ITERATED FUNCTION SYSTEMS WITH NON-DISTINCT FIXED POINTS

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Abstract. The dimension theory of self-similar sets is quite well understood in the cases when some separation conditions (open set condition or weak separation condition) or the so-called transversality condition hold. Otherwise the study of the Hausdorff dimension is far from well understood. We investigate the properties of the Hausdorff dimension of self-similar sets such that some functions in the corresponding iterated function system share the same fixed point. Then it is not possible to apply directly known techniques. In this paper we are going to calculate the Hausdorff dimension for almost every contracting parameters and calculate the proper dimensional Hausdorff measure of the attractor.

1. Introduction and Statements

Let us denote the Hausdorff dimension of a compact subset Λ of $\mathbb{R}$ by $\dim_H \Lambda$, and respectively denote the Box dimension by $\dim_B \Lambda$. For the definition and basic properties of Hausdorff and Box dimension we refer the reader to [3] or [4].

Let $\{f_0, \dots, f_{m-1}\}$ be a family of contracting similarity map such that $|f_i(x) - f_i(y)| = |\lambda_i||x-y|$ for all x, y and for some $-1 < \lambda_i < 1$. Then there exists a unique, nonempty compact subset Λ of $\mathbb{R}$ which satisfies

$$\Lambda = \bigcup_{i=0}^{m-1} f_i(\Lambda)$$

We call this set Λ the attractor of the iterated function system (IFS) $\{f_0(x), \dots, f_{m-1}(x)\}$. In this case we say that the attractor Λ (or the IFS itself) is self-similar.

It is well known that the Hausdorff dimension and the Box dimension of the attractor is the unique solution of

$$\sum_{i=0}^{m-1} |\lambda_i|^s = 1, \tag{1.1}$$

if the open set condition (OSC) holds, for precise details see for example [5]. Even if the OSC does not hold, the solution of equation (1.1) is called similarity dimension of the IFS. The similarity dimension is always an upper bound for the Hausdorff dimension of the attractor, see [3]. In the case when the IFS has overlapping structure, i.e. the open set condition does not hold, the Hausdorff dimension of the attractor Λ of IFS $\{f_i(x) = \lambda_i x + d_i\}_{i=0}^{m-1}$ is

$$\dim_B \Lambda = \dim_H \Lambda = \min\{s, 1\} \text{ for a.e. } (d_0, \dots, d_{m-1}) \in \mathbb{R}^m$$

where s is the unique solution of (1.1), see [10].

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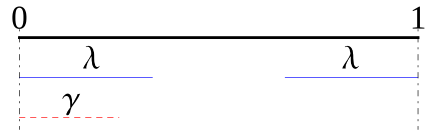

FIGURE 1. The simplest example of IFS with some of the functions share the same fixed point, considered in [2].

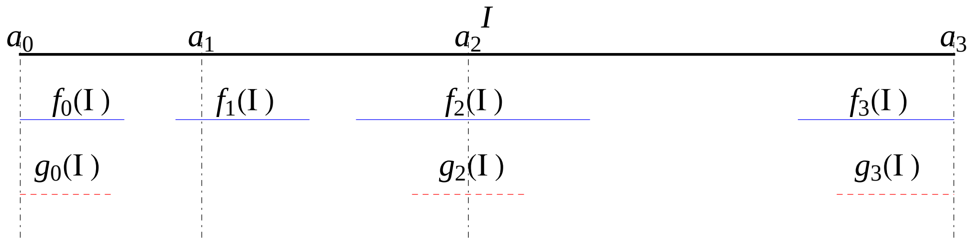

FIGURE 2. $a_0 = \mathrm{Fix}(f_0) = \mathrm{Fix}(g_0)$, $a_2 = \mathrm{Fix}(f_2) = \mathrm{Fix}(g_2)$, $a_3 = \mathrm{Fix}(f_3) = \mathrm{Fix}(g_3)$

In [2] we considered the IFS $\{\gamma x, \lambda x, \lambda x + 1\}$, $\gamma < \lambda$ on the real line. Let I be the convex hull of the attractor Λ. See Figure 1 for the image of I by the functions of this IFS. The problem of the computation of the dimension of this IFS was raised by Pablo Shmerkin at the conference in Greifswald in 2008. The novelty of the result obtained in [2] about the dimension of Λ was to tackle the difficulty which comes from the fact that the first two maps have the same fixed point.

In this paper we consider an IFS $\mathcal{S}$ which does not satisfy the OSC, but we can partition $\mathcal{S}$ into two disjoint subfamilies $\mathcal{F}, \mathcal{G}$ such that the first "cylinders" of $\mathcal{F}$ are disjoint and for every $g \in \mathcal{G}$ there exists exactly one $f \in \mathcal{F}$ such that $\mathrm{Fix}(g) = \mathrm{Fix}(f)$. For an example see Figure 2. More precisely,

Principal Assumptions:

(A1) $\mathcal{S} = \mathcal{F} \cup \mathcal{G}$

(A2) $\mathcal{F} = \{f_i(x) = \lambda_i x + a_i(1-\lambda_i)\}_{i=0}^{N-1}$ where $0 < \lambda_i < 1$ and the fixed points satisfy: $a_0 < a_1 < \cdots < a_{N-1}$.

(A3) Let $I = [a_0, a_{N-1}]$ (the convex hull of the attractor). We require that $f_{i-1}(I) < f_i(I)$ that is

$$f_{i-1}(a_{N-1}) < f_i(a_0) \text{ for every } i = 1, \dots, N-1. \tag{1.2}$$

(A4) We write $\mathcal{I} = \{0, \dots, N-1\}$ and let $\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{G} = \{g_i(x) = \gamma_i x + a_i(1-\gamma_i)\}_{i \in \mathcal{J}}$ such that $0 < \gamma_i < \lambda_i$ for every $i \in \mathcal{J}$.

Observe that for every $i \in \mathcal{J}$, $\mathrm{Fix}(f_i) = \mathrm{Fix}(g_i) = a_i$.

Denote $\underline{\gamma} \in (0,1)^{\sharp \mathcal{J}}$ the vector of contraction ratios of $\mathcal{G}$ and $\underline{\lambda} \in (0,1)^N$ the vector of contraction ratios of $\mathcal{F}$. Moreover, let $\underline{a} \in \mathbb{R}^N$ be the vector of fixed points and denote the attractor of $\mathcal{S}$ by Λ.

The main theorem of this paper is an almost all type result about the dimension of the attractor Λ, assuming that the contractions γ_i are sufficiently small compare to the contraction ratios and the gaps of the first cylinders of the functions from $\mathcal{F}$.

Theorem 1.1. *Let $\mathcal{S}$ as in (A1)-(A4) then the attractor Λ of $\mathcal{S}$ satisfies that*

$$\dim_B \Lambda = \dim_H \Lambda = \min\{1, s\}, \tag{1.3}$$

where s is the unique solution of

$$\sum_{i=0}^{N-1} \lambda_i^s + \sum_{i\in\mathcal{J}} \gamma_i^s - \sum_{i\in\mathcal{J}} \lambda_i^s \gamma_i^s = 1, \tag{1.4}$$

for Lebesgue almost every $\underline{\gamma}$ in

$$\left\{ \underline{\gamma} : 0 < \gamma_i < \min\left\{ \lambda_i, \frac{2}{(1+\sqrt{2})(\alpha_i^2 \lambda_{\max} + 2)} \right\} \right\}, \tag{1.5}$$

where $\lambda_{\max} = \max_i \{\lambda_i\}$ and

$$\alpha_i = \frac{\max\{a_{N-1} - a_i, a_i - a_0\}}{\min\{f_{i+1}(a_0) - a_i, a_i - f_{i-1}(a_{n-1})\}} \text{ for every } i \in \mathcal{I}.$$

Moreover $\mathcal{L}(\Lambda) > 0$ for Lebesgue almost every $\underline{\gamma}$ such that $\underline{\gamma}$ satisfies (1.5) and $s > 1$.

In the proof of Theorem 1.1 we are going to show that s is always an upper bound for the Hausdorff and Box dimension. Moreover we will prove that the s dimensional Hausdorff measure of the attractor is zero.

Theorem 1.2. *Assume that $\mathcal{S}$ satisfies (A1)-(A4) and let s be the unique solution of (1.4) then*

$$\mathcal{H}^s(\Lambda) = 0.$$

To prove the main result of this paper, Theorem 1.1, we are going to use the so-called transversality method. Note, that our original system does not satisfy the transversality condition (see later the precise arguments), but some well-chosen subsystems of the sufficiently high iterations of $\mathcal{S}$ do so. To verify this we use two methods of checking the transversality condition. One of them was introduced by Simon, Solomyak and Urbański [11], [12] and the other one is due to [8], [9]. For the convenience of the reader in Section 2 we summarize these methods.

In Section 3 we prove Theorem 1.1. This Section is decomposed into three parts. In 3.1 we introduce some notations about the natural projections. In 3.2 we prove the transversality condition for the approximating subsystems and in 3.3 the Hausdorff dimension is calculated.

In Section 4 we prove our Theorem 1.2 about the Hausdorff measure of Λ. The method of the proof is similar to that of [7, Theorem 1.1] obtained by a modification of the Brandt, Graf method [1].

In Section 5 we show a higher dimensional application of Theorem 1.1. We will calculate the Hausdorff and Box dimension of some overlapping diagonally self-affine sets, for almost every contraction coefficients.

2. Transversality methods

First let us introduce the *transversality condition* for self-similar IFS on the real line with d dimensional parameter-space. The definition corresponds to the definition in [11],[12] which was introduced for much more general IFS.

Let U be an open, bounded subset of $\mathbb{R}^d$ with smooth boundary and $\mathcal{I}$ a finite set of symbols. Let $\Psi_{\underline{t}} = \left\{\psi_i^{\underline{t}}(x) = \lambda_i(\underline{t})x + d_i(\underline{t})\right\}_{i\in\mathcal{I}}$, where $\lambda_i, d_i \in C^1(\overline{U})$ and $0 < \alpha \leq \lambda_i(\underline{t}) \leq \beta < 1$ for every $i \in \mathcal{I}$ and $\underline{t} \in \overline{U}$ and for some $\alpha, \beta \in (0,1)$. Let $\Lambda^{\underline{t}}$ be the attractor of $\Psi_{\underline{t}}$ and $\pi_{\underline{t}}$ is the natural projection from the symbolic space $\Sigma = \mathcal{I}^{\mathbb{N}}$ to $\Lambda^{\underline{t}}$. More precisely, for $\mathbf{i} = (i_0 i_1 \dots) \in \Sigma$ we write

$$\pi_{\underline{t}}(\mathbf{i}) = \lim_{n\to\infty} \psi_{i_0}^{\underline{t}} \circ \psi_{i_1}^{\underline{t}} \circ \cdots \circ \psi_{i_n}^{\underline{t}}(0). \tag{2.1}$$

It is well-known that the limit exists and independent of the base point 0. Moreover, $\pi_{\underline{t}}$ is a continuous, surjective function from Σ onto $\Lambda^{\underline{t}}$. Denote σ the left-shift operator on Σ. That is $\sigma : (i_0 i_1 \dots) \mapsto (i_1 i_2 \dots)$. It is easy to see that

$$\pi_{\underline{t}}(\mathbf{i}) = \psi_{i_0}^{\underline{t}}(\pi_{\underline{t}}(\sigma\mathbf{i})).$$

Definition 2.1. *We say that $\Psi_{\underline{t}}$ satisfies the* **transversality condition** *on an open, bounded set $U \subset \mathbb{R}^d$, if for any $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$ there exists a constant $C = C(i_0, j_0)$ such that*

$$\mathcal{L}_d(\underline{t} \in U : |\pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})| \leq r) \leq Cr \text{ for every } r > 0,$$

where $\mathcal{L}_d$ is the d dimensional Lebesgue measure.

In short, we say that there is transversality if the transversality condition holds. This definition is equivalent to the ones given in e.g. [11], [12]. As a special case of [11, Theorem 3.1] we obtain:

Theorem 2.2 (Simon, Solomyak, Urbański)**.** *Suppose that $\Psi_{\underline{t}}$ satisfies the transversality condition on an open, bounded set $U \subset \mathbb{R}^d$. Then*

1. $\dim_H \Lambda^{\underline{t}} = \min\{s(\underline{t}), 1\}$ *for Lebesgue-a.e. $\underline{t} \in U$,*
2. $\mathcal{L}_1(\Lambda^{\underline{t}}) > 0$ *for Lebesgue-a.e. $\underline{t} \in U$ such that $s(\underline{t}) > 1$,*

where $s(\underline{t})$ is the similarity dimension of $\Psi_{\underline{t}}$. More precisely, $s(\underline{t})$ satisfies the equation

$$\sum_{i\in\mathcal{I}} \lambda_i(\underline{t})^{s(\underline{t})} = 1. \tag{2.2}$$

We can use the following Lemma to prove transversality which follows from [11, Lemma 7.3].

Lemma 2.3. *Let $U \subset \mathbb{R}^d$ be an open, bounded set with smooth boundary and $f_{\mathbf{i},\mathbf{j}}(\underline{t}) = \pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})$. If for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$ and for every $\underline{t}_0 \in U$*

$$f_{\mathbf{i},\mathbf{j}}(\underline{t}_0) = 0 \Rightarrow \|\mathrm{grad}_{\underline{t}} f_{\mathbf{i},\mathbf{j}}\big|_{\underline{t}=\underline{t}_0}\| > 0 \tag{2.3}$$

then there is transversality on any open subset V whose closure is contained in U.

There is an other Lemma which is useful to prove transversality by controlling the double roots of infinite series. The proof of the Lemma below depends on the so-called $(*)$-functions which were introduced by Solomyak [13] and further developed by Peres and Solomyak [8] and [9]. Although, the following Lemma was not proved explicitly in [9] but one can easily see that a simple modification of the proofs [9, Lemma 5.1], [9, Corollary 5.2] yields:

Lemma 2.4. *Let the function $g : [0,1) \mapsto \mathbb{R}$ be given in the following form:*

$$g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k.$$

Let us suppose that $a_1 \in (-d,d)$ and for every $k \geq 2$, $a_k \in (-b,b)$, where $d,b>0$. Then

$$g(x_0) = 0 \Rightarrow g'(x_0) < 0 \text{ for every } x_0 \in \left(0, \frac{1}{1+\sqrt{b}}\right).$$

3. Proof of Theorem 1.1

3.1. **Natural Projection.** Because of the special nature of the IFS $\mathcal{S} = \mathcal{F} \cup \mathcal{G}$ under consideration, it is reasonable to modify the way as the elements of $\mathcal{S}$ are labeled. Namely, we label the functions of $\mathcal{S}$ by pairs of integers like (i,κ), where $\kappa = 1$ if the function is from $\mathcal{F}$ and $\kappa = 2$ when the function is from $\mathcal{G}$. In both cases $i \in \{0,\dots,N-1\}$, where we recall that N was defined in our Principal Assumptions as the cardinality of $\mathcal{F}$. From now on we always write $\mathcal{I} = \{(0,1),(1,1),\dots,(N-1,1)\}$ for $N \geq 2$. According to this new notation the contraction ratio and the fixed point of the functions from $\mathcal{F}$ are $0 < \lambda_{(i,1)} < 1$, and $a_{(i,1)} \in \mathbb{R}$, $(i,1) \in \mathcal{I}$. That is

$$f_{(i,1)}(x) = \lambda_{(i,1)}x + a_{(i,1)}(1-\lambda_{(i,1)}), \quad (i,1) \in \mathcal{I}. \tag{3.1}$$

Let $\mathcal{J} \subseteq \{(0,2),\dots,(N-1,2)\}$ and denote $\mathcal{N} = \{i : (i,2) \in \mathcal{J}\}$. Like above, the contraction ratio and the fixed point of the functions from $\mathcal{G}$ are $0 < \lambda_{(i,2)} < 1$ and $a_{(i,2)} \in \mathbb{R}$, $(i,2) \in \mathcal{J}$. That is

$$f_{(i,2)}(x) = \lambda_{(i,2)}x + a_{(i,2)}(1-\lambda_{(i,2)}) \text{ for } i \in \mathcal{N}. \tag{3.2}$$

So

$$\mathcal{F} = \left\{f_{(i,1)}\right\}_{i=0}^{N-1} \text{ and } \mathcal{G} = \left\{f_{(i,2)}\right\}_{i\in\mathcal{N}}.$$

According to our principal assumptions (A1)-(A4) in between the fixed points and contraction ratios we have the following relations:

$$a_i := a_{(i,1)} = a_{(i,2)} \text{ and } 0 < \lambda_{(i,2)} < \lambda_{(i,1)} < 1 \text{ for every } i \in \mathcal{N}.$$

Moreover, by definition $a_0 < a_1 < \dots < a_{N-1}$ and we also assumed that satisfies

$$f_{(i-1,1)}(a_{N-1}) < f_{(i,1)}(a_0), \tag{3.3}$$

see (1.2). For simplicity denote $\underline{\lambda}_1$ the vector of contraction ratios of $\mathcal{F}$ and similarly $\underline{\lambda}_2$ the vector of contraction ratios of $\mathcal{G}$. We denote the attractor of $\mathcal{S}$ by $\Lambda(\underline{\lambda},\underline{a})$, where $\underline{\lambda} = \underline{\lambda}_1 \times \underline{\lambda}_2$ and the vector of the distinct fixed points of the functions of $\mathcal{S}$ is $\underline{a} = (a_0,\dots,a_{N-1})$. As usual we write

$$\underline{\gamma}^{\underline{k}} := \prod_{i=1}^{m} \gamma_i^{k_i}, \quad \underline{k} = (k_1,\dots,k_m) \in \mathbb{N}^m, \quad \underline{\gamma} \in \mathbb{R}^m. \tag{3.4}$$

The symbolic space is

$$\Sigma := (\mathcal{I} \cup \mathcal{J})^{\mathbb{N}}.$$

The the natural projection $\pi_{\underline{\lambda},\underline{a}}$ from the symbolic space Σ to the attractor Λ is defined exactly as in (2.1).

For an $\mathbf{i}=((i_0,\kappa_0)(i_1,\kappa_1)(i_2,\kappa_2)\cdots)\in\Sigma$ we write $\mathbf{i}(k)$ for the sequence of the first k elements of $\mathbf{i}$. In particular, $\mathbf{i}(0)$ is empty sequence. We denote the number of $(i,\kappa)\in\mathcal{I}\cup\mathcal{J}$ in $\mathbf{i}(k)$ by $\sharp_{(i,\kappa)}\mathbf{i}(k)$. We form the vector $\sharp\mathbf{i}(k)\in\{0,\dots,k\}^{\sharp\mathcal{I}+\sharp\mathcal{J}}$

$$\sharp\mathbf{i}(k):=\left(\sharp_{(0,1)}\mathbf{i}(k),\sharp_{(1,1)}\mathbf{i}(k),\dots,\sharp_{(N-1,1)}\mathbf{i}(k),\sharp_{(\min\mathcal{J},2)}\mathbf{i}(k),\dots,\sharp_{(\max\mathcal{J},2)}\mathbf{i}(k)\right).$$

Using the notation introduced in (3.4), clearly,

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i})=\sum_{k=0}^{\infty}a_{i_k}(1-\lambda_{(i_k,\kappa_k)})\underline{\lambda}^{\sharp\mathbf{i}(k)}. \tag{3.5}$$

Equivalently,

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i})=a_{i_0}+\sum_{k=0}^{\infty}\left(a_{i_{k+1}}-a_{i_k}\right)\underline{\lambda}^{\sharp\mathbf{i}(k+1)}. \tag{3.6}$$

In this way only those elements of the some above have non-zero contribution for which $a_{i_{k+1}}\neq a_{i_k}$. Now we partition the elements of $\mathbf{i}$ into blocks to rewrite the natural projection. The block which consists the element (i_l,κ_l) is the maximal subsequence of $\mathbf{i}$ satisfying $(i_u,\kappa_u)\cdots(i_l,\kappa_l)\cdots(i_v,\kappa_v)$ such that $u\leq l\leq v$ and $i_u=\cdots=i_l=\cdots=i_v$. Therefore all functions which correspond to any symbols in a block share the same fixed point.

We write $b_l^{\mathbf{i}}$ for the l-th block and $k_l^{\mathbf{i}}$ for the length of the l-th block. The length of the first l blocks is denoted by $p_l^{\mathbf{i}}$, that is $p_l^{\mathbf{i}}=\sum_{j=0}^{l}k_j^{\mathbf{i}}$.

In this way the decomposition of $\mathbf{i}$ into blocks is as follows:

$$\mathbf{i}=(\underbrace{(i_0,\kappa_0)\cdots(i_{k_0^{\mathbf{i}}-1},\kappa_{k_0^{\mathbf{i}}-1})}_{b_0^{\mathbf{i}}}\cdots\underbrace{(i_{p_l^{\mathbf{i}}},\kappa_{p_l^{\mathbf{i}}})\cdots(i_{p_l^{\mathbf{i}}+k_{l+1}^{\mathbf{i}}-1},\kappa_{p_l^{\mathbf{i}}+k_{l+1}^{\mathbf{i}}-1})}_{b_{l+1}^{\mathbf{i}}}\cdots)$$

or simply $\mathbf{i}=b_0^{\mathbf{i}}b_1^{\mathbf{i}}b_2^{\mathbf{i}}\dots$. Let $a_{b_l^{\mathbf{i}}}$ be the common fixed point of all the functions $f_{(i,\kappa)}$, where $(i,\kappa)\in b_l^{\mathbf{i}}$. That is

$$a_{b_l^{\mathbf{i}}}:=a_{i_{p_{l-1}^{\mathbf{i}}}}=a_{i_{p_{l-1}^{\mathbf{i}}+1}}=\cdots=a_{i_{p_{l-1}^{\mathbf{i}}+k_l^{\mathbf{i}}-1}}.$$

For a block $b=((i_u,\kappa_u),\dots,(i_v,\kappa_v))$ we define

$$f_b:=f_{(i_u,\kappa_u)}\circ\cdots\circ f_{(i_v,\kappa_v)}. \tag{3.7}$$

By the notations above we have

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i})=\lim_{l\to\infty}f_{b_0^{\mathbf{i}}}\circ\cdots\circ f_{b_l^{\mathbf{i}}}(0)=a_{b_0^{\mathbf{i}}}+\sum_l(a_{b_{l+1}^{\mathbf{i}}}-a_{b_l^{\mathbf{i}}})\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}. \tag{3.8}$$

We define both the empty sum, and for every $0<\alpha<1$, α^{∞} as 0. Let us assume about the first element (i_0,κ_0) of $\mathbf{i}$ that $i_0\in\mathcal{N}$. To find the exponent of $\lambda_{i_0,2}$ we introduce a set $Q^{\mathbf{i}}$ as follows: First for every $l\geq 0$ we assign an integer $m(l)$ which is the total number of the appearances of $(i_0,2)$ in the union of the first l blocks. Observe we always assign the same $m(l)$ to more than one consecutive l. Among these, the smallest one is called $r_m^{\mathbf{i}}$ and the biggest one is $o_m^{\mathbf{i}}\geq 1+r_m^{\mathbf{i}}$ The collection of the distinct integers $m(l)$ assigned in this way to some $l\geq 0$ is the set $Q^{\mathbf{i}}$. That is

$$Q^{\mathbf{i}}=\left\{m\geq 0:\exists l\geq 0,\ m=\sharp_{(i_0,2)}(p_l^{\mathbf{i}})\right\}. \tag{3.9}$$

and

$$o_m^{\mathbf{i}}=\sup\left\{l:\sharp_{(i_0,2)}(p_l^{\mathbf{i}})=m\right\},\ r_m^{\mathbf{i}}=\inf\left\{l:\sharp_{(i_0,2)}(p_l^{\mathbf{i}})=m\right\}. \tag{3.10}$$

It is possible that $o_{\mathbf{i}}^m = \infty$. Now we partition the sum in (3.8) according to the exponent of $(i_0, 2)$:

$$\begin{aligned}\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) &= a_{b_0^{\mathbf{i}}} + \sum_{l=0}^{\infty}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})} \\ &= a_{b_0^{\mathbf{i}}} + \sum_{m\in Q^{\mathbf{i}}}\sum_{l=r_m^{\mathbf{i}}}^{o_m^{\mathbf{i}}}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})} \\ &= a_{b_0^{\mathbf{i}}} + \sum_{m\in Q^{\mathbf{i}}} d_{\mathbf{i}}^m \lambda_{(i_0,2)}^m, \end{aligned} \tag{3.11}$$

where

$$d_{\mathbf{i}}^m = \sum_{l=r_m^{\mathbf{i}}}^{o_m^{\mathbf{i}}}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\frac{\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}}{\lambda_{(i_0,2)}^{\sharp_{(i_0,2)}\mathbf{i}(p_l^{\mathbf{i}})}} = \sum_{l=r_m^{\mathbf{i}}}^{o_m^{\mathbf{i}}}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\frac{\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}. \tag{3.12}$$

Note that for $l = r_m^{\mathbf{i}}, \dots, o_m^{\mathbf{i}}$ the ratio $\frac{\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}$ is independent of $\lambda_{(i_0,2)}$, by the definition of m.

Lemma 3.1. *Let* $\mathbf{i} = ((i_0,\kappa_0)(i_1,\kappa_1)\cdots) \in \Sigma$ *such that* $i_0 \in \mathcal{N}$. *Then for every* $m \in Q^{\mathbf{i}}$ *we have*

$$|d_{\mathbf{i}}^m| \leq \frac{\underline{\lambda}^{\sharp\mathbf{i}(p_{r_m^{\mathbf{i}}}^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m} \max\left\{a_{N-1} - a_{i_0}, a_{i_0} - a_0\right\}. \tag{3.13}$$

Moreover if $0 \in Q^{\mathbf{i}}$ *then*

$$|d_{\mathbf{i}}^0| \geq \lambda_{(i_0,1)}^{k_0^{\mathbf{i}}} \min\left\{f_{(i_0+1,1)}(a_0) - a_{i_0}, a_{i_0} - f_{(i_0-1,1)}(a_{N-1})\right\}. \tag{3.14}$$

Proof. The statement of the lemma follows easily from the following observation:

$$d_{\mathbf{i}}^m = \frac{\underline{\lambda}^{\sharp\mathbf{i}(p_{r_m^{\mathbf{i}}}^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}\left(f_{\underline{i}}(a_{i_0}) - a_{i_0}\right), \tag{3.15}$$

where $\underline{i} := (b_{r_m^{\mathbf{i}}+1}^{\mathbf{i}} \cdots b_{o_m^{\mathbf{i}}}^{\mathbf{i}})$ and using the notation of (3.7) we define

$$f_{\underline{i}} = f_{b_{r_m^{\mathbf{i}}+1}^{\mathbf{i}}} \circ \cdots \circ f_{b_{o_m^{\mathbf{i}}}^{\mathbf{i}}}.$$

To verify (3.15) we fix an $\mathbf{i} = ((i_0,\kappa_0)(i_1,\kappa_1)\cdots) \in \Sigma$ and $m \in Q^{\mathbf{i}}$. Using that $a_{b_{o_m^{\mathbf{i}}+1}^{\mathbf{i}}} = a_{b_{r_m^{\mathbf{i}}}^{\mathbf{i}}} = a_{i_0}$ by definition we have

$$f_{\underline{i}}(a_{i_0}) = a_{b_{r_m^{\mathbf{i}}+1}^{\mathbf{i}}} + \sum_{l=r_m^{\mathbf{i}}+1}^{o_m^{\mathbf{i}}-1}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})-\sharp\mathbf{i}(p_{r_m^{\mathbf{i}}}^{\mathbf{i}})} + (a_{b_{o_m^{\mathbf{i}}+1}^{\mathbf{i}}} - a_{b_{o_m^{\mathbf{i}}}^{\mathbf{i}}})\underline{\lambda}^{\sharp\mathbf{i}(p_{o_m^{\mathbf{i}}}^{\mathbf{i}})-\sharp\mathbf{i}(p_{r_m^{\mathbf{i}}}^{\mathbf{i}})}$$

and

$$d_{\mathbf{i}}^m = \sum_{l=r_m^{\mathbf{i}}}^{o_m^{\mathbf{i}}}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\frac{\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m} = \frac{\underline{\lambda}^{\sharp\mathbf{i}(p_{r_m^{\mathbf{i}}}^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}\left(f_{\underline{i}}(a_{i_0}) - a_{b_{r_m^{\mathbf{i}}}^{\mathbf{i}}}\right).$$

Which completes the proof of (3.15). Therefore

$$|d_{\mathbf{i}}^m| \leq \frac{\underline{\lambda}^{\sharp\mathbf{i}(p^{\mathbf{i}}_{r^{\mathbf{i}}_m})}}{\lambda^m_{(i_0,2)}} \max\left\{a_{N-1}-a_{i_0}, a_{i_0}-a_0\right\}.$$

Now let us suppose that $0 \in Q^{\mathbf{i}}$ then $r_0^{\mathbf{i}} = 0$. Moreover $b_0^{\mathbf{i}}$ contains only $(i_0, 1)$. Then by $|b_0^{\mathbf{i}}| = k_0^{\mathbf{i}}$ we have

$$d_{\mathbf{i}}^m = \lambda_{(i_0,1)}^{k_0^{\mathbf{i}}}\left(f_{\underline{i}'}(a_{i_0}) - a_{i_0}\right),$$

where $\underline{i}' = (b_1^{\mathbf{i}} \cdots b^{\mathbf{i}}_{o^{\mathbf{i}}_0})$. By definition, $b_1^{\mathbf{i}}$ does not contain elements from $\{(i_0,1),(i_0,2)\}$. Then by (3.3) and $\lambda_{(i,2)} < \lambda_{(i,1)}$ we have

$$|f_{\underline{i}'}(a_{i_0}) - a_{i_0}| \geq \min\left\{f_{(i_0+1,1)}(a_0) - a_{i_0}, a_{i_0} - f_{(i_0-1,1)}(a_{N-1})\right\}$$

which completes the proof. □

3.2. **Proof of transversality condition.** Since, for every $i \in \mathcal{N}$ and every $\mathbf{i},\mathbf{j} \in \{(i,1),(i,2)\}^{\mathbb{N}}$ with $(i,\kappa_0) \neq (i,\tau_0)$ we have $\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) \equiv \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}) \equiv a_i$, the IFS $\mathcal{S}$ does not satisfy transversality condition. The main tool of the paper is to prove that we can find a series of suitable subsystems of $\mathcal{S}$ which satisfy the transversality and well approximates the attractor of $\mathcal{S}$ (in terms of Hausdorff dimension). For $k \geq 2$ let

$$\mathcal{U}_k = \mathcal{I} \bigcup \left(\bigcup_{l=0}^{k-2} \bigcup_{\underline{i}\in\mathcal{J}^l} \bigcup_{u\in\mathcal{N}} \bigcup_{v=0,u\neq v}^{N-1} \left\{\underline{i}(u,2)(v,1)\right\}\right). \tag{3.16}$$

For a $k \geq 2$ we define

$$\Psi_k = \left\{f_{\underline{i}}\right\}_{\underline{i}\in\mathcal{U}_k}. \tag{3.17}$$

We prove in Lemma 3.2 below that for every $k \geq 2$ the IFS Ψ_k satisfies transversality on a certain parameter domain $R_{\underline{\varepsilon}}$. Using this, in Proposition 3.4, we verify that the transversality holds on a domain which approximates the parameter domain that appears in Theorem 1.1. First we introduce the corresponding notation. Let us denote the attractor of Ψ_k by $\Lambda_k^{\underline{\lambda}}$ and the natural projection from $\Sigma_k := \mathcal{U}_k^{\mathbb{N}}$ onto $\Lambda_k^{\underline{\lambda}}$ by $\pi_k^{\underline{\lambda}}$. Denote the elements of Σ_k by $\mathbf{i}' = (\underline{i}_0\underline{i}_1\cdots)$.

Lemma 3.2. *Let $0 < \varepsilon_i < \lambda_{(i,1)}$ for every $i = 0,\dots,N-1$. Then for every $k \geq 2$ and every $\mathbf{i}' = (\underline{i}_0\underline{i}_1\cdots), \mathbf{j}' = (\underline{j}_0\underline{j}_1\cdots) \in \Sigma_k$ such that $\underline{i}_0 \neq \underline{j}_0 \in \mathcal{U}_k$,*

$$\pi_k^{\widetilde{\underline{\lambda}}}(\mathbf{i}') = \pi_k^{\widetilde{\underline{\lambda}}}(\mathbf{j}') \implies \left|\frac{\partial}{\partial\lambda_{(i,2)}}\left(\pi_k^{\underline{\lambda}}(\mathbf{i}') - \pi_k^{\underline{\lambda}}(\mathbf{j}')\right)\Big|_{\underline{\lambda}=\widetilde{\underline{\lambda}}}\right| > 0, \tag{3.18}$$

for some i and for every

$$\widetilde{\underline{\lambda}_2} \in R_{\underline{\varepsilon}} = \prod_{i\in\mathcal{N}}\left(\varepsilon_i, \min\left\{\lambda_{(i,1)}, \frac{1}{1+\sqrt{\lambda_{\max}\alpha_i\left(1+\frac{\alpha_i}{\varepsilon_i}\right)}}\right\}\right), \tag{3.19}$$

if it exists, where $\lambda_{\max} = \max_{i=0,\dots,N-1}\left\{\lambda_{(i,1)}\right\}$ and

$$\alpha_i = \frac{\max\left\{a_{N-1}-a_i, a_i-a_0\right\}}{\min\left\{f_{(i+1,1)}(a_0)-a_i, a_i - f_{(i-1,1)}(a_{N-1})\right\}}.$$

To prove Lemma 3.2 we need the following Sublemma:

Sublemma 3.3. *Let $\underline{i}, \underline{j}$ finite length word of symbols such that*

$$\underline{i} = \overbrace{(i,1)\cdots(i,1)}^{k_1}(l_1,\kappa_1)$$

$$\underline{j} = \overbrace{(i,2)\cdots(i,2)}^{k_2}(l_2,\kappa_2)$$

where $l_1, l_2 \neq i$. If $f_{\underline{i}}([a_0,a_{N-1}]) \cap f_{\underline{j}}([a_0,a_{N-1}]) \neq \emptyset$ then

$$\frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}} \leq \alpha_i.$$

Proof. Since for every $(i,2) \in \mathcal{J}$, $\lambda_{(i,2)} < \lambda_{(i,1)}$, we have that $f_{\underline{i}}([a_0,a_{N-1}]) \cap f_{\underline{j}}([a_0,a_{N-1}]) \neq \emptyset$ implies

$$\lambda_{(i,1)}^{k_1}\lambda_{(l_1,\kappa_1)}a_0 + \lambda_{(i,1)}^{k_1}a_{l_1}(1-\lambda_{(l_1,\kappa_1)}) + a_i(1-\lambda_{(i,1)}^{k_1}) \leq \\ \lambda_{(i,2)}^{k_2}\lambda_{(l_2,\kappa_2)}a_{N-1} + \lambda_{(i,2)}^{k_2}a_{l_2}(1-\lambda_{(l_2,\kappa_2)}) + a_i(1-\lambda_{(i,2)}^{k_2}),$$

$$\lambda_{(i,2)}^{k_2}\lambda_{(l_2,\kappa_2)}a_0 + \lambda_{(i,2)}^{k_2}a_{l_2}(1-\lambda_{(l_2,\kappa_2)}) + a_i(1-\lambda_{(i,2)}^{k_2}) \leq \\ \lambda_{(i,1)}^{k_1}\lambda_{(l_1,\kappa_1)}a_{N-1} + \lambda_{(i,1)}^{k_1}a_{l_1}(1-\lambda_{(l_1,\kappa_1)}) + a_i(1-\lambda_{(i,1)}^{k_1}).$$

Using the fact that $\mathcal{F}$ satisfies (3.3), we have $l_1, l_2 > i$ or $l_1, l_2 < i$. One can finish the proof by some obvious algebraic manipulations. □

Proof of Lemma 3.2. Let $0 < \varepsilon_i < \lambda_{(i,1)}$ and suppose that $\varepsilon_i < \lambda_{(i,2)}$ for every $i \in \mathcal{N}$. Let $\mathbf{i}', \mathbf{j}' \in \Sigma_k$ such that $\underline{i}_0 \neq \underline{j}_0$ and $\pi_k^{\underline{\lambda}}(\mathbf{i}') = \pi_k^{\underline{\lambda}}(\mathbf{j}')$. Divide $\underline{i}_0$ and $\underline{j}_0$ into blocks such that $\underline{i}_0 = (b_0^{i_0}\cdots b_l^{i_0})$ and $\underline{j}_0 = (b_0^{j_0}\cdots b_q^{j_0})$. By definition, a block consists of such pairs which share the same first component. If u is the common first element in the case of the block $b_0^{i_0}$ and v for $b_0^{j_0}$ then applying (3.3) we obtain that $u = v$. That is the first elements of all of the pairs that are contained either in $b_0^{i_0}$ or in $b_0^{j_0}$ are the same. First let us assume that both of $\underline{i}_0$ and $\underline{j}_0$ begin with $(i,2)$. Then by the definition of $\mathcal{U}_k$ (see (3.16)), $b_0^{i_0}, b_0^{j_0}$ contain only $(i,2)$. Since $\mathcal{S}$ satisfies (3.3) we have that $|b_0^{i_0}| = |b_0^{j_0}| = n$. This implies that

$$0 = \pi_k^{\underline{\lambda}}(\mathbf{i}') - \pi_k^{\underline{\lambda}}(\mathbf{j}') = \lambda_{(i,2)}^n\left(\pi_k^{\underline{\lambda}}(\mathbf{i}'^*) - \pi_k^{\underline{\lambda}}(\mathbf{j}'^*)\right)$$

where the first element of $\mathbf{i}'^*$ is $(b_1^{i_0}\cdots b_l^{i_0}) \in \Sigma_k$ and the first element of $\mathbf{j}'^*$ is $(b_1^{j_0}\cdots b_q^{j_0}) \in \Sigma_k$. Since $\lambda_{(i,2)} > \varepsilon_i$, without loss of generality we can assume that $\underline{i}_0 = (i,1)$ and $b_0^{j_0}$ contains only $(i,2)$ for an $i \in \mathcal{N}$. Let us write $\mathbf{i}, \mathbf{j}$ for the elements of $\Sigma = (\mathcal{I} \cup \mathcal{J})^{\mathbb{N}}$ that correspond to $\mathbf{i}', \mathbf{j}'$ respectively. Then $\pi_k^{\underline{\lambda}}(\mathbf{i}') \equiv \pi_{\underline{\lambda},\underline{a}}(\mathbf{i})$ and $\pi_k^{\underline{\lambda}}(\mathbf{j}') \equiv \pi_{\underline{\lambda},\underline{a}}(\mathbf{j})$.

If $\sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}}) \geq \sharp_{(i,2)}\mathbf{j}(k_0^{\mathbf{j}})$ then by (3.3), $\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) \neq \pi_{\underline{\lambda},\underline{a}}(\mathbf{j})$ therefore without loss of generality we assume that $\sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}}) < \sharp_{(i,2)}\mathbf{j}(k_0^{\mathbf{j}})$. Then

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) - \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}) = \lambda_{(i,2)}^{\sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}})}\left(\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}^*) - \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}^*)\right),$$

where

$$\mathbf{i}^* = (\overbrace{(i,1)\cdots(i,1)}^{\sharp_{(i,1)}\mathbf{i}(k_0^{\mathbf{i}})} b_1^{\mathbf{i}}\cdots) \text{ and } \mathbf{j}^* = (\overbrace{(i,2)\cdots(i,2)}^{\sharp_{(i,2)}\mathbf{j}(k_0^{\mathbf{j}})-\sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}})} b_1^{\mathbf{j}} b_2^{\mathbf{j}}\cdots).$$

Since $\lambda_{(i,2)} > \varepsilon_i > 0$ it is enough to prove that

$$f(\underline{\lambda}) = 0 \implies \|\mathrm{grad} f(\underline{\lambda})\| > 0, \tag{3.20}$$

where $f(\underline{\lambda}) = \pi_{\underline{\lambda},\underline{a}}(\mathbf{i}^*) - \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}^*)$. Let $m = \min Q^{\mathbf{j}^*}$ then by (3.11) we have

$$f(\underline{\lambda}) = d_{\mathbf{i}^*}^0\left(1 + \sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d_{\mathbf{i}^*}^k}{d_{\mathbf{i}^*}^0}\lambda_{(i,2)}^k - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d_{\mathbf{j}^*}^k}{d_{\mathbf{i}^*}^0}\lambda_{(i,2)}^k\right) =$$

$$d_{\mathbf{i}^*}^0\left(1 + \sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d_{\mathbf{i}^*}^k}{d_{\mathbf{i}^*}^0}\lambda_{(i,2)}^k - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d_{\mathbf{j}^*}^k\lambda_{(i,2)}^m}{d_{\mathbf{i}^*}^0\lambda_{(i,2)}}\lambda_{(i,2)}^{k-m+1}\right).$$

Now we give upper bound for the absolute value of the coefficients. It is easy to see by Lemma 3.1 and Sublemma 3.3 that

$$\begin{aligned}
&\left|\frac{d_{\mathbf{i}^*}^k}{d_{\mathbf{i}^*}^0}\right| \le \lambda_{\max}\alpha_i && \text{for every } k\in Q_i^{\mathbf{i}^*}\setminus\{0\}\\
&\left|\frac{d_{\mathbf{j}^*}^m\lambda_{(i,2)}^m}{d_{\mathbf{i}^*}^0\lambda_{(i,2)}}\right| \le \frac{\alpha_i^2}{\varepsilon_i} && \text{and}\\
&\left|\frac{d_{\mathbf{j}^*}^k\lambda_{(i,2)}^m}{d_{\mathbf{i}^*}^0\lambda_{(i,2)}}\right| \le \lambda_{\max}\frac{\alpha_i^2}{\varepsilon_i} && \text{for every } k\in Q_i^{\mathbf{j}^*}\setminus\{m\}.
\end{aligned}$$

Therefore absolute value of the coefficient of $\lambda_{(i,2)}$ is at most $\lambda_{\max}\alpha_i + \frac{\alpha_i^2}{\varepsilon_i}$ and the absolute value of the coefficient of $\lambda_{(i,2)}^k$ for $k\ge 2$ is at most $\lambda_{\max}\alpha_i + \lambda_{\max}\frac{\alpha_i^2}{\varepsilon_i}$. If $f(\widetilde{\underline{\lambda}}) = 0$ then

$$\frac{\partial f}{\partial\lambda_{(i,2)}}(\underline{\lambda}) = d_{\mathbf{i}^*}^0\left(\sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d_{\mathbf{i}^*}^k}{d_{\mathbf{i}^*}^0}k\lambda_{(i,2)}^{k-1} - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d_{\mathbf{j}^*}^k\lambda_{(i,2)}^m}{d_{\mathbf{i}^*}^0\lambda_{(i,2)}}(k-m+1)\lambda_{(i,2)}^{k-m}\right.$$

$$\left. - \sum_{k\in Q^{\mathbf{j}^*}}(m-1)\frac{d_{\mathbf{j}^*}^k\lambda_{(i,2)}^{m-2}}{d_{\mathbf{i}^*}^0}\lambda_{(i,2)}^{k-m+1}\right),$$

and by Lemma 2.4 we obtain that for $\lambda_{(i,2)} \in \left(\varepsilon_i, \frac{1}{1+\sqrt{\lambda_{\max}\alpha_i\left(1+\frac{\alpha_i}{\varepsilon_i}\right)}}\right)$ the following inequality holds:

$$\sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d_{\mathbf{i}^*}^k}{d_{\mathbf{i}^*}^0}k\lambda_{(i,2)}^{k-1} - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d_{\mathbf{j}^*}^k\lambda_{(i,2)}^m}{d_{\mathbf{i}^*}^0\lambda_{(i,2)}}(k-m+1)\lambda_{(i,2)}^{k-m} < 0. \tag{3.21}$$

On the other hand, (3.15) yields that for suitable $\underline{i}', \underline{j}'$ we have

$$\frac{d_{\mathbf{j}^*}^m}{d_{\mathbf{i}^*}^0} = \frac{\frac{\underline{\lambda}^{\sharp\mathbf{j}(p_{r_m^{\mathbf{j}}}^{\mathbf{j}})}}{\lambda_{(i,2)}^m}\left(f_{\underline{j}'}(a_i) - a_i\right)}{\lambda_{(i,1)}^{k_0^{\mathbf{i}^*}}\left(f_{\underline{i}'}(a_i) - a_i\right)}.$$

Let i'_0 and j'_0 be the first element of the first component of $\underline{i}'$, $\underline{j}'$. Then by (3.3), $i'_0, j'_0 > i$ or $i'_0, j'_0 < i$ which implies that $\frac{d^m_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}} > 0$. Therefore by Lemma 3.1 we have for $\lambda_{(i,2)} < \frac{1}{1+\lambda_{\max}\alpha_i}$ that

$$\sum_{k\in Q^{\mathbf{j}^*}} (m-1)\frac{d^k_{\mathbf{j}^*}\lambda^{m-2}_{(i,2)}}{d^0_{\mathbf{i}^*}}\lambda^{k-m+1}_{(i,2)} = (m-1)\frac{d^m_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}}\lambda^{m-1}_{(i,2)}\left(1+\sum_{k\in Q^{\mathbf{j}^*}\setminus\{m\}}\frac{d^k_{\mathbf{j}^*}}{d^m_{\mathbf{j}^*}}\lambda^{k-m}_{(i,2)}\right) \geq$$

$$(m-1)\frac{d^m_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}}\lambda^{m-1}_{(i,2)}\left(1-\sum_{k=1}^{\infty}\lambda_{\max}\alpha_i\lambda^k_{(i,2)}\right) \geq 0. \quad (3.22)$$

Observe that $\frac{1}{1+\sqrt{\lambda_{\max}\alpha_i\left(1+\frac{\alpha_i}{\varepsilon_i}\right)}} < \frac{1}{1+\lambda_{\max}\alpha_i}$ holds for every $0 < \varepsilon_i < 1$. Using this (3.21) and (3.22) we have

$$f(\underline{\widetilde{\lambda}}) = 0 \implies \frac{\partial f}{\partial\lambda_{(i,2)}}(\underline{\widetilde{\lambda}}) < 0$$

which was to be proved. □

Proposition 3.4. *For every $k \geq 2$, the IFS Ψ_k satisfies the transversality condition on*

$$\underline{\lambda}_2 \in \mathcal{T}_N(\xi) = \prod_{i\in\mathcal{N}}(\xi, \min\left\{\lambda_{(i,1)}, \frac{2}{(1+\sqrt{2})(\alpha_i^2\lambda_{\max}+2)}\right\} - \xi) \quad (3.23)$$

where $\xi > 0$ is arbitrary small and

$$\alpha_i = \frac{\max\{a_{N-1}-a_i, a_i-a_0\}}{\min\{f_{i+1}(a_0)-a_i, a_i-f_{i-1}(a_{N-1})\}} \text{ for } i \in \mathcal{N}.$$

Proof. Let

$$g_i(x) = \frac{1}{1+\sqrt{\lambda_{\max}\alpha_i\left(1+\frac{\alpha_i}{x}\right)}}.$$

We can extend g_i onto $[0,\infty)$ as $g_i(0) = 0$, which is a fixed point of g_i. It is easy to see by simple calculations that g_i is strictly monotone increasing and has a unique positive fixed point ε_i^*.

Hence, we can cover the rectangle $\prod_{i\in\mathcal{N}}(0, \min\{\lambda_{(i,1)}, \varepsilon_i^*\})$ by countable many rectangles in the type $R_{\underline{\varepsilon}}$, see (3.19).

It follows from Lemma 3.2 that for every $k \geq 2$ and $\mathbf{i}', \mathbf{j}' \in \Sigma_k$ with $\underline{i}_0 \neq \underline{j}_0$ the function $\pi^{\lambda}_{k}(\mathbf{i}') - \pi^{\lambda}_{k}(\mathbf{j}')$ satisfies (2.3) on the rectangle $\prod_{i\in\mathcal{N}}(0, \min\{\lambda_{(i,1)}, \varepsilon_i^*\})$.

Now we are going to prove that

$$\frac{2}{(\sqrt{2}+1)(\alpha_i^2\lambda_{\max}+2)} \leq \varepsilon_i^*. \quad (3.24)$$

To verify this, observe that

$$\varepsilon_i^* = \frac{2}{\sqrt{(\alpha_i^2\lambda_{\max}+2)^2+4(\alpha_i\lambda_{\max}-1)}+\alpha_i^2\lambda_{\max}+2}.$$

If the second term under the square root is non-positive, that is if $\alpha_i\lambda_{\max} \leq 1$ then clearly (3.24) holds. Otherwise, $\alpha_i\lambda_{\max} > 1$. Then $\alpha_i > 1$. A simple calculation yields: $4(\alpha_i\lambda_{\max}-1) \leq (\alpha_i^2\lambda_{\max}+2)^2$ which follows that (3.24) holds. To complete the proof we apply Lemma 2.3 for the rectangle on the right hand side of (3.23) with $\xi = 0$. □

3.3. **Hausdorff dimension.** Before we prove the theorems we have to introduce a sequence of functions. For every $k \geq 2$ we introduce the function $h_{\underline{\lambda},k}(s)$ which is defined as the sum of the s-power of the contraction ratios of the IFS Ψ_k. That is

$$h_{\underline{\lambda},k}(s) = \sum_{i=0}^{N-1} \lambda_{(i,1)}^s + \sum_{l=0}^{k-2} \left(\sum_{i \in \mathcal{N}} \lambda_{(i,2)}^s \right)^l \sum_{i \in \mathcal{N}} \sum_{j=0, j \neq i}^{N-1} \lambda_{(i,2)}^s \lambda_{(j,1)}^s. \tag{3.25}$$

Let $s_k(\underline{\lambda})$ be the unique solution of $h_{\underline{\lambda},k}(s) = 1$. Therefore $\dim_H \Lambda_k^{\underline{\lambda}} \leq \min\{1, s_k(\underline{\lambda})\}$, where $\Lambda_k^{\underline{\lambda}}$ is the attractor of Ψ_k.

Since the sequence $s_k(\underline{\lambda})$ is monotone increasing and bounded, it is convergent. It is easy to see by some algebraic manipulation that the limit of $s_k(\underline{\lambda})$ is the unique solution of

$$\sum_{i=0}^{N-1} \lambda_{(i,1)}^s + \sum_{i \in \mathcal{N}} \lambda_{(i,2)}^s \left(1 - \lambda_{(i,1)}^s \right) = 1.$$

This equation corresponds to (1.4).

Moreover, we need to introduce a sequence of subsets of Σ^*. Let

$$\mathcal{C}_1 = \mathcal{I} = \{(0,1), \dots, (N-1,1)\} \tag{3.26}$$

and by induction let

$$\mathcal{C}_{k+1} = \bigcup_{j=0}^{N-1} \bigcup_{\underline{i} \in \mathcal{C}_k} \{(j,1)\underline{i}\} \cup \bigcup_{j \in \mathcal{N}} \bigcup_{\substack{\underline{i} \in \mathcal{C}_k \\ (i_0,\kappa_0) \neq (j,1)}} \{(j,2)\underline{i}\}. \tag{3.27}$$

Then we can look at the elements of $\mathcal{C}_k$ either as certain sequences of length k of symbols from $\mathcal{I} \cup \mathcal{J}$ or juxtapositions of at most k elements of $\mathcal{U}_k$.

Lemma 3.5. *Let $\widetilde{s}_k(\underline{\lambda})$ be the unique solution of*

$$\sum_{\underline{i} \in \mathcal{C}_k} \lambda_{\underline{i}}^s = 1,$$

and let $\widetilde{s}(\underline{\lambda}) = \sup_k \widetilde{s}_k(\underline{\lambda})$ then

$$\dim_H \Lambda_{\underline{\lambda},\underline{a}} \leq \min\{1, \widetilde{s}(\underline{\lambda})\}.$$

Moreover

$$\mathcal{H}^{\widetilde{s}(\underline{\lambda})}(\Lambda_{\underline{\lambda},\underline{a}}) \leq (a_{N-1} - a_0)^{\widetilde{s}(\underline{\lambda})}$$

Note that $\widetilde{s}_k(\underline{\lambda})$ is bounded since $\mathcal{C}_k \subset (\mathcal{I} \cup \mathcal{J})^k$.

Proof. Using that for every $i \in \mathcal{N}$

$$f_{(i,1)} \circ f_{(i,2)} \equiv f_{(i,2)} \circ f_{(i,1)},$$

and $0 < \lambda_{(i,2)} < \lambda_{(i,1)} < 1$ we have that the set of closed intervals

$$\left\{ f_{\underline{i}}([a_0, a_{N-1}]) \right\}_{\underline{i} \in \mathcal{C}_k}$$

gives a cover of $\Lambda_{\underline{\lambda},\underline{a}}$ with diameter at most $\lambda_{\max}^k$. Then

$$\mathcal{H}^{\widetilde{s}(\underline{\lambda})}_{\lambda^k_{\max}}(\Lambda_{\underline{\lambda},\underline{a}}) \leq \sum_{\underline{i}\in\mathcal{C}_k} \left|f_{\underline{i}}([a_0,a_{N-1}])\right|^{\widetilde{s}(\underline{\lambda})} = (a_{N-1}-a_0)^{\widetilde{s}(\underline{\lambda})} \sum_{\underline{i}\in\mathcal{C}_k} f'_{\underline{i}}(0)^{\widetilde{s}(\underline{\lambda})} \leq$$

$$(a_{N-1}-a_0)^{\widetilde{s}(\underline{\lambda})} \underbrace{\sum_{\underline{i}\in\mathcal{C}_k} f'_{\underline{i}}(0)^{\widetilde{s}_k(\underline{\lambda})}}_{1} = (a_{N-1}-a_0)^{\widetilde{s}(\underline{\lambda})}.$$

This proves the upper bound of the dimension and the measure claim of the Lemma. □

Proof of Theorem 1.1. Let $\xi > 0$. By the definition of $\mathcal{C}_k$ we have that for every $k \geq 1$

$$\mathcal{C}_k \subset \bigcup_{l=1}^{k} \mathcal{U}_k^l. \tag{3.28}$$

As it was mentioned above, every $\underline{i} \in \mathcal{C}_k$ can be decomposed as a juxtaposition $\underline{i} = \underline{j}_1 \cdots \underline{j}_r$, where each $\underline{j}_l$ is in $\mathcal{U}_k$ and $1 \leq r \leq k$. By using this fact and Proposition 3.4 we have that the system $\widetilde{\Psi}_k = \left\{f_{\underline{i}}\right\}_{\underline{i}\in\mathcal{C}_k}$ satisfies transversality on $\mathcal{T}_N(\xi)$. By Theorem 2.2 we have

$$\dim_H \widetilde{\Lambda}_k^{\underline{\lambda}} = \min\left\{1, \widetilde{s}_k(\underline{\lambda})\right\} \text{ for } \mathcal{L}\text{-a.e. } \underline{\lambda}_2 \in \mathcal{T}_N(\xi), \tag{3.29}$$

where $\widetilde{\Lambda}_k^{\underline{\lambda}}$ denotes the attractor of $\left\{f_{\underline{i}}\right\}_{\underline{i}\in\mathcal{C}_k}$. Using (3.28)

$$\dim_H \widetilde{\Lambda}_k^{\underline{\lambda}} \leq \dim_H \Lambda_k^{\underline{\lambda}}.$$

Moreover by Proposition 3.4 and Theorem 2.2 we have

$$\dim_H \Lambda_k^{\underline{\lambda}} = \min\left\{1, s_k(\underline{\lambda})\right\} \text{ for } \mathcal{L}\text{-a.e. } \underline{\lambda}_2 \in \mathcal{T}_N(\xi).$$

Since $\widetilde{\Lambda}_k^{\underline{\lambda}}, \Lambda_k^{\underline{\lambda}} \subseteq \Lambda_{\underline{\lambda},\underline{a}}$ for every $k \geq 2$ by Lemma 3.5 we have

$$\min\left\{1, \widetilde{s}_k(\underline{\lambda})\right\} \leq \min\left\{1, s_k(\underline{\lambda})\right\} \leq \min\left\{1, \widetilde{s}(\underline{\lambda})\right\}.$$

Since $s_k(\underline{\lambda})$ is strictly monotone increasing $\lim_{k\to\infty} s_k(\underline{\lambda}) = \sup_k s_k(\underline{\lambda})$. This implies that $\min\left\{1, s(\underline{\lambda})\right\} = \min\left\{1, \widetilde{s}(\underline{\lambda})\right\}$, moreover

$$\dim_H \Lambda_{\underline{\lambda},\underline{a}} = \min\left\{1, s(\underline{\lambda})\right\}.$$

To complete the proof of the last assertion of Theorem 1.1 first observe that whenever $s(\underline{\lambda}) > 1$ then there exists a $k \geq 2$ such that $s_k(\underline{\lambda}) > 1$. Therefore, by Theorem 2.2 and Proposition 3.4, $\mathcal{L}\left(\Lambda_{\underline{\lambda},\underline{a}}\right) \geq \mathcal{L}\left(\Lambda_k^{\underline{\lambda}}\right) > 0$ for a.e. $\underline{\lambda}_2 \in \mathcal{T}_N(\xi) \cap \left\{\underline{\lambda}_2 : s(\underline{\lambda}) > 1\right\}$. Since ξ was arbitrary, this completes the proof. □

3.4. **Example.** To visualize the behavior of the vector of contracting ratios we consider an easy example, where the functions of $\mathcal{F}$ are uniformly distributed with uniform contracting ratio, that is

$$\mathcal{F} = \left\{f_i(x) = \lambda x + i(1-\lambda)\right\}_{i=0}^{N-1},$$

where $0 < \lambda < \frac{1}{N}$. Let us add to the system the following N functions:

$$\mathcal{G} = \left\{g_i(x) = \gamma_i x + i(1-\gamma_i)\right\}_{i=0}^{N-1}.$$

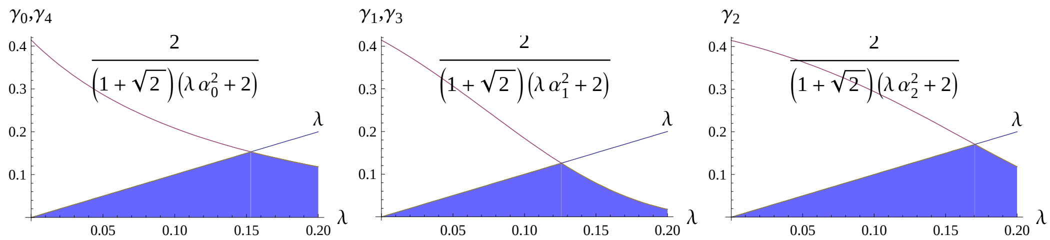

FIGURE 3. Transversality region for $N=5$ fixed points

Note that the fixed point of both f_i and g_i is i, $i=0,\dots,N-1$. It is easy to see that for every $i=1,\dots,N-2$

$$\alpha_i=\alpha_{N-1-i}=\frac{\max\{N-1-i,i\}}{\min\{1-(i+1)\lambda,1-(N-i)\lambda\}}\text{ and }\alpha_0=\alpha_{N-1}=\frac{N-1}{1-\lambda},$$

where α_i is as in Theorem 1.1. To satisfy the assumptions of Theorem 1.1 it is enough to require that

$$0<\gamma_i<\min\left\{\lambda,\frac{2}{(1+\sqrt{2})(\alpha_i^2\lambda+2)}\right\} \tag{3.30}$$

holds for $i=0,\dots,N-1$. For example, when $N=5$ then we can choose γ_i from the appropriate shaded region of Figure 3. In general, first we observe that

$$\alpha_i\le\alpha_1=\alpha_{N-2}=\frac{N-2}{1-(N-1)\lambda},$$

holds for every $i=0,\dots,N-1$. So by (3.30) the assumptions of Theorem 1.1 hold if we assume that

$$0<\gamma_i<\min\left\{\lambda,\frac{2}{(1+\sqrt{2})\left(\left(\frac{N-2}{1-(N-1)\lambda}\right)^2\lambda+2\right)}\right\},\quad 0\le i\le N-1. \tag{3.31}$$

We know that $0<\lambda$ must be smaller than $1/N$. By (3.31) we obtain that whenever $\lambda<0.4764/N$ holds then the assumptions of Theorem 1.1 are satisfied for $\gamma_i<\lambda$.

4. HAUSDORFF MEASURE

To prove Theorem 1.2 we use the method of Bandt and Graf [1] in the line as it was used by Peres, Simon and Solomyak, [7] with some modifications.

Without loss of generality we may assume that $s(\underline{\lambda})\le 1$. (Otherwise $\mathcal{H}^s(\Lambda)=0$ holds obviously.) Let us denote the local inverse of the left-shift operator σ on $\Sigma=(\mathcal{I}\cup\mathcal{J})^{\mathbb{N}}$ by $\sigma_{(i,\kappa)}^{-1}$. More precisely, for every $\mathbf{i}\in\Sigma$ let $\sigma_{(i,\kappa)}^{-1}\mathbf{i}=(i,\kappa)\mathbf{i}$. Denote $\sigma_{\underline{i}}^{-1}:=\sigma_{(i_0,\kappa_0)}^{-1}\circ\dots\circ\sigma_{(i_n,\kappa_n)}^{-1}$ for an $\underline{i}\in\Sigma^*$. Let

$$\widehat{\Sigma}=\bigcup_{k=0}^{\infty}\bigcup_{\underline{i}\in(\mathcal{I}\cup\mathcal{J})^k}\left\{\sigma_{\underline{i}}^{-1}\mathcal{J}^{\mathbb{N}}\right\},$$

which is the subset of Σ such that every $\mathbf{i} \in \widehat{\Sigma}$ contains only finitely many symbols of $\mathcal{I}$. Then

$$\Lambda_{\underline{\lambda},\underline{a}} = \pi_{\underline{\lambda},\underline{a}}\left(\widehat{\Sigma}\right) \bigcup \pi_{\underline{\lambda},\underline{a}}\left(\Sigma \setminus \widehat{\Sigma}\right).$$

Let

$$\mathcal{U}_\infty = \mathcal{I} \bigcup \left(\bigcup_{l=0}^{\infty} \bigcup_{\underline{i}\in\mathcal{J}^l} \bigcup_{i\in\mathcal{N}} \bigcup_{j=0, j\neq i}^{N-1} \{\underline{i}(i,2)(j,1)\} \right).$$

Cf. to (3.16) the definition of $\mathcal{U}_k$.

Lemma 4.1.

$$\pi_{\underline{\lambda},\underline{a}}\left(\Sigma \setminus \widehat{\Sigma}\right) \subseteq \pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right).$$

Proof. For every $\mathbf{i} \in \Sigma \setminus \widehat{\Sigma}$ there are at most two possibilities, it contains finitely or infinitely many blocks. If $\mathbf{i}$ contains an infinite length block (which is equivalent to $\mathbf{i}$ contains finitely many blocks) then we can change in the last block every element to a suitable $i \in \mathcal{I}$ without change the image by the natural projection.

The fact $f_{(i,1)} \circ f_{(i,2)} \equiv f_{(i,2)} \circ f_{(i,1)}$ completes the proof. □

Since Hausdorff dimension of $\pi_{\underline{\lambda},\underline{a}}\left(\widehat{\Sigma}\right)$ is equal to the Hausdorff dimension of the attractor of $\mathcal{G}$, which is the unique solution of $\sum_{i\in\mathcal{N}} \lambda_{(i,2)}^s = 1$, we have

$$\mathcal{H}^{s(\underline{\lambda})}(\Lambda_{\underline{\lambda},\underline{a}}) = \mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right). \tag{4.1}$$

We say that $\overline{i}$ and $\overline{j}$ elements of $\mathcal{U}_\infty^*$ (the set of finite length symbols of $\mathcal{U}_\infty$) are *incomparable* if there are no $\overline{\eta} \in \widetilde{\Sigma}_\infty^*$ such that $\overline{i} = \overline{j}\overline{\eta}$ or $\overline{j} = \overline{i}\overline{\eta}$ holds.

We define an outer measure. Let

$$\mu^s(K) = \inf\left\{ \sum_{k\in I} |U_k|^s : \text{ open, } K \subseteq \bigcup_{k\in I} U_k \right\}.$$

Lemma 4.2. *For measurable* $K \subseteq \pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)$, $\mathcal{H}^{s(\underline{\lambda})}(K)$ *coincides with the outer measure* $\mu^{s(\underline{\lambda})}(K)$. *Moreover,*

$$\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right) \cap f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\right) = 0$$

for every $\underline{i}, \underline{j} \in \mathcal{U}_\infty^*$ *such that* $\underline{i}$ *and* $\underline{j}$ *are incomparable.*

The proof of this Lemma coincides with the proof of [1, Proposition 3].

Proof of Theorem 1.2. Without loss of generality we can assume that for every $i \in \mathcal{N}$ the quotient $\frac{\log \lambda_{(i,2)}}{\log \lambda_{(i,1)}}$ is irrational. Otherwise $\dim_H \Lambda_{\underline{\lambda},\underline{a}} < s(\underline{\lambda})$ trivially.

Let $\underline{i} = (i,1)\cdots(i,1)(j,\kappa_1)$ and $\underline{j} = (i,2)\cdots(i,2)(j,\kappa_2)$ such that $\sharp_{(i,1)}(\underline{i}) = k_1$, $\sharp_{(i,2)}(\underline{j}) = k_2$ and $j \neq i$. Then

$$f_{\underline{i}}^{-1} \circ f_{\underline{j}}(x) = \frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}} x + \left(1 - \frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}}\right)\left(a_j(1 - \frac{1}{\lambda_{(i,1)}}) + \frac{a_i}{\lambda_{(i,1)}}\right).$$

Therefore for every $\delta > 0$ there exists $\underline{i}, \underline{j} \in \mathcal{U}_\infty^*$ incomparable words such that

$$\sup_{x\in[a_{(0,1)}, a_{(n-1,1)}]} \left\{ \left| x - f_{\underline{i}}^{-1} \circ f_{\underline{j}}(x) \right| \right\} < \delta. \tag{4.2}$$

Indirectly, let us suppose that $\mathcal{H}^{s(\underline{\lambda})}(\Lambda_{\underline{\lambda},\underline{a}})>0$ and let $\xi\in(1,\frac{3}{2})$. Since $\Lambda_{\underline{\lambda},\underline{a}}$ is compact, there exists $U_1,\dots,U_l$ finite cover of $\Lambda_{\underline{\lambda},\underline{a}}$ such that

$$\sum_{m=1}^{l}|U_l|^{s(\underline{\lambda})}<\xi\mathcal{H}^{s(\underline{\lambda})}(\Lambda_{\underline{\lambda},\underline{a}})=\xi\mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty,n}^{\mathbb{N}}\right)\right) \tag{4.3}$$

by (4.1). Let

$$\delta=\inf\left\{|a-x|:a\in\Lambda_{\underline{\lambda},\underline{a}},x\notin\bigcup_{m=1}^{l}U_m\right\}\le\inf\left\{|a-x|:a\in\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right),x\notin\bigcup_{m=1}^{l}U_m\right\}. \tag{4.4}$$

Let $\underline{i},\underline{j}\in\mathcal{U}_{\infty}^{*}$ such that

$$\sup_{x\in[a_0,a_{N-1}]}\left\{\left|x-f_{\underline{i}}^{-1}\circ f_{\underline{j}}(x)\right|\right\}<\delta$$

and $\frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}}>2-\xi$. Therefore by (4.4) we have

$$f_{\underline{i}}^{-1}\circ f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\subseteq\bigcup_{m=1}^{l}U_m$$

and

$$f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\bigcup f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\subseteq f_{\underline{i}}\left(\bigcup_{m=1}^{l}U_m\right).$$

So, we have by Lemma 4.2 that

$$\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\right)+\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\right)=$$
$$\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\bigcup f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\right)$$

which is less than or equal to

$$\sum_{m=1}^{l}|f_{\underline{i}}(U_m)|^{s(\underline{\lambda})}=\lambda_{(i,1)}^{k_1 s(\underline{\lambda})}\sum_{m=1}^{l}|U_m|^{s(\underline{\lambda})}<\lambda_{(i,1)}^{k_1 s(\underline{\lambda})}\xi\mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)$$

In the last inequality we have used (4.3) and (4.1).

However, by the definition of Hausdorff measure,

$$\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\right)+\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)\right)=$$
$$\lambda_{(i,1)}^{k_1 s(\underline{\lambda})}\mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right)+\lambda_{(i,2)}^{k_2 s(\underline{\lambda})}\mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty}^{\mathbb{N}}\right)\right).$$

Since we assumed that $\mathcal{H}^{s(\underline{\lambda})}\left(\Lambda_{\underline{\lambda},\underline{a}}\right)>0$ and by Lemma 3.5 $\mathcal{H}^{s(\underline{\lambda})}\left(\Lambda_{\underline{\lambda},\underline{a}}\right)$ is finite, by (4.1) we have $2-\xi<\xi-1$ which is a contradiction. □

5. Applications for higher dimensional self-affine sets

In this section we are going to show an application of the results for two dimensional, diagonally self-affine iterated function systems.

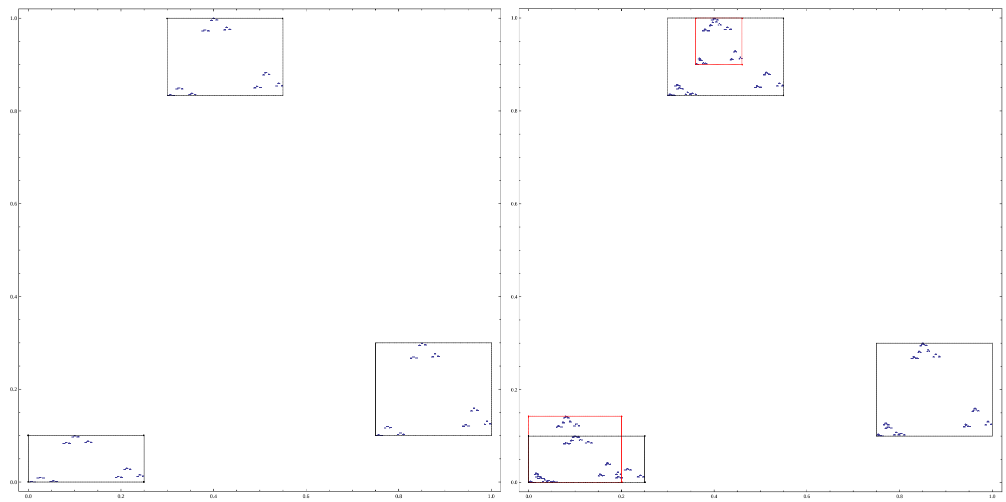

FIGURE 4. Diagonally self-affine fractals with fixed point correspondence

5.1. **Overlapping self-affine sets.** Let $\mathcal{M} = \{0, \dots, N-1\}$ and $\mathcal{N} \subseteq \{0, \dots, n-1\}$. Define the following diagonal matrices

$$\underline{\underline{\lambda}}_{(i,1)} = \begin{pmatrix} \lambda^1_{(i,1)} & 0 \\ 0 & \lambda^2_{(i,1)} \end{pmatrix} \text{ for } i \in \mathcal{M}$$

and

$$\underline{\underline{\lambda}}_{(i,2)} = \begin{pmatrix} \lambda^1_{(i,2)} & 0 \\ 0 & \lambda^2_{(i,2)} \end{pmatrix} \text{ for } i \in \mathcal{N}.$$

Let us suppose that

$$\begin{aligned} &0 < \lambda^2_{(i,1)} \leq \lambda^1_{(i,1)} < 1 \text{ for } i \in \mathcal{M} \\ &0 < \lambda^2_{(i,2)} \leq \lambda^1_{(i,2)} < \lambda^1_{(i,1)} < 1 \text{ for } i \in \mathcal{N}. \end{aligned} \tag{5.1}$$

Let $\underline{a}_i \in \mathbb{R}^2$ vectors for $i \in \mathcal{M}$ such that the IFS $\left\{\lambda^1_{(i,1)}x + a^1_i(1 - \lambda^1_{(i,1)})\right\}_{i \in \mathcal{M}}$ satisfies the strong separation condition. Let

$$G_1 = \left\{ f_{(i,1)}(\underline{x}) = \underline{\underline{\lambda}}_{(i,1)}\underline{x} + (\underline{\underline{I}} - \underline{\underline{\lambda}}_{(i,1)})\underline{a}_i \right\}_{i \in \mathcal{M}} \tag{5.2}$$

and

$$G_2 = \left\{ f_{(i,2)}(\underline{x}) = \underline{\underline{\lambda}}_{(i,2)}\underline{x} + (\underline{\underline{I}} - \underline{\underline{\lambda}}_{(i,2)})\underline{a}_i \right\}_{i \in \mathcal{N}}, \tag{5.3}$$

see Figure 4.

Theorem 5.1. *Let G_1 and G_2 as in (5.2) and (5.3). Let us assume that (5.1) holds. Moreover the IFS $\left\{h_i(x) = \lambda^1_{(i,1)}x + a^1_i(1 - \lambda^1_{(i,1)})\right\}_{i \in \mathcal{M}}$ satisfies*

$$h_{i-1}(a_N) < h_i(a_0). \tag{5.4}$$

If $G_1 \cup G_2$ satisfies

$$\sum_{i=0}^{n-1} \lambda^1_{(i,1)} + \sum_{i \in \mathcal{N}} \lambda^1_{(i,2)} - \sum_{i \in \mathcal{N}} \lambda^1_{(i,2)}\lambda^1_{(i,1)} < 1 \tag{5.5}$$

then

$$\dim_H \Lambda = s \text{ Lebesgue-a.e. } 0 < \lambda^1_{(i,2)} < \min\left\{\lambda^1_{(i,1)}, \frac{2}{(\sqrt{2}+1)(\alpha_i^2\lambda^1_{\max}+2)}\right\}, \tag{5.6}$$

where Λ *denotes the attractor of* $G_1 \cup G_2$, $\lambda^1_{\max} = \max_{i\in\mathcal{M}}\left\{\lambda^1_i\right\}$ *and*

$$\alpha_i = \frac{\max\left\{a^1_{N-1} - a^1_i, a^1_i - a^1_0\right\}}{\min\left\{h_{i+1}(a^1_0) - a^1_i, a^1_i - h_{i-1}(a^1_{N-1})\right\}}$$

and s *is the unique solution of*

$$\sum_{i=0}^{n}\left(\lambda^1_{(i,1)}\right)^s + \sum_{i\in\mathcal{N}}\left(\lambda^1_{(i,2)}\right)^s - \sum_{i\in\mathcal{N}}\left(\lambda^1_{(i,2)}\lambda^1_{(i,1)}\right)^s = 1. \tag{5.7}$$

Proof. First, we prove the upper bound. Let $\mathcal{C}_k$ as in (3.26) and (3.27). Let $r_{\underline{i}} = \prod_{i\in\mathcal{M}}\left(\lambda^1_{(i,1)}\right)^{\sharp_{(i,1)}\underline{i}}\prod_{i\in\mathcal{N}}\left(\lambda^1_{(i,2)}\right)^{\sharp_{(i,2)}\underline{i}}$ for every finite length word $\underline{i}$ of symbols from $\mathcal{I}\cup\mathcal{J}$. Using (5.1) and the fact that $i \in \mathcal{N}$, $f_{(i,1)} \circ f_{(i,2)} \equiv f_{(i,2)} \circ f_{(i,1)}$, it is easy to see that the attractor Λ can be covered by cubes with side length $r_{\underline{i}}$, $\underline{i} \in \Sigma^*$. The proof of the claim that

$$\dim_{\mathrm{H}} \Lambda \leq s$$

can be carried out in the same way as it was done in the proof of Lemma 3.5. Further, it immediately follows from (5.5) that $s < 1$.

Denote proj_x the projection onto the x axis. Then $\dim_H \mathrm{proj}_x\Lambda \leq \dim_H \Lambda$. However, by using Theorem 1.1 we have $\dim_H \mathrm{proj}_x\Lambda = s$ for Lebesgue-a.e. $\lambda^1_{(i,2)} \in (0, \min\left\{\lambda^1_{(i,1)}, \frac{2}{(\sqrt{2}+1)(\alpha_i^2\lambda^1_{\max}+2)}\right\})$. This completes the proof. $\square$

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