THE ABSOLUTE CONTINUITY OF THE INVARIANT MEASURE OF RANDOM ITERATED FUNCTION SYSTEMS WITH OVERLAPS

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Abstract. We consider iterated function systems on the interval with random perturbation. Let Y_ε be uniformly distributed in $[1-\varepsilon, 1+\varepsilon]$ and let $f_i \in C^{1+\alpha}$ be contractions with fixpoints a_i. We consider the iterated function system $\{Y_\varepsilon f_i + a_i(1-Y_\varepsilon)\}_{i=1}^n$, were each of the maps are chosen with probability p_i. It is shown that the invariant density is in L^2 and the L^2-norm does not grow faster than $1/\sqrt{\varepsilon}$, as ε vanishes.

The proof relies on defining a piecewise hyperbolic dynamical system on the cube, with an SRB-measure with the property that its projection is the density of the iterated function system.

1. Introduction and Statements of Results

Let $\{f_1, \ldots, f_l\}$ be an iterated function system (IFS) on the real line, where the maps are applied according to the probabilities $(p_1, \ldots, p_l)$, with the choice of the map random and independent at each step. We assume that for each i, f_i maps $[-1, 1)$ into itself and $f_i \in C^{1+\alpha}([-1, 1))$. Let ν be the invariant measure of our IFS, namely,

$$\nu = \sum_{i=1}^{l} p_i \nu \circ f_i^{-1}. \tag{1.1}$$

Let $\mu = (p_1, \ldots, p_l)^{\mathbb{N}}$ be a Bernoulli measure on the space $\sum = \{1, \ldots, l\}^{\mathbb{N}}$. Let $h(\underline{p}) = -\sum_{i=1}^{l} p_i \log p_i$ be the entropy of the underlying Bernoulli process μ. It was proved in [7] for non-linear contracting on average IFSs (and later extended in [3]) that

$$\dim_{\mathrm{H}}(\nu) \leq \frac{h}{|\chi|},$$

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where $\dim_{\mathrm{H}}(\nu)$ is the Hausdorff dimension of the measure ν and χ is the Lyapunov exponent of the IFS associated to the Bernoulli measure μ.

One can expect that, at least "typically", the measure ν is absolutely continuous when $h/|\chi| > 1$. Essentially the only known approach to this is transversality. For example, in linear case with uniform contracting ratios see [8],[10]. In the linear case for non-uniform contracting ratios, see [5], [6]. In the non-linear case, see for example [12], [1]. We note that there is an other direction in the study of IFSs with overlaps, which is concerned with concrete, but not-typical systems, often of arithmetic nature, for which there is a dimension drop, see, for example [4].

Trough this paper we are interested in to study absolute continuity with density in L^2. We study a modification of the problem, namely we consider a random perturbation of the functions. The linear case was studied by Peres, Simon and Solomyak in [9]. They proved absolute continuity for random linear IFS, with non-uniform contracting ratios and also L^2 and continuous density in the uniform case. We would like to extend this result by proving L^2 density with non-uniform contracting ratios and in non-linear case.

We consider two cases. First let us suppose that for each $i \in \{1, \ldots, l\}$, f_i maps $[-1, 1)$ into itself, $f_i \in C^{1+\alpha}([-1, 1))$ and

$$0 < \lambda_{i,\min} \leq |f_i'(x)| \leq \lambda_{i,\max} < 1 \tag{1.2}$$

for every $x \in [-1, 1)$. Moreover let us suppose that for every i the fix point of f_i is $a_i \in [-1, 1]$, and

$$i \neq j \;\Rightarrow\; a_i \neq a_j. \tag{1.3}$$

Let Y_ε be uniformly distributed on $[1 - \varepsilon, 1 + \varepsilon]$. Let us denote the probability measure of Y_ε by η_ε. Let

$$f_{i,Y_\varepsilon}(x) = Y_\varepsilon f_i(x) + a_i(1 - Y_\varepsilon) \tag{1.4}$$

for every $i \in \{1, \ldots, l\}$. The iterated maps are applied randomly according to the stationary measure μ, with the sequence of independent and identically distributed errors $y_1, y_2, \ldots$, distributed as Y_ε, independent of the choice of the function. The Lyapunov exponent of the IFS is defined by

$$\chi(\mu, \eta_\varepsilon) = \mathbb{E}(\log(Y_\varepsilon f'))$$

and

$$\chi(\mu, \eta_\varepsilon) < \sum_{i=1}^{l} p_i \log((1 + \varepsilon)\lambda_{i,\max}) < 0,$$

for sufficiently small $\varepsilon > 0$. Let Z_ε be the following random variable

$$Z_\varepsilon := \lim_{n\to\infty} f_{i_1,y_{1,\varepsilon}} \circ f_{i_2,y_{2,\varepsilon}} \circ \cdots \circ f_{i_n,y_{n,\varepsilon}}(0), \tag{1.5}$$

where the numbers i_k are i.i.d., with the distribution μ on $\{1,\dots,l\}$, and y_k are pairwise independent with distribution of Y_ε and also independent of the choice of i_k. Let ν_ε be the distribution of Z_ε.

One can easily prove the following theorem.

Theorem 1.1. *The measure ν_ε converges weakly to the measure ν as $\varepsilon \to 0$, see (1.1).*

Theorem 1.2. *Let ν_ε be the distribution of the limit (1.5). We assume that (1.2), (1.3) hold, and*

$$\sum_{i=1}^{l} p_i^2 \frac{\lambda_{i,\max}}{\lambda_{i,\min}^2} < 1. \tag{1.6}$$

Then for every sufficiently small $\varepsilon > 0$, we have that $\nu_\varepsilon \ll \mathcal{L}_1$ with density in L^2. For the L^2-norm of the density we have the following estimate

$$\|\nu_\varepsilon\|_2 \le \frac{C'_\varepsilon}{\sqrt{\varepsilon}},$$

where

$$C'_\varepsilon = \sqrt{\frac{32}{\left(1 - \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\right) C''_\varepsilon}}$$

and

$$C''_\varepsilon = \min_{i\neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1-\varepsilon^2} \right\}.$$

We can draft an easy corollary of the theorem.

Corollary 1.3. *Let $\{\lambda_i Y_\varepsilon x + a_i(1-\lambda_i Y_\varepsilon)\}_{i=1}^{l}$ be a random iterated function system. We assume that (1.3) holds, and*

$$\sum_{i=1}^{l} \frac{p_i^2}{\lambda_i} < 1. \tag{1.7}$$

Then for every sufficiently small $\varepsilon > 0$, we have that $\nu_\varepsilon \ll \mathcal{L}_1$ with density in L^2, the L^2-norm of the density satisfies

$$\|\nu_\varepsilon\|_2 \le \frac{C'_\varepsilon}{\sqrt{\varepsilon}},$$

where

$$C'_\varepsilon = \sqrt{\frac{32}{\left(1 - \sum_{i=1}^{l} p_i^2 \frac{1+\varepsilon}{(1-\varepsilon)^2\lambda_i}\right) C''_\varepsilon}}$$

and

$$C''_\varepsilon = \min_{i\neq j}\left\{\frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1-\varepsilon^2}\right\}.$$

We study an other case of random perturbation, namely let $\widetilde{\lambda}_{i,\varepsilon}$ be uniformly distributed on $[\lambda_i - \varepsilon, \lambda_i + \varepsilon]$. Let $\left\{\widetilde{\lambda}_{i,\varepsilon}x + a_i(1-\widetilde{\lambda}_{i,\varepsilon})\right\}_{i=1}^{l}$ be our random iterated function system, where $a_i \neq a_j$ for every $i \neq j$. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_l)$, and $X_{\underline{\lambda},\varepsilon}$ be the following random variable

$$X_{\underline{\lambda},\varepsilon} = \sum_{k=1}^{\infty}(a_{i_k}(1-\widetilde{\lambda}_{i_k,\varepsilon}))\prod_{j=1}^{k-1}\widetilde{\lambda}_{i_j,\varepsilon} \tag{1.8}$$

where the numbers i_k are i.i.d., with the distribution μ on $\{1,\ldots,l\}$, and $\widetilde{\lambda}_{i_k,\varepsilon}$ are pairwise independent. Let $\nu_{\underline{\lambda},\varepsilon}$ denote the distribution of the random variable $X_{\underline{\lambda},\varepsilon}$. Moreover let $\nu_{\underline{\lambda}}$ be the invariant measure of the the iterated function system $\{\lambda_i x + a_i(1-\lambda_i)\}_{i=1}^{l}$ according to μ.

Theorem 1.4. *The measure $\nu_{\underline{\lambda},\varepsilon}$ converges weakly to the measure $\nu_{\underline{\lambda}}$ as $\varepsilon \to 0$.*

To have a similar statement as in Theorem 1.2 we need a technical assumption, namely

$$\min_{i\neq j}\left|\frac{a_j\lambda_i - a_i\lambda_j}{\lambda_i - \lambda_j}\right| > 1. \tag{1.9}$$

Theorem 1.5. *Let us suppose that (1.9) and (1.3) hold, and moreover that*

$$\sum_{i=1}^{l}\frac{p_i^2}{\lambda_i} < 1. \tag{1.10}$$

Then for every sufficiently small $\varepsilon > 0$, the measure $\nu_{\underline{\lambda},\varepsilon}$ is absolutely continuous with density in L^2, and the L^2-norm of the density satisfies

$$\|\nu_{\underline{\lambda},\varepsilon}\|_2 \leq \frac{C'_\varepsilon}{\sqrt{\varepsilon}},$$

where

$$C'_\varepsilon = \sqrt{\frac{32}{\left(1 - \sum_{i=1}^{l} p_i^2 \frac{\lambda_i+\varepsilon}{(\lambda_i-\varepsilon)^2}\right)C''_\varepsilon}}$$

and

$$C''_\varepsilon = \sigma\min_{i\neq j}\frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i\lambda_j}.$$

where $0 < \sigma < 1$.

The main difference between Theorem 1.5 and Corollary 1.3 is the random perturbation. Namely, in Theorem 1.5 we choose the contracting ratio uniformly in the ε neighborhood of λ_i, but in Corollary 1.3 we choose the contraction ratio uniformly in the $\lambda_i\varepsilon$ neighborhood of λ_i.

Throughout this paper we will use the method in [11].

2. Proof of Theorem 1.2

Let $Q = [-1, 1)^3$ and $m \in \mathbb{N}$. We partition the cube Q into the rectangles $\{Q_{1,k}, \ldots, Q_{l,k}\}_{k=0}^{2^m-1}$, where

$$Q_{i,k} = \Bigg\{ (x, y, z) \in Q : -1 + 2\sum_{j=1}^{i-1} p_j \leq y < -1 + 2\sum_{j=1}^{i} p_i, \\ -1 + k2^{-m+1} \leq z < -1 + (k+1)2^{-m+1} \Bigg\},$$

where we use the convention that an empty sum is 0. Hence we slice Q in 2^m slices along the z-axis and l slices along the y-axis. We thereby get $2^m l$ pieces which we call $Q_{i,k}$, according to the definition above.

Let

$$Q_i = \bigcup_{k=0}^{2^m-1} Q_{i,k}.$$

For $(x, y, z) \in Q_i$, define $g_{\varepsilon,m} \colon Q \to Q$ by

$$g_{\varepsilon,m} \colon (x, y, z) \mapsto \left(d(z)f_i(x) + a_i(1 - d(z)),\ \frac{1}{p_i}y + b(y),\ 2^m z + c(z) \right),$$

where

$$\begin{aligned} d(z) &= 1 + 2^m\varepsilon(z - (-1 + (k + \tfrac{1}{2})2^{-m+1}), &&\text{for } (x, y, z) \in Q_{i,k}, \\ b(y) &= 1 - \frac{1}{p_i}\left(-1 + 2\sum_{j=1}^{i} p_j\right), &&\text{for } (x, y, z) \in Q_{i,k}, \\ c(z) &= 2^m - 2k - 1, &&\text{for } (x, y, z) \in Q_{i,k}. \end{aligned}$$

Hence $g_{\varepsilon,m}$ maps each of the pieces $Q_{i,j}$ so that it s contracted in the x-direction and fully expanded in the y- and z-directions.

Let $\mathcal{L}_3$ be the normalised Lebesgue measure on Q. The measures

$$\gamma_{\varepsilon,m,n} = \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_3 \circ g_{\varepsilon,m}^{-k}$$

converge weakly to an SRB-measure $\gamma_{\varepsilon,m}$ as $n \to \infty$. The measure $\gamma_{\varepsilon,m}$ is clearly ergodic. Moreover, let $\nu_{\varepsilon,m}$ be the projection of $\gamma_{\varepsilon,m}$ onto the first coordinate. More precisely, if $E \subset [-1,1)$ is a measurable set, then $\nu_{\varepsilon,m}(E) = \gamma_{\varepsilon,m}(E \times [-1,1) \times [-1,1))$.

The measure $\nu_{\varepsilon,m}$ is the distribution of the limit

$$\lim_{n\to\infty} f_{i_1,y_{1,\varepsilon}} \circ f_{i_2,y_{2,\varepsilon}} \circ \cdots \circ f_{i_n,y_{n,\varepsilon}}(0),$$

where $y_{i,\varepsilon}$ are uniformly distributed on $[1-\varepsilon, 1+\varepsilon]$, but not independent. However, one can easily prove the following lemma.

Lemma 2.1. *The measure $\nu_{\varepsilon,m}$ converges weakly to ν_ε as $m \to \infty$.*

Let

$$A_i = \{(i,0),(i,1),\ldots,(i,2^m-1)\}$$

and

$$A = \bigcup_{i=1}^{l} A_i.$$

Let $\Theta_0 = A^{\mathbb{N}\cup\{0\}}$. If $p \in Q$ then there is a unique sequence $\rho_0(p) = \{\rho_0(p)_k\}_{k=0}^\infty \in \Theta_0$ such that

$$g_{\varepsilon,m}^k(p) \in Q_{\rho_0(p)_k}, \ k = 0,1,\ldots$$

The map $\rho_0 \colon Q \to \Theta_0$ is not injective.

We can transfer the measures $\gamma_{\varepsilon,m}$ to a measure γ_{Θ_0} by $\gamma_{\Theta_0} = \gamma_{\varepsilon,m} \circ \rho_0^{-1}$.

We let Θ denote the natural extension of Θ_0. That is, Θ is the set of all two sides infinite sequences such that any one sided infinite subsequence of sequence in Θ is a sequence in Θ_0. The measures γ_{Θ_0} defines an ergodic measure γ_Θ on Θ in a natural way. If $\xi \colon \Theta \to \Theta_0$ is defined by $\xi(\{i_k\}_{k\in\mathbb{Z}}) = \{i_k\}_{k\in\mathbb{N}\cup\{0\}}$, then $\gamma_{\Theta_0}(E) = \gamma_\Theta(\xi^{-1}E)$. We can define a map $\rho^{-1} \colon \Theta \to Q$ such that $\rho^{-1}(\sigma(\boldsymbol{a})) = g_{\varepsilon,m}(\rho^{-1}(\boldsymbol{a}))$ holds for any sequence $\boldsymbol{a} \in \Theta$.

We note that the L^2 norm of the density $\nu_{\varepsilon,m}$ is not larger than twice that of the density of $\gamma_{\varepsilon,m}$. If $h_{\nu_{\varepsilon,m}}(x)$ and $h_{\gamma_{\varepsilon,m}}(x,y,z)$ denote the density of $\nu_{\varepsilon,m}$ and $\gamma_{\varepsilon,m}$ respectively, then by Lyapunov's inequality

$$\begin{aligned}\|\nu_{\varepsilon,m}\|_2^2 &\leq \int_{-1}^1 h_{\nu_{\varepsilon,m}}(x)^2\,dx = 32\int_{-1}^1 \left(\int_{-1}^1\int_{-1}^1 h_{\gamma_{\varepsilon,m}}(x,y,z)\,\frac{dy}{2}\frac{dz}{2}\right)^2 \frac{dx}{2} \\ &\leq 32\int_{-1}^1\int_{-1}^1\int_{-1}^1 h_{\gamma_{\varepsilon,m}}(x,y,z)^2\,\frac{dy}{2}\frac{dz}{2}\frac{dx}{2} = 4\|\gamma_{\varepsilon,m}\|_2^2.\end{aligned}$$

This proves that if $\gamma_{\varepsilon,m}$ has L^2 density, then so has $\nu_{\varepsilon,m}$, and

$$\|\nu_{\varepsilon,m}\|_2 \leq 2\|\gamma_{\varepsilon,m}\|_2. \tag{2.1}$$

Lemma 2.2. *Let*

$$C_p = \left\{ (u,v,w) \in T_pQ : \left|\frac{u}{w}\right|, \left|\frac{v}{w}\right| < \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right\},$$

where $p \in Q$ *and* $\lambda_{\max,\max} = \max_i \lambda_{i,\max} = \max_i \sup_{x\in[-1,1)} |f_i'(x)|$. *The cones* C_p *defines a family of unstable cones, that is* $d_p g_{\varepsilon,m}(C_p) \subset C_{g_{\varepsilon,m}(p)}$.

Moreover, for sufficiently large m *and every* $0 < \varepsilon < \min_{i\neq j} \frac{|a_i - a_j|}{2+|a_i+a_j|}$, *if* $\zeta_1 \subset Q_{\xi_1}$ *and* $\zeta_2 \subset Q_{\xi_2}$ *are two curves segments with tangents in* C_p *such that* $\xi_1 \in A_i$ *and* $\xi_2 \in A_j$, $i \neq j$, *then if* $g_{\varepsilon,m}(\zeta_1)$ *and* $g_{\varepsilon,m}(\zeta_2)$ *intersects, and if* $(u_1, v_1, 1)$ *and* $(u_2, v_2, 1)$ *are tangents to* $g_{\varepsilon,m}(\zeta_1)$ *and* $g_{\varepsilon,m}(\zeta_2)$ *respectively, it holds* $|u_1 - u_2| > C_{\varepsilon,m}\varepsilon$, *where*

$$C_{\varepsilon,m} = \min_{i\neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1-\varepsilon^2} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right\}.$$

Proof of Lemma 2.2. The Jacobian of $g_{\varepsilon,m}$ is

$$d_p g_{\varepsilon,m} = \begin{pmatrix} d(z)f_i'(x) & 0 & 2^m\varepsilon(f_i(x) - a_i) \\ 0 & \frac{1}{p_i} & 0 \\ 0 & 0 & 2^m \end{pmatrix}$$

where $p = (x,y,z) \in Q_{i,k}$. If $(u,v,w) \in C_p$, then

$$d_p g_{\varepsilon,m}(u,v,w) = \begin{pmatrix} d(z)f_i'(x)u + 2^m\varepsilon(f_i(x) - a_i)w \\ \frac{1}{p_i}v \\ 2^m w \end{pmatrix}$$

The estimates

$$\begin{aligned} \frac{|d(z)f_i'(x)u + 2^m\varepsilon(f_i(x) - a_i)w|}{|2^m w|} &\leq \frac{(1+\varepsilon)\lambda_{i,\max}}{2^m}\frac{|u|}{|w|} + 2\varepsilon \\ &\leq \frac{(1+\varepsilon)\lambda_{i,\max}}{2^m}\frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}} + 2\varepsilon \leq \frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}} \end{aligned}$$

and

$$\frac{|\frac{1}{p_i}v|}{|2^m w|} \leq \frac{1}{p_i 2^m}\frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}} \leq \frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}}$$

proves that $d_p g_{\varepsilon,m}(C_p) \subset C_{g_{\varepsilon,m}(p)}$ if m is sufficiently large, so that $2^m - (1+\varepsilon)\lambda_{\max,\max} > 0$ and $p_i 2^m > 1$.

To prove the other statement of the Lemma, assume that $p = (x_p, y_p, z_p) \in Q_i$ and $q = (x_q, y_q, z_q) \in Q_j$, $i \neq j$, are such that $g_{\varepsilon,m}(p) = g_{\varepsilon,m}(q) = (x,y,z)$. Then, if $p \in Q_i$

$$d_p g_{\varepsilon,m} \colon (u,v,1) \mapsto 2^m \left(\frac{d(z_p)f_i'(x_p)}{2^m}u + (f_i(x_p) - a_i)\varepsilon,\ \frac{v}{p_i},\ 1 \right)$$

Then

$$f_i(x_p) = \frac{x - a_i(1 - d(z_p))}{d(z_p)} \quad \text{and} \quad f_j(x_q) = \frac{x - a_j(1 - d(z_q))}{d(z_q)}.$$

Without loss of generality, let us assume that $a_i > a_j$. For simplicity we study the case $x \geq a_i > a_j$. The proof of the other cases $a_i \geq x \geq a_j$ and $a_i > a_j \geq x$ is similar. Then

$$\begin{aligned} d_p g_{\varepsilon,m}(C_p) \subset \Bigg\{ w(u,v,1) : \frac{x - a_i}{1+\varepsilon}\varepsilon - \frac{2(1+\varepsilon)\lambda_{i,\max}\varepsilon}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \\ \leq u \leq \frac{x - a_i}{1-\varepsilon}\varepsilon + \frac{2(1+\varepsilon)\lambda_{i,\max}\varepsilon}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \Bigg\} \end{aligned}$$

Therefore

$$\begin{aligned} |u_2 - u_1| &\geq \frac{x - a_j}{1+\varepsilon}\varepsilon - \frac{x - a_i}{1-\varepsilon}\varepsilon - \frac{2(1+\varepsilon)(\lambda_{i,\max} + \lambda_{j,\max})\varepsilon}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \\ &\geq \left(\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right)\varepsilon \end{aligned}$$

for every $x \geq a_i > a_j$. Since $0 < \varepsilon < \min_{i \neq j} \frac{|a_i - a_j|}{2 + |a_i + a_j|}$,

$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} > 0$$

Therefore

$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)} > 0$$

for sufficiently large m. By similar methods, we have for $a_i \geq x \geq a_j$

$$|u_2 - u_1| \geq \left(\frac{a_i - a_j}{1+\varepsilon} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right)\varepsilon$$

and for $a_i > a_j \geq x$

$$|u_2 - u_1| \geq \left(\frac{a_i - a_j - \varepsilon(a_i + a_j + 2)}{1 - \varepsilon^2} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right)\varepsilon$$

Therefore we can choose $C_{\varepsilon,m}$ as

$$C_{\varepsilon,m} = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1 - \varepsilon^2} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right\}. \qquad \square$$

The rest of the proof follows Tsujii's article [13].

Proof of Theorem 1.2. For any $r > 0$ we define the bilinear form $(\cdot,\cdot)_r$ of signed measures on $\mathbb{R}$ by

$$(\rho_1, \rho_2)_r = \int_{\mathbb{R}} \rho_1(B_r(x))\rho_2(B_r(x))\, dx$$

where $B_r(x) = [x-r, x+r]$. It is easy to see that if

$$\liminf_{r\to 0} \frac{1}{r^2}(\rho,\rho)_r < \infty$$

then the measure ρ has density in L^2, moreover

$$\|\rho\|_2^2 \le \liminf_{r\to 0} \frac{1}{r^2}(\rho,\rho)_r.$$

Let γ_z denote the conditional measure of $\gamma_{\varepsilon,m}$ on the set $R_z = \{ (u,v,w) \in Q : v = y, w = z \}$. Note that γ_z is independent of y almost everywhere. Let

$$J(r) := \frac{1}{r^2}\int_{-1}^{1} (\gamma_z,\gamma_z)_r \, dz.$$

It is easy to see that

$$\|\gamma_{\varepsilon,m}\|_2^2 = \int_{-1}^{1} \|\gamma_z\|_2^2 \, dz. \tag{2.2}$$

By the invariance of $\gamma_{\varepsilon,m}$ it follows that

$$\gamma_z = 2^{-m}\sum_{i=1}^{l} p_i \sum_{a\in A_i} \gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}, \tag{2.3}$$

where $g^{-a}_{\varepsilon,m}$ denotes the inverse branch of $g_{\varepsilon,m}$ such that the image of $g^{-a}_{\varepsilon,m}$ is in the cylinder $[a]$. Then by (2.3) and the definition of $J(r)$

$$J(r) = \frac{1}{2^{2m}r^2}\sum_{i=1}^{l}\sum_{j=1}^{l} p_i p_j \sum_{a\in A_i}\sum_{b\in A_j}\int_{-1}^{1} (\gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}, \gamma_{g^{-b}_{\varepsilon,m}(z)} \circ g^{-b}_{\varepsilon,m})_r \, dz. \tag{2.4}$$

For fixed $a, b \in A_i$ it holds,

$$\begin{aligned}
&(\gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}, \gamma_{g^{-b}_{\varepsilon,m}(z)} \circ g^{-b}_{\varepsilon,m})_r \\
&\le (\gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}, \gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m})_r^{\frac12} (\gamma_{g^{-b}_{\varepsilon,m}(z)} \circ g^{-b}_{\varepsilon,m}, \gamma_{g^{-b}_{\varepsilon,m}(z)} \circ g^{-b}_{\varepsilon,m})_r^{\frac12} \\
&\le (1+\varepsilon)\lambda_{i,\max} (\gamma_{g^{-a}_{\varepsilon,m}(z)}, \gamma_{g^{-a}_{\varepsilon,m}(z)})^{\frac12}_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}} \times (\gamma_{g^{-b}_{\varepsilon,m}(z)}, \gamma_{g^{-b}_{\varepsilon,m}(z)})^{\frac12}_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}} \\
&\le (1+\varepsilon)\lambda_{i,\max} \frac{(\gamma_{g^{-a}_{\varepsilon,m}(z)}, \gamma_{g^{-a}_{\varepsilon,m}(z)})_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}} + (\gamma_{g^{-b}_{\varepsilon,m}(z)}, \gamma_{g^{-b}_{\varepsilon,m}(z)})_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}}{2}.
\end{aligned} \tag{2.5}$$

Moreover, if $a \in A_i$ and $b \in A_j$, $i \ne j$, then

$$\begin{aligned}
&(\gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}, \gamma_{g^{-b}_{\varepsilon,m}(z)} \circ g^{-b}_{\varepsilon,m})_r \\
&\qquad = \int \gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}(B_r(x)) \gamma_{g^{-b}_{\varepsilon,m}(z)} \circ g^{-b}_{\varepsilon,m}(B_r(x)) dx \\
&\qquad = \iiint \mathbb{I}_{\{ s:|s-x|<r \}}(s)\mathbb{I}_{\{ t:|t-x|<r \}}(t) \\
&\qquad\qquad\qquad d\gamma_{g^{-a}_{\varepsilon,m}(z)} \circ g^{-a}_{\varepsilon,m}(s) d\gamma_{g^{-b}_{\varepsilon,m}(z)} \circ g^{-b}_{\varepsilon,m}(t)dx
\end{aligned}$$

$$\leq \iint 2r\mathbb{I}_{\{(s,t):|s-t|<2r\}}(s,t)\,d\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m}(s)d\gamma_{g^{-b}_{\varepsilon,m}(z)}\circ g^{-b}_{\varepsilon,m}(t)$$
$$= \iint \mathbb{I}_{\{(\boldsymbol{c},\boldsymbol{d}):|\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z))-\rho^{-1}(d_{-2}d_{-1}b\rho_0(z))|<2r\}}(\boldsymbol{c},\boldsymbol{d})$$
$$d\gamma_\Theta(\boldsymbol{c})d\gamma_\Theta(\boldsymbol{d}). \tag{2.6}$$

Therefore by Lemma 2.2 and (2.6) we get that

$$\int_{-1}^{1}(\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m},\gamma_{g^{-b}_{\varepsilon,m}(z)}\circ g^{-b}_{\varepsilon,m})_r dz$$
$$\leq 2r\iint \mathcal{L}_1(\{\,z:|\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z))-\rho^{-1}(d_{-2}d_{-1}b\rho_0(z))|<2r\,\})$$
$$d\gamma_\Theta(\boldsymbol{c})d\gamma_\Theta(\boldsymbol{d})$$
$$\leq \frac{8r^2}{C_{\varepsilon,m}\varepsilon}. \tag{2.7}$$

Then by using (2.4) we have

$$J(r)=\frac{1}{2^{2m}r^2}\sum_{i=1}^{l}p_i^2\sum_{a,b\in A_i}\int_{-1}^{1}(\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m},\gamma_{g^{-b}_{\varepsilon,m}(z)}\circ g^{-b}_{\varepsilon,m})_r\,dz$$
$$+\frac{1}{2^{2m}r^2}\sum_{i\neq j}p_ip_j\sum_{a\in A_i}\sum_{b\in A_j}\int_{-1}^{1}(\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m},\gamma_{g^{-b}_{\varepsilon,m}(z)}\circ g^{-b}_{\varepsilon,m})_r\,dz.$$

Then we can give an upper bound for the first part of the sum using (2.5) and an integral transformation

$$\frac{1}{2^{2m}r^2}\sum_{i=1}^{l}p_i^2\sum_{a,b\in A_i}\int_{-1}^{1}(\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m},\gamma_{g^{-b}_{\varepsilon,m}(z)}\circ g^{-b}_{\varepsilon,m})_r\,dz$$
$$\leq \frac{1}{2^{2m}r^2}\sum_{i=1}^{l}p_i^2(1+\varepsilon)\lambda_{i,\max}2^m\sum_{a\in A_i}\int_{-1}^{1}(\gamma_{g^{-a}_{\varepsilon,m}(z)},\gamma_{g^{-a}_{\varepsilon,m}(z)})_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz$$
$$\leq \frac{1}{2^{2m}r^2}\sum_{i=1}^{l}p_i^2(1+\varepsilon)\lambda_{i,\max}2^m\sum_{k=0}^{2^m-1}2^m\int_{-1+k2^{-m+1}}^{-1+(k+1)2^{-m+1}}(\gamma_z,\gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz$$
$$\leq \sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\frac{1}{\left(\frac{r}{(1-\varepsilon)\lambda_{i,\min}}\right)^2}\int_{-1}^{1}(\gamma_z,\gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz$$
$$\leq \max_i J\left(\frac{r}{\lambda_{i,\min}(1-\varepsilon)}\right)\sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}. \tag{2.8}$$

For the second part of the sum, we use (2.7), to prove that it is bounded by

$$\frac{1}{2^{2m}r^2}\sum_{i\neq j}p_ip_j\sum_{a\in A_i}\sum_{b\in A_j}\int_{-1}^{1}(\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m},\gamma_{g^{-b}_{\varepsilon,m}(z)}\circ g^{-b}_{\varepsilon,m})_r\,dz$$
$$\leq \frac{1}{2^{2m}r^2}\sum_{i\neq j}p_ip_j\sum_{a\in A_i}\sum_{b\in A_j}\frac{8r^2}{C_{\varepsilon,m}\varepsilon}\leq\frac{8}{C_{\varepsilon,m}\varepsilon}. \quad (2.9)$$

By using (2.8) and (2.9) we have

$$J(r)\leq\frac{8}{C_{\varepsilon,m}\varepsilon}+b\max_i J\left(\frac{r}{\lambda_{i,\min}(1-\varepsilon)}\right) \quad (2.10)$$

where $b=\sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}$ is less than 1 by (1.6). for sufficiently small $\varepsilon>0$. We define a strictly monotone decreasing series r_k. Let $r_0<1/2$ be fixed and $r_k=r_0(1-\varepsilon)^k\prod_{n=1}^{k}(\lambda_{i_n,\min})$ such that

$$\max_i J\left(\frac{r_k}{(1-\varepsilon)\lambda_{i,\min}}\right)=J(r_{k-1}).$$

We note that r_k is a well defined series. Then by induction and by using (2.10), we have

$$J(r_k)\leq\frac{8}{C_{\varepsilon,m}\varepsilon}\frac{1-b^k}{1-b}+b^kJ(r_0) \quad (2.11)$$

for every $k\geq 1$. Hence by (2.1), (2.2) and (2.11) we get

$$\|\nu_{\varepsilon,m}\|_2^2\leq 4\liminf_{r\to 0}J(r)\leq 4\liminf_{k\to\infty}J(r_k)$$
$$\leq\frac{32}{C_{\varepsilon,m}\varepsilon}\frac{1}{1-\sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}}. \quad (2.12)$$

Since $\nu_{\varepsilon,m}$ converges weakly to ν_ε we get that

$$\|\nu_\varepsilon\|_2\leq\frac{1}{\sqrt{\varepsilon}}C'_\varepsilon \quad (2.13)$$

where

$$C'_\varepsilon=\sqrt{\frac{32}{\left(1-\sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\right)C''_\varepsilon}}$$

and

$$C''_\varepsilon=\lim_{m\to\infty}C_{\varepsilon,m}=\min_{i\neq j}\left\{\frac{|a_i-a_j|+\varepsilon(-|a_i+a_j|-2)}{1-\varepsilon^2}\right\}.$$

□

3. Proof of Theorem 1.5

We do not notify the proof of Theorem 1.5, because it is similar to the proof of Theorem 1.2. We notify only the modification of Lemma 2.2, which is important as it proves transversality.

First we define a new dynamical system. Let $Q_{i,k}$ and $A_{i,k}$ be as in Section 2. Let $\widetilde{g}_{\varepsilon,m} \colon Q \to Q$ be defined by

$$\widetilde{g}_{\varepsilon,m} \colon (x,y,z) \mapsto \left(\widetilde{d}(z)x + a_i(1-\widetilde{d}(z)),\ \frac{1}{p_i}y + b(y),\ 2^m z + c(z) \right),$$

for $(x,y,z) \in Q_i$, where

$$\begin{aligned} \widetilde{d}(z) &= \lambda_i + 2^m \varepsilon (z - (-1 + (k + \frac{1}{2})2^{-m+1})), && \text{for } (x,y,z) \in Q_{i,k}, \\ b(y) &= 1 - \frac{1}{p_i}\left(-1 + 2\sum_{j=1}^{i} p_j \right), && \text{for } (x,y,z) \in Q_{i,k}, \\ c(z) &= 2^m - 2k - 1, && \text{for } (x,y,z) \in Q_{i,k}. \end{aligned}$$

Lemma 3.1. *Let us suppose that (1.9) holds. Let*

$$C_p = \left\{ (u,v,w) \in T_p Q : \left|\frac{u}{w}\right|, \left|\frac{v}{w}\right| < \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max} - \varepsilon} \right\},$$

where $p \in Q$ and $\lambda_{\max} = \max_i \lambda_i$. The cones C_p defines a family of unstable cones, that is $d_p\widetilde{g}_{\varepsilon,m}(C_p) \subset C_{\widetilde{g}_{\varepsilon,m}(p)}$.

Moreover, for sufficiently large m and every sufficiently small $0 < \varepsilon$, if $\zeta_1 \subset Q_{\xi_1}$ and $\zeta_2 \subset Q_{\xi_2}$ are two line segments with tangents in C_p such that $\xi_1 \in A_i$ and $\xi_2 \in A_j$, $i \neq j$, then if $\widetilde{g}_{\varepsilon,m}(\zeta_1)$ and $\widetilde{g}_{\varepsilon,m}(\zeta_2)$ intersects, and if $(u_1,v_1,1)$ and $(u_2,v_2,1)$ are tangents to $\widetilde{g}_{\varepsilon,m}(\zeta_1)$ and $\widetilde{g}_{\varepsilon,m}(\zeta_2)$ respectively, there exists a constant $C_{\varepsilon,m}$, depending on ε and m, but bounded away from 0 and infinity, such that $|u_1 - u_2| > C_{\varepsilon,m}\varepsilon$.

Proof of Lemma 3.1. The Jacobian of $\widetilde{g}_{\varepsilon,m}$

$$d_p\widetilde{g}_{\varepsilon,m} = \begin{pmatrix} \widetilde{d}(z) & 0 & 2^m\varepsilon(x-a_i) \\ 0 & \frac{1}{p_i} & 0 \\ 0 & 0 & 2^m \end{pmatrix},$$

where $p = (x,y,z) \in Q_{i,k}$. If $(u,v,w) \in C_p$, then

$$d_p\widetilde{g}_{\varepsilon,m}(u,v,w) = \begin{pmatrix} \widetilde{d}(z)u + 2^m\varepsilon(x-a_i)w \\ \frac{1}{p_i}v \\ 2^m w \end{pmatrix}.$$

The estimate

$$\frac{|\widetilde{d}(z)u + 2^m\varepsilon(x-a_i)w|}{|2^m w|} \le \frac{\widetilde{d}(z)|u|}{2^m|w|} + 2\varepsilon$$
$$\le \frac{\lambda_i+\varepsilon}{2^m}\frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max} - \varepsilon} + 2\varepsilon \le \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max} - \varepsilon}$$

shows that $d_p\widetilde{g}_{\varepsilon,m}(C_p) \subset C_{\widetilde{g}_{\varepsilon,m}(p)}$. Now we prove the other statement of the Lemma. Assume that $p = (x_p, y_p, z_p) \in Q_i$ and $q = (x_q, y_q, z_q) \in Q_j$, $i \neq j$, are such that $\widetilde{g}_{\varepsilon,m}(p) = \widetilde{g}_{\varepsilon,m}(q) = (x, y, z)$. Then

$$p \in Q_i \quad \Rightarrow \quad d_p\widetilde{g}_{\varepsilon,m} : (u, v, 1) \mapsto 2^m \left(\frac{\widetilde{d}(z_p)}{2^m}u + (x_p - a_i)\varepsilon,\ \frac{v}{p_i},\ 1 \right).$$

Then

$$x_p = \frac{x - a_i(1-\widetilde{d}(z_p))}{\widetilde{d}(z_p)}, \quad x_q = \frac{x - a_j(1-\widetilde{d}(z_q))}{\widetilde{d}(z_q)}$$

and

$$d_p\widetilde{g}_{\varepsilon,m}(C_p) \subset \left\{ w(u,v,1) : \frac{x-a_i}{\widetilde{d}(z_p)}\varepsilon - \frac{2(\lambda_i+\varepsilon)\varepsilon}{2^m - \lambda_{\max} - \varepsilon} \right.$$
$$\left. \le u \le \frac{x-a_i}{\widetilde{d}(z_p)}\varepsilon + \frac{2(\lambda_i+\varepsilon)\varepsilon}{2^m - \lambda_{\max} - \varepsilon} \right\}.$$

Therefore

$$|u_2 - u_1| \ge \left(\left| \frac{x-a_i}{\widetilde{d}(z_p)} - \frac{x-a_j}{\widetilde{d}(z_q)} \right| - \frac{2(\lambda_i+\lambda_j+2\varepsilon)}{2^m - \lambda_{\max} - \varepsilon} \right)\varepsilon.$$

The term

$$\left| \frac{x-a_i}{\widetilde{d}(z_p)} - \frac{x-a_j}{\widetilde{d}(z_q)} \right|$$

can be estimated by

$$\left| \frac{x-a_i}{\widetilde{d}(z_p)} - \frac{x-a_j}{\widetilde{d}(z_q)} \right| \ge \left| \frac{|\widetilde{d}(z_p) - \widetilde{d}(z_q)||x| - |a_j\widetilde{d}(z_p) - a_i\widetilde{d}(z_q)|}{\widetilde{d}(z_p)\widetilde{d}(z_q)} \right|.$$

Hence, this term is positive provided that

$$|a_j\widetilde{d}(z_p) - a_i\widetilde{d}(z_q)| > |\widetilde{d}(z_p) - \widetilde{d}(z_q)|.$$

Since $\lambda_i - \varepsilon \le \widetilde{d}(z_p) \le \lambda_i + \varepsilon$ and $\lambda_j - \varepsilon \le \widetilde{d}(z_q) \le \lambda_j + \varepsilon$, this is implied by (1.9) if ε is sufficiently small.

If we let

$$C_{\varepsilon,m} = \frac{1}{2}\min_{i\neq j} \frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i\lambda_j},$$

then

$$|u_2 - u_1| \ge C_{\varepsilon,m}\varepsilon,$$

provided that ε is small and m large.

In fact we can let

$$C_{\varepsilon,m} = \sigma \min_{i \neq j} \frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i\lambda_j},$$

for $0 < \sigma < 1$. □

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