

Description of the free motion with momentums in Gödel's universe *

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Abstract

We study the geodesic motion in Gödel's universe, using conserved quantities. We give a necessary and sufficient condition for curves to be geodesic curves in terms of conserved quantities, which can be computed from the initial values of the curve. We check our result with numerical simulations too.

1 Introduction

Gödel's solution [2] for the Einstein equation gives a cosmological model of a rotating universe. This solution has many interesting properties, for example it contains closed timelike curves, geodesically complete, and has neither a singularity nor a horizon. The free motion of particles in this cosmological model was analyzed first by Kundt [6] and Chandrasekhar and Wright [1]. A physical model for the geodesic motion was given by Novello et. al. [7] using the effective potential. In this paper we describe the free motion with conserved quantities.

In the flat spacetime $(\mathbb{R}^4, g = \text{Diag}(1, -1, -1, -1))$ the equation for a geodesic curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^4$ is given by

$$\ddot{\gamma} = 0. \tag{1}$$

The solution is obvious $\gamma(u) = \underline{v}t + \underline{x}$. The vectors $\underline{v}, \underline{x}$ are given as initial values. In this case we can consider \underline{v} as a conserved quantity, since a curve γ describes a free motion if and only if $\dot{\gamma}(u) = \underline{v}$ for every $u \in \text{Dom}(\gamma)$. In this setting the physical meaning of \underline{v} is well-known. In the next section we give a similar description for geodesic motion in the Gödel spacetime and we check our result using numerical simulations. During the computation we use the Einstein summation and the usual differential geometrical formalism.

2 Momentums in the Gödel's universe

2.1 Geodesic lines

Gödel presented the line-element of his universe in the following form [2], [3, p. 195], [4, p. 275].

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Definition 2.1. We call the Riemannian manifold (\mathcal{M}, g) to *Gödel spacetime*, where the manifold is \mathcal{M} expressed in cylindrical coordinates $\mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi[\times \mathbb{R}$ and the Riemannian metric at a point $(t, r, \varphi, z) \in \mathcal{M}$ is

$$g(t, r, \varphi, z) = 4a^2 \begin{pmatrix} 1 & 0 & \sqrt{2} \sinh^2 r & 0 \\ 0 & -1 & 0 & 0 \\ \sqrt{2} \sinh^2(r) & 0 & \sinh^4(r) - \sinh^2(r) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (2)$$

where a is positive parameter.

The parameter a can be interpreted as $a = \frac{1}{\sqrt{2}}\omega$, where ω is a measure of the constant rotation [3, p. 191].

To get the differential equations of the geodesic curve we compute the second order Christoffel symbols, using the Equation

$$\Gamma_{ij}^m = \frac{1}{2} g^{km} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ji}), \quad (3)$$

where g^{km} denotes the (k, m) element of the inverse matrix of g . The nonzero symbols are the following (and their counterparts $\Gamma_{ij}^k = \Gamma_{ji}^k$).

$$\Gamma_{12}^1 = \frac{2 \sinh(r)}{\cosh(r)} \quad \Gamma_{23}^1 = \frac{\sqrt{2} \sinh^3(r)}{\cosh(r)} \quad (4)$$

$$\Gamma_{13}^2 = \frac{\sqrt{2} \sinh(r) \cosh(r)}{r^2} \quad \Gamma_{23}^1 = \frac{\sinh(r) \cosh(r) (2 \cosh^2(r) - 3)}{r^2} \quad (5)$$

$$\Gamma_{12}^3 = -\frac{\sqrt{2}}{\sinh(r) \cosh(r)} \quad \Gamma_{23}^3 = \frac{1}{\sinh(r) \cosh(r)} \quad (6)$$

A curve $\gamma : \mathbb{R} \rightarrow M$ is a geodesic line if

$$\ddot{\gamma}^k + \Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j = 0 \quad (7)$$

holds for $k = 1, 2, 3, 4$. If we write the curve as $\gamma(u) = (t(u), r(u), \varphi(u), z(u))$ then γ is a geodesic line if and only if the following equations hold.

$$\ddot{t} + \frac{4 \sinh(r)}{\cosh(r)} \dot{t} \dot{r} + \frac{2\sqrt{2} \sinh^3(r)}{\cosh(r)} \dot{\varphi} \dot{r} = 0 \quad (8)$$

$$\ddot{r} + 2\sqrt{2} \sinh(r) \cosh(r) \dot{t} \dot{\varphi} + \sinh(r) \cosh(r) (2 \cosh^2(r) - 3) (\dot{\varphi})^2 = 0 \quad (9)$$

$$\ddot{\varphi} \sinh(r) \cosh(r) - 2\sqrt{2} \dot{t} \dot{r} + 2\dot{\varphi} \dot{r} = 0 \quad (10)$$

$$\ddot{z} = 0, \quad (11)$$

where the dot $\dot{\cdot}$ denotes the derivation with respect to u . It is known from the theory of differential equations that, there exists a unique solution for every initial value [8].

2.2 Description of the free motion with conserved quantities

In this subsection we derive a 4×4 matrix M , which mixes the components of $\dot{\gamma}$, where γ is a curve in the Gödel spacetime, such that γ is a geodesic curve, if and only if $M\dot{\gamma}$ is constant. Since during the geodesic motion $M\dot{\gamma}$ is constant, like the impulse momentum in the classical case, we call this matrix M to momentum matrix. First we define this matrix, and then we prove its mentioned properties.

Definition 2.2. For parameters $\rho_1, \rho_2 \in \mathbb{R}$, $r \in \mathbb{R}^+$ and $\varphi \in [0, 2\pi[$ let us define the *momentum matrix* $M(\rho_1, \rho_2, r, \varphi)$

$$\begin{pmatrix} \rho_2 + \frac{\rho_1 \cosh^2(r)}{2} & 0 & \frac{(\rho_1 \cosh^2(r) + 4\rho_2) \sinh^2(r)}{2\sqrt{2}} & 0 \\ -\sqrt{2} \cos(\varphi) \sinh(2r) & \sin(\varphi) & -\frac{\cos(\varphi)(\cosh(2r) - 2) \sinh(2r)}{2} & 0 \\ \sqrt{2} \sin(\varphi) \sinh(2r) & \cos(\varphi) & \frac{\sin(\varphi)(\cosh(2r) - 2) \sinh(2r)}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (12)$$

Theorem 2.3. For every geodesic line $\gamma(u) = (t(u), r(u), \varphi(u), z(u))$ and for every real parameters ρ_1, ρ_2 and there exist real parameters c_1, c_2, c_3 and $f(\rho_1, \rho_2)$ depending just on the initial conditions $\gamma(u_0)$ and $\dot{\gamma}(u_0)$, such that

$$M(\rho_1, \rho_2, r, \varphi) \cdot \begin{pmatrix} \dot{t} \\ \dot{r} \\ \dot{\varphi} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} f(\rho_1, \rho_2) \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad (13)$$

holds at every point of $\text{Dom}\gamma$. If a curve $\gamma : \mathbb{R} \rightarrow M$ satisfies Equation (13) at every point of $\text{Dom}\gamma$, then γ is a geodesic curve.

Proof. Since the Theorem is obviously valid for the last component z , we skip it during the proof. Consider the left hand side of the differential Equations (8, 9, 10) as a vector $\underline{X} = ((8), (9), (10))$. Assume that $D(t, r, \varphi)$ is a 3×3 invertible matrix, which entries are smooth functions of t, r and φ . Assume that the equation

$$\frac{d}{du} \left(D(t(u), r(u), \varphi(u)) \cdot \dot{\gamma}(u) \right) = D(t(u), r(u), \varphi(u)) \cdot \underline{X}(u) \quad (14)$$

holds.

If D consists of the column-vectors $\underline{d}_1, \underline{d}_2$ and \underline{d}_3 , then the Equation (14) can be written as

$$\begin{aligned} & \underline{d}_1 \ddot{t} + \underline{d}_2 \ddot{r} + \underline{d}_3 \ddot{\varphi} + \frac{\partial \underline{d}_1}{\partial t} (\dot{t})^2 + \frac{\partial \underline{d}_2}{\partial r} (\dot{r})^2 + \frac{\partial \underline{d}_3}{\partial \varphi} (\dot{\varphi})^2 + \\ & + \left(\frac{\partial \underline{d}_1}{\partial r} + \frac{\partial \underline{d}_2}{\partial t} \right) \dot{t} \dot{r} + \left(\frac{\partial \underline{d}_1}{\partial \varphi} + \frac{\partial \underline{d}_3}{\partial t} \right) \dot{t} \dot{\varphi} + \left(\frac{\partial \underline{d}_2}{\partial \varphi} + \frac{\partial \underline{d}_3}{\partial r} \right) \dot{\varphi} \dot{r} = \\ & = \underline{d}_1 \ddot{t} + \underline{d}_2 \ddot{r} + \underline{d}_3 \ddot{\varphi} + \sinh(r) \cosh(r) (2 \cosh^2(r) - 3) \underline{d}_2 (\dot{\varphi})^2 + \\ & + \left(4 \frac{\sinh(r)}{\cosh(r)} \underline{d}_1 - \frac{2\sqrt{2}}{\sinh(r) \cosh(r)} \underline{d}_3 \right) \dot{t} \dot{r} + 2\sqrt{2} \sinh(r) \cosh(r) \underline{d}_2 \dot{t} \dot{\varphi} + \\ & + \left(2\sqrt{2} \frac{\sinh^3(r)}{\cosh(r)} \underline{d}_1 + \frac{2}{\sinh(r) \cosh(r)} \underline{d}_3 \right) \dot{\varphi} \dot{r}. \end{aligned} \quad (15)$$

After elementary calculations we have the set of differential equations for the vectors $(\underline{d}_i)_{i=1,2,3}$.

$$\frac{\partial \underline{d}_1}{\partial t} = 0 \quad (16)$$

$$\frac{\partial \underline{d}_1}{\partial r} = -\frac{\partial \underline{d}_2}{\partial t} - \frac{2\sqrt{2}}{\sinh(r) \cosh(r)} \underline{d}_3 + 4\underline{d}_1 \tanh(r) \quad (17)$$

$$\frac{\partial \underline{d}_1}{\partial \varphi} = -\frac{\partial \underline{d}_3}{\partial t} + \frac{2 \sinh(2r)}{\sqrt{2}} \underline{d}_2 \quad (18)$$

$$\frac{\partial \underline{d}_2}{\partial r} = 0 \quad (19)$$

$$\frac{\partial \underline{d}_2}{\partial \varphi} = -\frac{\partial \underline{d}_3}{\partial r} + \frac{2}{\sinh(r) \cosh(r)} \underline{d}_3 + 2\sqrt{2} \cosh(r) \sinh(r) \underline{d}_1 - 2\sqrt{2} \underline{d}_1 \tanh(r) \quad (20)$$

$$\frac{\partial \underline{d}_3}{\partial \varphi} = \frac{1}{2} \underline{d}_2 (\cosh(2r) - 2) \sinh(2r) \quad (21)$$

Let us note, that the same set of differential equations are valid for all components of the vectors $\underline{d}_1, \underline{d}_2, \underline{d}_3$. Therefore it is enough to consider just the first component of the vectors, which will be denoted with d_1, d_2 and d_3 . (That is for index $i \in \{1, 2, 3\}$, d_i denotes the first component of the vector \underline{d}_i .)

Equations (16,19) imply that $d_1 = d_1(r, \varphi)$ and $d_2 = d_2(t, \varphi)$. If we compute the derivative of the Equation (18) with respect to φ and t , using Equation (21) we have the following equation for d_2 .

$$\left(\frac{\cosh(2r) - 2}{2} \right) \frac{\partial^2 d_2}{\partial t^2} = \sqrt{2} \frac{\partial^2 d_2}{\partial t \partial \varphi} \quad (22)$$

Computing the derivative of the previous Equation with respect to r , and taking into account the Equation (19) we have the following form for d_2 .

$$d_2(t, \varphi) = b_1(\varphi)t + b_2(\varphi) \quad (23)$$

Using this form for d_2 Equation (21) gives

$$d_3(t, r, \varphi) = \frac{1}{2} \left(t \int b_1(\varphi) d\varphi + \int b_2(\varphi) d\varphi \right) \cdot (\cosh(2r) - 2) \sinh(2r) + e(r, t), \quad (24)$$

where $e(r, t)$ is a smooth function. Substituting this form of d_3 to the Equation (17) and computing the derivative with respect t , we have

$$0 = -2\sqrt{2}(\cosh(2r) - 2) \int b_1(\varphi) d\varphi - \frac{2\sqrt{2}}{\cosh(r) \sinh(r)} \frac{\partial e(r, t)}{\partial t}. \quad (25)$$

This means, that there exists a real number $\lambda \in \mathbb{R}$, such that

$$\int b_1(\varphi) d\varphi = \lambda, \quad \text{and} \quad \frac{1}{(\cosh(2r) - 2) \cosh(r) \sinh(r)} \frac{\partial e(r, t)}{\partial t} = -\lambda \quad (26)$$

holds. This means that $b_1(\varphi) = 0$. From this $d_2 = b_2(\varphi)$ and $e(r, t) = g(r)$ follows. So far, we have the following forms for the functions $(d_i)_{i=1,2,3}$.

$$d_1 = \sqrt{2} \sinh(2r) \int b_2(\varphi) d\varphi + f(r) \quad (27)$$

$$d_2 = b_2(\varphi) \quad (28)$$

$$d_3 = \frac{1}{2} \left(\int b_2(\varphi) d\varphi \right) (\cosh(2r) - 2) \sinh(2r) + g(r) \quad (29)$$

Where Equation (27) comes from Equation (18). Let us substitute the Equations (27,28,29) into the Equation (20).

$$\begin{aligned} \frac{d b_2}{d \varphi} &= -\frac{1}{2} \left(\int b_2(\varphi) d\varphi \right) \left(2(\cosh(2r) - 2) \cosh(2r) + 2 \sinh^2(2r) \right) - \frac{d g}{d r}(r) \\ &+ \frac{2}{\cosh(r) \sinh(r)} \left(\frac{1}{2} \left(\int b_2(\varphi) d\varphi \right) (\cosh(2r) - 2) \sinh(2r) + g(r) \right) \\ &+ 2\sqrt{2} \left(\sqrt{2} \sinh(2r) \int b_2(\varphi) d\varphi + f(r) \right) \cosh(r) \sinh(r) \\ &- 2\sqrt{2} \left(\sqrt{2} \sinh(2r) \int b_2(\varphi) d\varphi + f(r) \right) \frac{\sinh(r)}{\cosh(r)}. \end{aligned} \quad (30)$$

After simplifications we get the following equation.

$$\begin{aligned} \frac{d b_2(\varphi)}{d \varphi} + \int b_2(\varphi) d\varphi \\ = -\frac{d g(r)}{d r} + \frac{2g(r)}{\cosh(r) \sinh(r)} + 2\sqrt{2} f(r) \cosh(r) \sinh(r) - 2\sqrt{2} f(r) \tanh(r) \end{aligned} \quad (31)$$

This implies that

$$\frac{d b_2(\varphi)}{d \varphi} + \int b_2(\varphi) d\varphi = \alpha \quad (32)$$

holds, for a real constant α . This gives the explicit form of b_2

$$b_2(\varphi) = k_1 \cos(\varphi) + k_2 \sin(\varphi), \quad (33)$$

where $k_1, k_2 \in \mathbb{R}$. After this form of b_2 we rewrite the Equations (27,28,29).

$$d_1 = \sqrt{2} \left(k_1 \sin(\varphi) - k_2 \cos(\varphi) \right) \sinh(2r) + f^*(r) \quad (34)$$

$$d_2 = k_1 \cos(\varphi) + k_2 \sin(\varphi) \quad (35)$$

$$d_3 = \frac{1}{2} \left(k_1 \sin(\varphi) - k_2 \cos(\varphi) \right) \left(\cosh(2r) - 2 \right) \sinh(2r) + g^*(r) \quad (36)$$

If we substitute Equations (34,35,36) into Equation (17), after some simplification we have

$$\frac{d f^*}{d r}(r) = \frac{-4\sqrt{2}g^*(r)}{\sinh(2r)} + 4f^*(r) \tanh(r). \quad (37)$$

And if we substitute Equations (34,35,36) into Equation (20), then after simplifications we get

$$\frac{d g^*}{d r}(r) = \frac{4(g^*(r) + \sqrt{2}f^*(r) \sinh^4(r))}{\sinh(2r)}. \quad (38)$$

If we express the function $g^*(r)$ from Equation (37) and substitute it to the Equation (38) then we get a solvable differential equation.

$$2 \cosh(2r) \frac{d f^*}{d r}(r) = \sinh(2r) \frac{d^2 f^*}{d r^2}(r) \quad (39)$$

The general solution is

$$f^*(r) = \rho_2 + \frac{1}{2} \rho_1 \cosh(r)^2, \quad (40)$$

where $\rho_1, \rho_2 \in \mathbb{R}$. After this, we can compute the function g^* .

$$g^*(r) = \frac{(\rho_1 + 8\rho_2 + \rho_1 \cosh(2r)) \sinh^2(r)}{4\sqrt{2}}. \quad (41)$$

We determined the first components of the vectors $\underline{d}_1, \underline{d}_2$ and \underline{d}_3 . This means, that the column-vectors of the matrix D are of the following form.

$$\underline{d}_1 = \sqrt{2}(\underline{k}_1 \sin(\varphi) - \underline{k}_2 \cos(\varphi)) \sinh(2r) + \underline{1}(\rho_2 + \frac{1}{2} \rho_1 \cosh^2(r)) \quad (42)$$

$$\underline{d}_2 = \underline{k}_1 \cos(\varphi) + \underline{k}_2 \sin(\varphi) \quad (43)$$

$$\begin{aligned} \underline{d}_3 = & \frac{1}{2}(\underline{k}_1 \sin(\varphi) - \underline{k}_2 \cos(\varphi))(\cosh(2r) - 2) \sinh(2r) + \\ & + \underline{1} \frac{(g_1 + 8g_2 + g_1 \cosh(2r)) \sinh^2(r)}{4\sqrt{2}}, \end{aligned} \quad (44)$$

where $\underline{1} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and the vectors $\underline{k}_1, \underline{k}_2$ and $\underline{1}$ are linearly independent, since we assumed, that the matrix D is invertible. Moreover we can assume that the vectors $\underline{k}_1, \underline{k}_2$ and $\underline{1}$ form an orthonormal basis in \mathbb{R}^3 . According to this, if we multiply from left the matrix \underline{D} with the matrix L which has row-vectors $(\underline{1}, \underline{k}_2, \underline{k}_1)$ we get the matrix $M(\rho_1, \rho_2, r, \varphi)$, which is independent of t . This proves the Theorem. \square

2.3 Numerical verification of the Theorem

Now we give a numerical verification of the previous Theorem. First we note that enough to consider two linearly independent constants vectors $(\rho_1, \rho_2) \in \mathbb{R}^2$ and $(\rho'_1, \rho'_2) \in \mathbb{R}^2$ in the matrix M , since easy to check that the function $f(\rho_1, \rho_2)$ is linear

$$f(\rho_1, \rho_2) + f(\rho'_1, \rho'_2) = f(\rho_1 + \rho'_1, \rho_2 + \rho'_2). \quad (45)$$

We choose the matrices $M(0, 1, t, r, \varphi)$ and $M\left(1, -\frac{1}{4}, t, r, \varphi\right)$. For these M matrices the Equation (13) can be written in the following form. (We skip the z component, since this part of the Theorem

is obvious.)

$$\dot{t}(u) + \sqrt{2} \sinh^2(r) \dot{\varphi}(u) = f(0, 1) \quad (46)$$

$$\frac{\cosh(2r)}{4} \dot{t}(u) + \frac{\sinh^4(r)}{2\sqrt{2}} \dot{\varphi}(u) = f\left(1, -\frac{1}{4}\right) \quad (47)$$

$$\begin{aligned} & -\sqrt{2} \cos(\varphi) \sinh(2r) \dot{t}(u) + \sin(\varphi) \dot{r}(u) \\ & - \frac{\cos(\varphi)(\cosh(2r) - 2) \sinh(2r)}{2} \dot{\varphi}(u) = c_1 \end{aligned} \quad (48)$$

$$\begin{aligned} & \sqrt{2} \sin(\varphi) \sinh(2r) \dot{t}(u) + \cos(\varphi) \dot{r}(u) \\ & + \frac{\sin(\varphi)(\cosh(2r) - 2) \sinh(2r)}{2} \dot{\varphi}(u) = c_2 \end{aligned} \quad (49)$$

Since the set of differential equations for the geodesic line is invariant with respect to the translation for variables t , z and φ , we can assume that $t(0) = 0$, $\varphi(0) = 0$ and $z(0) = 0$. We choose the other initial values for the geodesic lines randomly. For example we can choose the initial values $r(0) = 1.33$, $\dot{r}(0) = 0.78$, $\dot{t}(0) = 0.81$ and $\dot{\varphi}(0) = 0.56$. From these values we have the exact numbers $f(0, 1)$, $f(1, -\frac{1}{4})$, c_1 and c_2 from Equations (46-49). We use the MAPLE software, *dsolve* package, *dverk78* method to solve numerically the geodesic equations with these initial values on the interval $[0, 2]$. We set the absolute and relative error for the solution to be 10^{-8} . We compute the quantities of the left hand side in the Equations (46-49) at the points $u = k \cdot 0.005$ ($1 \leq k \leq 400$), and we compare the computed values to the exact ones. In this case the biggest error for Equation (46) is $E_1 = 0.5 \times 10^{-8}$, for Equation (47) is $E_2 = 0.5 \times 10^{-8}$, for Equation (48) is $E_3 = 0.4 \times 10^{-7}$ and for Equation (49) is $E_4 = 0.54 \times 10^{-8}$. The results of 20 simulations are shown in Table 1.

It is clear from the table, that the errors are acceptable with respect to the computational precision. This gives a numerical verification of the Theorem.

2.4 Some simple consequences

If we choose the parameters in the matrix M to be $(\rho_1, \rho_2) = (2, 0)$ and $(\rho_1, \rho_2) = (0, 1)$ then the first components of the Equation (13) are

$$\dot{t} + \frac{\sinh^2(r)}{\sqrt{2}} \dot{\varphi} = \frac{f(2, 0)}{\cosh^2(r)} \quad (50)$$

$$\dot{t} + \sqrt{2} \sinh^2(r) \dot{\varphi} = f(0, 1). \quad (51)$$

Solving these Equations for \dot{t} and $\dot{\varphi}$ we have

$$\dot{t} = \frac{2f(2, 0)}{\cosh^2(r)} - f(0, 1) \quad (52)$$

$$\dot{\varphi} = \frac{\sqrt{2}}{\sinh^2(r)} \left(f(0, 1) - \frac{f(2, 0)}{\cosh^2(r)} \right). \quad (53)$$

So the functions \dot{t} and $\dot{\varphi}$ depend just on the initial conditions and on r . Moreover we can define a critical radius $R_t = \text{arch} \sqrt{\frac{2f(2,0)}{f(0,1)}}$ (if exists) from the initial conditions, where $\dot{t} = 0$. If $r > R_t$ then the particle moving backward in time and if $r < R_t$ then moving forward.

This leads us to the following Lemma.

Table 1:

$r(0)$	$\dot{r}(0)$	$\dot{t}(0)$	$\dot{\phi}(0)$	$E_1 \times 10^8$	$E_2 \times 10^8$	$E_3 \times 10^8$	$E_4 \times 10^8$
1.33	0.78	0.81	0.56	0.5	0.5	4	0.54
1.14	-0.21	0.26	-1.37	0.7	0.6	2.5	0.42
0.5	-0.97	0.67	0.14	0.25	0.13	0.73	3
0.68	-1.05	0.97	-0.34	0.2	0.13	6.4	4
0.1	1.43	1.29	-0.37	0.4	0.19	0.38	0.8
0.84	-0.56	1.36	1.11	0.2	0.2	0.8	0.28
0.62	0.96	0.3	-0.69	0.07	0.04	26.13	6
1.26	0.13	0.75	0.37	0.3	0.3	2	0.2
0.29	0.32	1.26	0.67	0.1	0.1	0.15	0.06
0.11	-0.52	1.23	-0.03	0.1	0.1	0.1	0.11
1.28	-0.31	0.67	-0.58	0.4	0.26	1.7	0.6
0.14	-0.82	1.4	-0.51	0.1	0.2	0.17	0.19
0.35	0.92	0.02	0.43	0.04	0.02	5.32	1
1.24	0.79	0.69	1.25	0.8	0.8	5	0.8
0.15	-0.17	1.49	0.14	0.1	0.1	0.15	0.3
0.3	0.45	0.35	0.97	0.09	0.06	0.11	0.28
0.41	0.69	1.49	-0.08	0.1	0.1	0.4	0.19
0.7	-0.2	1.41	1.48	0.1	0.2	0.06	0.04
0.2	0.17	1.14	-1.06	0.1	0.1	0.12	0.04
0.36	-0.84	1.31	-1.08	0.2	0.1	0.3	0.18

Lemma 2.4. *If $f(2,0) \geq 0$ and $f(0,1) \leq 0$ then $\dot{t}(u) \geq 0$ for every $u \in \text{Dom}\gamma$. If $f(2,0) \leq 0$ and $f(0,1) \geq 0$ then $\dot{t}(u) \leq 0$ for every $u \in \text{Dom}\gamma$.*

Let us compute the inverse of the matrix M

$$M^{-1} = \frac{1}{\kappa} \begin{pmatrix} 8 - 4 \cosh(2r) & -\frac{\sigma \cos(\varphi) \tanh(r)}{\sqrt{2}} & \frac{\sigma \sin(\varphi) \tanh(r)}{\sqrt{2}} & 0 \\ 0 & \kappa \sin(\varphi) & \kappa \cos(\varphi) & 0 \\ 8\sqrt{2} & \frac{2\sigma \cos(\varphi)}{\sinh(2r)} & \frac{-2\sigma \sin(\varphi)}{\sinh(2r)} & 0 \\ 0 & 0 & 0 & \kappa \end{pmatrix}, \quad (54)$$

where

$$\kappa = \rho_1 + (\rho_1 + 4\rho_2) \cosh(2r) \quad \text{and} \quad \sigma = \rho_1 + 8\rho_2 + \rho_1 \cosh(2r). \quad (55)$$

From these Equations proves the following Lemma.

Lemma 2.5. *We have correspondences between the functions r and φ*

$$\dot{r} = c_1 \sin(\varphi) + c_2 \cos(\varphi) \quad (56)$$

$$\dot{\varphi} = \frac{2}{\sinh(2r)} (c_1 \cos(\varphi) - c_2 \sin(\varphi)) + f(2,0) \frac{2\sqrt{2}}{\cosh^2(r)}. \quad (57)$$

Which means that the function \dot{r} is bounded

$$|\dot{r}| \leq \sqrt{c_1^2 + c_2^2} \quad (58)$$

and we have a bound for $\dot{\varphi}$

$$|\dot{\varphi}| \leq \frac{2\sqrt{c_1^2 + c_2^2}}{\sinh(2r)} + f(2, 0) \frac{2\sqrt{2}}{\cosh^2(r)}. \quad (59)$$

From Equations (53,56) we get

$$\frac{dr}{d\varphi} = \frac{c_1 \sin(\varphi) + c_2 \cos(\varphi)}{\frac{\sqrt{2}}{\sinh^2(r)} \left(f(0, 1) - \frac{f(2, 0)}{\cosh^2(r)} \right)}. \quad (60)$$

After integration we have the following Lemma.

Lemma 2.6. *There is a real parameter m such that*

$$c_2 \sin(\varphi) - c_1 \cos(\varphi) = \frac{2\sqrt{2}}{\sinh(2r)} (f(2, 0) \cosh(2r) - f(0, 1) \cosh^2(r)) + m \quad (61)$$

holds.

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