ON THE ABSOLUTE CONTINUITY OF THE BLACKWELL MEASURE

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Abstract. In 1957, Blackwell expressed the entropy of hidden Markov chains using a measure which can be characterised as an invariant measure for linear fractional transformations with a certain class of rational weights. This measure, called the Blackwell measure, plays a central role in understanding the entropy rate and other important characteristics of fundamental models in information theory. We show that for a suitable set of parameter values the Blackwell measure is absolutely continuous for almost every parameter in the case of binary symmetric channels.

1. Introduction and Statements

Blackwell [1] expressed the entropy for hidden Markov chains using a measure which is called the *Blackwell measure* and can be characterised as an invariant measure of an Iterated Function System (IFS). The properties of the Blackwell measure are examined by several papers, for example [6, 7, 10, 13] etc. Blackwell showed some examples, where the support of the measure is at most countable, hence, the measure is singular, see [1, Section 3]. In our paper we focus on the Blackwell measure defined by the binary-symmetric channel with crossover probability ε. Bárány, Pollicott and Simon showed a set of parameters, where the measure is singular, see [3, Theorem 1]. Our goal is to give a set of parameters for which the Blackwell measure is absolutely continuous (a.c.) with respect to the Lebesgue measure. To the best of our knowledge, absolute continuity of the Blackwell measure has not been proved for any example before.

Let us introduce the basic notations for the binary symmetric channel. Let $X := \{X_i\}_{i=-\infty}^{\infty}$ be a binary, symmetric, stationary, ergodic Markov chain source, $X_i \in \{0,1\}$ with probability transition matrix

$$\Pi := \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

Then it is well known that the entropy $H(X)$ is given by

$$H(X) = -p\log p - (1-p)\log(1-p).$$

By adding to X a binary independent and identically distributed (i.i.d.) noise $E = \{E_i\}_{i=-\infty}^{\infty}$ with

$$\mathbb{P}(E_i = 0) = 1-\varepsilon, \qquad \mathbb{P}(E_i = 1) = \varepsilon,$$

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we get a Markov chain $Y := \{Y_i\}_{i=-\infty}^{\infty}$, $Y_i = (X_i, E_i)$ with states $\{(0,0),(0,1),(1,0),(1,1)\}$ and transition probabilities:

$$M := \begin{bmatrix} p(1-\varepsilon) & p\varepsilon & (1-p)(1-\varepsilon) & (1-p)\varepsilon \\ p(1-\varepsilon) & p\varepsilon & (1-p)(1-\varepsilon) & (1-p)\varepsilon \\ (1-p)(1-\varepsilon) & (1-p)\varepsilon & p(1-\varepsilon) & p\varepsilon \\ (1-p)(1-\varepsilon) & (1-p)\varepsilon & p(1-\varepsilon) & p\varepsilon \end{bmatrix}.$$

Let $\Psi : \{(0,0),(0,1),(1,0),(1,1)\} \mapsto \{0,1\}$ be a surjective map such that

$$\Psi(0,0) = \Psi(1,1) = 0 \text{ and } \Psi(0,1) = \Psi(1,0) = 1.$$

We consider the ergodic stationary process $Z = \{Z_i = \Psi(Y_i)\}_{i=-\infty}^{\infty}$, is the corrupted output of the channel. Equivalently, Z is the stationary stochastic process

$$Z_i = X_i \bigoplus E_i,$$

where $\bigoplus$ denotes the binary addition. According to [6, Example 4.1] and [3, Section 3.1,3.2], the entropy of Z can be characterized as

$$H(Z) = -\int_0^1 p_0^{\varepsilon,p}(x)\log p_0^{\varepsilon,p}(x) + p_1^{\varepsilon,p}(x)\log p_1^{\varepsilon,p}(x) d\mu_{\varepsilon,p}(x),$$

where the Blackwell measure $\mu_{\varepsilon,p}$ can be obtained as follows. Let $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$ be a set of functions on the interval $[0,1]$,

$$S_0^{\varepsilon,p}(x) := \frac{x\cdot p\cdot(1-\varepsilon) + (1-x)\cdot(1-p)\cdot(1-\varepsilon)}{x\cdot[p(1-\varepsilon)+(1-p)\cdot\varepsilon] + (1-x)\cdot[(1-p)(1-\varepsilon)+p\cdot\varepsilon]}, \tag{1.1}$$

$$S_1^{\varepsilon,p}(x) := \frac{x\cdot p\cdot\varepsilon + (1-x)\cdot(1-p)\cdot\varepsilon}{x\cdot[p\varepsilon+(1-p)\cdot(1-\varepsilon)] + (1-x)\cdot[(1-p)\varepsilon+p\cdot(1-\varepsilon)]}. \tag{1.2}$$

We call $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$ an iterated function system (IFS) on $[0,1]$. Further, let us define two other functions on the interval $[0,1]$

$$p_0^{\varepsilon,p}(x) = x\cdot[p(1-\varepsilon)+(1-p)\cdot\varepsilon] + (1-x)\cdot[(1-p)(1-\varepsilon)+p\cdot\varepsilon], \tag{1.3}$$

$$p_1^{\varepsilon,p}(x) = x\cdot[p\varepsilon+(1-p)\cdot(1-\varepsilon)] + (1-x)\cdot[(1-p)\varepsilon+p\cdot(1-\varepsilon)]. \tag{1.4}$$

Since for every $x \in [0,1]$, $p_0^{\varepsilon,p}(x), p_1^{\varepsilon,p}(x) > 0$ and $p_0^{\varepsilon,p}(x) + p_1^{\varepsilon,p}(x) \equiv 1$, the functions $(p_0^{\varepsilon,p}, p_1^{\varepsilon,p})$ can be interpreted as a place-dependent probability vector. Then the Blackwell measure $\mu_{\varepsilon,p}$ is the unique measure that satisfies the following relation for every Borel set B with the conditions above (see [5, Theorem 1.1])

$$\mu_{\varepsilon,p}(B) = \int_{(S_0^{\varepsilon,p})^{-1}B} p_0^{\varepsilon,p}(x) d\mu_{\varepsilon,p}(x) + \int_{(S_1^{\varepsilon,p})^{-1}B} p_1^{\varepsilon,p}(x) d\mu_{\varepsilon,p}(x). \tag{1.5}$$

As we have mentioned before, our main result shows a set of parameters $(\varepsilon,p) \in (0,1)^2$ such that the Blackwell measure $\mu_{\varepsilon,p} \ll \mathcal{L}$ (shown in Figure 1), where $\mathcal{L}$ denotes the Lebesgue measure on the real line.

Theorem 1.1 (Main Theorem)**.** *The Blackwell measure $\mu_{\varepsilon,p}$ is singular in the blue region and absolutely continuous for every $\varepsilon \neq 1/2$ and Lebesgue almost every p in the red region marked on Figure 1.*

Remark 1.2. *The singularity region of the measure was already showed in* [3, Theorem 2]*. We will prove the absolute continuity part of the theorem and precisely characterize the region of absolute continuity later, see Sections 3 and 4.*

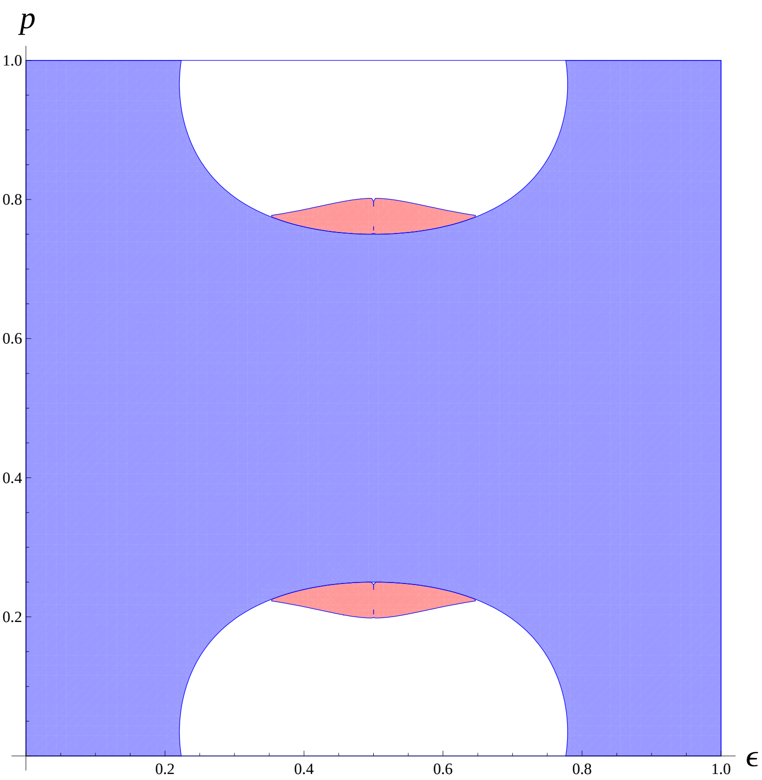

FIGURE 1. The absolute continuity (red region) and the singularity region (blue region) of the Blackwell measure $\mu_{\varepsilon,p}$.

We note that the IFS $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$ is not contracting for every $0 < \varepsilon, p < 1$. The IFS is contracting if $\sup_{x\in[0,1]} |\left(S_i^{\varepsilon,p}\right)'(x)| < 1$ for every $i = 0, 1$. We will restrict ourselves to the set of parameters ε, p such that the IFS is strictly contracting later. Then there exists a unique non-empty compact set $\Lambda_{\varepsilon,p}$ such that $\Lambda_{\varepsilon,p} = S_0^{\varepsilon,p}(\Lambda_{\varepsilon,p}) \cup S_1^{\varepsilon,p}(\Lambda_{\varepsilon,p})$, see [4]. $\Lambda_{\varepsilon,p}$ is the attractor of the IFS $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$. The measure $\mu_{\varepsilon,p}$ is an invariant measure of the IFS $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$ with place-dependent probabilities $\{p_0^{\varepsilon,p}(\cdot), p_1^{\varepsilon,p}(\cdot)\}$ and the support of $\mu_{\varepsilon,p}$ is $\Lambda_{\varepsilon,p}$.

Corollary 1.3. *The Blackwell measure $\mu_{\varepsilon,p}$ is equivalent to the measure $\mathcal{L}|_{\Lambda_{\varepsilon,p}}$ for every $\varepsilon \neq 1/2$ and Lebesgue almost every p in the red region marked on Figure 1.*

Proof. The statement follows immediately from Theorem 1.1 and [8, Theorem 1.1]. □

The properties of invariant measures of iterated function systems have been studied by several authors, for example [11, 15], etc. They considered a family of parameterised IFSs and used the so-called transversality condition, introduced by Pollicott and Simon in [12] (see precise definition in Section 2) to prove absolute continuity or to calculate the Hausdorff dimension of invariant measures. However, the studied invariant measures were not place-dependent probability measures. There were no tools for proving absolute continuity in place-dependent case in lately. In [2] there was given a sufficient condition for calculating the Hausdorff-dimension and for proving absolute continuity for place-dependent invariant probability measures, which used also the transversality condition.

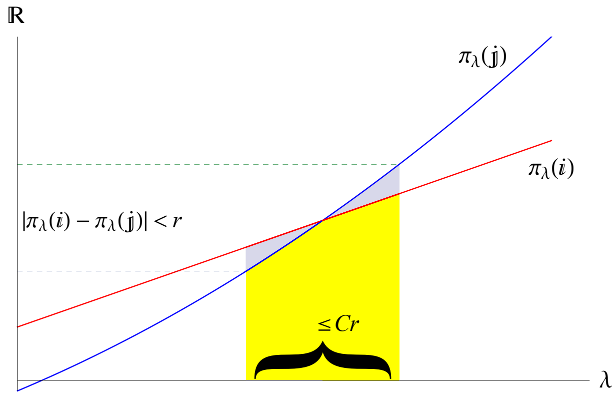

FIGURE 2. The transversality condition for one parameter.

Structure of the paper. In Section 2 we give a short overview of the main tool of the proof, the transversality condition. A sketch of the proof is also given. Section 3 determines the set of parameters $(\varepsilon, p) \subset (0,1)^2$ for which the transversality condition holds and finally Theorem 1.1 is proved in Section 4.

2. Transversality methods for place-dependent invariant measures

This section is devoted to introduce the definition of transversality condition and state the results about place-dependent probability measures.

Denote by $\mathcal{S} = \{1, \dots, k\}$ the set of symbols. Let X be a compact interval on the real line and $U \subset \mathbb{R}^d$ be an open, bounded set. Let us consider a parameterized family of IFSs $\Psi_\lambda = \left\{f_i^\lambda : X \mapsto X\right\}_{i \in \mathcal{S}}$, $\lambda \in \overline{U}$, such that there exist $0 < \alpha < \beta < 1$ that $\alpha < \left|(f_i^\lambda)'(x)\right| < \beta$ for every $x \in X$, $\lambda \in \overline{U}$ and $i \in \mathcal{S}$.

Let $\Sigma = S^{\mathbb{N}}$ be the symbolic space. Let us define the natural projection from the symbolic space to the compact interval X as follows

$$\pi_\lambda(\mathbf{i}) := \lim_{n \to \infty} f_{i_0}^\lambda \circ \cdots \circ f_{i_n}^\lambda(x) \text{ for } \mathbf{i} = (i_0, i_1, \dots) \in \Sigma.$$

Since the functions f_i are uniformly contracting, the function $\pi : \Sigma \times \overline{U} \mapsto X$ is well defined. Moreover, let us assume that the functions $\lambda \mapsto f_i^\lambda$ are uniformly continuous from $\overline{U}$ to $C^{1+\theta}(X)$.

Definition 2.1. *We say that Ψ_λ satisfies the* **transversality condition** *on the open, bounded set $U \subset \mathbb{R}^d$, if there exists a constant $C > 0$ such that for any $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$*

$$\mathcal{L}_d\left(\lambda \in U : |\pi_\lambda(\mathbf{i}) - \pi_\lambda(\mathbf{j})| < r\right) < Cr \text{ for every } r > 0, \tag{2.1}$$

where $\mathcal{L}_d$ is the d-dimensional Lebesgue measure.

Let $P_\lambda = \{p_i^\lambda : X \mapsto (0,1)\}_{i \in \mathcal{S}}$ be a parameterized family of Hölder continuous place-dependent probabilities, i.e. $\sum_{i \in \mathcal{S}} p_i^\lambda(x) \equiv 1$ for every $\lambda \in \overline{U}$. Moreover, suppose that the the functions $\lambda \mapsto p_i^\lambda$ are uniformly continuous from $\overline{U}$ to $C^\theta(X, (0,1))$. There exists a unique corresponding place-dependent invariant measure μ_λ which satisfies

$$\mu_\lambda(B) = \sum_{i\in\mathcal{S}} \int_{\left(f_i^\lambda\right)^{-1}(B)} p_i^\lambda(x) d\mu_\lambda(x) \text{ for every Borel set } B.$$

The existence and uniqueness of such measure follows from [5]. Let us define the entropy $h(\mu_\lambda)$ and Lyapunov exponent $\chi(\mu_\lambda)$ of measure μ_λ as

$$h(\mu_\lambda) = -\int \sum_{i\in\mathcal{S}} p_i^\lambda(x) \log p_i^\lambda(x) d\mu_\lambda(x), \tag{2.2}$$

$$\chi(\mu_\lambda) = -\int \sum_{i\in\mathcal{S}} p_i^\lambda(x) \log \left|(f_i^\lambda)'(x)\right| d\mu_\lambda(x). \tag{2.3}$$

According to the result of Jaroszewska and Rams [9], the quotient $h(\mu_\lambda)/\chi(\mu_\lambda)$ is an upper bound for the Hausdorff dimension of the measure μ_λ for every $\lambda \in U$. Therefore, $h(\mu_\lambda)/\chi(\mu_\lambda) > 1$ is a necessary condition to prove absolute continuity of μ_λ.

Theorem 2.2. [2, Theorem 1.1(2)] *Suppose that all of the conditions above are satisfied. Then μ_λ is absolutely continuous w.r.t. the Lebesgue measure for $\mathcal{L}_d$ almost every $\lambda \in \{\lambda \in U : h(\mu_\lambda)/\chi(\mu_\lambda) > 1\}$.*

In general, to prove that the measure μ_λ is absolutely continuous it is sufficient to show that the Radon-Nykodim derivative of μ_λ w.r.t. Lebesgue measure exists for μ_λ almost-every point. The standard technique to prove that fact for $\mathcal{L}_d$-a.e. λ is to prove

$$\iint \liminf_{r\to 0} \frac{\mu_\lambda(B_r(x))}{2r} d\mu_\lambda(x) d\lambda \leq \liminf_{r\to 0} \frac{1}{2r} \iiint \mathbb{I}_{\{|x-y|<r\}} d\mu_\lambda(y) d\mu_\lambda(x) d\lambda < \infty.$$

The measure μ_λ is a push-down measure of some measure ν_λ on the symbolic space Σ, i.e. $\mu_\lambda = \nu_\lambda \circ \pi_\lambda^{-1}$, see [2, Lemma 2.3]. The difficulty of the proof of Theorem 2.2 comes from the fact that since the measure μ_λ is place-dependent, the measure ν_λ itself depends on λ as well. To avoid this difficulty, the author integrates a function instead of the indicator in the previous inequality, which function is constant over the cylinder sets of Σ and majoring the indicator.

Sketch of proof. In point of view of Theorem 2.2, to prove absolute continuity of the Blackwell measure $\mu_{\varepsilon,p}$ (1.5) it is enough to show a region $R_{\text{trans}} \subset (0,1)^2$ where the IFS $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$ satisfies the transversality condition (2.1) and another region R_{ratio} where the quotient $h(\mu_{\varepsilon,p})/\chi(\mu_{\varepsilon,p})$ is strictly greater than 1.

A region for R_{ratio} can be shown using the results of [3, Section 4], see (4.2). It is much harder to find a region R_{trans} (3.7), especially in the case of non-linear functions. We show such a method in Section 3, similar to [3, Section 7.1]. First, the original IFS $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$ is transformed to an equivalent IFS $\{H_0^{\varepsilon,2p-1}, H_1^{\varepsilon,2p-1}\}$ for easier handling. To show transversality some technical assumptions need to be made, which are collected in R_{region} (3.5), also see Figure 3.

Firstly, as mentioned before the parameters need to be picked so that the IFS is strictly contracting which yields R_{contr} (3.2). Secondly, for an IFS with two functions it is easy to see that its attractor is either an interval or a Cantor-set. Since the Lebesgue measure of the support of an absolutely continuous measure needs to be positive, we will consider those parameters where $\Lambda_{\varepsilon,p}$ is an interval, which gives R_{overlap} (3.3). A final purely technical assumption gives R_{pos} (3.4).

The key lemma to prove transversality is the following.

Lemma 2.3. [14, Lemma 7.3] *Let $U \subset \mathbb{R}^d$ be an open, bounded set. Suppose that f is a C^1 real-valued function defined in a neighborhood of $\overline{U}$ such that there exists an $\eta > 0$ satisfying*

$$|f(\lambda)| < \eta \Rightarrow \|\mathrm{grad}_\lambda f(\lambda)\| > \eta \text{ for every } \lambda \in U.$$

Then there exists $C = C(\eta)$ such that

$$\mathcal{L}_d\left(\lambda \in U : |f(\lambda)| < r\right) < Cr \text{ for every } r > 0.$$

For a visualization of Definition 2.1 and Lemma 2.3 see Figure 2. As a result we can prove absolute continuity in the region $R_{\mathrm{ratio}} \cap R_{\mathrm{trans}} \cap R_{\mathrm{region}}$.

3. Transversality region

In this section we are going to show a region of (ε, p) parameters, where the IFS $\{S_0^{\varepsilon,p}, S_1^{\varepsilon,p}\}$ satisfies the transversality condition using Lemma 2.3. Because of some technical reasons, we are going to modify our original IFS. That is, we are going to prove transversality for an IFS which is equivalent to the original one. By symmetrical reasons, without loss of generality suppose $1/2 < p < 1$ and let $q := 2p - 1$.

Lemma 3.1. *For every $0 < \varepsilon, q < 1$, $\varepsilon \neq 1/2$, there exists an $f_{\varepsilon,q}$ linear function such that $f_{\varepsilon,q} \circ H_i^{\varepsilon,q} \circ (f_{\varepsilon,q})^{-1} \equiv S_i^{\varepsilon,(q+1)/2}$ for $i = 0, 1$, i.e. the IFS $\{H_0^{\varepsilon,q}, H_1^{\varepsilon,q}\}$ is equivalent to the IFS $\left\{S_0^{\varepsilon,(q+1)/2}, S_1^{\varepsilon,(q+1)/2}\right\}$, where*

$$H_0^{\varepsilon,q}(x) = -\frac{2 + (-1 + 3q + c_{\varepsilon,q})x}{-3 + q + c_{\varepsilon,q} + 2(-1+q)(-1+q+c_{\varepsilon,q})x},$$
$$H_1^{\varepsilon,q}(x) = \frac{(1 + q - c_{\varepsilon,q})x}{1 + q + c_{\varepsilon,q} + 2(-1+q)(-1+q+c_{\varepsilon,q})x},$$

and $c_{\varepsilon,q} = \sqrt{1 + 2(1 - 8\varepsilon + 8\varepsilon^2)q + q^2}$.

Proof. Let

$$L_i^{\varepsilon,q}(x) = S_i^{\varepsilon,(q+1)/2}(x + 1/2) - 1/2.$$

Since $\{S_0, S_1\}$ maps [0,1] into itself, $\{L_0, L_1\}$ maps $[-1/2, 1/2]$ into itself. Moreover, $S_0 + S_1(1 - x) = 1$ implies $L_0(x) = -L_1(-x)$. Let $y_{\varepsilon,q}$ be the fixed point of L_0 in $[-1/2, 1/2]$. That is

$$y_{\varepsilon,q} := -\frac{-1 + q + \sqrt{1 + 2(1 - 8\varepsilon + 8\varepsilon^2)q + q^2}}{4(-1 + 2\varepsilon)q}.$$

We define $y_{1/2,q} = 0$. So when $\varepsilon \neq 1/2$ the following transformation of the function is valid.

$$Q_i^{\varepsilon,q}(x) := L_i^{\varepsilon,q}(y_{\varepsilon,q}x)/y_{\varepsilon,q}.$$

Finally, we do the last manipulation:

$$H_i^{\varepsilon,q}(x) := \frac{Q_i^{\varepsilon,q}(2(1-q)x - 1) + 1}{2(1-q)}.$$

By the definition $f_{\varepsilon,q}(x) := 2(1-q)y_{\varepsilon,q}x + (1-q)y_{\varepsilon,q} - 1$ the statement of the lemma follows. ∎

The importance of the modification of our original IFS comes from the fact that $\lim_{\varepsilon\to1/2} H_0^{\varepsilon,q}(x)=qx+1$ and $\lim_{\varepsilon\to1/2} H_1^{\varepsilon,q}(x)=qx$. Peres and Solomyak [11] proved that the IFS $\{qx+1,qx\}$ satisfies the transversality condition for $q\in(0.5,0.65)$. Therefore one can claim that the transversality holds for the IFS $\{H_0^{\varepsilon,q},H_1^{\varepsilon,q}\}$ in a neighborhood of $\varepsilon=1/2$. For the proof we will use the technique of [3, Section 7].

It is easy to check that

$$\text{the functions } H_0^{\varepsilon,q} \text{ and } H_1^{\varepsilon,q} \text{ are mon. increasing for every } 0<\varepsilon,q<1. \tag{3.1}$$

Furthermore, $Q_0^{\varepsilon,q}(1)=1$ and $Q_1^{\varepsilon,q}(-1)=-1$, therefore $H_0^{\varepsilon,q}(\frac{1}{1-q})=\frac{1}{1-q}$ and $H_1^{\varepsilon,q}(0)=0$. This fact and 3.1 implies that the functions $H_0^{\varepsilon,q}$ and $H_1^{\varepsilon,q}$ map the interval $[0,\frac{1}{1-q}]$ into itself.

As we mentioned earlier, the IFS $\{H_0^{\varepsilon,q},H_1^{\varepsilon,q}\}$ is not contracting, just eventually contracting. Let $\kappa(\varepsilon,q)$ denote the contraction ratio of the IFS,

$$\kappa(\varepsilon,q):=\max\left\{(H_0^{\varepsilon,q})'(0),(H_0^{\varepsilon,q})'\left(\frac{1}{1-q}\right),(H_1^{\varepsilon,q})'(0),(H_1^{\varepsilon,q})'\left(\frac{1}{1-q}\right)\right\},$$

and let

$$R_{\mathrm{contr}}:=\left\{(\varepsilon,q)\in[0,1]^2:\kappa(\varepsilon,q)<1\right\}. \tag{3.2}$$

Because of (3.1), R_{contr} is exactly the region of parameters, where the IFS is contracting.

Let $\pi_{\varepsilon,q}$ denote the usual natural projection from the symbolic space $\Sigma=\{0,1\}^{\mathbb{N}}$ to $[0,\frac{1}{1-q}]$, that is

$$\pi_{\varepsilon,q}(i_0,i_1,i_2,\dots)=\lim_{n\to\infty}H_{i_0}^{\varepsilon,q}\circ H_{i_1}^{\varepsilon,q}\circ\cdots\circ H_{i_n}^{\varepsilon,q}(0).$$

Since the functions $H_i^{\varepsilon,q}$ are contractions for $(\varepsilon,q)\in R_{\mathrm{contr}}$, the function $\pi_{\varepsilon,q}$ is well defined.

To prove absolute continuity and in particular, transversality, it is necessary that the maps are overlapping. That is, if $H_0^{\varepsilon,q}([0,\frac{1}{1-q}])\cap H_1^{\varepsilon,q}([0,\frac{1}{1-q}])=\emptyset$ then the attractor of the IFS (the unique nonempty compact set $\Lambda'_{\varepsilon,q}=H_0^{\varepsilon,q}(\Lambda'_{\varepsilon,q})\cup H_1^{\varepsilon,q}(\Lambda'_{\varepsilon,q})$) is a Cantor set with zero Lebesgue measure, which implies that any measure with support $\Lambda'_{\varepsilon,q}$ is singular. On the other hand, if $H_0^{\varepsilon,q}([0,\frac{1}{1-q}])\cap H_1^{\varepsilon,q}([0,\frac{1}{1-q}])\neq\emptyset$ then $\Lambda'_{\varepsilon,q}=[0,\frac{1}{1-q}]$. However, the transversality condition works if the overlap is "weak". Therefore, we give a technical assumption for the set of parameters. Let us consider the following region of parameters

$$\begin{aligned}R_{\mathrm{overlap}}:=\Bigg\{&(\varepsilon,q)\in[0,1]^2:H_0^{\varepsilon,q}(0)<H_1^{\varepsilon,q}\left(\frac{1}{1-q}\right),\\ &H_0^{\varepsilon,q}(H_0^{\varepsilon,q}(0))>H_1^{\varepsilon,q}\left(\frac{1}{1-q}\right)\text{ and }H_1^{\varepsilon,q}\left(H_1^{\varepsilon,q}\left(\frac{1}{1-q}\right)\right)<H_0^{\varepsilon,q}(0)\Bigg\}.\end{aligned} \tag{3.3}$$

For the parameters in R_{overlap} we have

$$\pi_{\varepsilon,q}(\mathbf{i})=\pi_{\varepsilon,q}(\mathbf{j})\text{ and }i_0\neq j_0\Rightarrow i_1\neq j_1.$$

As a technical condition we need also that the functions $\frac{\partial H_0^{\varepsilon,q}}{\partial q}$ and $\frac{\partial H_1^{\varepsilon,q}}{\partial q}$ are monotone increasing. Unfortunately, this is not true for every parameters. Let R_{pos} be the set of parameters, where this is true. Precisely, simple calculations

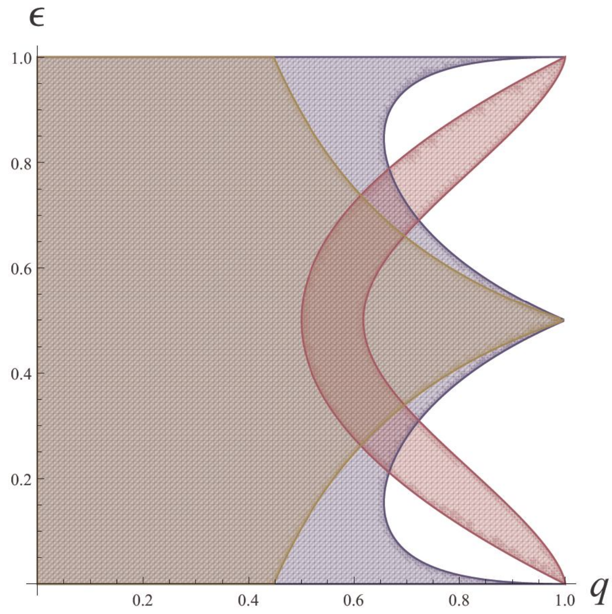

FIGURE 3. The regions R_{contr} (blue region), R_{overlap} (red) and R_{pos} (brown). The intersection of the three regions is R_{region}.

show that the functions $\frac{\partial H_i^{\varepsilon,q}}{\partial q}, H_i^{\varepsilon,q} : [0, \frac{1}{1-q}] \mapsto \mathbb{R}$ are smooth functions for every $0 < \varepsilon, q < 1$. Denote $x_{\varepsilon,q}^i$ the unique root of the function

$$\frac{\partial\,(H_i^{\varepsilon,q})'}{\partial q}(x_{\varepsilon,q}^i) = 0.$$

Let

$$R_{\text{pos}} := \left\{(\varepsilon, q) \in [0,1]^2 : \frac{\partial (H_i^{\varepsilon,q})'}{\partial q}(0) > 0 \text{ and } x_{\varepsilon,q}^i \notin \left[0, \frac{1}{1-q}\right] \text{ for } i = 0, 1\right\}. \tag{3.4}$$

From now we focus our study for the set of parameters R_{region}, where

$$R_{\text{region}} := R_{\text{contr}} \cap R_{\text{overlap}} \cap R_{\text{pos}}, \tag{3.5}$$

see Figure 3. The definition of R_{region} implies that it is open.

Define $\omega(\varepsilon, q)$ for $(\varepsilon, q) \in R_{\text{region}}$ as

$$\omega(\varepsilon, q) := \max\left\{\frac{\partial H_0^{\varepsilon,q}}{\partial q}\left(\frac{1}{1-q}\right), \frac{\partial H_1^{\varepsilon,q}}{\partial q}\left(\frac{1}{1-q}\right)\right\}.$$

Lemma 3.2. *For every $(\varepsilon, q) \in R_{\text{region}}$ and $\mathbf{i} \in \Sigma$*

$$0 \le \frac{\partial}{\partial q}\pi_{\varepsilon,q}(\mathbf{i}) \le \frac{\omega(\varepsilon, q)}{1 - \kappa(\varepsilon, q)}.$$

Proof. One can check that for every $(\varepsilon, q) \in R_{\text{region}}$, $\frac{\partial H_0^{\varepsilon,q}}{\partial q}(0), \frac{\partial H_1^{\varepsilon,q}}{\partial q}(0) \ge 0$. Since $H_0^{\varepsilon,q}, H_1^{\varepsilon,q}$ and $\frac{\partial H_0^{\varepsilon,q}}{\partial q}, \frac{\partial H_1^{\varepsilon,q}}{\partial q}$ are monotone increasing, the first inequality holds.

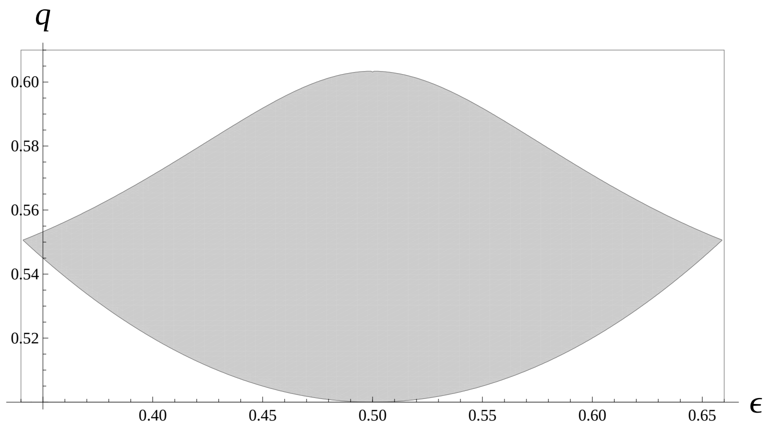

FIGURE 4. The region $R_{\text{trans}} \cap R_{\text{region}}$.

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial q}\pi_{\varepsilon,q}(\mathbf{i}) &= \frac{\partial}{\partial q}(H_{i_0}^{\varepsilon,q}(\pi_{\varepsilon,q}(\sigma\mathbf{i}))) \\ &= \frac{\partial H_{i_0}^{\varepsilon,q}}{\partial q}(\pi_{\varepsilon,q}(\sigma\mathbf{i})) + (H_{i_0}^{\varepsilon,q})'(\pi_{\varepsilon,q}(\sigma\mathbf{i}))\frac{\partial}{\partial q}\pi_{\varepsilon,q}(\sigma\mathbf{i}) \\ &\leq \omega(\varepsilon,q) + \kappa(\varepsilon,q)\frac{\partial}{\partial q}\pi_{\varepsilon,q}(\sigma\mathbf{i}).\end{aligned}$$

The second inequality follows by induction. □

Since the functions $H_0^{\varepsilon,q}, H_1^{\varepsilon,q}$ are strictly monotone increasing on $[0, \frac{1}{1-q}]$, they are invertible. Denote the inverse functions by

$$\mathcal{H}_0^{\varepsilon,q} := (H_0^{\varepsilon,q})^{-1} \quad \text{and} \quad \mathcal{H}_1^{\varepsilon,q} := (H_1^{\varepsilon,q})^{-1}.$$

For simplicity, denote $H_{10}^{\varepsilon,q} := H_1^{\varepsilon,q} \circ H_0^{\varepsilon,q}$ and $H_{01}^{\varepsilon,q} := H_0^{\varepsilon,q} \circ H_1^{\varepsilon,q}$. Then easy calculations show that the function

$$\mathbb{H}^{\varepsilon,q}(x) := \frac{\partial H_{10}^{\varepsilon,q}}{\partial q} \circ \mathcal{H}_0^{\varepsilon,q} \circ \mathcal{H}_1^{\varepsilon,q}(x) - \frac{\partial H_{01}^{\varepsilon,q}}{\partial q} \circ \mathcal{H}_1^{\varepsilon,q} \circ \mathcal{H}_0^{\varepsilon,q}(x) \tag{3.6}$$

is a convex polynomial of second degree. Denote the minimum of it by $z_{\varepsilon,q}$.

Lemma 3.3. *For every $(\varepsilon_0, q_0) \in R_{\text{trans}} \cap R_{\text{region}}$ and for every $\mathbf{i}, \mathbf{j} \in \Sigma$ such that $i_0 \neq j_0$ we have*

$$\pi_{\varepsilon_0,q_0}(\mathbf{i}) = \pi_{\varepsilon_0,q_0}(\mathbf{j}) \Rightarrow \left| \frac{\partial}{\partial q}\left(\pi_{\varepsilon_0,q}(\mathbf{i}) - \pi_{\varepsilon_0,q}(\mathbf{j})\right)\Big|_{q=q_0} \right| > 0,$$

where

$$R_{\text{trans}} := \left\{(\varepsilon,q) \in [0,1]^2 : \mathbb{H}^{\varepsilon,q}(z_{\varepsilon,q}) - \frac{\omega(\varepsilon,q)\kappa(\varepsilon,q)^2}{1-\kappa(\varepsilon,q)} > 0\right\}. \tag{3.7}$$

One can see the region of parameters $R_{\text{trans}} \cap R_{\text{region}}$ on Figure 4.

Proof. Suppose that $\pi_{\varepsilon_0,q_0}(\mathbf{i}) = \pi_{\varepsilon_0,q_0}(\mathbf{j})$ and $i_0 \neq j_0$ then $(\varepsilon_0, q_0) \in R_{\text{overlap}}$ implies

$$0 = \pi_{\varepsilon_0,q_0}(\mathbf{i}) - \pi_{\varepsilon_0,q_0}(\mathbf{j}) = H_1^{\varepsilon_0,q_0} \circ H_0^{\varepsilon_0,q_0}(\pi_{\varepsilon_0,q_0}(\sigma^2\mathbf{i})) - H_0^{\varepsilon_0,q_0} \circ H_1^{\varepsilon_0,q_0}(\pi_{\varepsilon_0,q_0}(\sigma^2\mathbf{j})).$$

So it is enough to show that the partial derivative by q of the right-hand side is positive. Then from Lemma 3.2 it follows that

$$\begin{aligned}&\frac{\partial}{\partial q}\left(H_{10}^{\varepsilon,q}(\pi_{\varepsilon,q}(\sigma^2\mathbf{i}))-H_{01}^{\varepsilon,q}(\pi_{\varepsilon,q}(\sigma^2\mathbf{j}))\right)\\&\geq\frac{\partial H_{10}^{\varepsilon,q}}{\partial q}(\pi_{\varepsilon,q}(\sigma^2\mathbf{i}))-\frac{\partial H_{01}^{\varepsilon,q}}{\partial q}(\pi_{\varepsilon,q}(\sigma^2\mathbf{j}))-\frac{\kappa(\varepsilon,q)^2\omega(\varepsilon,q)}{1-\kappa(\varepsilon,q)}.\end{aligned}$$

Hence by the definition of $\mathbb{H}^{\varepsilon,q}$

$$\begin{aligned}&\frac{\partial H_{10}^{\varepsilon_0,q}}{\partial q}(\pi_{\varepsilon_0,q}(\sigma^2\mathbf{i}))-\frac{\partial H_{01}^{\varepsilon_0,q}}{\partial q}(\pi_{\varepsilon_0,q}(\sigma^2\mathbf{j}))\Big|_{q=q_0}\\&=\frac{\partial H_{10}^{\varepsilon_0,q}}{\partial q}(\mathcal{H}_0^{\varepsilon_0,q}\circ\mathcal{H}_1^{\varepsilon_0,q}(\pi_{\varepsilon_0,q}(\mathbf{i})))-\frac{\partial H_{01}^{\varepsilon_0,q}}{\partial q}(\mathcal{H}_1^{\varepsilon_0,q}\circ\mathcal{H}_0^{\varepsilon_0,q}(\pi_{\varepsilon_0,q}(\mathbf{j})))\Big|_{q=q_0}\\&\geq\mathbb{H}^{\varepsilon_0,q_0}(z_{\varepsilon_0,q_0}),\end{aligned}$$

so the statement follows. □

For the sake of completeness, finally, we give a compactness argument for proving transversality condition.

Proposition 3.4. *For every $\varepsilon>0$ the IFS $\{H_0^{\varepsilon,q},H_1^{\varepsilon,q}\}$ satisfies the transversality condition on any open interval $V\subset\mathbb{R}$ such that $\overline{V}\subset R_{\text{trans}}\cap R_{\text{region}}\cap([0,1]\times\{\varepsilon\})$.*

Proof. Let $V\subset\mathbb{R}$ an open set such that the closure is contained in $R_{\text{trans}}\cap R_{\text{region}}\cap[0,1]\times\{\varepsilon\}$ and let

$$\eta_1:=\min_{q\in\overline{V}}\left\{\mathbb{H}^{\varepsilon,q}(x)(z_{\varepsilon,q})-\frac{\omega(\varepsilon,q)\kappa(\varepsilon,q)^2}{1-\kappa(\varepsilon,q)}\right\},$$

where $\mathbb{H}^{\varepsilon,q}$ was defined in (3.6). It is easy to see that the space $\Sigma\times\Sigma\times\overline{V}$ is compact and the function $(\mathbf{i},\mathbf{j},q)\mapsto\frac{\partial}{\partial q}(\pi_{\varepsilon,q}(\mathbf{i})-\pi_{\varepsilon,q}(\mathbf{j}))$ is continuous. The function $(\mathbf{i},\mathbf{j},q)\mapsto\pi_{\varepsilon,q}(\mathbf{i})-\pi_{\varepsilon,q}(\mathbf{j})$ is continuous as well. Therefore, for every $\eta\geq0$, the set $L_\eta=\{(\mathbf{i},\mathbf{j},q):|\pi_{\varepsilon,q_0}(\mathbf{i})-\pi_{\varepsilon,q_0}(\mathbf{j})|\leq\eta\}$ is compact. Since

$$\left|\frac{\partial}{\partial q}(\pi_{\varepsilon,q_0}(\mathbf{i})-\pi_{\varepsilon,q_0}(\mathbf{j}))\right|\geq\eta_1\text{ for every }(\mathbf{i},\mathbf{j},q)\in L_0,$$

there exists an $\eta_2>0$ depending only on ε such that for every $q_0\in V$ and any $\mathbf{i},\mathbf{j}\in\Sigma$, $i_0\neq j_0$ we have

$$|\pi_{\varepsilon,q_0}(\mathbf{i})-\pi_{\varepsilon,q_0}(\mathbf{j})|<\eta_1\Rightarrow\left|\frac{\partial}{\partial q}(\pi_{\varepsilon,q}(\mathbf{i})-\pi_{\varepsilon,q}(\mathbf{j}))\Big|_{q=q_0}\right|>\frac{\eta_2}{2}.$$

This implies the statement of the proposition by Lemma 2.3. □

4. Proof of Theorem 1.1

The last section of our paper is devoted to prove the absolute continuity of the Blackwell measure. In order to apply Theorem 2.2 we recall a result of [3] to find the region where the quotient entropy over Lyapunov exponent is strictly greater than 1. Let

$$\mathfrak{h}_{\varepsilon,q}(x)=-\left(p_0^{\varepsilon,(q+1)/2}(x)\log p_0^{\varepsilon,(q+1)/2}(x)+p_1^{\varepsilon,(q+1)/2}(x)\log p_1^{\varepsilon,(q+1)/2}(x)\right).$$

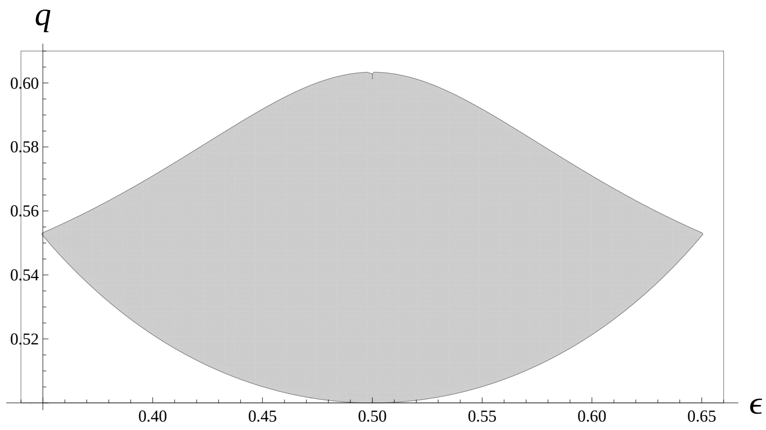

FIGURE 5. The region $R_{\text{ratio}} \cap R_{\text{trans}} \cap R_{\text{region}}$.

Define the Perron-Frobenius operator corresponding to measure $\mu_{\varepsilon,p}$ as follows

$$(T_{\varepsilon,p}f)(x) := p_0^{\varepsilon,p}(x) \cdot f(S_0^{\varepsilon,p}(x)) + p_1^{\varepsilon,p}(x) \cdot f(S_1^{\varepsilon,p}(x)),$$

where the functions and probabilities were defined in (1.1), (1.2), (1.3) and (1.4).

According to the result [3, Corollary 12, Proposition 14, Proposition 18]

$$3(T^{10}_{\varepsilon,(q+1)/2}\mathfrak{h}_{\varepsilon,q})(0) + \log(\varepsilon(1-\varepsilon)q) > 0 \Rightarrow \frac{h(\mu_{\varepsilon,(q+1)/2})}{\chi(\mu_{\varepsilon,(q+1)/2})} > 1, \tag{4.1}$$

where $h(\mu_{\varepsilon,p})$ is the entropy (2.2) and $\chi(\mu_{\varepsilon,p})$ denotes the Lyapunov exponent (2.3) of the measure $\mu_{\varepsilon,p}$. Define R_{ratio} as the region where the ratio is strictly greater than 1

$$R_{\text{ratio}} := \left\{ (\varepsilon, q) \in [0,1]^2 : 3(T^{10}_{\varepsilon,(q+1)/2}\mathfrak{h}_{\varepsilon,q})(0) + \log(\varepsilon(1-\varepsilon)q) > 0 \right\}. \tag{4.2}$$

Proof of Theorem 1.1. For every fixed $\varepsilon \neq 1/2$, the IFS $\left\{S_0^{\varepsilon,(q+1)/2}, S_1^{\varepsilon,(q+1)/2}\right\}$ satisfies the transversality condition by Lemma 3.1 and Proposition 3.4 for $(\varepsilon, q) \in R_{\text{trans}} \cap R_{\text{region}}$. It follows from Theorem 2.2 and (4.1) that for every $\varepsilon \neq 1/2$ and Lebesgue-a.e q in $R_{\text{ratio}} \cap R_{\text{trans}} \cap R_{\text{region}}$, the measure $\mu_{\varepsilon,(q+1)/2}$ is absolutely continuous w.r.t Lebesgue measure. Using the symmetrical properties of $\mu_{\varepsilon,p}$, one can finish the proof. □

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