathbf{A}^\top)} \{\mathrm{Im}(\widehat{G}(A^\top))\} \subseteq \mathcal{P}\left(\left(\wedge^1 \mathbb{R}^d \otimes \cdots \otimes \wedge^{k-1} \mathbb{R}^d \otimes \mathrm{span}\{u_*^{\wedge k}\}^\perp \otimes \wedge^{k+1} \mathbb{R}^d \otimes \cdots \otimes \wedge^{d-1} \mathbb{R}^d\right) \cap W\right) \\ \bigcup \mathcal{P}\left(\left(\wedge^1 \mathbb{R}^d \otimes \cdots \otimes \wedge^{k} \mathbb{R}^d \otimes \mathrm{span}\{u_*^{\wedge k+1}\}^\perp \otimes \wedge^{k+2} \mathbb{R}^d \otimes \cdots \otimes \wedge^{d-1} \mathbb{R}^d\right) \cap W\right).\end{aligned}$$

By Lemma 5.3, $\bigcup_{A^\top \in \mathcal{S}_0(\mathbf{A}^\top)} \{\mathrm{Im}(\widehat{G}(A^\top))\}$ is dense in $\mathcal{L}(\mathbf{A}^\top)$ and so by taking the closure,

$$\begin{aligned}\mathcal{L}(\mathbf{A}^\top) \subseteq \mathcal{P}\left(\left(\wedge^1 \mathbb{R}^d \otimes \cdots \otimes \wedge^{k-1} \mathbb{R}^d \otimes \mathrm{span}\{u_*^{\wedge k}\}^\perp \otimes \wedge^{k+1} \mathbb{R}^d \otimes \cdots \otimes \wedge^{d-1} \mathbb{R}^d\right) \cap W\right) \\ \bigcup \mathcal{P}\left(\left(\wedge^1 \mathbb{R}^d \otimes \cdots \otimes \wedge^{k} \mathbb{R}^d \otimes \mathrm{span}\{u_*^{\wedge k+1}\}^\perp \otimes \wedge^{k+2} \mathbb{R}^d \otimes \cdots \otimes \wedge^{d-1} \mathbb{R}^d\right) \cap W\right),\end{aligned}$$

which is a contradiction by Lemma 5.4. $\square$

Let $A_{\mathtt{i}} \in \mathcal{S}_0(\mathbf{A})$ as in the Claim and let $m = |\,\mathtt{i}\,|$. Now, let $\mathtt{j} \in \bigcup_{k=0}^\infty \Sigma_{km}$ be arbitrary. Then for every sufficiently large n, $q_n := \frac{K_n! - |\,\mathtt{j}\,|}{m} \in \mathbb{N}$, and so

$$\begin{aligned}0 = \left\langle v_{K_n!}^{\wedge k}, A_{\mathtt{j}}^{\wedge k} \frac{(A_{\mathtt{i}}^{\wedge k})^{q_n}}{\|\,(A_{\mathtt{i}}^{\wedge k})^{q_n}\,\|_k} u_{K_n!}^{\wedge k} \right\rangle_k \left\langle v_{K_n!}^{\wedge k+1}, A_{\mathtt{j}}^{\wedge k+1} \frac{\left(A_{\mathtt{i}}^{\wedge k+1}\right)^{q_n}}{\|\left(A_{\mathtt{i}}^{\wedge k+1}\right)^{q_n}\|_{k+1}} u_{K_n!}^{\wedge k+1} \right\rangle_{k+1} \\ \to \left\langle v_*^{\wedge k}, A_{\mathtt{j}}^{\wedge k} G_k(A_{\mathtt{i}}^{\wedge k}) u_*^{\wedge k} \right\rangle_k \left\langle v_*^{\wedge k+1}, A_{\mathtt{j}}^{\wedge k+1} G_{k+1}(A_{\mathtt{i}}^{\wedge k+1}) u_*^{\wedge k+1} \right\rangle_{k+1}\end{aligned}$$

as $n \to \infty$. In particular, for every $\mathtt{j} \in \bigcup_{k=0}^\infty \Sigma_{km}$

$$\left\langle v_*^{\wedge k}, A_{\mathtt{j}}^{\wedge k} G_k(A_{\mathtt{i}}^{\wedge k}) u_*^{\wedge k} \right\rangle_k \left\langle v_*^{\wedge k+1}, A_{\mathtt{j}}^{\wedge k+1} G_{k+1}(A_{\mathtt{i}}^{\wedge k+1}) u_*^{\wedge k+1} \right\rangle_{k+1} = 0$$

Since $G_k(A_{\mathtt{i}}^{\wedge k})u_*^{\wedge k}$ and $G_{k+1}(A_{\mathtt{i}}^{\wedge k+1})u_*^{\wedge k+1}$ are nonzero vectors, by Lemma 5.1 we have that $\mathbf{A}^m$ is neither strongly k-irreducible nor strongly $(k+1)$-irreducible, which together with Lemma 5.2 and Lemma 5.4 is a contradiction. □

Remark 5.5. To show the claim of Proposition 2.5 for a particular $s \notin \mathbb{N} \cap (0, d)$ (or $s \in \mathbb{N} \cap (0, d)$), it is enough to assume that the induced action of the tuple $\mathbf{A}$ on $\wedge^{\lfloor s \rfloor} \mathbb{R}^d \otimes \wedge^{\lceil s \rceil} \mathbb{R}^d$ (on $\wedge^s \mathbb{R}^d$) is proximal and strongly irreducible. However, in our situation to prove Theorem 2.2 we need a condition which holds for every $s > 0$ since the value s_0, for which $P(s_0) = \alpha(s_0)$, is unknown.

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