Master Thesis

Dimension Theory of non-conformal attractors

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Contents

1 Introduction

One of the most important things concerning the attractors of dynamical system generated by an IFS is the evualuation of the Hausdorff- and Box-dimension of the set. Moreover it is also an interesting question, what the Hausdorff-dimension of the invariant measure of the attractor is. In my thesis I would like to study the dimensions of different iterated function systems. I decompose my thesis into three parts. In the first part I study the subadditive pressure.

The subadditive pressure, which is definied by K. Falconer [2] and L. Barreira, is a tool to estimate the Box- and Hausdorff-dimension. It is well-known in conformal case with some condition that the zero of the subadditive pressure is equal to the Hausdorff-dimenion. In non-conformal case with some special condition the zero of the subadditive pressure is greater than or equal to the Hausdorff-dimension. I examine some important properties of the pressure for a special IFS.

In the second part of the thesis I consider a family of self-affine dynamical system. In the dimension theory of such system there is an important tool, the Lyapunov charts. This is the most basic ingredient of the Ledrappier-Young Theory. Their theorems establish connection between Lyapunov-exponents, entropy, and pointwise dimension. The Ledrappier-Young Theory concerning the dimension theory of the invariant measures of C^2-diffeomorphisms do not cover the cases, when singularities appear. However all of the machinery works in process of Lyapunov charts. In this section my aim is to verify the existence of Lyapunov charts in order to prove the Ledrappier-Young Theorem for some maps with singularities induced by a self-affine IFS.

In the last part I examine a family of non-linear iterated function scheme with many parameters. We would like to estimate the Hausdorff-dimension of the invariant measure. Károly Simon and Mark Pollicott introduced a special property, namely the transversality condition. There are a lot of articles in linear and non-linear cases, too, which use this condition and prove absolute continuity. In this section my aim is to prove that this condition holds and to estimate the Lyapunov-exponent.

2 Subadditive pressure

In $\mathbb{R}^n$, where $n > 1$, we consider iterated function systems which are non-conformal. (We say that a map is conformal if the derivative is a similarity in every point) The dimension theory of non-conformal IFS is very difficult and there are only very few results. The most important tool of this field is the subadditive pressure, which is used to estimate the dimension of the attractors (and to compute it into a few cases when we can compute the dimension). Unfortunately, we know very little about subadditive pressure itself. This pressure is the generalization of the usual topological pressure, see for example [14, Chapter 9]. When we compute the topological pressure we take the exponential growrate over the sum of the values of a certain function evaluated on each cylinder. In the theory of standard top. pressure it turns out that the sum mentioned above can be evaluated at arbitrary points of the cylinders while the value of the pressure will be the same. Therefore we say that the top. pressure is not sensitive to the places where the function is evaluated. The same has not been verified for the sub. pressure yet. In this section we prove that the sub. pressure is not sensitive at least in the case when our IFS is given by maps, which derivative matrices at every point are triangular matrices. I generalize the result of K. Simon and A. Manning [6]. They proved in two dimension. I proved the same theorem in $\mathbb{R}^n$. My result is also a generalization of K. Falconer's and J. Miao's article [1]. They have a formula to estimate the Hausdorff-dimension of self-affine fractals generated by upper-triangular matrices. I show a formula to estimate the subadditive pressure in non-conformal case. In this section I use the methods in K. Falconer's and J. Miao's article [1].

2.1 Definitions

In this section we define our iterated function system and the subadditive pressure.

Throughout the section we will always assume the following, let $M \subset \mathbb{R}^n$

be non-empty, open and let $F_i : M \mapsto M$ contractive maps for every $i = 1, ..., l$. For an $\mathbf{i} = i_1 i_2 ... i_k$, $i_j \in \{1, ..., l\}$, we define $F_{\mathbf{i}}(\underline{x}) = F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_n}(\underline{x})$. Assume about F_i, $i = 1, ..., l$ the following:

$$F_i(x_1, ..., x_n) = \left(f_i^1(x_1), f_i^2(x_1, x_2), ..., f_i^n(x_1, ..., x_n)\right), \tag{1}$$

and $F_i(x_1, ..., x_n) \in C^{1+\varepsilon}$ for every $i = 1, ..., l$. Moreover $D_{\underline{x}} F_{\mathbf{i}}$ for every $\underline{x} \in M$ and every $\mathbf{i} \in \{1, ..., l\}^*$ finite sequence is regular. Denote the elements of $D_{\underline{x}} F_{\mathbf{i}}$ by $x_{ij}(\mathbf{i}, \underline{x})$.

Proposition 2.1.1. *There is a $0 < C < \infty$ real constant that*

$$C^{-1} < \frac{|x_{ii}(\mathbf{i}, \underline{x})|}{|x_{ii}(\mathbf{i}, \underline{y})|} < C \tag{2}$$

for every $\underline{x}, \underline{y} \in M$ and for every $\mathbf{i} \in \{1, ..., l\}^*$.

Proof. Let $G_i^{(m)} : \mathbb{R}^m \mapsto \mathbb{R}^m$ for every integer m between 1 and n, is the restriction of F_i to the first m component, i.e.:

$$G_i^{(m)}(x_1, ..., x_m) := \left(f_i^1(x_1), f_i^2(x_1, x_2), ..., f_i^m(x_1, ..., x_m)\right).$$

From [8] it follows that for every $\underline{x}, \underline{y} \in M$, for every $\mathbf{i} \in \{1, ..., l\}^*$ finite sequence, and for $1 \leq m \leq n$ there exist a real $0 < C_m < \infty$ constant that

$$C_m^{-1} < \frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})} < C_m$$

Since for every m the $D_{\underline{x}} G_{\mathbf{i}}^{(m)}$ matrix is in lower triangular matrix form, the jacobian is the following

$$\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x}) = |x_{11}(\mathbf{i}, \underline{x}) \cdots x_{mm}(\mathbf{i}, \underline{x})|.$$

Therefore for every integer $1 \leq m \leq n$ and for every $\underline{x}, \underline{y} \in M$

$$\frac{C_m^{-1}}{C_{m+1}} < \frac{\frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})}}{\frac{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{y})}} < \frac{C_m}{C_{m+1}^{-1}}$$

and

$$\frac{\frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})}}{\frac{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{y})}} = \frac{\left|x_{m+1m+1}\left(\mathbf{i},\underline{y}\right)\right|}{\left|x_{m+1m+1}\left(\mathbf{i},\underline{x}\right)\right|}.$$

Then $C := \max_{1\leq m<n-1}\left\{\frac{C_m}{C_{m+1}^{-1}}, C_1\right\}$ choice completes the proof of the proposition. $\square$

The singular values of a linear contraction T are the positive square roots of the eigenvalues of TT^*, where T^* is the transpose of T. Let $\alpha_k(D_{\underline{x}}F_{\mathbf{i}})$ the kth greatest singular value of the $D_{\underline{x}}F_{\mathbf{i}}$ matrix and let

$$\overline{\alpha}_k(\mathbf{i}) := \max_{\underline{x}\in M}\alpha_k(D_{\underline{x}}F_{\mathbf{i}}),\ \underline{\alpha}_k(\mathbf{i}) := \min_{\underline{x}\in M}\alpha_k(D_{\underline{x}}F_{\mathbf{i}})$$

The singular value function ϕ^s is then defined for $0\leq s\leq n$ as

$$\phi^s(D_{\underline{x}}F_{\mathbf{i}}) := \alpha_1(D_{\underline{x}}F_{\mathbf{i}})...\alpha_{k-1}(D_{\underline{x}}F_{\mathbf{i}})\alpha_k(D_{\underline{x}}F_{\mathbf{i}})^{s-k+1}$$

where $k-1<s\leq k$ and k is positive integer. We define the maximum and the minimum of the singular value function analogously as above

$$\overline{\phi}^s(\mathbf{i}) := \max_{\underline{x}\in M}\phi^s(D_{\underline{x}}F_{\mathbf{i}})\ ,\ \underline{\phi}^s(\mathbf{i}) := \min_{\underline{x}\in M}\phi^s(D_{\underline{x}}F_{\mathbf{i}})$$

We define the subadditive pressure after K. Falconer 1994 and L. Barreira 1996:

$$P(s) := \lim_{k\to\infty}\frac{1}{k}\log\sum_{|\mathbf{i}|=k}\overline{\phi}^s(\mathbf{i})$$

and define the lower pressure:

$$\underline{P}(s) := \liminf_{k\to\infty}\frac{1}{k}\log\sum_{|\mathbf{i}|=k}\underline{\phi}^s(\mathbf{i})$$

2.2 Subadditive pressure for triangular maps

Theorem 2.2.1. *Let $0\leq s\leq n$. If $F_1,...,F_l$ contractive maps in form (1) and $F_i\in C^{1+\varepsilon}$ for every $1\leq i\leq l$ then*

$$P(s) = \underline{P}(s).$$

In the following we state some linear algebra definitions and lemmas, the proofs of which can be found in article [1].

The m-dimensional exterior algebra Φ^m consists of formal elements $v_1 \wedge ... \wedge v_m$ with $v_i \in \mathbb{R}^n$ such that $v_1 \wedge ... \wedge v_m = 0$ if $v_i = v_j$ for some $i \neq j$, and such that interchanging two different elements reverses the sign, i.e. $v_1 \wedge ...v_i...v_j... \wedge v_m = -v_1 \wedge ...v_j...v_i... \wedge v_m$, if $i \neq j$. Then Φ^m is a vector space of dimension $\binom{n}{m}$ with basis $\{e_{j_1} \wedge ... \wedge e_{j_m} : 1 \leq j_1 < ... < j_m \leq n\}$ where $e_1, ...e_n$ are a given set of orthonormal vectors in $\mathbb{R}^n$.

Then Φ^m becomes a normed space under the norm

$$\|v_1 \wedge ... \wedge v_m\| = |\text{m-dimensional volume of the parallelepiped spanned by } v_1, ...v_m|$$

We may also define a norm $\|.\|_\infty$ on Φ^m by

$$\left\| \sum_{1 \leq i_1 < ... < i_m \leq m} \lambda_{i_1...i_m}(e_{i_1} \wedge ... \wedge e_{i_m}) \right\|_\infty := \max |\lambda_{i_1...i_m}|$$

If $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear there is an induced linear mapping $\widetilde{T} : \Phi^m \mapsto \Phi^m$ given by

$$\widetilde{T}(v_1 \wedge ... \wedge v_m) := (Tv_1) \wedge ... \wedge (Tv_m)$$

The norms on Φ^m induce norms on the space of linear mappings $\mathfrak{L}(\Phi^m, \Phi^m)$ in the usual way by

$$\left\|\widetilde{T}\right\| = \sup_{w \in \Phi^m, w \neq 0} \frac{\left\|\widetilde{T}w\right\|}{\|w\|}$$

Then with respect to the norm $\|.\|$

$$\left\|\widetilde{T}\right\| = \phi^m(T) \tag{3}$$

and with respect to the $\|.\|_\infty$

$$\left\|\widetilde{T}\right\|_\infty = \max\left\{\left|T^{(m)}\right| : T^{(m)} \text{ is an } m \times m \text{ minor of } T \right\}, \tag{4}$$

Recall that the $m \times m$ minor $T^{(m)} \equiv T\binom{r_1,...r_m}{s_1,...,s_m}$ of the $n \times n$ matrix T is the determinant of the $m \times m$ matrix formed by the elements of T in the rows $1 \le r_1 < ... < r_m \le n$ and columns $1 \le s_1 < ... < s_m \le n$. The space of linear mappings $\mathfrak{L}(\Phi^m, \Phi^m)$ is of finite dimension $\binom{n}{m}^2$. Since any two norms on a finite dimensional normed space are equivalent, there are constants $0 < c_1 < c_2 < \infty$ depending only on n and m such that

$$c_1 \left\|\widetilde{T}\right\|_\infty \le \left\|\widetilde{T}\right\| \le c_2 \left\|\widetilde{T}\right\|_\infty \tag{5}$$

Now we notice several lemmas relating to minors of matrices. We will need some well-known inequalities.

Lemma 2.2.1. *Let $x_i \ge 0$, $i = 1, ..., m$ and $p \in \mathbb{R}^+$.*

1. *If $p > 1$, then $(x_1^p + ... + x_m^p) \le (x_1 + ... + x_m)^p \le m^{p-1}(x_1^p + ... + x_m^p)$*
2. *If $0 < p \le 1$, then $m^{p-1}(x_1^p + ... + x_m^p) \le (x_1 + ... + x_m)^p \le (x_1^p + ... + x_m^p)$.*

Lemma 2.2.2. *Let a_n a sequence of real numbers such that $a_{n+m} \le a_n + a_m$. Then there exists $\lim_{n\to\infty} \frac{a_n}{n}$ and it equals to $\inf_n \frac{a_n}{n}$.*

We first look at the expansion of $m \times m$ minors of the product of k matrices $A = A_1 A_2 \cdots A_k$, where for $i = 1, ..., k$

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i & ... & a_{1n}^i \\ a_{21}^i & a_{22}^i & ... & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^i & a_{n2}^i & ... & a_{nn}^i \end{bmatrix}$$

Lemma 2.2.3. *For $1 \le m \le n$, the $m \times m$ minors of $A = A_1 \cdots A_k$ have formal expansions in terms of the entries of the A_i of the form*

$$A\binom{r_1, ... r_m}{s_1, ..., s_m} = \sum_{c_1, ..., c_k} \pm a_{1(c_1)}^1 \cdots a_{m(c_1)}^1 a_{1(c_2)}^2 \cdots a_{m(c_2)}^2 \cdots a_{1(c_k)}^k \cdots a_{m(c_k)}^k$$

such that for each $i = 1...k$, the $a_{1(c_i)}^i ... a_{m(c_i)}^i$ are distinct entries a_{rs}^i of A_i. In particular, for each i, $1(c_i), ..., m(c_i)$ denote pairs (r, s) corresponding to entries in m different rows and columns of the ith matrix A_i, and the sum is over all such entry combinations $(c_1, ..., c_k)$ with appropriate sign $\pm$.

The proof of this Lemma can be found on [1, Lemmma 2.2]. Now we consider lower triangular matrices. For $i = 1, ..., k$, let

$$U_i = \begin{bmatrix} u_1^i & 0 & ... & 0 \\ u_{21}^i & u_2^i & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1}^i & u_{n2}^i & ... & u_n^i \end{bmatrix}$$

We consider the product

$$U = U_1 \cdots U_k = \begin{bmatrix} u_1 & 0 & ... & 0 \\ u_{21} & u_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & ... & u_n \end{bmatrix}$$

We note that

$$u_{rs} = \sum_{r \geq r_1 \geq ... \geq r_{k-1} \geq s} u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s} \quad 1 \leq r \leq s \leq n \tag{6}$$

since all other products are 0.

Lemma 2.2.4. *With notations as in above, let $U_1, ..., U_k$ be lower triangular matrices and $U = U_1 \cdots U_k$. Then*

1. *If $r < s$, $u_{rs} = 0$*
2. *If $r = s$, $u_{rs} \equiv u_r = u_r^1 \cdots u_r^k$*
3. *If $r > s$, then the sum (6) for u_{rs} has at most $k^{r-s} \leq k^{n-1}$ non-zero terms. Moreover, each non-zero summand $u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}$ has at most $n-1$ non-diagonal terms in the product, i.e. terms with $r \neq r_1$ or $r_i \neq r_{i+1}$ or $r_{k-1} \neq s$.*

The proof can also be found in [1, Lemma 2.3] for upper-triangular matrices. Now we extend the estimate of Lemma 2.2.4 to minors.

Lemma 2.2.5. *Let $U_1, ..., U_k$ and U be lower triangular matrices as in above. Then each $m \times m$ minor of U has an expansion of the form*

$$U\begin{pmatrix} r_1, \dots r_m \\ s_1, \dots, s_m \end{pmatrix} = \sum_{c_1,\dots,c_k} \pm u^1_{1(c_1)} u^2_{1(c_2)} \cdots u^k_{1(c_k)} \cdots u^1_{m(c_1)} u^2_{m(c_2)} \cdots u^k_{m(c_k)}$$

where $1(c_i), ..., m(c_i)$ are as in Lemma 2.2.3 and

1. *there are at most $m!k^{m(n-1)}$ terms in the sum which are non-zero,*
2. *each summand contains at most $(n-1)^m$ non-diagonal elements in the product.*

The proof is equivalent to the proof of[1, Lemma 2.4]. Before we prove the Theorem 2.2.1, we define two sums.

$$H(s,r) = \max_{\substack{j_1,\dots,j_{m-1} \\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} (d_{j_1 j_1}(\mathbf{i}) \cdots d_{j_{m-1} j_{m-1}}(\mathbf{i}))^{m-s} (d_{j'_1 j'_1}(\mathbf{i}) \cdots d_{j'_m j'_m}(\mathbf{i}))^{s-m+1} \tag{7}$$

where $m-1 < s \leq m$ and $d_{jj}(\mathbf{i}) = \inf_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$. Moreover

$$T(s,r) = \max_{\substack{j_1,\dots,j_{m-1} \\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} (t_{j_1 j_1}(\mathbf{i}) \cdots t_{j_{m-1} j_{m-1}}(\mathbf{i}))^{m-s} (t_{j'_1 j'_1}(\mathbf{i}) \cdots t_{j'_m j'_m}(\mathbf{i}))^{s-m+1} \tag{8}$$

where $m-1 < s \leq m$ and $t_{jj}(\mathbf{i}) = \sup_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$. It is easy to see from Proposition 2.1.1 and the definition of the two sums that

$$H(s,r) \leq T(s,r) \leq C^s H(s,r). \tag{9}$$

Lemma 2.2.6. *For every positive integers r, z, $T(s, r+z) \leq T(s,r)T(s,z)$. Moreover $\lim_{r\to\infty} \frac{\log T(s,r)}{r}$ exists and equal with $\inf_r \frac{\log T(s,r)}{r}$.*

Proof. *of Lemma 2.2.6* From the definition $T(s,r)$ it follows

$$T(s,r+z)=\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}}\sum_{|\mathbf{i}|=r+z}(t_{j_1j_1}(\mathbf{i})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s}(t_{j'_1j'_1}(\mathbf{i})\cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1}\leq$$

$$\leq\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}}(\sum_{|\mathbf{i}|=r}\sum_{|\mathbf{h}|=z}((t_{j_1j_1}(\mathbf{i})t_{j_1j_1}(\mathbf{h})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i})t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}\times$$

$$\times(t_{j'_1j'_1}(\mathbf{i})t_{j'_1j'_1}(\mathbf{h})\cdots t_{j'_mj'_m}(\mathbf{i})t_{j'_mj'_m}(\mathbf{h}))^{s-m+1})=$$

$$=\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}}(\sum_{|\mathbf{i}|=r}(t_{j_1j_1}(\mathbf{i})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s}(t_{j'_1j'_1}(\mathbf{i})\cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1}\times$$

$$\times\sum_{|\mathbf{h}|=z}(t_{j_1j_1}(\mathbf{h})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}(t_{j'_1j'_1}(\mathbf{h})\cdots t_{j'_mj'_m}(\mathbf{h}))^{s-m+1}))\leq$$

$$\leq T(s,r)T(s,z)$$

The existence of the limit is following from Lemma 2.2.2. □

Proof. *of Theorem 2.2.1* We begin the proof by defining a new IFS.
Let $\{G_h\}_{h=1}^{l^r}=\{F_{i_1\dots i_r}\}_{i_1=1,\dots,i_r=1}^{l,\dots,l}$. In this case a h index is suit a $\mathbf{i}\in\{1,\dots,l\}^r$ finite sequence, length r. We define the singular value function $\phi^s(D_{\underline{x}}G_{\mathbf{h}}),\overline{\phi'}^s(\mathbf{h}),\underline{\phi'}^s(\mathbf{h})$, $\mathbf{h}\in\{1,\dots,l^r\}^*$, for $\{G_h\}_{h=1}^{l^r}$, exatly the same way. It is easy to see that

$$\sum_{|\mathbf{i}|=kr}\phi^s(\mathbf{i})=\sum_{|\mathbf{h}|=k}\phi'^s(\mathbf{h}).\tag{10}$$

The elements of $D_{\underline{x}}G_h$, denote by $y_{ij}\,(h,\underline{x})$, are equal with $x_{ij}\,(\mathbf{i},\underline{x})$ for a suit finite sequence $\mathbf{i}$, length r. It is very simple to see that
$\phi^s(D_{\underline{x}}G_{\mathbf{h}})=(\phi^{m-1}(D_{\underline{x}}G_{\mathbf{h}}))^{m-s}(\phi^m(D_{\underline{x}}G_{\mathbf{h}}))^{s-m+1}$, where $m-1<s\leq m$. By using relations (3), (4) and (5) it follows that

$$\phi^m(D_{\underline{x}}G_{\mathbf{h}})\geq c_2\max\left\{\left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right|:D_{\underline{x}}G_{\mathbf{h}}^{(m)}\text{ is an }m\times m\text{ minor of }D_{\underline{x}}G_{\mathbf{h}}\right\}$$

The maximum $m\times m$ minor of $D_{\underline{x}}G_{\mathbf{h}}$ is at least the largest product of m distinct diagonal elements of $D_{\underline{x}}G_{\mathbf{h}}$, since such products are themselves minors of triangular matrices. Therefore

$$\underline{\phi'}^s(\mathbf{h})\geq c_2^s\left(\inf_{\underline{x}}\left|y_{j_1j_1}\,(\mathbf{h},\underline{x})\cdots y_{j_{m-1}j_{m-1}}\,(\mathbf{h},\underline{x})\right|\right)^{m-s}\left(\inf_{\underline{x}}\left|y_{j'_1j'_1}\,(\mathbf{h},\underline{x})\cdots y_{j'_mj'_m}\,(\mathbf{h},\underline{x})\right|\right)^{s-m+1}$$

for every $j_1, ..., j_{m-1}, j'_1, ..., j'_m$.

Since $D_{\underline{x}}G_{\mathbf{h}} = D_{G_{h_2...h_k}(\underline{x})}G_{h_1}D_{G_{h_3...h_k}(\underline{x})}G_{h_2}...D_{\underline{x}}G_{h_k}$,
$y_{jj}(\mathbf{h}, \underline{x}) = y_{jj}(h_1, G_{h_2...h_k}(\underline{x}))\, y_{jj}(h_2, G_{h_3...h_k}(\underline{x}))\, ...y_{jj}(h_k, \underline{x})$. It follows with the notation $\inf_{\underline{x}} |y_{jj}(h, \underline{x})| = d'_{jj}(h)$ that

$$\inf_{\underline{x}} \left| y_{j_1j_1}(\mathbf{h}, \underline{x}) \cdots y_{j_{m-1}j_{m-1}}(\mathbf{h}, \underline{x}) \right|^{m-s} \inf_{\underline{x}} \left| y_{j'_1j'_1}(\mathbf{h}, \underline{x}) \cdots y_{j'_mj'_m}(\mathbf{h}, \underline{x}) \right|^{s-m+1} \geq$$

$$\geq (d'_{j_1j_1}(h_1) \cdots d'_{j_1j_1}(h_k) d'_{j_2j_2}(h_1) \cdots d'_{j_{m-1}j_{m-1}}(h_1) \cdots d'_{j_{m-1}j_{m-1}}(h_k))^{m-s} \times$$

$$\times (d'_{j'_1j'_1}(h_1) \cdots d'_{j'_1j'_1}(h_k) d'_{j'_2j'_2}(h_1) \cdots d'_{j'_mj'_m}(h_1) \cdots d'_{j'_mj'_m}(h_k))^{s-m+1}$$

The next inequality follows from the rearrangement of the product

$$\sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \geq c_2^s \sum_{|\mathbf{h}|=k} (d'_{j_1j_1}(h_1) \cdots d'_{j_{m-1}j_{m-1}}(h_1))^{m-s} (d'_{j'_1j'_1}(h_1) \cdots d'_{j'_mj'_m}(h_1))^{s-m+1} \cdots$$

$$\cdots (d'_{j_1j_1}(h_k) \cdots d'_{j_{m-1}j_{m-1}}(h_k))^{m-s} (d'_{j'_1j'_1}(h_k) \cdots d'_{j'_mj'_m}(h_k))^{s-m+1} =$$

$$= c_2^s((d'_{j_1j_1}(1) \cdots d'_{j_{m-1}j_{m-1}}(1))^{m-s} (d'_{j'_1j'_1}(1) \cdots d'_{j'_mj'_m}(1))^{s-m+1} + \cdots$$

$$\cdots + (d'_{j_1j_1}(l^r) \cdots d'_{j_{m-1}j_{m-1}}(l^r))^{m-s} (d'_{j'_1j'_1}(l^r) \cdots d'_{j'_mj'_m}(l^r))^{s-m+1})^k$$

The inequality in above is true for every $j_1, ..., j_{m-1}, j'_1, ..., j'_m$, therefore we can receive the maximum. From definition of $\{G_h\}_{h=1}^{l^r}$ and $H(s, r)$ it follows

$$\sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \geq c_2^s H(s, r)^k \tag{11}$$

By using relations (3), (4) and (5) it follows similarly that

$$\phi^m(D_{\underline{x}}G_{\mathbf{h}}) \leq c_1 \max \left\{ \left| D_{\underline{x}}G_{\mathbf{h}}^{(m)} \right| : D_{\underline{x}}G_{\mathbf{h}}^{(m)} \text{ is an } m \times m \text{ minor of } D_{\underline{x}}G_{\mathbf{h}} \right\}$$

Therefore

$$\sum_{|\mathbf{h}|=k} \overline{\phi'}^s(\mathbf{i}) \leq c_1^2 \sum_{|\mathbf{h}|=k} \left(\sup_{\underline{x}} \max_{m-1 \times m-1 \text{ minor}} \left| D_{\underline{x}}G_{\mathbf{h}}^{(m-1)} \right| \right)^{m-s} \left(\sup_{\underline{x}} \max_{m \times m \text{ minor}} \left| D_{\underline{x}}G_{\mathbf{h}}^{(m)} \right| \right)^{s-m+1}$$

The supremum and the maximum are commutable in this situation, we can estimate the sum with

$$c_1^2 \binom{n}{m}^2 \binom{n}{m-1}^2 \max_{\left\{ \substack{r_1, \ldots, r_{m-1} \\ s_1, \ldots, s_{m-1}} \right\}} \max_{\left\{ \substack{r'_1, \ldots, r'_m \\ s'_1, \ldots, s'_m} \right\}} \sum_{|\mathbf{h}|=k} \left(\sup_{\underline{x}} \left| D_{\underline{x}}G_{\mathbf{h}}^{(m-1)} \right| \right)^{m-s} \left(\sup_{\underline{x}} \left| D_{\underline{x}}G_{\mathbf{h}}^{(m)} \right| \right)^{s-m+1}$$

where $r_1, ..., r_{m-1}$ are the rows and $s_1, ..., s_{m-1}$ are the columns of the $(m-1)\times(m-1)$ minor, and $r'_1, ..., r'_m$ are the rows and $s'_1, ..., s'_m$ are the columns of $m\times m$ minor. Since $D_{\underline{x}}G_{\mathbf{h}} = D_{G_{h_2...h_k}(\underline{x})}G_{h_1}D_{G_{h_3...h_k}(\underline{x})}G_{h_2}...D_{\underline{x}}G_{h_k}$, we obtain

$$D_{\underline{x}}G_{\mathbf{h}}\binom{r_1, ..., r_m}{s_1, ..., s_m} = \sum_{c_1,...,c_k} \pm y_{1(c_1)}(h_1, G_{h_2...h_k}(\underline{x}))...y_{1(c_k)}(h_k, \underline{x})...y_{m(c_1)}(h_1, G_{h_2...h_k}(\underline{x}))\times$$

$$\times\, y_{m(c_2)}(h_2, G_{h_3...h_k}(\underline{x}))...y_{m(c_k)}(h_k, \underline{x}) \tag{12}$$

Therefore

$$\sup_{\underline{x}}\left|D_{\underline{x}}G^{(m)}_{\mathbf{h}}\right| \leq \sum_{c_1,...,c_k} \sup_{\underline{x}}\left|y_{1(c_1)}(h_1,\underline{x})\right| ... \sup_{\underline{x}}\left|y_{1(c_k)}(h_k,\underline{x})\right| ... \sup_{\underline{x}}\left|y_{m(c_1)}(h_1,\underline{x})\right| \times$$

$$\times \sup_{\underline{x}}\left|y_{m(c_2)}(h_2,\underline{x})\right| ... \sup_{\underline{x}}\left|y_{m(c_k)}(h_k,\underline{x})\right| \tag{13}$$

Denote by $t'_{kl}(h) := \sup_{\underline{x}}|y_{kl}(h,\underline{x})|$ the supremums. It follows from the inequality (13) and the Lemma 2.2.1

$$\begin{aligned}
&\sum_{|\mathbf{h}|=k} \sup_{\underline{x}}\left|D_{\underline{x}}G^{(m-1)}_{\mathbf{h}}\right|^{m-s} \sup_{\underline{x}}\left|D_{\underline{x}}G^{(m)}_{\mathbf{h}}\right|^{s-m+1} \leq \\
&\sum_{\substack{c_1,...,c_k\\ c'_1,...,c'_k}} ((t'_{1(c_1)}(1)...t'_{m-1(c_1)}(1))^{m-s}(t'_{1(c'_1)}(1)...t'_{m(c'_1)}(1))^{s-m+1}+ \\
&... + (t'_{1(c_1)}(l^r)...t'_{m-1(c_1)}(l^r))^{m-s}(t'_{1(c'_1)}(l^r)...t'_{m(c'_1)}(l^r))^{s-m+1})\times \\
&... \times ((t'_{1(c_k)}(1)...t'_{m-1(c_k)}(1))^{m-s}(t'_{1(c'_k)}(1)...t'_{m(c'_k)}(1))^{s-m+1}+ \\
&... + (t'_{1(c_k)}(l^r)...t'_{m-1(c_k)}(l^r))^{m-s}(t'_{1(c'_k)}(l^r)...t'_{m(c'_k)}(l^r))^{s-m+1})
\end{aligned} \tag{14}$$

Lemma 2.2.5 implies that each non-zero term of the sum in above has at most $2(n-1)^m = b$ of the indices $1(c_1), ..., m-1(c_1), ..., 1(c_k), ..., m-1(c_k)$, $1(c'_1), ..., m(c'_1), ..., 1(c'_k), ..., m(c'_k)$ that are non-diagonal terms. Thus, for each set of indices $(c_1, ..., c_k, c'_1, ..., c'_k)$, we have at least $k-b$ of these indices such that $1(c_r), ..., m-1(c_r), 1(c'_r), ..., m(c'_r)$ are all diagonal entries.

For such c_r and c'_r

$$((t'_{1(c_r)}(1)...t'_{m-1(c_r)}(1))^{m-s}(t'_{1(c'_r)}(1)...t'_{m(c'_r)}(1))^{s-m+1}+...$$

$$...+(t'_{1(c_r)}(l^r)...t'_{m-1(c_1)}(l))^{m-s}(t'_{1(c'_r)}(l^r)...t'_{m(c'_r)}(l^r))^{s-m+1})\leq$$

$$\leq \max_{\{j_1,...,j_{m-1}\},\{j'_1,...,j'_m\}}((t'_{j_1j_1}(1)...t'_{j_{m-1}j_{m-1}}(1))^{m-s}(t'_{j'_1}(1)...t'_{j'_m}(1))^{s-m+1}+...$$

$$...+(t'_{j_1j_1}(l^r)...t'_{j_{m-1}j_{m-1}}(l^r))^{m-s}(t'_{j'_1}(l^r)...t'_{j'_mj'_m}(l^r))^{s-m+1})=T(s,r)$$

The last equality follows from definition $\{G_h\}_{h=1}^{l^r}$ and $T(s,r)$. Hence from (14)

$$\begin{aligned}&\sum_{|\mathbf{h}|=k}\sup_{\underline{x}}\left|D_{\underline{x}}G^{(m-1)}_{\mathbf{h}}\right|^{m-s}\sup_{\underline{x}}\left|D_{\underline{x}}G^{(m)}_{\mathbf{h}}\right|^{s-m+1}\leq\\&\leq\sum_{\substack{c_1,...,c_k\\c'_1,...,c'_k}}\left(T(s,r)^{k-b}(l^r)^b\right)\leq c''k^ql^{rb}T(s,r)^{k-b},\end{aligned}\tag{15}$$

where, using Lemma 2.2.5, $c''=m!(m-1)!$ and $q=(2m-1)(n-1)$.

By using (9), (10), (11) and (15)

$$\begin{aligned}&\sum_{|\mathbf{i}|=kr}\overline{\phi}^s(\mathbf{i})=\sum_{|\mathbf{h}|=k}\overline{\phi'}^s(\mathbf{h})\leq c''k^ql^{rb}T(s,r)^{k-b}\leq c''(C^s)^kk^ql^{rb}T(s,r)^{-b}H(s,r)^k\leq\\&\leq c'''(C^s)^kk^ql^{rb}T(s,r)^{-b}\sum_{|\mathbf{h}|=k}\underline{\phi}'^s(\mathbf{h})=c'''k^ql^{rb}T(s,r)^{-b}\sum_{|\mathbf{i}|=kr}\underline{\phi}^s(\mathbf{i}).\end{aligned}\tag{16}$$

We apply both sides of the inequality logarithm and we divide by kr, then

$$\begin{aligned}&\frac{\log\sum_{|\mathbf{i}|=kr}\overline{\phi}^s(\mathbf{i})}{kr}\leq\\&\leq\frac{\log c'''}{kr}+\frac{q\log k}{kr}+\frac{rb\log l}{kr}+\frac{(kb)\log(C^s)}{kr}+\frac{-b\log T(s,r)}{kr}+\frac{\log\sum_{|\mathbf{i}|=kr}\underline{\phi}^s(\mathbf{i})}{kr}\end{aligned}\tag{17}$$

is true for every positive k,r integer. We apply limit inferior for both sides of the inequality. The limit exists in the left-hand side of the inequality and in the right-hand side the limit of every term exists and equals zero except the last term. Therefore

$$P(s) \le \underline{P}(s)$$

While the opposite relation is trivial this completes the proof. $\square$

The next theorem is a consequence of the last proof.

Theorem 2.2.2. *For $0 \le s \le n$. If $F_1, ..., F_l$ contractive maps in form (1) and $F_i \in C^{1+\varepsilon}$ for every $1 \le i \le l$ then*

$$P(s) = \lim_{r\to\infty} \frac{1}{r} \log(\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} \left(|x_{j_1 j_1}(\mathbf{i}, \underline{x})| \dots \left|x_{j_{m-1} j_{m-1}}(\mathbf{i}, \underline{x})\right|\right)^{m-s} \times$$
$$\times \left(\left|x_{j'_1 j'_1}(\mathbf{i}, \underline{x})\right| \dots \left|x_{j'_m j'_m}(\mathbf{i}, \underline{x})\right|\right)^{s-m+1})$$

for every $\underline{x} \in M$.

Proof. It follows from inequality (9) that the $\lim_{r\to\infty} \frac{\log H(s,r)}{r}$ exists and $\lim_{r\to\infty} \frac{\log H(s,r)}{r} = \lim_{r\to\infty} \frac{\log T(s,r)}{r}$. It is clear by (**??**) that $\lim_{r\to\infty} \frac{\log T(s,r)}{r} = P(s)$. Because of the definition $H(s,r), T(s,r)$, this is exactly what we want to prove. $\square$

2.3 Some examples

In this subsection we show some examples to calculate the box dimension, and the upper bound of Hausdorff dimension. It follows from [2], the box dimension is equal with s_0 if $P(s_0) = 0$, and the Hausdorff dimension is less or equal then s_0 in our cases.

The easiest example is the perturbated Sierpinski-triangular. Let

$$T = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and $T_i\underline{x} = T\underline{x} + \underline{v}_i$ for $i = 1, 2, 3$, where $v_1 = \binom{0}{0}, v_2 = \binom{\frac{2}{3}}{0}, v_3 = \binom{\frac{1}{3}}{\frac{1}{2}}$. This is not the usual Sierpinski-triangular, because we must handle the open-set condition care by the perturbation. The image of this self-similar fractal can be showed in Figure 1.

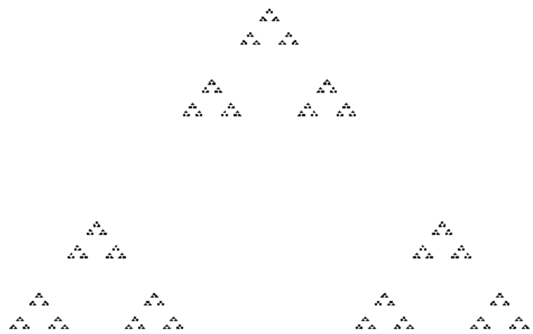

Figure 1: The image of Sierpinski-triangular

The Hausdorff and box dimension is $\frac{\ln 3}{\ln 3} = 1$. Now let $f(x) = \sin(\pi x)/6$ and

$$F_i \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix} + \begin{pmatrix} v_i \\ w_i \end{pmatrix}$$

for $i = 1, 2, 3$, where $\binom{v_1}{w_1} = \binom{0}{0}$, $\binom{v_2}{w_2} = \binom{\frac{2}{3}}{0}$, $\binom{v_3}{w_3} = \binom{\frac{1}{3}}{\frac{1}{2}}$. We can consider it a perturbated Sierpinski-triangular. The F_i functions make the $[0,1]^2$ cube into itself like in the Figure 2, and the picture of this fractal is Figure 3.

Our proposition is the two fractal's box dimension is equal. We use Theorem 2.2.2 to prove. From the definition in this case it is easy to see that $x_{11}(\mathbf{i}, \underline{x}) = \frac{1}{3}^{|\mathbf{i}|}$ and $x_{22}(\mathbf{i}, \underline{x}) = \frac{1}{3}^{|\mathbf{i}|}$. We can suppose that $1 \leq s < 2$. Then

$$P(s) = \lim_{r \to \infty} \frac{1}{r} \log \left(\max_{\substack{j_1, \\ j_1', j_2'}} \sum_{|\mathbf{i}| = r} \left(|x_{j_1 j_1}(\mathbf{i}, \underline{x})| \right)^{2-s} \times \left(\left| x_{j_1' j_1'}(\mathbf{i}, \underline{x}) \right| \left| x_{j_2' j_2'}(\mathbf{i}, \underline{x}) \right| \right)^{s-2+1} \right) =$$

$$\lim_{r \to \infty} \frac{1}{r} \log \left(\sum_{|\mathbf{i}| = r} \left(\frac{1}{3}^{|\mathbf{i}|} \right)^{2-s} \left(\frac{1}{3}^{|\mathbf{i}|} \frac{1}{3}^{|\mathbf{i}|} \right)^{s-1} \right) = \lim_{r \to \infty} \frac{1}{r} \log \left(3^r \frac{1}{3}^{sr} \right) = \log 3 - s \log 3$$

It is easy to see that $P(s) = 0$ if and only if $s = 1$, which is the box dimension of the fractal in above. This follows from [2]. In general it is easy to see, that for every $f_i : \mathbb{R} \to \mathbb{R}$, if $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ are contractions for $i = 1, 2, 3$, where

$$F_i \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f_i(x) \end{pmatrix} + \begin{pmatrix} v_i \\ w_i \end{pmatrix}$$

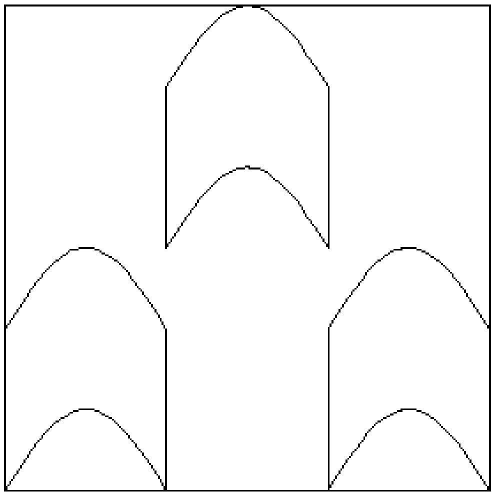

Figure 2: The image of F_i $i = 1, 2, 3$ functions

Figure 3: The image of fractal

satisfies the open-set condition, and constitute $[0,1]^2$ into itself, the box dimension is equal to 1.

Now we see an other example. Let $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ for every $i = 1, 2, 3, 4$:

$$F_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{4} + \frac{\sin(\pi x)}{4} + \frac{1}{2} \end{pmatrix}, F_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ \frac{y}{4} - \frac{\sin(\pi x)}{4} + \frac{1}{4} \end{pmatrix}$$
$$F_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} + \frac{1}{2} \\ \frac{y}{4} + \frac{\sin(\pi x)}{4} + \frac{1}{2} \end{pmatrix}, F_4\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} + \frac{1}{2} \\ \frac{y}{4} - \frac{\sin(\pi x)}{4} + \frac{1}{4} \end{pmatrix}$$

These functions are contractions and constitute $[0,1]^2$ into itself as in Figure 4. The image of fractal is in Figure 5.

Then it is easy to see that $x_{11}(\mathbf{i}, \underline{x}) = \frac{1}{2}^{|\mathbf{i}|}$ and $x_{22}(\mathbf{i}, \underline{x}) = \frac{1}{4}^{|\mathbf{i}|}$. We can suppose that $1 \leq s < 2$. In this case

$$\max_{\substack{j_1, \\ j_1', j_2'}} \sum_{|\mathbf{i}|=r} \left(|x_{j_1 j_1}(\mathbf{i}, \underline{x})|\right)^{2-s} \times \left(\left|x_{j_1' j_1'}(\mathbf{i}, \underline{x})\right| \left|x_{j_2' j_2'}(\mathbf{i}, \underline{x})\right|\right)^{s-2+1} =$$
$$4^r \frac{1}{2}^{r(2-s)} \left(\frac{1}{4}\frac{1}{2}\right)^{r(s-1)} = 2^{(3-2s)r}$$

Then $P(s) = (3 - 2s)\ln 2$, and $P(s) = 0$ if $s = \frac{3}{2}$. It means that the box dimension of the fractal in above is $\frac{3}{2}$ and the Hausdorff dimension is less or equal then $\frac{3}{2}$.

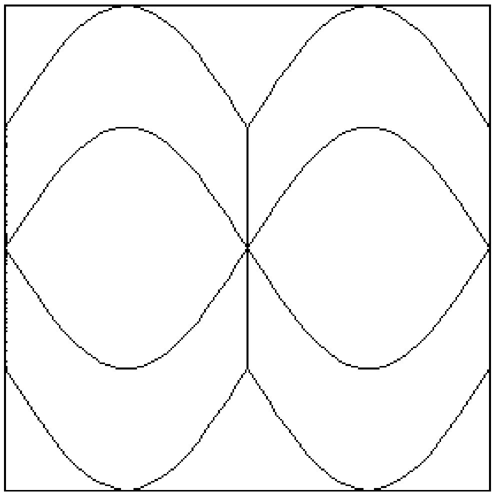

Figure 4: The image of F_i $i = 1, 2, 3, 4$ functions

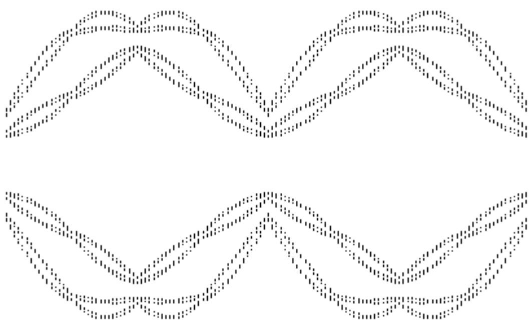

Figure 5: The image of fractal

3 Ledrappier-Young Theorem for self-affine IFS

In their article in 1985 F. Ledrappier and L.-S. Young solved, [4], [5], an important problem of dynamical systems, which finds a connection between entropy and Lyapunov-exponents. Jörg Neunhäuserer proved that the Ledrappier-Young Theorem can be applied for self-affine IFS with diagonal matrices in the special case when we work on the plane and an IFS consists of two-maps. In this section we extend Neunhäuserer's result for every n-dimension without the restriction of the number of functions in the IFS.

3.1 Regular hyperbolic measures and Lyapunov-charts

Let A_i diagonal matrices $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ for $i = 0, ..., m-1$, where the diagonal elements are $0 < a_j^i < 1$. For these matrices we define our iterated function system. Let $g_i : [0,1]^n \mapsto [0,1]^n$

$$g_i(\underline{x}) = A_i\underline{x} + \underline{t}_i,$$

where $\underline{t}_i \in \mathbb{R}^n$ for $i = 0, ..., m-1$ such that $g_i([0,1]^n) \cap g_j([0,1]^n) = \emptyset$ if $i \neq j$.

Let Ω the following compact set

$$\Omega = \bigcap_{n=1}^{\infty} \bigcup_{i_1,...,i_n} g_{i_1} \circ \cdots \circ g_{i_n}([0,1]^n)$$

then we say that Ω is the attractor of IFS $\{g_0, ..., g_{m-1}\}$.

Moreover let $\sum = \{0, ..., m-1\}^{\mathbb{N}}$ and $\sum^* = \{0, ..., m-1\}^*$. For every $n \geq 1$ and $\mathbf{i} \in \{0, ..., m-1\}^n$ with the notation $g_{\mathbf{i}} = g_{i_1} \circ \cdots \circ g_{i_n}$ we can define the natural projection $\pi : \sum \mapsto \Omega$:

$$\pi(\mathbf{i}) = \lim_{n\to\infty} g_{\mathbf{i}_n}(0) \tag{18}$$

where $\mathbf{i}_n$ is the first n elements of $\mathbf{i}$.

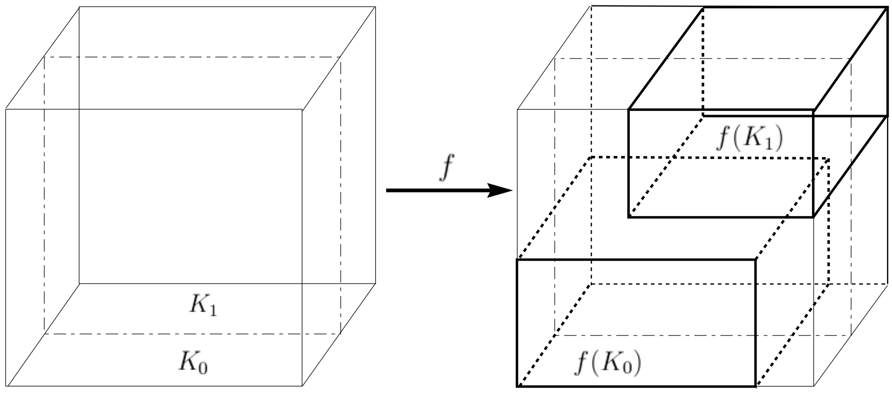

Figure 6: The map of f function

We define for these IFS a dynamical system in $[0,1]^{n+1}$.

Let $M = [0,1]^{n+1}$, $K = (0,1)^{n+1}$ and $K_i = (0,1)^n \times (\frac{i}{m}, \frac{i+1}{m})$ for $i = 0,...,m-1$. Define the $N = M \setminus \cup_{i=0}^{m-1} K_i$ closed subset. We define our dynamical system $f : K \setminus N \mapsto K$.

Let $\widetilde{A}_i \in \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ the following diagonal matrices

$$\widetilde{A}_i = \begin{bmatrix} A_i & \underline{0} \\ \underline{0}^{\mathrm{T}} & m \end{bmatrix}$$

for every $i = 0,...,m-1$, where $\underline{0}$ is the zero vector of $\mathbb{R}^n$, then our discrete dynamical system is:

$$f(\underline{x}) := \widetilde{A}_i \underline{x} + \underline{v}_i \text{ if } \underline{x} \in K_i, \tag{19}$$

where $\underline{v}_i \in \mathbb{R}^{n+1}$ for $i = 0,...,m-1$ and $\underline{v}_i = \binom{t_i}{-i}$. A special case of f can be found in Figure 6.

It is easy to see that $f(K_i) \cap f(K_j) = \emptyset$ and f can be extended to a $C^{1+\alpha}$-diffeomorphism $f : \overline{K_i} \mapsto \overline{f(K_i)}$, $i = 1,...,m$ for some $\alpha > 0$. We sometimes write f_i for $f \mid_{\overline{K_i}}$.

We can write $f : [0,1]^{n+1} \mapsto [0,1]^{n+1}$ in an other form. Namely, let $\varphi : [0,1] \mapsto [0,1]$ the following function:

$$\varphi(x) = mx \mod 1$$

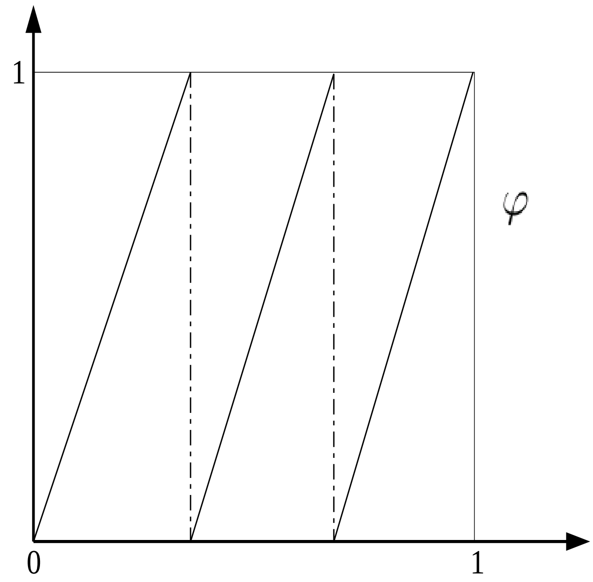

Figure 7: The function φ in the $(n+1)$ coordinate

Then $f : [0,1]^n \times [0,1] \mapsto [0,1]^n \times [0,1]$ and

$$f(\underline{y}, x) = \begin{pmatrix} g_{\lfloor mx \rfloor}(\underline{y}) \\ \varphi(x) \end{pmatrix}$$

Let

$$N^- = \{\underline{x} \in M : \exists z \in N, z_n \in M \backslash N \text{ if } z_n \to z, f(z_n) \to \underline{x}\}$$

Moreover

$$\begin{aligned} &M^+ = \{\underline{x} \in M : f^n(\underline{x}) \notin N, n = 0, 1, 2, ...\} \\ &D = \bigcap_{n \geq 0} f^n(M^+) \\ &\Lambda = \overline{D}. \end{aligned}$$

Λ is called the attractor of f. Obviously D is f-invariant. In this case $M^+ = \left\{\underline{x} \in M : x_n \neq \frac{i}{m^j}, i = 0, 1, ..., m^j, \text{ and } j = 0, 1, 2, ...\right\}$. Evidently, in this situation $\Lambda = \Omega \times [0,1]$, moreover $f \mid_\Lambda$ is one-to-one map.

Let $\Theta = \{0, ..., m-1\}^{\mathbb{Z}}$. The natural projection $\pi' : \Theta \mapsto \Lambda \subset \mathbb{R}^n \times \mathbb{R}$ is defined in such a way that the negative indices correspond the first n coordinates and the non-negative coordinates determine the $(n+1)$ coordinate in $\mathbb{R}^{n+1}$. Namely,

$$\pi'(...i_{-n}...i_{-2}i_{-1}; i_0 i_1 ... i_n ...) = \begin{pmatrix} \pi(i_{-1}i_{-2}...i_{-n}...) \\ \sum_{k=0}^{\infty} \frac{i_k}{m^{k+1}} \end{pmatrix} \tag{20}$$

one-to-one mapping. It is easy to see that $f(\pi'(\mathbf{i})) = \pi'(\sigma\mathbf{i})$, where σ is the left-shift operator on Θ.

For every $\varepsilon > 0$ and $l = 1, 2, ...$ let

$$\begin{aligned} D^+_{\varepsilon,l} &= \left\{\underline{x} \in M^+ \cap \Lambda : d(f^n(\underline{x}), N) \geq l^{-1} e^{-\varepsilon n}, n = 0, 1, 2, ...\right\} \\ D^-_{\varepsilon,l} &= \left\{\underline{x} \in M^+ \cap \Lambda : d(f^{-n}(\underline{x}), N) \geq l^{-1} e^{-\varepsilon n}, n = 0, 1, 2, ...\right\} \\ D^+_{\varepsilon} &= \bigcup_{l=1}^{\infty} D^+_{\varepsilon,l},\ D^-_{\varepsilon} = \bigcup_{l=1}^{\infty} D^-_{\varepsilon,l} \\ D_{\varepsilon} &= D^+_{\varepsilon} \cap D^-_{\varepsilon} \end{aligned} \tag{21}$$

Moreover let

$$U(\delta, N) := \{\underline{x} \in \Lambda : d(\underline{x}, N) \leq \delta\}$$

Definition 3.1.1. *We call a point $\underline{x} \in D_\varepsilon$ **regular** if there exist numbers $\chi_1(\underline{x}) > \cdots > \chi_{k(\underline{x})}(\underline{x})$ (called Lyapunov exponents) and a decomposition $T_{\underline{x}}M = \oplus_{i=1}^{k(\underline{x})} E^i(\underline{x})$ composed of the vector spaces*

$$E^i(\underline{x}) = \left\{ v \in T_{\underline{x}}M \setminus \{0\} : \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_{\underline{x}} f^n v\| = \chi_i(\underline{x}) \right\} \cup \{0\}$$

such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det D_{\underline{x}} f^n| = \sum_{i=1}^{k(\underline{x})} \dim E^i(\underline{x})$$

Note that $\chi_i(f(\underline{x})) = \chi_i(\underline{x})$ and $DfE^i(\underline{x}) = E^i(f(\underline{x}))$ for each i. Let $s(\underline{x}) = \min\{i : \chi_i(\underline{x}) < 0\}$ and $u(\underline{x}) = \max\{i : \chi_i(\underline{x}) > 0\}$. Let ν be a σ-invariant probability measure on Θ. Then $\mu = \nu \circ \pi'^{-1}$ is an f-invariant Borel probability measure. The Oseledec theorem tells us that μ-almost every point is regular. Moreover if μ is ergodic then $k(\underline{x}) = n+1$, $\chi_i(\underline{x}) = \chi_i$

and $\dim E^i(\underline{x}) = \dim E^i$ are μ-almost everywhere constant. The examination of χ_i is easy if μ is a Bernoulli-probability measure, then $\chi_i = \sum_{k=0}^{m-1} p_k \ln a_{s_i}^k$, where s_i is the coordinate which belongs to the ith Lyapunov-exponent. In our situation the $E^i(\underline{x})$ subspaces of $T_{\underline{x}}M$ are orthogonal. For every $\underline{x} \in M^+$ the expansion in the last coordinate is m therefore $\chi_u = \ln m$ for μ-almost every point.

Definition 3.1.2. *We say that an invariant Borel probability measure is **regular hyperbolic** if $\mu(D_\varepsilon) = 1$ for $\varepsilon > 0$ sufficiently small and $u(\underline{x}) = s(\underline{x}) + 1$.*

Proposition 3.1.1. *Let μ be an f-invariant probability measure. If there exist $C > 0$ and $q > 0$ such that for every $\delta > 0$*

$$\mu(U(\delta, N)) \leq C\delta^q \tag{22}$$

then μ is regular hyperbolic for every $\varepsilon > 0$.

Proof. It will be sufficient to show that $\mu(\Lambda \setminus D_\varepsilon) = 0$.

It is easy to see that

$$\Lambda \setminus D^+_{\varepsilon,l} \subset \left\{\underline{x} \in \Lambda : \exists m \in \mathbb{N} \text{ such that } \ f^m(\underline{x}) \in U(l^{-1}e^{-\varepsilon m}, N)\right\}.$$

Since μ is f-invariant

$$\mu(\Lambda \setminus D^+_{\varepsilon,l}) \leq \sum_{m=0}^{\infty} \mu(f^{-m}(U(l^{-1}e^{-\varepsilon m}, N))) = \sum_{m=0}^{\infty} \mu(U(l^{-1}e^{-\varepsilon m}, N)) \leq$$
$$\sum_{m=0}^{\infty} C(l^{-1}e^{-\varepsilon m})^q = C\frac{1}{l^q}\frac{1}{1-e^{-\varepsilon q}}$$

If $l_1 < l_2$ then $D^+_{\varepsilon,l_1} \subset D^+_{\varepsilon,l_2}$., therefore

$$\mu(\Lambda \setminus D^+_\varepsilon) = \mu(\bigcap_{l=1}^{\infty} \Lambda \setminus D^+_{\varepsilon,l}) = \lim_{l\to\infty} \mu(\Lambda \setminus D^+_{\varepsilon,l}) \leq \lim_{l\to\infty} C\frac{1}{l^q}\frac{1}{1-e^{-\varepsilon q}} = 0$$

By similar arguments we have $\mu(\Lambda \setminus D^-_\varepsilon) = 0$, and therefore $\mu(D_\varepsilon) = 1$. $\square$

It is clear that if ν is a Bernoulli-probability measure on Θ, then $\mu = \nu \circ \pi'^{-1}$ satisfies (22). Let

$$S_n = \bigcup_{i=0}^{m-1} \bigcup_{j \in \{0, m-1\}} [; i \overbrace{jj \cdots j}^{n-1}].$$

union of cylinder sets. It is easy to see that

$$\pi'(S_n) = U(\frac{1}{m^n}, N).$$

Then for arbitrary $\delta > 0$ let $n(\delta) = \lfloor \frac{-\ln \delta}{\ln m} \rfloor$ and therefore

$$\mu(U(\delta, N)) \leq \mu(U(\frac{1}{m^{n(\delta)}}, N)) = \nu(S_{n(\delta)}) = p_0^{n(\delta)-1} + p_{m-1}^{n(\delta)-1} \leq$$
$$p_1^{\frac{-\ln \delta}{\ln m}-2} + p_{m-1}^{\frac{-\ln \delta}{\ln m}-2} = p_1^{-2} \delta^{\frac{-\ln p_1}{\ln m}} + p_{m-1}^{-2} \delta^{\frac{-\ln p_{m-1}}{\ln m}} \leq \max\left\{p_1^{-2}, p_{m-1}^{-2}\right\} \delta^{\min\left\{\frac{-\ln p_1}{\ln m}, \frac{-\ln p_{m-1}}{\ln m}\right\}}.$$

By Proposition 3.1.1 a Bernoulli-measure is regular hyperbolic.

There is an other Proposition about regular hyperbolicity.

Proposition 3.1.2. *Let ν be an ergodic, left-shift invariant probability measure on Θ. If $m \geq 3$ and $\nu([; 1] \cup \cdots \cup [; m-2]) > 0$ then $\mu = \nu \circ \pi'^{-1}$ is regular hyperbolic.*

Proof. We begin the proof by defining a metric ρ on Θ.

$$\rho(\mathbf{i}, \mathbf{j}) = \sum_{k=-\infty}^{\infty} \frac{| i_k - j_k |}{m^k}$$

It is trivial to see by (20) that

$$d(\underline{x}, \underline{y}) \leq \rho(\mathbf{i}, \mathbf{j})$$

where $\pi'(\mathbf{i}) = \pi'(\mathbf{j})$. We need to prove that

$$\mu(\left\{\underline{x} \in \Lambda : \exists n_k \to \infty d(f^{n_k}(\underline{x}), N) \leq e^{-\varepsilon n_k}\right\}) = 0$$

It is enough to prove that

$$\nu(\left\{\mathbf{i} \in \Theta : \exists n_k \to \infty ; \rho(\sigma^{n_k}(\mathbf{i}), S) \leq e^{-\varepsilon n_k}\right\}) = 0,$$

where $S=\bigcup_{i=0}^{m-1}\bigcup_{j\in\{0,m-1\}}[;ijjj...j...]$ and $\pi'(S)=N$.

If $\rho(\sigma^{n_k}(\mathbf{i}),S)\leq e^{-\varepsilon n_k}$ then

$$\sigma^{n_k}(\mathbf{i})\in[;i\overbrace{jjj...j}^{\lfloor\frac{\varepsilon}{\ln m}n_k\rfloor}]$$

for some $i=0...m-1$, $j=0,m-1$. Therefore $\sigma^i(\mathbf{i})\notin[;1]\cup\cdots\cup[;m-2]$ for $n_k+1\leq i\leq\lfloor\frac{\varepsilon}{\ln m}n_k\rfloor+n_k+1$. We can apply Lemma 7.1 of [10] for the following set

$$\left\{\mathbf{i}:\exists(n_k)_{k\in\mathbb{N}}\ \forall k>0\ \sigma^i(\mathbf{i})\notin[;1]\cup\cdots\cup[;m-2];n_k+1\leq i\leq\lfloor\frac{\varepsilon}{\ln m}n_k\rfloor+n_k+1\right\}.$$

Therefore

$$\mu(\left\{\underline{x}\in\Lambda:\exists n_k\to\infty d(f^{n_k}(\underline{x}),N)\leq e^{-\varepsilon n_k}\right\})=0$$

□

The $m=2$ case was proved in [7, Lemma 5.1.3].

In the following we assume that $\mu=\nu\circ\pi'^{-1}$ is ergodic and regular hyperbolic.

Now we define the Lyapunov charts

Definition 3.1.3. *For a regular point $\underline{x}$ let $e_i(\underline{x})=\dim E^i(\underline{x})$. Let $\underline{y}=(y_1,...,y_{n+1})\in\mathbb{R}^{n+1}$, $\mid\underline{y}\mid=\max\mid y_i\mid$ and $R(\rho)=\left\{\underline{y}\in\mathbb{R}^n:\mid\underline{y}\mid<\rho\right\}$. We fix $\varepsilon>0$ small. Then for $\delta>0$ sufficiently small there exists a measurable function $r:D_\varepsilon\mapsto(1,\infty)$ with $r(f^{\pm1}\underline{x})\leq e^\delta r(\underline{x})$ and an embedding $\Phi_{\underline{x}}:R(r(\underline{x})^{-1})\mapsto M$ such that the following conditions hold:*

1. *$\Phi_{\underline{x}}(0)=\underline{x}$ and $D_0\Phi_{\underline{x}}$ maps $\{0\}\times\cdots\times\{0\}\times\mathbb{R}^{e_i}\times\{0\}\times\cdots\times\{0\}$ to $E^i(\underline{x})$*
2. *$\exp_{\underline{x}}^{-1}\circ\Phi_{\underline{x}}$ coincides with $D_0\Phi_{\underline{x}}$ on $R(r(\underline{x})^{-1})$*
3. *For $\widetilde{f}_{\underline{x}}=\Phi_{\underline{x}}^{-1}\circ f\circ\Phi_{\underline{x}}$ and $v\in\{0\}\times\cdots\times\{0\}\times\mathbb{R}^{e_i}\times\{0\}\times\cdots\times\{0\}$*

$$e^{\chi_i-\delta}\mid v\mid\leq\mid D_0\widetilde{f}_{\underline{x}}v\mid\leq e^{\chi_i+\delta}\mid v\mid$$

4. *The Lipsitz constants L satisfies*

$$L(\widetilde{f}_{\underline{x}} - D_0\widetilde{f}_{\underline{x}}) \leq \delta$$

$$L(\widetilde{f}_{\underline{x}}^{-1} - D_0\widetilde{f}_{\underline{x}}^{-1}) \leq \delta$$

$$L(D\widetilde{f}_{\underline{x}}) \leq r(\underline{x}), \quad L(D\widetilde{f}_{\underline{x}}^{-1}) \leq r(\underline{x})$$

5. *For all* $\underline{y}, \underline{y}' \in R(r(\underline{x})^{-1})$

$$C^{-1} d(\Phi_{\underline{x}}\underline{y}, \Phi_{\underline{x}}\underline{y}') \leq \mid \underline{y} - \underline{y}' \mid \leq r(\underline{x}) d(\Phi_{\underline{x}}\underline{y}, \Phi_{\underline{x}}\underline{y}')$$

The system of local charts $\{\Phi_{\underline{x}}\}$, $\underline{x}$ *a regular point, is called* ***Lyapunov chart system***

Lyapunov charts give control over stretching and contracting in the first step of iterating f while Lyapunov exponents are effective only asymptotically. A illustration of the action of Lyapunov charts can be found in Figure 8.

From [10, p. 4], [3, Part 1., Lemma 3.1] follows the next proposition.

Proposition 3.1.3. *It* μ *is regular hyperbolic invariant measure (that is* μ *is invariant and* $\mu(D_\varepsilon) = 1$*) then Lyapunov charts exist for a.e.* $\underline{x} \in D_\varepsilon$.

In our case there is one Lyapunov exponent which is positive.

Now we define the stable and unstable manifolds

$$W^u(\underline{x}) = \left\{ \underline{y} \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}(\underline{x}), f^{-n}(\underline{y})) \leq -\chi_1 \right\} \tag{23}$$

$$W^i(\underline{x}) = \left\{ \underline{y} \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{n}(\underline{x}), f^{n}(\underline{y})) \leq \chi_i \right\} \tag{24}$$

where $d(.,.)$ is the Euclidian metric in $M = [0,1]^{n+1}$ manifold. Evidently $W^{i+1}(\underline{x}) \subset W^i(\underline{x})$ for $i = 2, ..., n$.

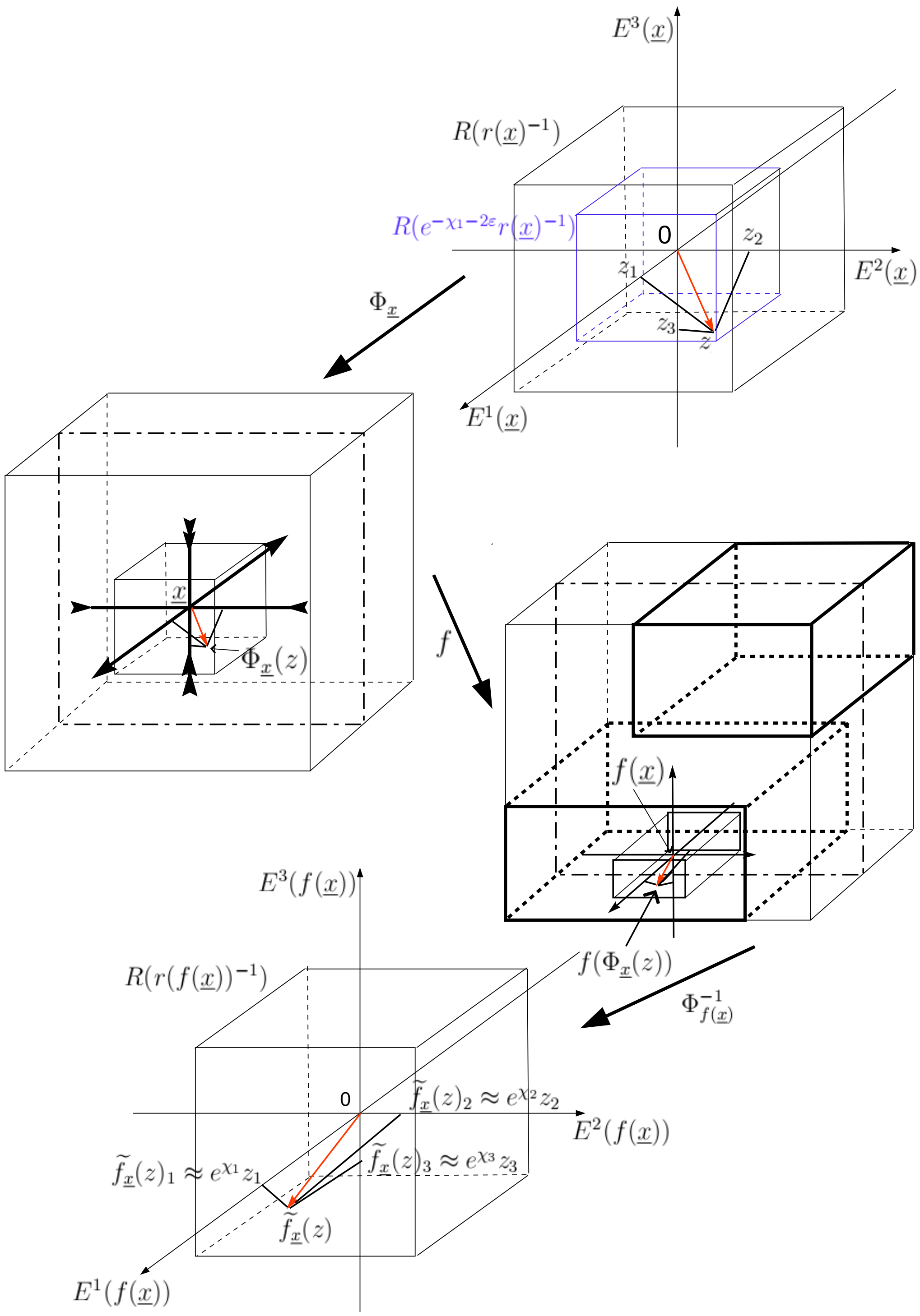

Figure 8: The operation of Lyapunov-charts

3.2 The construction of Lyapunov charts

In general it is very difficult to write down explicitly what the Lyapunov charts are. However, for the simplicity of our system, using [4, p. 536, Appendix], in this subsection we describe it precisely.

First, for every sufficiently small $\varepsilon > 0$ we construct a measurable function $C(\underline{x})$ such that

1. For every $\underline{x}$ regular points and $n \geq 0$
$$\left\|D_{\underline{x}} f^{-n} v\right\| \leq C(\underline{x}) e^{-(\chi_1 - \varepsilon/2)n} \|v\| \quad \text{for all } \ v \in E^1(\underline{x})$$
$$\left\|D_{\underline{x}} f^{n} v\right\| \leq C(\underline{x}) e^{-(\chi_j - \varepsilon/2)n} \|v\| \quad \text{for all } \ v \in E^j(\underline{x}) \ \text{ and } \ 2 \leq j \leq n+1 \tag{25}$$
2. $C(\underline{x}) \geq 1$
3. $C(f^{\pm 1}(\underline{x})) \leq e^{\varepsilon} C(\underline{x})$

We can write this C function explicitly. Namely, let

$$C_{1j}(\underline{x}) = \max\left\{1, \max_{n\geq 1}\left\{e^{-n\chi_j + \sum_{k=0}^{n-1} \ln a_{s_j}^{i_k} - n\varepsilon/2}\right\}, \max_{n\leq -1}\left\{e^{-n\chi_j - \sum_{k=n}^{-1} \ln a_{s_j}^{i_k} + n\varepsilon/2}\right\}\right\} \tag{26}$$

if $\underline{x} = \pi'(\mathbf{i})$. We assume that the empty sum is equal to zero. It is easy to see that

$$\begin{aligned} C_{1j}(f(\underline{x})) &\leq e^{\chi_j - \ln a_{s_j}^{i_0} + \varepsilon/2} C_{1j}(\underline{x}) \\ C_{1j}(f^{-1}(\underline{x})) &\leq e^{-\chi_j + \ln a_{s_j}^{i_{-1}} + \varepsilon/2} C_{1j}(\underline{x}) \end{aligned} \tag{27}$$

Namely,

$$C_{1j}(f(\underline{x})) = \max\left\{1, \max_{n\geq 1}\left\{e^{-n\chi_j + \sum_{k=0}^{n-1} \ln a_{s_j}^{i_k+1} - n\varepsilon/2}\right\}, \max_{n\leq -1}\left\{e^{-n\chi_j - \sum_{k=n}^{-1} \ln a_{s_j}^{i_k+1} + n\varepsilon/2}\right\}\right\} =$$

$$\max\left\{1, e^{\chi_j - \ln a_{s_j}^{i_0} + \varepsilon/2} \max_{n\geq 1}\left\{e^{-(n+1)\chi_j + \sum_{k=0}^{n} \ln a_{s_j}^{i_k} - (n+1)\varepsilon/2}\right\},\right.$$

$$\left. e^{\chi_j - \ln a_{s_j}^{i_0} - \varepsilon/2} \max_{n\leq -1}\left\{e^{-(n+1)\chi_j - \sum_{k=n+1}^{-1} \ln a_{s_j}^{i_k} + (n+1)\varepsilon/2}\right\}\right\} \leq$$

$$e^{\chi_j - \ln a_{s_j}^{i_0} + \varepsilon/2} \max\left\{e^{-\chi_j + \ln a_{s_j}^{i_0} - \varepsilon/2}, \max_{n\geq 2}\left\{e^{-n\chi_j + \sum_{k=0}^{n-1} \ln a_{s_j}^{i_k} - n\varepsilon/2}\right\}, \max_{n\leq 0}\left\{e^{-n\chi_j - \sum_{k=n}^{-1} \ln a_{s_j}^{i_k} + n\varepsilon/2}\right\}\right\}$$

$$= e^{\chi_j - \ln a_{s_j}^{i_0} + \varepsilon/2} C_{1j}(\underline{x})$$

And similarly:

$$C_{1j}(f^{-1}(\underline{x})) = \max\left\{1, \max_{n\geq 1}\left\{e^{-n\chi_j+\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k-1}-n\varepsilon/2}\right\}, \max_{n\leq -1}\left\{e^{-n\chi_j-\sum_{k=n}^{-1}\ln a_{s_j}^{i_k-1}+n\varepsilon/2}\right\}\right\} =$$

$$\max\left\{1, e^{-\chi_j+\ln a_{s_j}^{i_{-1}}-\varepsilon/2}\max_{n\geq 1}\left\{e^{-(n-1)\chi_j+\sum_{k=0}^{n-2}\ln a_{s_j}^{i_k}-(n-1)\varepsilon/2}\right\},\right.$$

$$\left. e^{-\chi_j+\ln a_{s_j}^{i_{-1}}+\varepsilon/2}\max_{n\leq -1}\left\{e^{-(n-1)\chi_j-\sum_{k=n-1}^{-1}\ln a_{s_j}^{i_k}+(n-1)\varepsilon/2}\right\}\right\} \leq$$

$$e^{-\chi_j+\ln a_{s_j}^{i_{-1}}+\varepsilon/2}\max\left\{e^{\chi_j-\ln a_{s_j}^{i_{-1}}-\varepsilon/2}, \max_{n\geq 0}\left\{e^{-n\chi_j+\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k}-n\varepsilon/2}\right\}, \max_{n\leq -2}\left\{e^{-n\chi_j-\sum_{k=n}^{-1}\ln a_{s_j}^{i_k}+n\varepsilon/2}\right\}\right\}$$

$$\leq e^{-\chi_j+\ln a_{s_j}^{i_{-1}}+\varepsilon/2}C_{1j}(\underline{x})$$

Moreover let

$$C_{2j}(\underline{x}) = 1 + \sum_{n=1}^{\infty} e^{n\chi_j-\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k}-n\varepsilon/2} + \sum_{k=-\infty}^{-1} e^{n\chi_j+\sum_{k=n}^{-1}\ln a_{s_j}^{i_k}+n\varepsilon/2}. \tag{28}$$

By similar argument like $C_{1j}(\underline{x})$ we can prove that

$$\begin{aligned} C_{2j}(f(\underline{x})) &\leq e^{-\chi_j+\ln a_{s_j}^{i_0}+\varepsilon/2}C_{2j}(\underline{x}) \\ C_{2j}(f^{-1}(\underline{x})) &\leq e^{\chi_j-\ln a_{s_j}^{i_{-1}}+\varepsilon/2}C_{2j}(\underline{x}) \end{aligned} \tag{29}$$

Therefore let $C(\underline{x})$ be the following function

$$C(\underline{x}) := \max_j \{C_{1j}(\underline{x}) \cdot C_{2j}(\underline{x})\} \tag{30}$$

The inequalities (27) and (29) imply that $C(f^{\pm 1}(\underline{x})) \leq e^{\varepsilon}C(\underline{x})$ and by the definition $C_{1j}(\underline{x})$, in (26), the property (25) is also true. This completes the construction of the function $C(\underline{x})$.

Continuing the construction of the Lyapunov chart we define $\Phi_{\underline{x}}$ and $r(\underline{x})$. First we introduce a new inner product $\langle\langle .,. \rangle\rangle'_{\underline{x}}$ on $T_{\underline{x}}M$ for every $\underline{x}$ regular points.

$$\langle\langle u, v\rangle\rangle'_{\underline{x}} = \begin{cases} \dfrac{\sum_{n=0}^{\infty}\langle D_{\underline{x}}f^{-n}u, D_{\underline{x}}f^{-n}v\rangle}{e^{-2n(\chi_1-\varepsilon)}} & \text{for } u, v \in E^1(\underline{x}) \\ \dfrac{\sum_{n=0}^{\infty}\langle D_{\underline{x}}f^{n}u, D_{\underline{x}}f^{n}v\rangle}{e^{2n(\chi_j+\varepsilon)}} & \text{for } u, v \in E^j(\underline{x}) \text{ and } 2 \leq j \leq n-1 \end{cases} \tag{31}$$

It is clear that if $\pi'(\mathbf{i}) = \underline{x}$, where $\mathbf{i} \in \Theta$ then $D_{\underline{x}} f^n u = e^{\sum_{k=0}^{n-1} \ln a_{s_j}^{i_k}} u$, where $u \in E^j(\underline{x})$, $2 \leq j \leq n+1$. Therefore

$$\langle\langle u, v \rangle\rangle'_{\underline{x}} = \langle u, v \rangle \sum_{n=0}^{\infty} e^{2n(-\chi_j - \varepsilon + \frac{1}{n}\sum_{k=0}^{n-1} \ln a_{s_j}^{i_k})}$$

if $u, v \in E^j(\underline{x})$. This sum is convergent for μ-a.e. $\underline{x}$, since μ is ergodic. It is easy to see that if $u, v \in E^1(\underline{x})$ then

$$\langle\langle u, v \rangle\rangle'_{\underline{x}} = \langle u, v \rangle \sum_{n=0}^{\infty} e^{-2n\varepsilon} = \langle u, v \rangle \frac{1}{1 - e^{-2\varepsilon}}$$

Let $L_{\underline{x}} : T_{\underline{x}} M \mapsto \mathbb{R}^n$ be a linear map satisfying

$$\langle L_{\underline{x}} u, L_{\underline{x}} v \rangle = \langle\langle u, v \rangle\rangle'_{\underline{x}}$$

for every $u, v \in T_{\underline{x}} M$. Then $L_{\underline{x}}$ is a diagonal matrix with elements:

$$L_{\underline{x}} = \begin{bmatrix} \sqrt{\sum_{k=0}^{\infty} e^{2k(-\chi_{S_1} - \varepsilon + \frac{1}{k}\sum_{j=0}^{k-1} \ln a_1^{i_j})}} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \sqrt{\sum_{k=0}^{\infty} e^{2k(-\chi_{S_n} - \varepsilon + \frac{1}{k}\sum_{j=0}^{k-1} \ln a_n^{i_j})}} & 0 \\ 0 & \cdots & 0 & \frac{1}{\sqrt{1-e^{-2\varepsilon}}} \end{bmatrix} \tag{32}$$

where S_i is the index of the Lyapunov-exponent in the ith coordinate. Setting

$$\Phi_{\underline{x}} = \exp_{\underline{x}} \circ L_{\underline{x}}^{-1} \tag{33}$$

Therefore

$$\Phi_{\underline{x}}(z) = L_{\underline{x}}^{-1} z + \underline{x}$$

which completes the construction of the Lyapunov chart. Now we check that each the conditions of Definition 3.1.3 hold. By the definition of $\Phi_{\underline{x}}$ in (33) the first and the second point of Definition 3.1.3 is obviously holds.

Since $E^i(\underline{x})$ subspaces of the tangent space $T_{\underline{x}} M = \oplus_{i=1}^{n+1} E^i(\underline{x})$ are orthogonal and $L_{\underline{x}}$ is a diagonal matrix, $\|v\|'_{\underline{x}} \geq \|v\|$ for every $v \in T_{\underline{x}} M$, where $\|.\|'_{\underline{x}}$

is the norm derived from $\langle\langle .,. \rangle\rangle'_{\underline{x}}$. From the first property of $C(\underline{x})$ function, follows immediately that if $v \in E^j(\underline{x})$ then

$$\|v\|'_{\underline{x}} \le C_0 C(\underline{x}) \|v\| \tag{34}$$

where $C_0 = \sqrt{2\sum_{i=0}^{\infty} e^{-\varepsilon i}}$. By similar arguments as in above it is easy to see that (34) satisfies for arbitrary $v \in T_{\underline{x}}M$.

Therefore if we choose

$$r_1(\underline{x}) = C_0 C(\underline{x}) \tag{35}$$

the 5th property of Definition 3.1.3 satisfies immediately with $r_1(\underline{x})$. We aim that

$$(r(\underline{x}))^{-1} \le \frac{d(\underline{x}, N)}{2\sqrt{1 - e^{-2\varepsilon}}}$$

In this case if $\underline{x} \in K_i$ then $\Phi_{\underline{x}}(z) \in K_i$ also for arbitrary $z \in R(r(\underline{x})^{-1})$. If $\underline{x} \in D_{\varepsilon,l}$ then $l^{-1} \le d(\underline{x}, N)$. Let $l(\underline{x})$ be the minimal l which satisfies that $\underline{x} \in D_{\varepsilon,l}$. Then

$$r(\underline{x}) := \max\left\{ r_1(\underline{x}), \frac{l(\underline{x})}{2\sqrt{1 - e^{-2\varepsilon}}} \right\}$$

and therefore the 5th property of Definition 3.1.3 holds also and by the construction of $D_{\varepsilon,l}$, which was defined in (21), $r(f^{\pm 1}(\underline{x})) \le e^{\varepsilon} r(\underline{x})$.

Since the derivatives of $\Phi_{\underline{x}}$ and f are diagonal matrices then the fourth item of Definition 3.1.3 is trivial.

We need only to check the third condition of Definition 3.1.3. To do so we note that for $z \in R(r(\underline{x})^{-1})$:

$$\widetilde{f}_{\underline{x}}(z) = \Phi^{-1}_{f(\underline{x})} \circ f \circ \Phi_{\underline{x}}(z) = L_{f(\underline{x})} \widetilde{A}_i L^{-1}_{\underline{x}} z$$

if $\underline{x} \in K_i$. If $v \in E^1(\underline{x})$ then $D_0 \widetilde{f}_{\underline{x}} v = mv$ clearly. In other cases, for $v \in E^j(\underline{x})$, we need only to prove that the diagonal elements, which correspond to the s_j coordinate, can be estimated by the following

$$(L_{f(\underline{x})} \widetilde{A}_i L^{-1}_{\underline{x}})_{s_j s_j} = \frac{\sqrt{\sum_{n=0}^{\infty} e^{2n(-\chi_j - \varepsilon + \frac{1}{n}\sum_{k=0}^{n-1} \ln a_{s_j}^{i_{k+1}})}}\, a_{s_j}^{i_0}}{\sqrt{\sum_{n=0}^{\infty} e^{2n(-\chi_j - \varepsilon + \frac{1}{n}\sum_{k=0}^{n-1} \ln a_{s_j}^{i_k})}}}$$

because $f(\pi'(\mathbf{i})) = \pi'(\sigma\mathbf{i})$. With simple transformations:

$$\frac{\sqrt{e^{2\chi_j+2\varepsilon}\sum_{n=1}^{\infty} e^{2n(-\chi_j-\varepsilon+\frac{1}{n}\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k})}}}{\sqrt{\sum_{n=0}^{\infty} e^{2n(-\chi_j-\varepsilon+\frac{1}{n}\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k})}}} \leq e^{\chi_j+\varepsilon}$$

Let

$$B = \sum_{n=0}^{\infty} e^{2n(-\chi_j-\varepsilon+\frac{1}{n}\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k})}$$

It is also true that

$$\frac{\sqrt{e^{2\chi_j+2\varepsilon}\sum_{n=1}^{\infty} e^{2n(-\chi_j-\varepsilon+\frac{1}{n}\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k})}}}{\sqrt{\sum_{n=0}^{\infty} e^{2n(-\chi_j-\varepsilon+\frac{1}{n}\sum_{k=0}^{n-1}\ln a_{s_j}^{i_k})}}} = \frac{\sqrt{e^{2\chi_j+2\varepsilon}(B-1)}}{\sqrt{B}}$$

We aim that

$$\frac{\sqrt{e^{2\chi_j+2\varepsilon}(B-1)}}{\sqrt{B}} \geq e^{\chi_j-\varepsilon} \tag{36}$$

With simple calculations it is equivalent to

$$B(1-e^{-4\varepsilon}) \geq 1$$

Since $B > 1 + e^{2(-\chi_j+\ln a_{s_j}^{i_0}-\varepsilon)}$ if $\varepsilon > 0$ sufficiently small then (36) satisfies. We expressed explicitly, what the Lyapunov charts are for our modell.

3.3 Partitions subordinated to the foliation

Definition 3.3.1. *A partition ξ is μ-measurable if and only if for μ-a.e. $\underline{x}$ there is a normalized measure $\mu_{\underline{x}}^{\xi}$ supported by the partition element $\xi(\underline{x})$ containing $\underline{x}$ such that for the sub-σ-algebra $\mathcal{B}_\xi$ consisting entirely of unions of atoms of the partition ξ and a measurable set A the function $\underline{x} \mapsto \mu_{\underline{x}}^{\xi}(A)$ is $\mathcal{B}_\xi$-measurable and $\mu(A) = \int \mu_{\underline{x}}^{\xi}(A)d\mu(\underline{x})$. The measures $\mu_{\underline{x}}^{\xi}$ are called* ***the conditional measures*** *of μ w. r. t. ξ. They are uniquely defined up to a set of measure 0. [4],[5],[10]*

Definition 3.3.2. *A μ-measurable partition ξ is* ***subordinate to the*** *W^i*-***foliation*** *if for μ-a.e. $\underline{x}$*

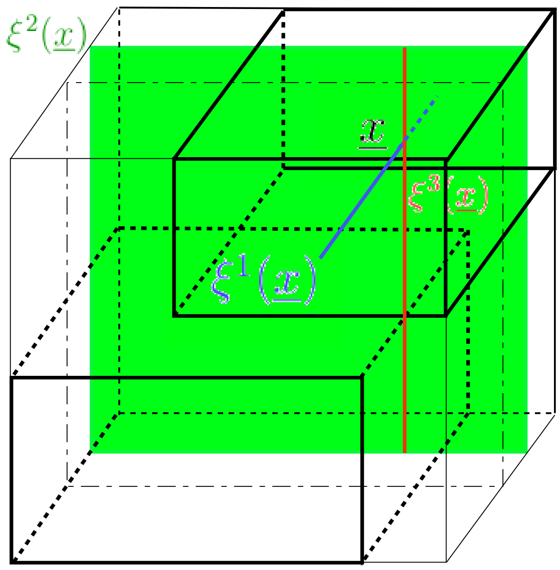

Figure 9: Subordinate partitions

1. $\xi(\underline{x}) \subset W^i(\underline{x})$
2. $\xi(\underline{x})$ *contains a neighborhood of* $\underline{x}$ *in* $W^i(\underline{x})$

For two partitions ξ, η we say that $\xi > \eta$ if for a.e. $\underline{x} \in M$ $\xi(\underline{x}) \subset \eta(\underline{x})$, and we say that a partition ξ is increasing (decreasing) if $\xi > f(\xi)$ ($\xi < f(\xi)$).

Proposition 3.3.1. *For* $1 \leq i \leq n+1$ *there exist measurable partitions* ξ^i *with the following properties:*

1. ξ^i *is subordinate to* $W^i(\underline{x})$
2. ξ^1 *is increasing and* ξ^i *are decreasing* $i = 2, ..., n+1$
3. $\xi^i > \xi^{i+1}$ *for* $i = 2, ..., n$
4. ξ^i *is generating - i. e.* $\bigvee_{n=0}^{\infty} f^{-n}(\xi^1)$ *or* $\bigvee_{n=0}^{\infty} f^n(\xi^i)$ *if* $i = 2, ..., n+1$ *is the partition into points.*

The proof of the existence of such partitions depends only on the existence of Lyapunov charts and can be found in [10, p. 6], [5, p. 554].

In our case it is easy to show such partitions. Namely,

$$\xi^1(\underline{x}) = \{x_1\} \times \cdots \times \{x_n\} \times [0,1] \tag{37}$$

and if $2 \leq i \leq n+1$ then

$$\xi^i(\underline{x}) = \prod_{j=1}^{n} H_j^i(\underline{x}) \times \{x_{n+1}\} \tag{38}$$

where

$$H_j^i(\underline{x}) = \begin{cases} [0,1] & \text{if } j \in \{s_i, ..., s_{n+1}\} \\ \{x_j\} & \text{else} \end{cases}$$

We remark that s_i is the coordinate of the ith Lyapunov-exponent and $\chi_1 > 0 > \chi_2 > \cdots > \chi_{n+1}$. Obviously, it is enough to define our partition for μ-a.e. point.

If we assume for the simplicity and for the better realization that $s_i = i-1$ then

$$\xi^i(\underline{x}) = \{x_1\} \times \cdots \times \{x_{i-2}\} \times [0,1]^{n+2-i} \times \{x_{n+1}\}$$

if $3 \leq i \leq n+1$ and

$$\xi^2(\underline{x}) = [0,1]^n \times \{x_{n+1}\}$$

A simple illustration of such partitions is Figure 9.

We define the pointwise entropy of the measure. Let $\left\{\mu^i_{\underline{x}}\right\}$, $1 \leq i \leq n+1$ be fixed versions of conditional measures associated to μ and ξ^i. For $\underline{x} \in M$ regular point $\gamma > 0$, $2 \leq i \leq n+1$ we define

$$\underline{h}_i(\underline{x}, \gamma, \xi^i) = \liminf_{n\to\infty} -\frac{1}{n} \log \mu^i_{\underline{x}} V^i(\underline{x}, n, \gamma) \tag{39}$$

$$\overline{h}_i(\underline{x}, \gamma, \xi^i) = \limsup_{n\to\infty} -\frac{1}{n} \log \mu^i_{\underline{x}} V^i(\underline{x}, n, \gamma) \tag{40}$$

with $V^i(\underline{x}, n, \gamma) = \left\{\underline{y} \in W^i(\underline{x}) : d^i(f^{-k}(\underline{x}), f^{-k}(\underline{y})) < \gamma, 0 \leq k \leq n\right\}$, where d^i is the Euclidean distance on W^i. We define also

$$\underline{h}_1(\underline{x}, \gamma, \xi^i) = \liminf_{n\to\infty} -\frac{1}{n} \log \mu^1_{\underline{x}} V^1(\underline{x}, n, \gamma) \tag{41}$$

$$\overline{h}_1(\underline{x},\gamma,\xi^i) = \limsup_{n\to\infty} -\frac{1}{n}\log \mu^1_{\underline{x}} V^1(\underline{x},n,\gamma) \tag{42}$$

with $V^1(\underline{x},n,\gamma) = \big\{\underline{y} \in W^1(\underline{x}) : d^1(f^k(\underline{x}), f^k(\underline{y})) < \gamma, 0 \le k \le n\big\}$.

In the following we interpret some propositions which were proved in [4] and [5] for C^2-diffeomorphism, but we constructed the Lyapunov charts of our model, therefore those proofs can be applied.

Proposition 3.3.2. *Then for μ-a.e. $\underline{x} \in M$*

$$\lim_{\gamma\to 0} \underline{h}_i(\underline{x},\gamma,\xi^i) = \lim_{\gamma\to 0} \overline{h}_i(\underline{x},\gamma,\xi^i) \equiv h_i(\underline{x},\xi^i)$$

Moreover $h_i(\underline{x},\xi^i)$ is μ-a.e. constant and independent of the choice of ξ^i.

The proof of this Proposition follows from Theorem 3.3.1.

We give a definition for the dimension of the measure along the stable and unstable directions. We consider for the ball $B^i(\underline{x},\gamma)$ in $W^i(\underline{x})$ centered at $\underline{x}$ of radius γ the quantities $1 \le i \le n+1$.

$$\underline{d}^i_\mu(\underline{x},\xi^i) = \liminf_{\gamma\to 0} \frac{\log \mu^i_{\underline{x}} B^i(\underline{x},\gamma)}{\log\gamma} \tag{43}$$

$$\overline{d}^i_\mu(\underline{x},\xi^i) = \limsup_{\gamma\to 0} \frac{\log \mu^i_{\underline{x}} B^i(\underline{x},\gamma)}{\log\gamma} \tag{44}$$

Proposition 3.3.3. *Then for μ-a.e. $\underline{x} \in M$*

$$\underline{d}^i_\mu(\underline{x},\xi^i) = \overline{d}^i_\mu(\underline{x},\xi^i) = d^i_\mu(\underline{x},\xi^i)$$

Moreover $d_i(\underline{x},\xi^i)$ is μ-a.e. constant and independent of the choice of ξ^i.

The proof of this Proposition follows also from Theorem 3.3.1

Theorem 3.3.1. Ledrappier-Young *With the assumptions and notations in above the following hold:*

1. $h_1 = \chi_1 d^1_\mu$

2. $h_k = -\chi_k d^k_\mu$

3. $h_i - h_{i+1} = -\chi_i(d_\mu^i - d_\mu^{i+1})$

4. $h_1 = h_2 = h_\mu(f)$

Here $h_\mu(f)$ is the entropy. Moreover for every ξ^i partitions subordinate to W^i-foliations $h_1 = h_1(f^{-1}, \xi^1) = H(\xi^1 \mid f\xi^1)$ and $h_i = h_i(f, \xi^i) = H(f\xi^i \mid \xi^i)$, $i = 2, ..., n+1$.

The proof of this Theorem coincide with the proof of Theorem C' in [5, p. 544]. It depends on the existence of Lyapunov charts and subordinate partitions to W^i-foliation, moreover on the existence of a partition $\mathcal{P}$ for every sufficiently small $\varepsilon' > 0$. We detail the proof with refer to [5].

Our first aim is $h_i = H(f\xi^i \mid \xi^i)$ for $i = 1, ..., n+1$. It can be found in [5, p. 555] (9.2) and (9.3) with the choose of partition $\mathcal{P}_\varepsilon^i$, $i = 1, ..., n+1$, $\varepsilon' > 0$:

$$\mathcal{P}_{\varepsilon'}^1(\underline{x}) = (0,1)^n \times (\frac{j_{n+1}}{2^{m+1}}, \frac{j_{n+1}+1}{2^{m+1}}) \tag{45}$$

$$\mathcal{P}_{\varepsilon'}^i(\underline{x}) = \prod_{k=1}^{n} (\frac{j_k}{2^{m+1}}, \frac{j_k+1}{2^{m+1}}) \times (0,1) \tag{46}$$

where $\frac{1}{2^{m+1}} < \varepsilon' \leq \frac{1}{2^m}$, $j_k = 0, ..., 2^{m+1}-1$ and $x_{n+1} \in (\frac{j_{n+1}}{2^{m+1}}, \frac{j_{n+1}+1}{2^{m+1}})$, $(x_1, ..., x_n) \in \prod_{k=1}^n (\frac{j_k}{2^{m+1}}, \frac{j_k+1}{2^{m+1}})$.

After that with the same partition $\mathcal{P}_{\varepsilon'}^i$ and ξ^i we use the (10.1) and (10.2) points and Section 11. of [5, p. 559-566].

4 A non-linear IFS with parameters

In this section we will study a special, non-conformal and non-linear iterated function scheme. Our purpose is to give a good parameter family, where the push-down measure is absolute continuous Lebesgue-almost everywhere. To prove it we will use the transversality condition, which was introduced by Karoly Simon [11] and [12]. Sze-Man Ngai and Yang Wang studied the absolute continuity in linear case [13]. Our result corresponds with it but in more general case.

4.1 Definitions

Let A_0 and A_1 two matrices from $\mathfrak{L}(\mathbb{R}^2)$, which is the set of the linear maps on $\mathbb{R}^2$. We assume in the following, that $\det(A_i) > 0$ for every $i = 0, 1$. Denote the four quadrants of the real plane Q_1, Q_2, Q_3, Q_4, namely

$$\begin{aligned}
Q_1 &= \left\{(x,y)^T : x \geq 0, y \geq 0\right\} \setminus \left\{(0,0)^T\right\} \\
Q_2 &= \left\{(x,y)^T : x \leq 0, y \geq 0\right\} \setminus \left\{(0,0)^T\right\} \\
Q_3 &= \left\{(x,y)^T : x \leq 0, y \leq 0\right\} \setminus \left\{(0,0)^T\right\} \\
Q_4 &= \left\{(x,y)^T : x \geq 0, y \leq 0\right\} \setminus \left\{(0,0)^T\right\}.
\end{aligned}$$

Proposition 4.1.1. *The following five expressions are equivalent*

1. $A_i^{-1} Q_2 \subset \mathrm{int} Q_2$
2. $A_i^{-1} Q_4 \subset \mathrm{int} Q_4$
3. $A_i Q_1 \subset \mathrm{int} Q_1$
4. $A_i Q_3 \subset \mathrm{int} Q_3$
5. *A_i has strictly positive elements*

Proof. We proof the equivalence of 1. and 5., and the equivalence of 2. and 5., the others are similar.

We suppose 1., then indirectly we assume that 5. is not true. If 5 is not true then there exists an element of the matrix which is non-positive. If

$$A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$$

then

$$A_i^{-1} = \frac{1}{\det A_i} \begin{bmatrix} d_i & -b_i \\ -c_i & a_i \end{bmatrix}.$$

Then for every $\underline{w} \in Q_2$

$$A_i^{-1}\underline{w} = \frac{1}{\det A_i} \begin{pmatrix} d_i w_1 - b_i w_2 \\ -c_i w_1 + a_i w_2 \end{pmatrix}$$

Our assumption is that $d_i w_1 - b_i w_2 < 0$ and $-c_i w_1 + a_i w_2 > 0$. If some of the elements of A_i, for example $d_i < 0$, is negative then the adequate coefficient of this element, which can be w_1 or w_2, tends to infinity or minus infinity, in our case $w_1 \to -\infty$, then there is a contradiction. If some of the elements of A_i is zero then we can choose $\underline{w}$ that $A_i^{-1}\underline{w} \notin \mathrm{int}Q_2$. Therefore A_i must have strictly positive elements.

Conversely, we suppose that 5. is true. In this case for every $\underline{w} = \binom{w_1}{w_2} \in Q_2$, where $w_1 \leq 0, w_2 \geq 0$, but at least one of the inequalities holds strictly, we have $d_i w_1 - b_i w_2 < 0$ and $-c_i w_1 + a_i w_2 > 0$, which was to be proved.

Now we suppose that 2. is true, then for every $\underline{w} \in Q_1$, where $w_1 \geq 0, w_2 \geq 0$

$$A_i\underline{w} = \begin{pmatrix} a_i w_1 + b_i w_2 \\ c_i w_1 + d_i w_2 \end{pmatrix}$$

and our assumption is $a_i w_1 + b_i w_2 > 0$ and $c_i w_1 + d_i w_2 > 0$. Similarly, if some of the elements of A_i, for example $d_i < 0$, is negative then the adequate coefficient of this element, which can be w_1 or w_2, tends to infinity, in our case $w_2 \to \infty$, then there is a contradiction. If some of the elements is zero then we can choose $\underline{w}$ that $A_i\underline{w} \notin \mathrm{int}Q_1$. Therefore A_i has positive elements.

Conversely, if 5. is true then the elements of $A_i\underline{w}$ are strictly positive, while at least one of the elements of $\underline{w}$ is strictly positive. □

In the following we assume that A_i has strictly positive elements. Now we define our iterated function scheme.

Let $\|\underline{x}\|_1 = |x|+|y|$ the norm in $\mathbb{R}^2$, $\underline{x} = (x,y)^T$. Let $B_1 = \{\underline{x} \in \mathbb{R}^2 : \|\underline{x}\|_1 = 1\}$, and $B_1^+ = \left\{\underline{x} \in B_1 : \underline{x} = (x,y)^T, x \geq 0, y \geq 0\right\}$. We define a function.

Definition 4.1.1. *For a matrix $S \in \mathfrak{L}(\mathbb{R}^2)$ let ψ_S be the following map*

$$\psi_S(\underline{x}) = \frac{1}{\|S\underline{x}\|_1} S\underline{x}$$

Then $\psi_S : B_1 \mapsto B_1$

Lemma 4.1.1. *For every $S_1, S_2, ..., S_n \in \mathfrak{L}(\mathbb{R}^2)$ matrices $\psi_{S_1 S_2 \cdots S_n} = \psi_{S_1} \circ \psi_{S_2} \circ \cdots \circ \psi_{S_n}$ and if a matrix $S \in \mathfrak{L}(\mathbb{R}^2)$ is invertible, ψ_S is also invertible on B_1 and $\psi_S^{-1} \equiv \psi_{S^{-1}}$.*

We do not notify the proof of this lemma, because it is very simple.

In the following we use the notation $A_{\mathbf{i}} = A_{i_1} \cdots A_{i_n}$ for every $\mathbf{i} \in \{0,1\}^n$ and $n \geq 1$ whole number. With the above assumptions $\psi_{A_0}, \psi_{A_1} : B_1^+ \mapsto B_1^+$. We can restrict these two functions into the axis x, let these functions $g_0, g_1 : [0,1] \mapsto [0,1]$. Then

$$g_0(x) = \frac{a_0 x + b_0(1-x)}{a_0 x + b_0(1-x) + c_0 x + d_0(1-x)}$$
$$g_1(x) = \frac{a_1 x + b_1(1-x)}{a_1 x + b_1(1-x) + c_1 x + d_1(1-x)}$$

Besides the hypotheses above, we assume that g_0 and g_1 are contractions, which means that the derivatives of these functions are less than 1, and there are overlap, namely $g_0((0,1)) \cap g_1((0,1)) \neq \emptyset$. It is easy to see that

$$g_0'(x) = \frac{\det A_0}{(a_0 x + b_0(1-x) + c_0 x + d_0(1-x))^2}$$
$$g_1'(x) = \frac{\det A_1}{(a_1 x + b_1(1-x) + c_1 x + d_1(1-x))^2}$$

These functions are monotone increasing or monotone decreasing on $(0,1)$, therefore if $\sup_{x \in (0,1)} g_0'(x) < 1$ and $\sup_{x \in (0,1)} g_1'(x) < 1$ then

$$\frac{\det A_i}{(a_i + c_i)^2} < 1 \quad \text{and} \quad \frac{\det A_i}{(b_i + d_i)^2} < 1$$

for every $i = 0, 1$. This implies that ψ_{A_0}, ψ_{A_1} are contractions, too. There exist two fix-points $\underline{x}_0, \underline{x}_1$ of ψ_{A_0} and ψ_{A_1}, and they are the eigenvectors of the matrices A_0, A_1 with strictly positive coordinates. Without loss of generality we can assume that $\underline{x}_0$ is the northern vector, which means that the first coordinate of $\underline{x}_0$ is less than the first coordinate of $\underline{x}_1$ ($\underline{x}_0, \underline{x}_1 \in B_1^+$). Moreover let us observe that for every $c > 0$ $\psi_{cA_i} = \psi_{A_i}$ for every i.

Let $S = [\underline{x}_1 \underline{x}_0]$ then $S^{-1}A_0S = \widetilde{A}_0$ and $S^{-1}A_1S = \widetilde{A}_1$, where $\widetilde{A}_0$ is a lower triangular matrix and $\widetilde{A}_1$ is an upper triangular matrix. It is easy to see that $\psi_{\widetilde{A}_0}\binom{0}{1} = \binom{0}{1}$ and $\psi_{\widetilde{A}_1}\binom{1}{0} = \binom{1}{0}$. For the sake of simplicity and in view of lemma 4.1.1 in the following we will study the matrices $\widetilde{A}_0, \widetilde{A}_1$. These matrices do not satisfy the condition 5., but they have non-negative elements and map B_1^+ into itself.

4.2 Transversality condition and absolute continuity

From the previous section if follows, that we can suppose that our two matrices are in the following form:

$$A_0 = \begin{bmatrix} a & 0 \\ 1-a & d_0 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} d_1 & b \\ 0 & 1-b \end{bmatrix}$$

where $0 < a < 1$, $0 < b < 1$ and $d_0, d_1 \in \mathbb{R}^+$. In this case our restricted functions to x-axis can be written as

$$g_0(x) = \frac{ax}{x + d_0(1-x)} \quad \text{and} \quad g_1(x) = \frac{d_1x + b(1-x)}{d_1x + (1-x)}.$$

Denote $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ the following function:

$$\phi(x, y) = \frac{x}{(1-y)x + y}$$

Then the functions g_0, g_1 can be expressed by ϕ.

$$g_0(x) = a\phi(x, d_0) \quad \text{and} \quad g_1(x) = 1 + (b-1)\phi(1-x, d_1) \tag{47}$$

Lemma 4.2.1. *For every $x \in (0, 1)$ and every $y \in \mathbb{R}^+$, $\phi'_x(x, y) \geq 0$, $\inf_{x\in(0,1)} \phi'_x(x, y) = \min\{y, 1/y\}$, $\|\phi'_x(., y)\| = \sup_{x\in(0,1)} |\phi'_x(x, y)| = \max\{y, 1/y\}$.*

Moreover $\phi(0,y) = 0$, $\phi(1,y) = 1$ and $\phi(x,1) = x$ for every $x \in [0,1]$ and $y \in \mathbb{R}^+$. If $\phi^{-1}(x,y)$ denote the inverse in the first variable for fixed y, then $\phi^{-1}(x,y) = \phi(x,1/y)$.

The proof of this lemma is trivial.

Now we define the natural projection and transversality condition. Let $\sum = \{0,1\}^{\mathbb{N}}$ and $\sum^* = \{0,1\}^*$. For every $\mathbf{i} = (i_1...i_n) \in \sum^*$ let

$$g_{\mathbf{i}} := g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_n}.$$

Definition 4.2.1. *Let $\pi_{\underline{t}} : \sum \mapsto [0,1]$ with parameters $\underline{t} = (a,b,d_0,d_1)$ the following function*

$$\pi_{\underline{t}}(\mathbf{i}) = \lim_{n\to\infty} g_{\mathbf{i}(n)}(0), \tag{48}$$

*where $\mathbf{i}(n)$ denote the first n elements of $\mathbf{i}$. We call $\pi_{\underline{t}}(\mathbf{i})$ the **natural projection**.*

It is easy to see that for every $\mathbf{i} \in \sum^*$ $g_{\mathbf{i}}$ is $C^{1+\varepsilon}$ in parameters $\underline{t} = (a,b,d_0,d_1) \in \mathbb{R}^4$, therefore for every $\mathbf{i} \in \sum$ the function $\pi_{\underline{t}}(\mathbf{i})$ is $C^{1+\varepsilon}$ in $\underline{t}$ also.

We would like to give an open set U of parameters $\underline{t} = (a,b,d_0,d_1)$, where the g_0, g_1 IFS has absolute continuous self-similar measure for Lebesgue-almost every $\underline{t} \in U$.

Let U_1 be the following open set of parameters

$$U_1 = \left\{(a,b,d_0,d_1) : b < a, a\max\left\{d_0, \frac{1}{d_0}\right\} < 1, (1-b)\max\left\{d_1, \frac{1}{d_1}\right\} < 1\right\} \tag{49}$$

Here, in view of lemma 4.2.1, U_1 is the set of parameters, where g_0 and g_1 overlap, namely $g_0([0,1]) \cap g_1([0,1]) \neq \emptyset$ and they are contractions. Therefore $\pi_{\underline{t}}$ is not one-to-one mapping.

Definition 4.2.2. *We say that the **transversality condition** holds on an open set $U \subset \mathbb{R}^4$ of the parameters, if there exists a constant C_1 such that for every $\mathbf{i}$ and $\mathbf{j} \in \sum$ with $i_1 \neq j_1$,*

$$\mathcal{L}_4(\underline{t} \in U : |\pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})| \leq r) \leq C_1 r \text{ for all } r > 0, \tag{50}$$

where $\mathcal{L}_4$ is the 4-dimensional Lebesgue-measure.

This definition is equivalent with [11, p. 448].

Before we prove the absolute continuity, we want to find an open set U where the IFS $\{g_0, g_1\}$ satisfies the transversality condition. Let $[i_1 i_2 ... i_n] = \{\mathbf{i} \in \sum : \mathbf{i}(n) = (i_1 i_2 ... i_n)\}$ the cylinder sets. We can prove a lemma, which helps the proof of transversality condition.

Lemma 4.2.2. *Suppose that $\underline{t} \in U_1$, moreover $a\phi(a, d_0) < b$ and $1 + (b - 1)\phi(1 - b, d_1) > a$. For every $\mathbf{i}, \mathbf{j} \in \sum$ with $i_1 \neq j_1$ if $\pi_{\underline{t}}(\mathbf{i}) = \pi_{\underline{t}}(\mathbf{j})$ then $i_2 \neq j_2$, too. In other words $\pi_{\underline{t}}(\mathbf{i}) = \pi_{\underline{t}}(\mathbf{j})$ implies that $\mathbf{i} \in [01]$ and $\mathbf{j} \in [10]$.*

Proof. To prove this lemma first we observe that in our case $g_0([0,1]) \cap g_1([0,1]) = [b, a]$, therefore if $i_1 \neq j_1$ and $\pi_t(\mathbf{i}) = \pi_t(\mathbf{j})$ then $\pi_t(\mathbf{i}) = \pi_t(\mathbf{j}) \in [b, a]$.

It is easy to see, that $g_0(g_0(1)) = a\phi(a, d_0)$ and if $g_0(g_0(1)) < b$ then $\pi_t([00]) \cap [b, a] = \emptyset$. It is also true that $g_1(g_1(0)) = 1 + (b - 1)\phi(1 - b, d_1)$ and if $g_1(g_1(0)) > a$ then $\pi_t([11]) \cap [b, a] = \emptyset$. These two previous statements complete the the proof of lemma. $\square$

Let U_2 be the following set of parameters:

$$U_2 = \{(a, b, d_0, d_1) : a\phi(a, d_0) < b, 1 + (b - 1)\phi(1 - b, d_1) > a\}. \tag{51}$$

On account of [11, p. 471] lemma 7.3, and [12, p. 5157] formula (5.1) the following lemma is true.

Lemma 4.2.3. *Assume that there exits an open set $U \subset \mathbb{R}^4$ such that for every $\mathbf{i}, \mathbf{j} \in \sum$ with $i_1 \neq j_1$ we have*

$$\|\nabla(\pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j}))\| > 0 \text{ whenever } \quad \pi_{\underline{t}}(\mathbf{i}) = \pi_{\underline{t}}(\mathbf{j}), \tag{52}$$

where ∇ denotes the gradient with respect to the parameters $\underline{t}$, then $\{g_0, g_1\}$ satisfies the transversality condition on U.

Finally, we can give the open set U, where $\{g_0, g_1\}$ satisfies the transversality condition.

Theorem 4.2.1. *Let $U_3, U_4 \subset \mathbb{R}^4$ the following sets:*

$$U_3 = \Bigg\{(a,b,d_0,d_1) \in \mathbb{R}^4 : \frac{(1-b)\phi(\frac{1-a}{1-b}, \frac{1}{d_1}) \min\left\{d_1, \frac{1}{d_1}\right\} - a}{a} > \frac{a(1-b)\max\left\{d_0, \frac{1}{d_0}\right\}\max\left\{d_1, \frac{1}{d_1}\right\}}{1 - a\max\left\{d_0, \frac{1}{d_0}\right\}}\Bigg\}$$

$$U_4 = \Bigg\{(a,b,d_0,d_1) \in \mathbb{R}^4 : \frac{1-b+a\max\left\{d_0, \frac{1}{d_0}\right\}\left(\phi(\frac{b}{a}, \frac{1}{d_0}) - 1\right)}{1-b} > \frac{a(1-b)\max\left\{d_0, \frac{1}{d_0}\right\}\max\left\{d_1, \frac{1}{d_1}\right\}}{1 - (1-b)\max\left\{d_1, \frac{1}{d_1}\right\}}\Bigg\}$$

Then on

$$U = U_1 \cap U_2 \cap (U_3 \cup U_4) \tag{53}$$

the IFS $\{g_0, g_1\}$ satisfies the transversality condition.

We recall that U_1, defined in (49), guarantees the overlap and contraction, moreover U_2, defined in (51), guarantees that $\underline{t}$ satisfies the assumptions of lemma 4.2.2.

Proof. We begin the proof by giving an upper and lower bound for $\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i})$ and $\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{i})$ for every $\mathbf{i} \in \sum$.

Let $\mathbf{i} \in \sum$ arbitrary and fixed. Moreover let $\lambda = \max\{\|g_0'\|, \|g_1'\|\}$. Here $0 < \lambda < 1$, because $\underline{t} \in U_1$. It immediate follows from chain rule that $\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i})$ and $\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{i})$ is less than or equal to the sum of different powers of λ. So

$$\left\|\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i})\right\|; \left\|\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{i})\right\| \leq \frac{1}{1-\lambda}$$

It is easy to see that $\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i}) \geq 0$ and $\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{i}) \geq 0$ hold. Namely, by Lemma 4.2.1, we have $g_1'(x) \geq 0$, $g_0'(x) \geq 0$, $\frac{\partial}{\partial a}g_0(x) \geq 0$ and $\frac{\partial}{\partial b}g_1(x) \geq 0$ for every $x \in [0,1]$.

Let n be the place of the first 0 element of $\mathbf{i}$, then

$$\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i}) = \frac{\partial}{\partial a}g_{\mathbf{i}(n-1)}(a\phi(\pi_{\underline{t}}(\sigma^n\mathbf{i}), d_0)) =$$
$$g_{\mathbf{i}(n-1)}'(a\phi(\pi_{\underline{t}}(\sigma^n\mathbf{i}), d_0)) \cdot \left(\phi(\pi_{\underline{t}}(\sigma^n\mathbf{i}), d_0) + a\phi_x'(\pi_{\underline{t}}(\sigma^n\mathbf{i}), d_0) \cdot \frac{\partial}{\partial a}\pi_{\underline{t}}(\sigma^n\mathbf{i})\right).$$

Therefore, using that $0 \leq \phi(x,y) \leq 1$, $0 \leq g_i'(x) \leq 1$ and $\phi_x'(x,y) \leq \max\{y, 1/y\}$ hold for every $x \in [0,1], y \in \mathbb{R}^+$ by lemma 4.2.1, we have

$$\left\| \frac{\partial}{\partial a} \pi_{\underline{t}}(\mathbf{i}) \right\| \leq 1 + a \max\left\{d_0, \frac{1}{d_0}\right\} \left\| \frac{\partial}{\partial a} \pi_{\underline{t}}(\sigma^n \mathbf{i}) \right\|.$$

Proceeding inductively we see that

$$\left\| \frac{\partial}{\partial a} \pi_{\underline{t}}(\mathbf{i}) \right\| \leq 1 + a \max\left\{d_0, \frac{1}{d_0}\right\} + \left(a \max\left\{d_0, \frac{1}{d_0}\right\}\right)^2 + \cdots = \frac{1}{1 - a \max\left\{d_0, \frac{1}{d_0}\right\}} \tag{54}$$

since $a \max\left\{d_0, \frac{1}{d_0}\right\} < 1$ for every $\underline{t} \in U$ and $\pi_{\underline{t}}(\mathbf{i})$ is bounded. By similar arguments the upper bound for $\frac{\partial}{\partial b} \pi_{\underline{t}}(\mathbf{i})$ is

$$\left\| \frac{\partial}{\partial b} \pi_{\underline{t}}(\mathbf{i}) \right\| \leq \frac{1}{1 - (1-b) \max\left\{d_1, \frac{1}{d_1}\right\}}. \tag{55}$$

Let $\underline{t} \in U$ and $\mathbf{i}, \mathbf{j} \in \sum$ with the following properties, $i_1 \neq j_1$ and $\pi_{\underline{t}}(\mathbf{i}) = \pi_{\underline{t}}(\mathbf{j})$. By Lemma 4.2.2, without loss of generality we can assume that $\mathbf{i} \in [01]$ and $\mathbf{j} \in [10]$.

$$\begin{aligned}
&\frac{\partial}{\partial a} \pi_{\underline{t}}(\mathbf{j}) - \frac{\partial}{\partial a} \pi_{\underline{t}}(\mathbf{i}) = \frac{\partial}{\partial a} \big(1 + (b-1)\phi(1 - a\phi(\pi_{\underline{t}}(\sigma^2 \mathbf{j}), d_0), d_1)\big) - \\
&- \frac{\partial}{\partial a} \big(a\phi(1 + (b-1)\phi(1 - \pi_{\underline{t}}(\sigma^2 \mathbf{i}), d_1), d_0)\big) = \\
&(1-b)\phi_x'(1 - \pi_{\underline{t}}(\sigma \mathbf{j}), d_1) \left(\phi(\pi_{\underline{t}}(\sigma^2 \mathbf{j}), d_0) + a\phi_x'(\pi_{\underline{t}}(\sigma^2 \mathbf{j}), d_0) \frac{\partial}{\partial a} \pi_{\underline{t}}(\sigma^2 \mathbf{j}) \right) - \\
&- \left(\phi(\pi_{\underline{t}}(\sigma \mathbf{i}), d_0) + a\phi_x'(\pi_{\underline{t}}(\sigma \mathbf{i}), d_0)(1-b)\phi_x'(1 - \pi_{\underline{t}}(\sigma^2 \mathbf{i}), d_1) \frac{\partial}{\partial a} \pi_{\underline{t}}(\sigma^2 \mathbf{i}) \right) = \\
&= (1-b)\phi_x'(1 - \pi_{\underline{t}}(\sigma \mathbf{j}), d_1) \frac{\pi_{\underline{t}}(\sigma \mathbf{j})}{a} - \frac{\pi_{\underline{t}}(\mathbf{i})}{a} + \\
&+ a(1-b)\phi_x'(1 - \pi_{\underline{t}}(\sigma \mathbf{j}), d_1)\phi_x'(\pi_{\underline{t}}(\sigma^2 \mathbf{j}), d_0) \frac{\partial}{\partial a} \pi_{\underline{t}}(\sigma^2 \mathbf{j}) - \\
&- a(1-b)\phi_x'(\pi_{\underline{t}}(\sigma \mathbf{i}), d_0)\phi_x'(1 - \pi_{\underline{t}}(\sigma^2 \mathbf{i}), d_1) \frac{\partial}{\partial a} \pi_{\underline{t}}(\sigma^2 \mathbf{i})
\end{aligned}$$

So, we have obtained that

$$\begin{aligned}\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{j})-\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i}) = (1-b)\phi_x'(1-\pi_{\underline{t}}(\sigma\mathbf{j}),d_1)\frac{\pi_{\underline{t}}(\sigma\mathbf{j})}{a}-\frac{\pi_{\underline{t}}(\mathbf{i})}{a}+ \\ + a(1-b)\phi_x'(1-\pi_{\underline{t}}(\sigma\mathbf{j}),d_1)\phi_x'(\pi_{\underline{t}}(\sigma^2\mathbf{j}),d_0)\frac{\partial}{\partial a}\pi_{\underline{t}}(\sigma^2\mathbf{j})- \\ - a(1-b)\phi_x'(\pi_{\underline{t}}(\sigma\mathbf{i}),d_0)\phi_x'(1-\pi_{\underline{t}}(\sigma^2\mathbf{i}),d_1)\frac{\partial}{\partial a}\pi_{\underline{t}}(\sigma^2\mathbf{i})\end{aligned} \tag{56}$$

We assumed that $\mathbf{j}\in[10]$. This follows that $\pi_{\underline{t}}(\sigma\mathbf{i}) = g_1^{-1}(\pi_{\underline{t}}(\mathbf{i}))$. By the definition of g_1 we have $g_1^{-1}(x) = \phi^{-1}(\frac{1-x}{1-b},d_1)$. From the last formula of lemma 4.2.1 we have $\pi_{\underline{t}}(\sigma\mathbf{i}) = g_1^{-1}(\pi_{\underline{t}}(\mathbf{i})) = \phi(\frac{1-\pi_{\underline{t}}(\mathbf{i})}{1-b},\frac{1}{d_1})$. We substitute this into the first line of (56). We can throw away the second line of (56) and we apply (54) in the third line of (56) to get:

$$\begin{aligned}\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{j})-\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i}) \geq \frac{(1-b)\phi_x'(1-\pi_{\underline{t}}(\sigma\mathbf{j}),d_1)\phi((1-\pi_{\underline{t}}(\mathbf{j}))/(1-b),d_1^{-1})-\pi_{\underline{t}}(\mathbf{i})}{a} \\ -\frac{a(1-b)\phi_x'(\pi_{\underline{t}}(\sigma\mathbf{i}),d_0)\phi_x'(1-\pi_{\underline{t}}(\sigma^2\mathbf{i}),d_1)}{1-a\max\left\{d_0,\frac{1}{d_0}\right\}}\end{aligned} \tag{57}$$

Now we use that $\phi_x'(x,d_1)\geq\min\left\{d_1,\frac{1}{d_1}\right\}$ and that ϕ is monotone increasing so $\phi(\frac{1-\pi_{\underline{t}}(\mathbf{i})}{1-b},d_1^{-1})\geq\phi(\frac{1-a}{1-b},d_1^{-1})$ Further we use for $y=d_0,d_1$ that $\phi_x'(x,y)\leq\max\{y,y^{-1}\}$ for every $x\in[0,1]$. In this way we get

$$\begin{aligned}\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{j})-\frac{\partial}{\partial a}\pi_{\underline{t}}(\mathbf{i}) \geq \\ \frac{(1-b)\min\left\{d_1,d_1^{-1}\right\}\phi(\frac{1-a}{1-b},d_1^{-1})-a}{a}-\frac{a(1-b)\max\left\{d_0,d_0^{-1}\right\}\max\left\{d_1,d_1^{-1}\right\}}{1-a\max\left\{d_0,d_0^{-1}\right\}}\end{aligned} \tag{58}$$

Since $\underline{t}\in U_3$ the right hand side of (58) is positive, the transversality condition holds. In the same way we can prove that:

$$\begin{aligned}\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{j})-\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{i}) \geq \frac{1-\pi_{\underline{t}}(\mathbf{j})+a\max\left\{d_0,d_0^{-1}\right\}\left(\phi(\pi_{\underline{t}}(\mathbf{i})/a,d_0^{-1})-1\right)}{1-b} \\ -\frac{a(1-b)\max\left\{d_0,d_0^{-1}\right\}\max\left\{d_0,d_0^{-1}\right\}}{1-(1-b)\max\left\{d_1,d_1^{-1}\right\}}\end{aligned} \tag{59}$$

We remind the reader that by our assumption

$$\pi_{\underline{t}}(\mathbf{j}) = \pi_{\underline{t}}(\mathbf{i}) \tag{60}$$

Let

$$h(z) = \frac{1 - z + a \max\left\{d_0, \frac{1}{d_0}\right\} \cdot \left(\phi(\frac{z}{a}, \frac{1}{d_0}) - 1\right)}{1 - b}.$$

Further, let

$$A = \frac{a(1-b)\max\left\{d_0, d_0^{-1}\right\}\max\left\{d_0, d_0^{-1}\right\}}{1 - (1-b)\max\left\{d_1, d_1^{-1}\right\}}.$$

Note that (59) is equivalent to

$$\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{j}) - \frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{i}) \geq h(\pi_{\underline{t}}(\mathbf{i})) - A \tag{61}$$

Our claim is to prove that

$$\frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{j}) - \frac{\partial}{\partial b}\pi_{\underline{t}}(\mathbf{i}) \geq h(b) - A$$

By (61) to see this we have to verify only that

$$h(z) \geq h(b) \quad \text{for every} \quad z \in [b, a]. \tag{62}$$

Since $\pi_{\underline{t}}(\mathbf{j}) = \pi_{\underline{t}}(\mathbf{i}) \in [b, a]$. Using that $\phi'_x(\frac{z}{a}, d_0^{-1}) \geq \min\left\{d_0, d_0^{-1}\right\}$ by differentiation of $h(z)$ we get immediately that (62) holds. Therefore the transversality condition holds by lemma 4.2.3 and by $\underline{t} \in U_4$, which is equivalent to $h(b) - A > 0$.

□

In the case $d_0 = d_1 = 1$, which is the linear case by lemma 4.2.1, the transversality domain U seems like in Figure 10.

We can represent the following hyper-planes: $d_0 = d_1 = d$ Figure 11, $d_0 = 1, d_1 = d$ Figure 12, and $d_0 = d, d_1 = 1$ Figure 13. The last two cases are if one of the function is linear.

Let μ be a shift-invariant ergodic Borel probability measure on $\sum$ with positive entropy. The definition of entropy, denote h_μ can be found in [9]. If

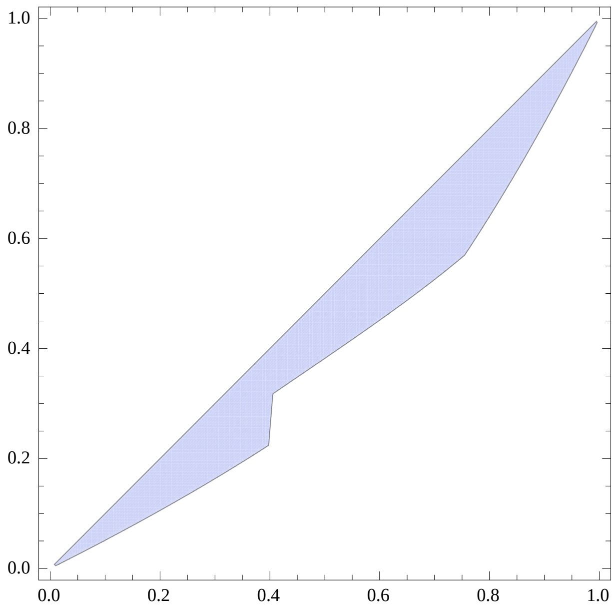

Figure 10: Transversality domain in linear case

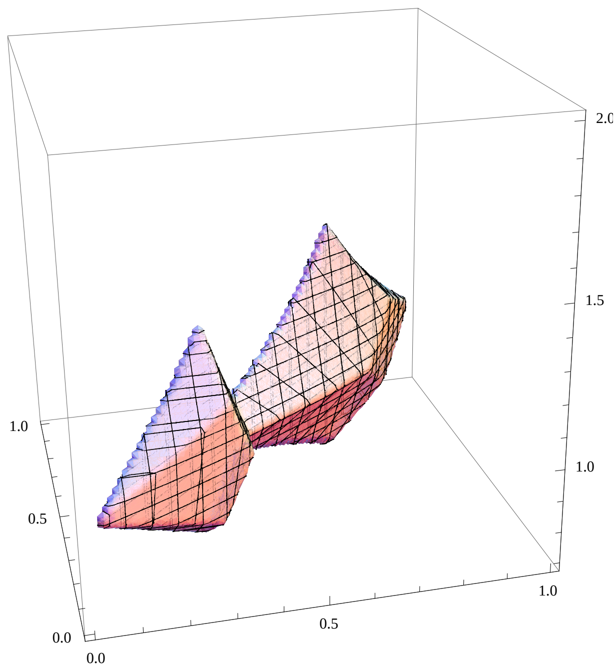

Figure 11: Transversality domain of $d_0 = d_1$ hyperplane

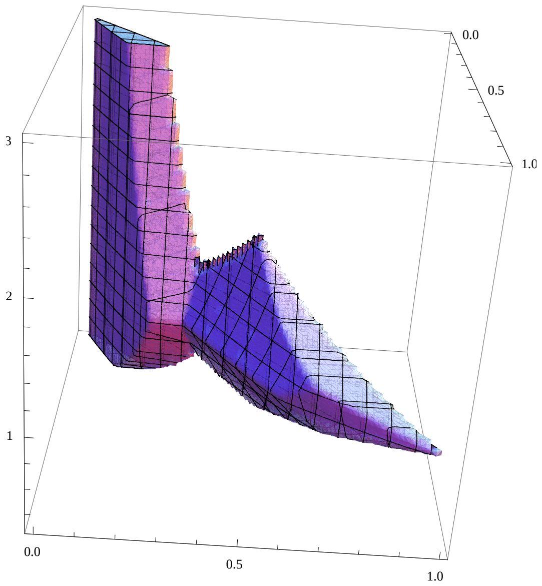

Figure 12: Transversality domain of $d_0 = 1$ hyperplane

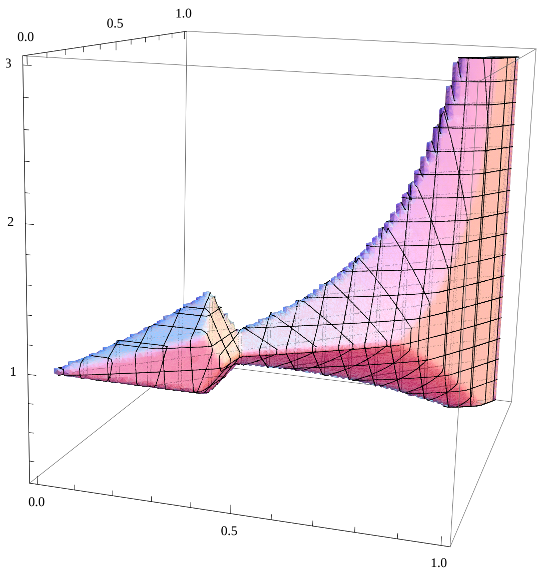

Figure 13: Transversality domain of $d_1 = 1$ hyperplane

μ is a Bernoulli-measure then $h_\mu = -p_0 \log p_0 - p_1 \log p_1$, where $p_0 + p_1 = 1$. Let $\nu_{\underline{t}} = \mu \circ \pi_{\underline{t}}^{-1}$.

The Lyapunov exponent of the IFS $\{g_0, g_1\}$ with parameter $\underline{t}$, corresponding to the measure μ is

$$\chi_\mu(\underline{t}) = -\int_{\sum} \log |g'_{i_1}(\pi_{\underline{t}}(\sigma \mathbf{i}))| d\mu(\mathbf{i})$$

In the important special case when μ is a Bernoulli-measure, the Lyapunov exponent can be rewritten as follows:

$$\chi_\mu(\underline{t}) = -p_0 \int_0^1 \log |g'_0(x)| d\nu_{\underline{t}}(x) - p_1 \int_0^1 \log |g'_1(x)| d\nu_{\underline{t}}(x)$$

In the next theorem we determine an open set U' s. t. for $\mathcal{L}_4$ a. e. $\underline{t} \in U'$ we have $\nu_{\underline{t}}$ is absolute continuous. The proof of the theorem can be found in [12, p. 5163].

Theorem 4.2.2. *We device the open set $U \subset \mathbb{R}^4$ as in Theorem 4.2.1, (53). Let μ be a shift-invariant ergodic Borel probability measure with positive entropy on $\sum$ and let $\nu_{\underline{t}} = \mu \circ \pi_{\underline{t}}^{-1}$. Then for Lebesgue-a. e. $\underline{t} \in U$,* $\dim_H(\nu_{\underline{t}}) = \min\left\{\frac{h_\mu}{\chi_\mu(\underline{t})}, 1\right\}$. *Moreover the measure $\nu_{\underline{t}}$ is absolute continuous for a. e. $\underline{t}$ in $\left\{\underline{t} \in U : \frac{h_\mu}{\chi_\mu(\underline{t})} > 1\right\}$.*

Proposition 4.2.1. *Let μ be a Bernoulli probability measure on $\sum$, and U_5 the following set:*

$$\begin{aligned} &U_5 = \{(a, b, d_0, d_1) : -p_0 \log p_0 - p_1 \log p_1 > \\ &> -p_0 \log \left(a \min\left\{d_0, \frac{1}{d_0}\right\}\right) - p_1 \log\left((1-b)\min\left\{d_1, \frac{1}{d_1}\right\}\right)\right\} \end{aligned} \tag{63}$$

Then $\nu_{\underline{t}} = \mu \circ \pi_{\underline{t}}^{-1}$ is absolute continuous for a. e. $\underline{t} \in U \cap U_5$.

Proof. By Lemma 4.2.1, it is easy to see that

$$\min_{\underline{x} \in (0,1)} g'_0(\underline{x}) = a \min\left\{d_0, \frac{1}{d_0}\right\}$$

and

$$\min_{\underline{x} \in (0,1)} g'_2(\underline{x}) = (1-b) \min\left\{d_1, \frac{1}{d_1}\right\}.$$

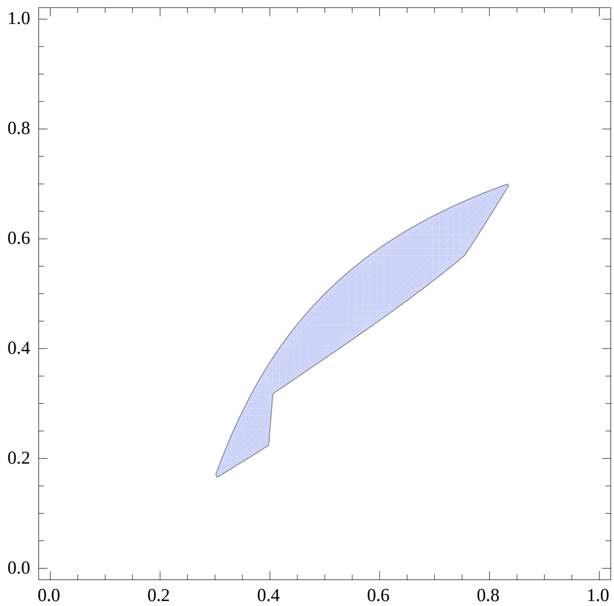

Figure 14: Absolute continuity region in linear case, $p_0 = p_1 = \frac{1}{2}$

Therefore

$$-p_0 \log\left(a \min\left\{d_0, \frac{1}{d_0}\right\}\right) - p_1 \log\left((1-b)\min\left\{d_1, \frac{1}{d_1}\right\}\right) > \chi_\mu(\underline{t})$$

Hence for every $\underline{t} \in U_5$, $\frac{h_\mu}{\chi_\mu(\underline{t})} > 1$, and by Theorem 4.2.2 we proved the proposition. $\square$

If $\mu = \left\{\frac{1}{2}, \frac{1}{2}\right\}^{\mathbb{N}}$ then by proposition 4.2.1 the open set, where $\nu_{\underline{t}}$ is absolute continuous, in linear case the image of the region is in Figure 14.

This set is smaller than what was proved in [13, p. 4.], but it is a little bit more general, and the proof of our set does not use $(*)$-functions. For $\mu = \left\{\frac{1}{3}, \frac{2}{3}\right\}^{\mathbb{N}}$ the $U \cap U_5$ set is in Figure 15.

We can show the absolute continuity domain for hyper-planes: $d_0 = d_1 = d$ Figure 16, $d_0 = 1, d_1 = d$ Figure 18 and $d_0 = d, d_1 = 1$ Figure 17, when $\mu = \left\{\frac{1}{2}, \frac{1}{2}\right\}^{\mathbb{N}}$.

Let us transform the absolute continuity domain $a \to a, b \to 1-b$, because $a, 1-b$ are the contracting ratios. Sze-Man Ngai and Yang Wang proved that

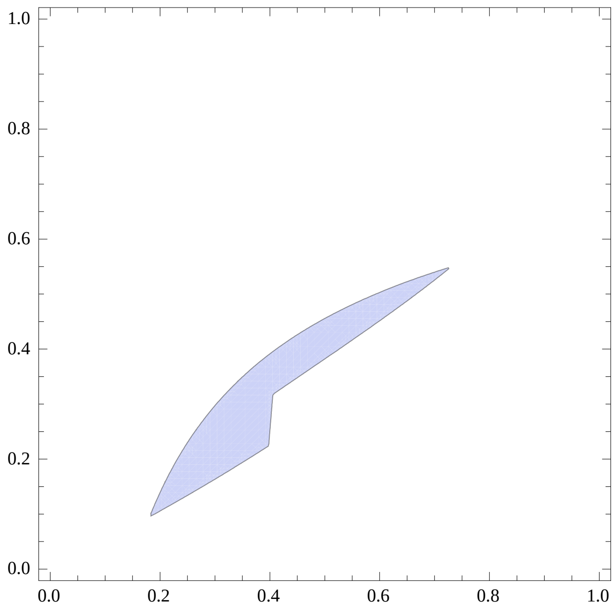

Figure 15: Absolute continuity region in linear case, $p_0 = \frac{1}{3}, p_1 = \frac{2}{3}$

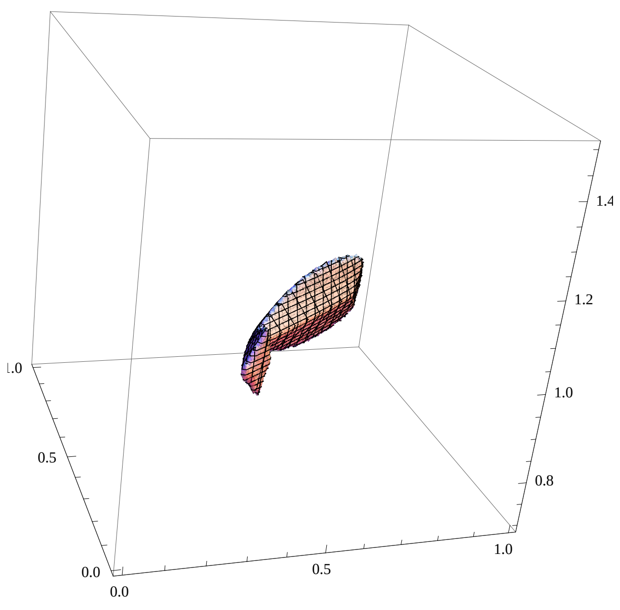

Figure 16: Absolute continuity domain of $d_0 = d_1$ hyperplane, $p_0 = p_1 = \frac{1}{2}$

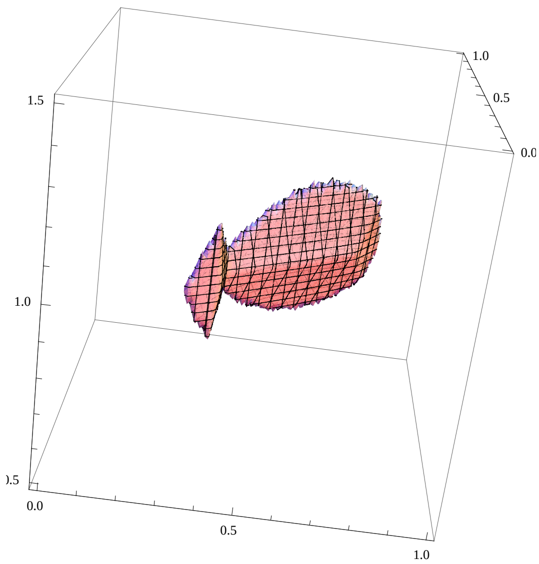

Figure 17: Absolute continuity domain of $d_1 = 1$ hyperplane, $p_0 = p_1 = \frac{1}{2}$

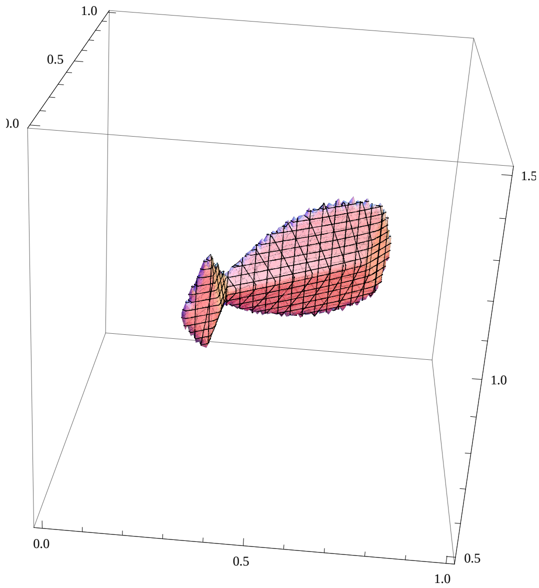

Figure 18: Absolute continuity domain of $d_0 = 1$ hyperplane, $p_0 = p_1 = \frac{1}{2}$

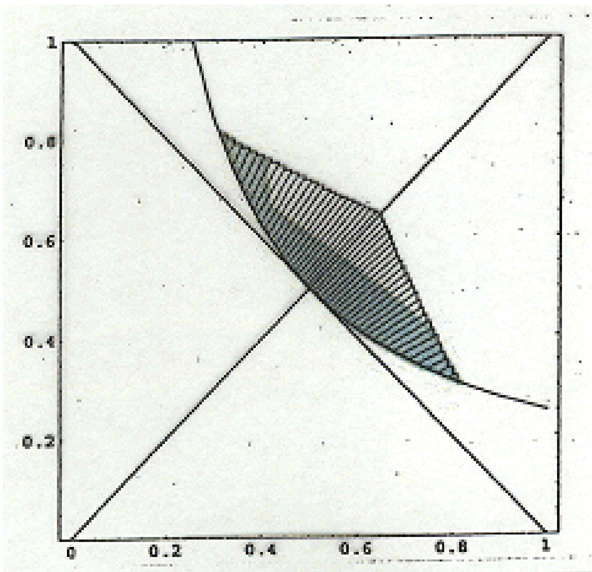

Figure 19: Compare the two regions

the μ self-similar measure corresponding to $S_1(x) = \rho_1 x, S_2(x) = \rho_2 x + 1$, $p_1 = p_2 = \frac{1}{2}$ is absolute continuous for Lebesgue almost all (ρ_1, ρ_2) in the region $\rho_1\rho_2 > \frac{1}{4}$ and $0 < \rho_1, \rho_2 < 0.6491$ [13, p. 3]. We proved an other region and we can compare this two regions in Figure 19.

Our a. c. region is contained in Sze-Man Ngai's and Yang Wang's result.

What do the results of the previous subsection mean for the original matrices and the original $\{g_0, g_1\}$ IFS?

Let $\underline{x}_0, \underline{x}_1 \in B_1^+$, where $B_1^+ = \left\{\underline{x} : \underline{x} = (x, y)^T, x \geq 0, y \geq 0, \|\underline{x}\|_1 = |x| + |y|\right\}$, and $S = [\underline{x}_1 \underline{x}_0]$. Let $c_0, c_1 > 0$. Let

$$\widetilde{A}_0 = \begin{bmatrix} a & 0 \\ 1-a & d_0 \end{bmatrix} \text{ and } \widetilde{A}_1 = \begin{bmatrix} d_1 & b \\ 0 & 1-b \end{bmatrix}$$

Moreover there exists a matrix S, depends on the elements of $\widetilde{A}_0, \widetilde{A}_1$ such that

$$A_0 := \begin{bmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{bmatrix} = c_0 S \widetilde{A}_0 S^{-1} \text{ and } A_1 := \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} = c_1 S \widetilde{A}_1 S^{-1} \quad (64)$$

matrices are in the original form, namely these matrices have positive elements. Let

$$X = \left\{(\underline{x}_0, \underline{x}_1) \in B_1^+ \times B_1^+ : A_0, A_1 \text{ have positive elements}\right\}$$

Then the transversality region of $\{\psi_{A_0}, \psi_{A_1}\}$ is the following 8-dimensional open set T.

$$\begin{aligned} T = \Big\{&(\alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1)^T \in \left(\mathbb{R}^+\right)^8 : \\ &c_0 > 0, c_1 > 0, (\underline{x}_0, \underline{x}_1) \in X, (a, b, d_0, d_1) \in U\} \end{aligned} \tag{65}$$

where U is defined in (53). Similarly the absolute continuity region T' is the following:

$$\begin{aligned} T' = \Big\{&(\alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1)^T \in \left(\mathbb{R}^+\right)^8 : \\ &c_0 > 0, c_1 > 0, (\underline{x}_0, \underline{x}_1) \in X, (a, b, d_0, d_1) \in U \cap U_5\} \end{aligned} \tag{66}$$

where U_5 is defined in (63).

The view of open sets T, T' is very difficult, because they are 8-dimensional.

5 Summary

In my thesis we studied three different iterated function systems in different methods.

In the second section we were interested in the estimate of Hausdorff-dimension for non-linear and non-conformal case. Our result is a generalization of K. Simon's and A. Manning's theorem. They proved in two dimension for such IFS, which functions have lower triangular derivative matrices, that the subadditive pressure is not sensitive to the choice of the points in every cylinders at which the singular value function is evaluated. We verified the same result in any dimension. Moreover K. Falconer and J. Miao gave a formula for the subadditive pressure and therefore for the Hausdorff-dimension of self-affine fractals generated by upper-triangular matrices. We gave a formula, too, in non-linear case. We showed some examples, where this formula can be used. This formula exactly gives the Box-dimension of the fractal, but for the Hausdorff-dimension it gives just an upper bound. We conjecture that the pressure is not sensitive for every IFS, which functions are at least $C^{1+\varepsilon}$. Maybe our result will help us to see this.

In the third section we examined a self-affine, diagonal, non-conformal IFS. We derived a dynamical system from these IFS. We aimed that we can apply the Ledrappier-Young Theorem in this case. The problem was that this theorem is true for C^2-diffeomorphisms. Fortunately, this theorem depends on the existence of Lyapunov charts. For special measures, namely the regular hyperbolic measures, Lyapunov charts exist. We constructed them explicitly, but in general it is not trivial what the Lyapunov charts are. In this section we wanted to demonstrate how one can use Lyapunov charts. This gives a better understanding of the dynamical systems whit singularities.

In the fourth section we studied a special group of IFS, which functions were derived from matrices with positive elements. We supposed that there is overlap between the two functions of the iterated function scheme. We were interested in giving an open set of parameters, where the invariant measure of the IFS is absolute continuous. K. Simon, B. Solomyak and M. Urbanski proved a theorem, which gave such a set of parameters, but they supposed, that the IFS satisfies the transversality condition, which was introduced by K. Simon and M. Pollicott. We checked this condition for our IFS and by using the Theorem of K. Simon, B. Solomyak and M. Urbanski we gave the absolute continuity region of parameters, too. Our result coincides with Sze-man Ngai's and Yang Wang's result in linear case, because our functions are linear with some very special choice of parameters. In the future we would like to extend in more general when the IFS is not given by triangular matrices.

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