# DIMENSION OF THE GENERALIZED 4-CORNER SET AND ITS PROJECTIONS

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Abstract. In the last two decades considerable attention has been paid to the dimension theory of self-affine sets. In the case of the generalized four corner sets (see Figure 1) the IFS obtain as the projection of the self-affine system have maps of common fixed points. In this paper we extend our result [3] which introduced a new method of computation of the box and Hausdorff dimension of self-similar families where some of the maps have common fixed point. The extended version of our method presented in this paper, makes it possible to determine the box dimension of the generalized four corner set for Lebesgue-typical contracting parameters.

## 1. Introduction and Statements

We call a set self-affine if it can be represented as a finite union of its affine copies. That is $\Lambda \subset \mathbb{R}^2$ is self-affine if there exists a finite list of contracting affine maps $\{f_i(x) = A_i x + a_i\}_{i=1}^m$ such that $\Lambda = \cup_{i=1}^m f_i(\Lambda)$, where $A_i$ are $2 \times 2$ real matrices on the plane. The dimension theory of self-affine sets is far from well understood even in the diagonal case, that is when all $A_i$ are diagonal matrices.

We consider the generalized four corner set $\Lambda(\underline{\alpha}, \underline{\beta})$ which is the attractor of the self-affine iterated function system (IFS) of Figure 1. (For a precise definition see Section 4.) The parameters $\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $\underline{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3)$ are chosen such that the rectangles $R_0, R_1, R_2, R_3$ on Figure 1 are disjoint. One of the main goals of the present paper is to determine the box dimension of this set for Lebesgue typical parameters.

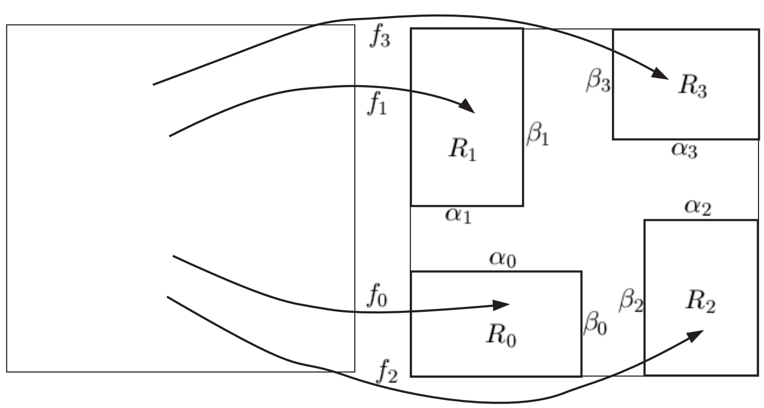


Figure 1. Maps of the generalized 4-corner set.

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The most natural upper bound of the box dimension is the subadditive-pressure formula which is called Falconer-dimension or singularity dimension and introduced by Falconer [4] and Barreira [2]. Namely, the Falconer Theorem (see [4]) states that the Hausdorff- and box dimension of a self-affine attractor coincide for almost every translation parameters and equal to the singularity dimension, whenever the norm of all the affine maps of IFS is smaller than 1/3. However, in our case we work with fixed translations and we modulate the multiplicative part of the affine maps. On the other hand, we do not impose any conditions of the norm involved the rectangles on Figure 1 are disjoint.

In the case of the generalized 4-corner set, the singularity dimension can be given by the following formula (see for example [9])

$$d_{\text{sing}} = \inf\left\{ s : \sum_{n=1}^{\infty} \sum_{i_1\cdots i_n \in \{0,\dots,3\}^n} \phi^s(i_1\cdots i_n) < \infty \right\} \tag{1.1}$$

where

$$\phi^s(i_1\cdots i_n) = \begin{cases} a(i_1\cdots i_n)^s & \text{if } 0 \le s \le 1 \\ a(i_1\cdots i_n)b(i_1\cdots i_n)^{s-1} & \text{if } 1 \le s \le 2 \end{cases}$$

where $a(i_1\cdots i_n) = \max\{\alpha_{i_1}\cdots\alpha_{i_n}, \beta_{i_1}\cdots\beta_{i_n}\}$ and $b(i_1\cdots i_n) = \min\{\alpha_{i_1}\cdots\alpha_{i_n}, \beta_{i_1}\cdots\beta_{i_n}\}$. For another method of calculation of the singularity dimension in our case see [7].

We will prove that for Lebesgue-typical parameters $\underline{\alpha}, \underline{\beta}$ the Hausdorff dimension and even the box dimension of the generalized 4-corner set is strictly smaller than the singularity dimension (1.1). The reason of this phenomena is the very special relative geometric position of the rectangles which generate the generalized 4-corner set. The speciality of the maps is that the fixed points are the corners of the unit square, so they do not move when we change the parameters $\underline{\alpha}, \underline{\beta}$. Therefore the orthogonal projection to the $x$-axis (and to the $y$-axis respectively) is an attractor of a special iterated function system of four similarities where the similarities derived from the maps having fixed points with same coordinate $y$ (and with same coordinate $x$) have common fixed points.

In [3] we considered the IFS $\{\gamma x, \lambda x, \lambda x + 1\}$, $\gamma < \lambda$ on the real line. See Figure 2 for the images of the convex hull of the attractor generated by the functions of this IFS. The novelty of the result obtained in [3] about the dimension of $\Lambda$ was to tackle the difficulty which comes from the fact that the first two maps have the same fixed point. In this paper we extend the scope of that result in the following way:

**Principal Assumptions:**

(A1) Let $\mathcal{F}$ be a finite set of linear, real functions such that for every $\varphi \in \mathcal{F}$, $\text{Fix}(\varphi) \in \{0, 1\}$ and $\varphi([0,1]) \subseteq [0,1]$.

(A2) For arbitrary $\varphi_i, \varphi_j \in \mathcal{F}$ suppose either $\varphi_i([0,1]) \cap \varphi_j([0,1]) = \emptyset$ or $\text{Fix}(\varphi_i) = \text{Fix}(\varphi_j)$.

By Theorem 1.1 we will be able to calculate the Hausdorff and box dimension of the attractor of iterated function schemes satisfying both of the assumptions (A1) and (A2).

Now we introduce some notation about our iterated function system. Let

$$\varphi_{0,1}(x) = \gamma_{0,1}x$$
$$\varphi_{0,2}(x) = \gamma_{0,2}x + (1 - \gamma_{0,2})$$

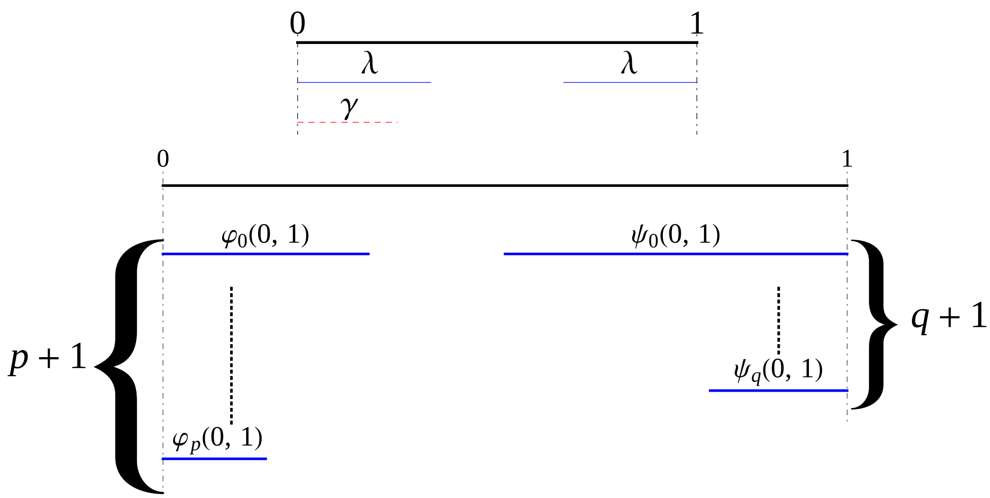


FIGURE 2. The first cylinder sets of the IFSs in [3] and of the extended version.

and suppose that $\gamma_{0,1}+\gamma_{0,2}<1$, which is equivalent with $\varphi_{0,1}([0,1])\cap\varphi_{0,2}([0,1])=\emptyset$.

Let $p,q$ be positive integers and let

$$\varphi_{i,1}(x)=\gamma_{i,1}x \text{ for } i=1,\dots,p$$
$$\varphi_{i,2}(x)=\gamma_{i,2}x+(1-\gamma_{i,2}) \text{ for } i=1,\dots,q.$$

Moreover suppose that $0<\gamma_{i,1}<\gamma_{0,1}$ for every $i=1,\dots,p$ and $0<\gamma_{i,2}<\gamma_{0,2}$ for every $i=1,\dots,q$.

**Theorem 1.1.** *Let $\mathcal{F}=\{\gamma_{i,1}x\}_{i=0}^{p}\cup\{\gamma_{i,2}x+(1-\gamma_{i,2})\}_{i=0}^{q}$ such that $0<\gamma_{i,1}<\gamma_{0,1}<1$ for $i=1,\dots,p$ and $0<\gamma_{j,2}<\gamma_{0,2}<1$ for $j=1,\dots,q$ (see Figure 1), then*

$$\dim_B\Lambda=\dim_H\Lambda=\min\{1,s\}, \tag{1.2}$$

*where $s$ is the unique solution of*

$$\prod_{i=0}^{p}(1-\gamma_{i,1}^{s})+\prod_{i=0}^{q}(1-\gamma_{i,2}^{s})=1 \tag{1.3}$$

*for Lebesgue almost every $(\underline{\gamma}_1,\underline{\gamma}_2)\in(0,\gamma_{0,1})^p\times(0,\gamma_{0,2})^q$, where $\underline{\gamma}_1=(\gamma_{1,1},\dots,\gamma_{p,1})$ and respectively $\underline{\gamma}_2=(\gamma_{1,2},\dots,\gamma_{q,2})$.*

*Moreover $\mathcal{L}(\Lambda)>0$ for Lebesgue almost every $(\underline{\gamma}_1,\underline{\gamma}_2)$ if $s>1$.*

Note that whenever $\gamma_{0,1}+\gamma_{0,2}\geq 1$ the attractor of $\mathcal{F}$ is an interval which immediately implies Theorem 1.1. In this way without loss of generality in the rest of the paper we may assume that $\gamma_{0,1}+\gamma_{0,2}<1$.

The following shows that we can calculate the box dimension of the generalized 4-corner set from the dimensions of its orthogonal projections to the axis which are calculable by Theorem 1.1 as we already mentioned.

**Theorem 1.2.** *Let $\Lambda(\underline{\alpha},\underline{\beta})$ be the attractor of the self-affine IFS of Figure 1. Then*

$$\dim_B \Lambda(\underline{\alpha},\underline{\beta}) = \max\left\{d_\alpha, d_\beta\right\}, \text{ for Lebesgue almost every } (\underline{\alpha},\underline{\beta}) \text{ such that}$$
$$\max\left\{\alpha_i+\alpha_{i+2}, \beta_i+\beta_{i+2}\right\} < 1 \text{ and } \min\left\{\alpha_i+\alpha_{3-i}, \beta_i+\beta_{3-i}\right\} < 1 \text{ for } i=0,1 \tag{1.4}$$

*where $d_\alpha$ and $d_\beta$ are defined in two steps. First we define two numbers $s_\alpha, s_\beta$ as the unique solution of the equations*

$$\alpha_0^{s_\alpha}+\alpha_1^{s_\alpha}+\alpha_2^{s_\alpha}+\alpha_3^{s_\alpha}-\alpha_0^{s_\alpha}\alpha_1^{s_\alpha}-\alpha_2^{s_\alpha}\alpha_3^{s_\alpha}=1$$
$$\beta_0^{s_\beta}+\beta_1^{s_\beta}+\beta_2^{s_\beta}+\beta_3^{s_\beta}-\beta_0^{s_\beta}\beta_2^{s_\beta}-\beta_1^{s_\beta}\beta_3^{s_\beta}=1.$$

*Then we can define $d_\alpha$ and $d_\beta$ as the unique real numbers such that*

$$\sum_{i=0}^{3}\alpha_i^{\min\{1,s_\alpha\}}\beta_i^{d_\alpha-\min\{1,s_\alpha\}}=1, \quad \sum_{i=0}^{3}\beta_i^{\min\left\{1,s_\beta\right\}}\alpha_i^{d_\beta-\min\left\{1,s_\beta\right\}}=1. \tag{1.5}$$

The condition in (1.4) is equivalent to that the rectangles $R_0, R_1, R_2, R_3$ are pairwise disjoint.

The same formula as in the equation (1.5) appeared in Gatzouras-Lalley [10] and also in Barański [1] for different kind of self-affine sets. The method of the proof of (1.5) follows the proof of Feng-Wang [8, Theorem 1] and Barański [1, Theorem B].

**Organization of the paper:**

In Section 2 we mention some method to prove the so-called transversality condition. In Section 3 we prove Theorem 1.1. We decompose this section into three parts. In Subsection 3.1 we introduce some notation about the natural projection. In Subsection 3.2 we prove the transversality condition and in Subsection 3.3 we calculate the Hausdorff dimension. Note, that our original system does not satisfy transversality (see later the precise arguments). The method of the proof is that we consider higher-order iterates of the system, we throw away some of the maps from it and then for this restricted family we apply the transversality condition. Taking higher and higher iterates we are approximating the original system.

In Section 4 we apply Theorem 1.1 to prove the formula of the box dimension of generalized 4-corner set. We obtain an almost all type result with respect to the contraction coefficients. Further, using the method of Barański [1] and Feng, Wang [8], we give a general formula (Theorem 4.1) for the box dimension of the self-affine sets on the plane which are constructed with axes parallel rectangles having disjoint interiors.

## 2. Transversality methods

First let us introduce the *transversality condition* for self-similar IFS on the real line with $d$ dimensional parameter-space. The technique of the transversality condition was first introduced in [11] to calculate the Hausdorff dimension of $\lambda$-expansions with deleted digits.

The definition corresponds to the definition of Simon, Solomyak and Urbański [13],[14] which was introduced for much more general IFS.

Let $U$ be an open, bounded subset of $\mathbb{R}^d$ with smooth boundary and $\mathcal{I}$ a finite set of symbols. Let $\Psi_{\underline{t}} = \left\{\psi_i^{\underline{t}}(x)=\lambda_i(\underline{t})x+d_i(\underline{t})\right\}_{i\in\mathcal{I}}$, where $\lambda_i, d_i \in C^1(\overline{U})$ and $0<\alpha\leq\lambda_i(\underline{t})\leq\beta<1$ for every $i\in\mathcal{I}$ and $\underline{t}\in\overline{U}$ and for some $\alpha,\beta\in(0,1)$. Let

$\Lambda^{\underline{t}}$ be the attractor of $\Psi_{\underline{t}}$ and $\pi_{\underline{t}}$ is the natural projection from the symbolic space $\Sigma = \mathcal{I}^{\mathbb{N}}$ to $\Lambda^{\underline{t}}$. More precisely, for $\mathbf{i} = (i_0 i_1 \dots) \in \Sigma$ we write

$$\pi_{\underline{t}}(\mathbf{i}) = \lim_{n\to\infty} \psi^{\underline{t}}_{i_0} \circ \psi^{\underline{t}}_{i_1} \circ \dots \circ \psi^{\underline{t}}_{i_n}(0). \tag{2.1}$$

It is well-known that the limit exists and is independent of the base point 0. Moreover, $\pi_{\underline{t}}$ is a continuous, surjective function from $\Sigma$ onto $\Lambda^{\underline{t}}$. Denote $\sigma$ the left-shift operator on $\Sigma$. That is $\sigma : (i_0 i_1 \dots) \mapsto (i_1 i_2 \dots)$. It is easy to see that

$$\pi_{\underline{t}}(\mathbf{i}) = \psi^{\underline{t}}_{i_0}(\pi_{\underline{t}}(\sigma \mathbf{i})).$$

**Definition 2.1.** *We say that $\Psi_{\underline{t}}$ satisfies the* **transversality condition** *on an open, bounded set $U \subset \mathbb{R}^d$, if for any $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$ there exists a constant $C = C(i_0, j_0)$ such that*

$$\mathcal{L}_d(\underline{t} \in U : |\pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})| \le r) \le Cr \text{ for every } r > 0,$$

*where $\mathcal{L}_d$ is the $d$ dimensional Lebesgue measure.*

In short, we say that there is transversality if the transversality condition holds. This definition is equivalent to the ones given in e.g. [13], [14]. As a special case of [13, Theorem 3.1] we obtain:

**Theorem 2.2** (Simon, Solomyak, Urbański)**.** *Suppose that $\Psi_{\underline{t}}$ satisfies the transversality condition on an open, bounded set $U \subset \mathbb{R}^d$. Then*

(1) $\dim_H \Lambda^{\underline{t}} = \min\{s(\underline{t}), 1\}$ *for Lebesgue-a.e. $\underline{t} \in U$,*
(2) $\mathcal{L}_1(\Lambda^{\underline{t}}) > 0$ *for Lebesgue-a.e. $\underline{t} \in U$ such that $s(\underline{t}) > 1$,*

*where $s(\underline{t})$ is the similarity dimension of $\Psi_{\underline{t}}$. More precisely, $s(\underline{t})$ satisfies the equation*

$$\sum_{i\in\mathcal{I}} \lambda_i(\underline{t})^{s(\underline{t})} = 1. \tag{2.2}$$

We can use the following Lemma to prove transversality which follows from [13, Lemma 7.3].

**Lemma 2.3.** *Let $U \subset \mathbb{R}^d$ be an open, bounded set with smooth boundary and $g_{\mathbf{i},\mathbf{j}}(\underline{t}) = \pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})$. If for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$ and for every $\underline{t}_0 \in U$*

$$g_{\mathbf{i},\mathbf{j}}(\underline{t}_0) = 0 \Rightarrow \|\mathrm{grad}_{\underline{t}} g_{\mathbf{i},\mathbf{j}}\big|_{\underline{t}=\underline{t}_0}\| > 0 \tag{2.3}$$

*then there is transversality on any open subset $V$ whose closure is contained in $U$.*

## 3. Proof of Theorem 1.1

3.1. **Natural projection.** Let $p, q$ be positive integers and let

$$\varphi_{i,1}(x) = \gamma_{i,1}x \text{ for } i = 0, \dots, p$$
$$\varphi_{i,2}(x) = \gamma_{i,2}x + (1 - \gamma_{i,2}) \text{ for } i = 0, \dots, q.$$

Then our main assumptions (A1), (A2) are equivalent to $0 < \gamma_{i,1} < \gamma_{0,1} < 1$ for every $i = 1, \dots, p$ and $0 < \gamma_{i,2} < \gamma_{0,2} < 1$ for every $i = 1, \dots, q$, moreover,

$$\gamma_{0,1} + \gamma_{0,2} < 1.$$

Therefore, without loss of generality we can assume that

$$\gamma_{i,1} = c_{i,1}\gamma_{0,1}$$
$$\gamma_{i,2} = c_{i,2}\gamma_{0,2},$$

where $0 < c_{i,1}, c_{j,2} < 1$ for $i = 1, \dots, p$ and $j = 1, \dots, q$. Then $\mathcal{F}$ can be written in the form

$$\mathcal{F} = \{\gamma_{0,1}x, \gamma_{0,2}x + (1-\gamma_{0,2})\} \bigcup \{c_{i,1}\gamma_{0,1}x\}_{i=1}^{p} \bigcup \{c_{i,2}\gamma_{0,2}x + (1 - c_{i,2}\gamma_{0,2})\}_{i=1}^{q}.$$

Let us introduce the vectors of parameters, namely, $\underline{c}_1 = (c_{1,1}, \dots, c_{p,1}) \in (0,1)^p$ and $\underline{c}_2 = (c_{1,2}, \dots, c_{q,2}) \in (0,1)^q$, moreover $\underline{c} = (\underline{c}_1, \underline{c}_2)$.

Denote the set of symbols of the functions with fixed point 0 by $A_1$, and similarly, denote the set of symbols of the functions with fixed point 1 by $A_2$. So

$$A_1 = \{(0,1), \dots, (p,1)\} \text{ and } A_2 = \{(0,2), \dots, (q,2)\}.$$

Let $\Sigma$ be the symbolic space generated by $A_1 \cup A_2$ and $\Sigma^*$ the set of finite words. That is, $\Sigma = (A_1 \cup A_2)^{\mathbb{N}}$ and $\Sigma^* = \bigcup_{n=0}^{\infty} (A_1 \cup A_2)^n$. For any $\underline{i} = ((i_0, \kappa_0)(i_1, \kappa_1) \cdots (i_n, \kappa_n)) \in \Sigma^*$ we use the notation

$$\varphi_{\underline{i}} = \varphi_{i_0,\kappa_0} \circ \varphi_{i_1,\kappa_1} \circ \cdots \circ \varphi_{i_n,\kappa_n} \text{ and } \gamma_{\underline{i}} = \gamma_{i_0,\kappa_0} \cdots \gamma_{i_n,\kappa_n}.$$

For an $\mathbf{i} \in \Sigma$ we write $\mathbf{i}(k)$ as the first $k$ elements of $\mathbf{i}$. In particular, $\mathbf{i}(k) = ((i_0, \kappa_0) \cdots (i_{k-1}, \kappa_{k-1}))$ and $\mathbf{i}(0) = \emptyset$. For $j = 1, 2$ and $i = 0, \dots, p$ or $q$, we define $\sharp_{i,j}\mathbf{i}(k)$ as the number of $(i,j)$ in $\mathbf{i}(k)$. Moreover, for $j = 1, 2$ we define $\sharp_j \mathbf{i}(k)$ as the number of symbols from $A_j$ in $\mathbf{i}(k)$. Clearly, $\sharp_1\mathbf{i}(k) = \sum_{i=0}^{p} \sharp_{i,1}\mathbf{i}(k)$ and respectively $\sharp_2\mathbf{i}(k) = \sum_{i=0}^{q} \sharp_{i,2}\mathbf{i}(k)$. Using this notations and the definition of the natural projection (2.1),

$$\pi_{\underline{c}}(\mathbf{i}) \quad = \quad \sum_{k=0}^{\infty} \left( \sum_{l=0}^{q} \delta_{(i_k,\kappa_k)}^{(l,2)} (1 - \gamma_{l,2}) \right) \gamma_{0,1}^{\sharp_1\mathbf{i}(k)} \gamma_{0,2}^{\sharp_2\mathbf{i}(k)} \prod_{i=1}^{p} c_{i,1}^{\sharp_{(i,1)}\mathbf{i}(k)} \prod_{i=1}^{q} c_{i,2}^{\sharp_{(i,2)}\mathbf{i}(k)}, \quad (3.1)$$

where

$$\delta_j^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}.$$

The set of $k$'s satisfying $(i_k, \kappa_k) \in A_2$ gives us non-zero elements in the infinite sum above. Hence it is useful to define $\beta_i^{\mathbf{i}}$ as the number of $(i,2)$ in $\mathbf{i}$ and $\beta^{\mathbf{i}}$ the number of symbols from $A_2$ in $\mathbf{i}$. Clearly, $\beta_i^{\mathbf{i}} = \lim_{k\to\infty} \sharp_{(i,2)}\mathbf{i}(k)$ and $\beta^{\mathbf{i}} = \sum_{l=0}^{q} \beta_l^{\mathbf{i}}$. Moreover, let $m_k^{\mathbf{i}}$ be the position of the $k$th symbol from $A_2$ in $\mathbf{i}$. Applying the notation $\sharp_2\mathbf{i}(m_k^{\mathbf{i}}) = k - 1$ and

$$\pi_{\underline{c}}(\mathbf{i}) \; = \; \sum_{k=1}^{\beta^{\mathbf{i}}} \left( \sum_{l=0}^{q} \delta_{(i_{m_k^{\mathbf{i}}}, \kappa_{m_k^{\mathbf{i}}})}^{(l,2)} (1 - \gamma_{l,2}) \right) \gamma_{0,2}^{k-1} \gamma_{0,1}^{\sharp_1\mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)}\mathbf{i}(m_k^{\mathbf{i}})}. \quad (3.2)$$

For every $i = 1, \dots, p$ we write (3.2) as the power series of $c_{i,1}$. So we collect all the different exponents of $c_{i,1}$ into the set $P_{\mathbf{i}}^i$. It is easy to see that if $\beta^{\mathbf{i}} = 0$ then $P_{\mathbf{i}}^i = \emptyset$, otherwise

$$P_{\mathbf{i}}^i = \left\{ m \geq 0 : \exists k \geq 1, \sharp_{(i,1)}\mathbf{i}(m_k^{\mathbf{i}}) = m \right\} \text{ for } i = 1, \dots, p.$$

Then we can write the natural projection in the following form

$$\pi_{\underline{c}}(\mathbf{i}) = \sum_{m \in P_{\mathbf{i}}^i} h_i^m(\mathbf{i}) c_{(i,1)}^m. \quad (3.3)$$

For every $m \in P^i_{\mathbf{i}}$ the coefficient $h^m_i(\mathbf{i})$ of $c^m_{i,1}$ is the sum of those elements of (3.2) divided by $c^m_{i,1}$ which's indexes $k$ satisfy $\sharp_{(i,1)}\mathbf{i}(m^{\mathbf{i}}_k) = m$. Precisely,

$$h^m_i(\mathbf{i}) = \sum_{k=\underline{s}^i_m(\mathbf{i})}^{\overline{s}^i_m(\mathbf{i})} \left( \sum_{l=0}^{q} \delta^{(l,2)}_{(i_{m^{\mathbf{i}}_k}, \kappa_{m^{\mathbf{i}}_k})} (1-\gamma_{l,2}) \right) \gamma_{0,2}^{k-1} \gamma_{0,1}^{\sharp_1 \mathbf{i}(m^{\mathbf{i}}_k)} \prod_{\substack{l=1 \\ l\neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m^{\mathbf{i}}_k)} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)}\mathbf{i}(m^{\mathbf{i}}_k)}. \tag{3.4}$$

where

$$\overline{s}^i_m(\mathbf{i}) = \sup\left\{ k : \sharp_{(i,1)}\mathbf{i}(m^{\mathbf{i}}_k) = m \right\} \text{ and } \underline{s}^i_m(\mathbf{i}) = \inf\left\{ k : \sharp_{(i,1)}\mathbf{i}(m^{\mathbf{i}}_k) = m \right\}.$$

**Lemma 3.1.** *Let $\mathbf{i} \in \Sigma$ then for every $i = 1, \dots, p$ and every $m \in P^i_{\mathbf{i}}$*

$$h^m_i(\mathbf{i}) \le \gamma_{0,2}^{\underline{s}^i_m(\mathbf{i})-1} \gamma_{0,1}^{\sharp_1 \mathbf{i}(m^{\mathbf{i}}_{\underline{s}^i_m(\mathbf{i})})} \prod_{\substack{l=1 \\ l\neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m^{\mathbf{i}}_{\underline{s}^i_m(\mathbf{i})})}.$$

*Moreover, if $0 \in P^i_{\mathbf{i}}$ then*

$$h^0_i(\mathbf{i}) \ge \gamma_{0,1}^{m^{\mathbf{i}}_1 - 1} \prod_{\substack{l=1 \\ l\neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m^{\mathbf{i}}_1)} (1-\gamma_{0,2}).$$

*Proof.* Let $\mathbf{i} \in \Sigma$ and for $m \in P^i_{\mathbf{i}}$ let $\underline{i}_m = ((i_{m_{\underline{s}^i_m(\mathbf{i})}}, \kappa_{m_{\underline{s}^i_m(\mathbf{i})}}) \cdots (i_{m_{\overline{s}^i_m(\mathbf{i})}}, \kappa_{m_{\overline{s}^i_m(\mathbf{i})}}))$. By the definition of $\overline{s}^i_m(\mathbf{i})$ and $\underline{s}^i_m(\mathbf{i})$, $\underline{i}_m$ is the segment of $\mathbf{i}$ corresponds to the coefficient $h^m_i(\mathbf{i})$. By (3.4)

$$h^m_i(\mathbf{i}) = \gamma_{0,2}^{\underline{s}^i_m(\mathbf{i})-1} \gamma_{0,1}^{\sharp_1 \mathbf{i}(m^{\mathbf{i}}_{\underline{s}^i_m(\mathbf{i})})} \prod_{\substack{l=1 \\ l\neq i}}^{p} c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m^{\mathbf{i}}_{\underline{s}^i_m(\mathbf{i})})} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)}\mathbf{i}(m^{\mathbf{i}}_{\underline{s}^i_m(\mathbf{i})})} \varphi_{\underline{i}_m}(0).$$

By the definition $\kappa_{m_{\underline{s}^i_m(\mathbf{i})}} = 2$ which implies that

$$1 - \gamma_{0,2} \le \varphi_{\underline{i}_m}(0) \le 1,$$

for every $m \in P^i_{\mathbf{i}}$.

If $0 \in P^i_{\mathbf{i}}$ then before the first $(i,1)$ there have to be at least one symbol from $A_2$. Therefore $\underline{s}^{\mathbf{i}}_0 = 1$. Moreover, before the place of the first symbol from $A_2$ the number of symbols from $A_1$ is $m^{\mathbf{i}}_1 - 1$. This proves the assertion of the Lemma. □

3.2. **Proof of the transversality condition.** For every $\mathbf{i}, \mathbf{j} \in A^{\mathbb{N}}_\kappa$ $(\kappa = 1,2)$ $\pi_{\underline{c}}(\mathbf{i}) \equiv \pi_{\underline{c}}(\mathbf{j})$ as the functions of $\underline{c}$. This implies the IFS $\mathcal{F}$ does not satisfy the transversality condition. The goal of this section is to introduce a sequence of iterated function system which satisfy the transversality and suitable to approximate the Hausdorff dimension of the attractor of $\mathcal{F}$.

Since $\varphi_{i_0,\kappa} \circ \varphi_{i_1,\kappa} = \varphi_{i_1,\kappa} \circ \varphi_{i_0,\kappa}$ holds for every $(i_0,\kappa), (i_1,\kappa) \in A_\kappa$ which is in the way of transversality. To eliminate this problem we choose a sequence of subsets of $\Sigma^*$ such that we order the symbols in each word by the first coordinate.

Define

$$\begin{aligned} &\mathcal{P}_0 = \{(0,1);(0,2)\} \text{ and} \\ &\mathcal{P}_1 = \{(1,2)(0,1);\dots;(q,2)(0,1);(1,1)(0,2);\dots;(p,1)(0,2)\} \end{aligned} \tag{3.5}$$

and by induction for $k \geq 2$

$$\mathcal{P}_k = \left( \bigcup_{j=1}^{p} \bigcup_{\substack{\underline{i} \in \mathcal{P}_{k-1} \\ \kappa_0 \neq 1 \vee j \leq i_0}} \{(j,1)\underline{i}\} \right) \bigcup \left( \bigcup_{j=1}^{q} \bigcup_{\substack{\underline{i} \in \mathcal{P}_{k-1} \\ \kappa_0 \neq 2 \vee j \leq i_0}} \{(j,2)\underline{i}\} \right). \tag{3.6}$$

and

$$\mathcal{U}_k = \bigcup_{l=0}^{k} \mathcal{P}_l. \tag{3.7}$$

Denote $\Sigma_k = \mathcal{U}_k^{\mathbb{N}}$ and the sequence of IFS's

$$\Psi_k = \left\{ \varphi_{\underline{i}} \right\}_{\underline{i} \in \mathcal{U}_k}. \tag{3.8}$$

**Proposition 3.2.** *Let $\xi > 0$ be arbitrary small, then the system $\Psi_k$ satisfies the transversality condition on $\underline{c} \in (\xi, 1-\xi)^{p+q}$ for every $k \geq 1$.*

*Proof.* Suppose that $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$ and let $\mathbf{i}', \mathbf{j}' \in \Sigma_k = \mathcal{U}_k^{\mathbb{N}}$ such that $\underline{i}_0 \neq \underline{j}_0 \in \mathcal{U}_k$. Denote $\mathbf{i}'$ (and $\mathbf{j}'$) as the element of $\Sigma$ by $\mathbf{i}$ (and $\mathbf{j}$ respectively). To prove transversality by Lemma 2.3 it is enough to show that

$$\pi_{\underline{c}}(\mathbf{i}) = \pi_{\underline{c}}(\mathbf{j}) \implies \mathrm{grad}_{\underline{c}} \left( \pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j}) \right) \neq 0. \tag{3.9}$$

Suppose that $\pi_{\underline{c}}(\mathbf{i}) = \pi_{\underline{c}}(\mathbf{j})$. Since $\gamma_{0,1} + \gamma_{0,2} < 1$, the first element of $\mathbf{i}$, $(i_0, \kappa_0)$, and the first element of $\mathbf{j}$, $(j_0, \tau_0)$, have to satisfy that $\kappa_0 = \tau_0$. Then $\mathbf{i}, \mathbf{j}$ can be written in the form

$$\mathbf{i} = \overbrace{(0,\kappa)\cdots(0,\kappa)}^{r_0}\overbrace{(1,\kappa)\cdots(1,\kappa)}^{r_1}\cdots\overbrace{(s,\kappa)\cdots(s,\kappa)}^{r_s}(l_1, 3-\kappa)\cdots$$
$$\mathbf{j} = \overbrace{(0,\kappa)\cdots(0,\kappa)}^{t_0}\overbrace{(1,\kappa)\cdots(1,\kappa)}^{t_1}\cdots\overbrace{(s,\kappa)\cdots(s,\kappa)}^{t_s}(l_2, 3-\kappa)\cdots,$$

where $r_i, t_i \geq 0$ for $i = 1, \ldots, s$, $s = p$ if $\kappa = 1$ and $s = q$ otherwise.

If $r_i \leq t_i$ for every $i = 0, \ldots, s$ and there exists an $1 \leq i \leq s$ such that $r_i < t_i$ then by $\gamma_{0,1} + \gamma_{0,2} < 1$, $\pi_{\underline{c}}(\mathbf{i}) \neq \pi_{\underline{c}}(\mathbf{j})$, which is a contradiction. Therefore there are two possibilities, there exist $i \neq j$ such that $r_i > t_i$ and $r_j < t_j$ or $r_i = t_i$ for every $i = 0, \ldots, s$. In the last case

$$0 = \pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j}) = \gamma_{0,\kappa}^{\sum_{i=0}^{s} r_i} \prod_{i=1}^{s} c_{i,\kappa}^{r_i} \left( \pi_{\underline{c}}(\sigma^{\sum_{i=0}^{s} r_i}\mathbf{i}) - \pi_{\underline{c}}(\sigma^{\sum_{i=0}^{s} r_i}\mathbf{j}) \right).$$

Since $c_{i,\kappa} > \xi/2$ for every $\kappa = 1, 2$ and $i = 1, \ldots, p$ or $q$ and moreover $\underline{i}_0 \neq \underline{j}_0$ without loss of generality we can assume the first case.

Firstly, let us suppose that $\kappa = 1$ then $\mathbf{i}$ and $\mathbf{j}$ are in the form

$$\mathbf{i} = \overbrace{(0,1)\cdots(0,1)}^{r_0}\overbrace{(1,1)\cdots(1,1)}^{r_1}\cdots\overbrace{(s,1)\cdots(s,1)}^{r_p}(l_1, 2)\cdots$$
$$\mathbf{j} = \overbrace{(0,1)\cdots(0,1)}^{t_0}\overbrace{(1,1)\cdots(1,1)}^{t_1}\cdots\overbrace{(s,1)\cdots(s,1)}^{t_p}(l_2, 2)\cdots,$$

and there exists $1 \leq j \leq p$ such that $r_j < t_j$. There exists also an $0 \leq i \leq p$ such that $r_i > t_i$ and $i \neq j$, but we prove transversality derivation in $c_{j,1}$.

Let

$$\mathbf{i}^* = \overbrace{(0,1)\cdots(0,1)}^{r_0}\cdots\overbrace{(j-1,1)\cdots(j-1,1)}^{r_{j-1}}\overbrace{(j+1,1)\cdots(j+1,1)}^{r_{j+1}}\cdots(l_1,2)\cdots$$

and

$$\mathbf{j}^* = \overbrace{(0,1)\cdots(0,1)}^{t_0}\cdots\overbrace{(j,1)\cdots(j,1)}^{t_j-r_j}\cdots(l_2,2)\cdots .$$

Then

$$\pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j}) = \gamma_{j,1}^{r_j} c_{j,1}^{r_j}\left(\pi_{\underline{c}}(\mathbf{i}^*) - \pi_{\underline{c}}(\mathbf{j}^*)\right).$$

Let $a(\underline{c}) = \pi_{\underline{c}}(\mathbf{i}^*) - \pi_{\underline{c}}(\mathbf{j}^*)$. Since $c_{j,1} > \xi/2$ to prove transversality it is enough to show that

$$a(\underline{c}) = 0 \implies \frac{\partial a}{\partial c_{j,1}}(\underline{c}) \neq 0$$

for every $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$. But instead of showing that we prove

$$\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = 0 \implies a(\underline{c}) > 0 \tag{3.10}$$

for every $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$. By (3.3) we have

$$a(\underline{c}) = h_j^0(\mathbf{i}^*) + \sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} h_j^m(\mathbf{i}^*)c_{j,1}^m - \sum_{m\in P^j_{\mathbf{j}^*}} h_j^m(\mathbf{j}^*)c_{j,1}^m.$$

Let $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$ such that $\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = 0$ then

$$0 = c_{j,1}\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = h_j^0(\mathbf{i}^*)\left(\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)} m c_{j,1}^m - \sum_{m\in P^j_{\mathbf{j}^*}} \frac{h_j^m(\mathbf{j}^*)}{h_j^0(\mathbf{i}^*)} m c_{j,1}^m\right) \leq$$

$$h_j^0(\mathbf{i}^*)\left(\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}(m-1)c_{j,1}^m + \sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}c_{j,1}^m - \sum_{m\in P^j_{\mathbf{j}^*}} \frac{h_j^m(\mathbf{j}^*)}{h_j^0(\mathbf{i}^*)}c_{j,1}^m\right).$$

It is enough to prove that

$$\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}(m-1)c_{j,1}^m < 1.$$

By Lemma 3.1 we have

$$\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}(m-1)c_{j,1}^m \leq$$

$$\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{\gamma_{0,2}^{\underline{s}^j_m(\mathbf{i}^*)-1}\gamma_{0,1}^{\sharp_1\mathbf{i}^*(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)})}\prod_{\substack{l=1\\l\neq j}}^p c_{l,1}^{\sharp_{(l,1)}\mathbf{i}^*(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)})}}{\gamma_{0,1}^{m_1^{\mathbf{i}^*}-1}\prod_{\substack{l=1\\l\neq j}}^p c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m_1^{\mathbf{i}^*})}(1-\gamma_{0,2})}(m-1)c_{j,1}^m. \tag{3.11}$$

Since $\mathbf{i}^*$ does not contain $(j,1)$ before the first element from $A_2$, $\underline{s}^j_0(\mathbf{i}^*) = 1$ and $\sharp_1\mathbf{i}^*(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)}) \geq m_1^{\mathbf{i}^*} + m - 1$ for every $m \in P^j_{\mathbf{i}^*}\setminus\{0\}$.

Let $q_1 = \min P^j_{\mathbf{i}^*} \setminus \{0\}$ and $q_2 = \min P^j_{\mathbf{i}^*} \setminus \{0, q_1\}$. We define the minimum of the empty set as infinity. Then $\underline{s}^j_{q_1}(\mathbf{i}^*) \geq 2$ and $\underline{s}^j_{q_2}(\mathbf{i}^*) \geq 3$. This implies that the right hand side of (3.11) is less than or equal to

$$\frac{\gamma_{0,1}^{q_1}\gamma_{0,2}}{1-\gamma_{0,2}}(q_1-1)c_{j,1}^{q_1} + \frac{\gamma_{0,1}^{q_2}\gamma_{0,2}^2}{1-\gamma_{0,2}}(q_2-1)c_{j,1}^{q_2} + \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}} \sum_{m \in P^j_{\mathbf{i}^*} \setminus \{0,q_1,q_2\}} \gamma_{0,1}^m (m-1) c_{j,1}^m. \tag{3.12}$$

Using that $(n-1)\gamma_{0,1}^n \leq \frac{-\gamma_{0,1}}{e \ln \gamma_{0,1}}$ for every $n \in \mathbb{N}$, we get that (3.12) is less than or equal to

$$\frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}} + \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}} \sum_{m=3}^{\infty}(m-1)\gamma_{0,1}^m = \frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}} + \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}} \frac{\gamma_{0,1}^3(2-\gamma_{0,1})}{(1-\gamma_{0,1})^2}.$$

Using the assumption $\gamma_{0,1} + \gamma_{0,2} < 1$ by some algebraic manipulation we get that

$$\frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}} + \frac{\gamma_{0,2}^3}{1-\gamma_{0,2}} \frac{\gamma_{0,1}^3(2-\gamma_{0,1})}{(1-\gamma_{0,1})^2} < 1,$$

which implies (3.10).

To prove transversality in the second case when $\kappa = 2$ we introduce the function $\eta(x) = -x + 1$. Let us observe that $\eta \circ \eta(x) = x$. Let

$$\widetilde{\varphi}_{i,1}(x) := \eta \circ \varphi_{i,1} \circ \eta(x) = \gamma_{i,1}x + (1-\gamma_{i,1}) \text{ for } i = 0, \dots, p$$
$$\widetilde{\varphi}_{i,2}(x) := \eta \circ \varphi_{i,2} \circ \eta(x) = \gamma_{i,2}x \text{ for } i = 0, \dots, q.$$

The IFS $\widetilde{\mathcal{F}} = \{\widetilde{\varphi}_{i,1}\}_{i=0}^p \cup \{\widetilde{\varphi}_{i,2}\}_{i=0}^q$ and $\mathcal{F}$ are equivalent. More precisely, let $\widetilde{\pi}_{\underline{c}}$ be the natural projection of $\widetilde{\mathcal{F}}$ then $\widetilde{\pi}_{\underline{c}}(\mathbf{i}) = -\pi_{\underline{c}}(\mathbf{i}) + 1$ for every $\mathbf{i} \in \Sigma$. Using this fact one can prove transversality in the case $\kappa = 2$ as in $\kappa = 1$.

The proof can be finished applying Lemma 2.3. □

3.3. **Hausdorff dimension.** In the first part of the section we calculate the Hausdorff dimension of the attractor of $\Psi_k$ (see 3.8) and in the second part we will prove that the limit will correspond with the dimension of the attractor of $\mathcal{F}$.

Let for $k \geq 0$

$$d_k(s) = \sum_{\underline{i} \in \mathcal{U}_k} \gamma_{\underline{i}}^s.$$

By the definition of $\mathcal{U}_k$ (see 3.7) for $k \geq 1$

$$d_k(s) = \gamma_{0,1}^s + \gamma_{0,2}^s + \gamma_{0,1}^s \sum_{l=1}^k \Phi_l + \gamma_{0,2}^s \sum_{l=1}^k \Upsilon_l$$

where

$$\Phi_k = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k,h_k)=(0,1)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,1}^s}$$

and

$$\Upsilon_k = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k,h_k)=(0,2)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,2}^s}.$$

**Lemma 3.3.** *Let us denote the attractor of* $\Psi_k$ *by* $\Lambda_k$. *Then*

$$\dim_H \Lambda_k = \min\{1, s_k\} \ \textit{ for } \mathcal{L}\textit{ebesgue-a.e. } \underline{c} \in (0,1)^{p+q}$$

*where* $s_k$ *is the unique solution of* $d_k(s) = 1$.

*Proof.* By Proposition 3.2, $\Psi_k$ satisfies the transversality condition on $\underline{c} \in (\xi, 1-\xi)^{p+q}$ for every arbitrary small $\xi > 0$. Since $d_k(s)$ is the sum of the contraction ratios of the functions in the IFS $\Psi_k$ to the power $s$, Theorem 2.2 implies that the Hausdorff dimension of $\Lambda_k$ is equal to $\min\{1, s_k\}$ where $s_k$ is the unique solution of

$$d_k(s) = 1 \tag{3.13}$$

for Lebesgue almost every $\underline{c} \in (\xi, 1-\xi)^{p+q}$. Since $\xi > 0$ was arbitrary the Lemma is proved. □

**Lemma 3.4.** *Let* $s_k$ *be the unique solution of* $d_k(s) = 1$. *Then the limit* $\lim_{k\to\infty} s_k = s$ *exists and* $s$ *is the unique solution of*

$$\prod_{i=0}^{p}(1-\gamma_{i,1}^s) + \prod_{i=0}^{q}(1-\gamma_{i,2}^s) = 1. \tag{3.14}$$

The proof of Formula (3.14) is a sequence of tedious algebraic manipulations carried out in the following pages.

*Proof of Lemma 3.4.* Without loss of generality we can assume that $p \le q$. Let

$$\Phi_k^{i,\kappa} = \sum_{\substack{\underline{i}\in\mathcal{P}_k \\ (i_k,\kappa_k)=(0,1) \\ (i_1,\kappa_1)=(i,\kappa)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,1}^s}, \qquad \Upsilon_k^{i,\kappa} = \sum_{\substack{\underline{i}\in\mathcal{P}_k \\ (i_k,\kappa_k)=(0,2) \\ (i_1,\kappa_1)=(i,\kappa)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,2}^s},$$

then $\Phi_k = \sum_{i=1}^p \Phi_k^{i,1} + \sum_{i=1}^q \Phi_k^{i,2}$ and $\Upsilon_k = \sum_{i=1}^p \Upsilon_k^{i,1} + \sum_{i=1}^q \Upsilon_k^{i,2}$. By the definition of $\mathcal{P}_k$ (see (3.5), (3.6)) we have

$$\begin{aligned} \Phi_1^{i,1} &= 0 \text{ for } i = 1,\dots,p, \\ \Phi_1^{i,2} &= \gamma_{i,2}^s \text{ for } i = 1,\dots,q, \\ \Upsilon_1^{i,1} &= \gamma_{i,1}^s \text{ for } i = 1,\dots,p, \\ \Upsilon_1^{i,2} &= 0 \text{ for } i = 1,\dots,q, \end{aligned} \tag{3.15}$$

moreover for $k \ge 2$

$$\begin{aligned} \Phi_k^{i,\kappa} &= \gamma_{i,\kappa}^s \left( \Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,\kappa} \right) \\ \Upsilon_k^{i,\kappa} &= \gamma_{i,\kappa}^s \left( \Upsilon_{k-1} - \sum_{l=1}^{i-1} \Upsilon_{k-1}^{l,\kappa} \right). \end{aligned} \tag{3.16}$$

Denote

$$\begin{aligned} a_{k,1} &= \sum_{1\le j_0<\cdots<j_{k-1}\le p} \gamma_{j_0,1}^s \cdots \gamma_{j_{k-1},1}^s \text{ for } i = 1,\dots,p, \\ a_{k,2} &= \sum_{1\le j_0<\cdots<j_{k-1}\le q} \gamma_{j_0,2}^s \cdots \gamma_{j_{k-1},2}^s \text{ for } i = 1,\dots,q. \end{aligned} \tag{3.17}$$

Applying (3.16) we have for $k \geq 2$

$$\Phi_k = \sum_{i=1}^{p} \Phi_k^{i,1} + \sum_{i=1}^{q} \Phi_k^{i,2} =$$
$$\sum_{i=1}^{p} \gamma_{i,1}^s \left( \Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,1} \right) + \sum_{i=1}^{q} \gamma_{i,2}^s \left( \Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,2} \right) =$$
$$a_{1,1}\Phi_{k-1} + a_{1,2}\Phi_{k-1} - \sum_{l=1}^{p-1} \sum_{i=l+1}^{p} \gamma_{i,1}^s \Phi_{k-1}^{l,1} - \sum_{l=1}^{q-1} \sum_{i=l+1}^{q} \gamma_{i,2}^s \Phi_{k-1}^{l,2}, \quad (3.18)$$

and similarly

$$\Upsilon_k \;=\; a_{1,1}\Upsilon_{k-1} \;+\; a_{1,2}\Upsilon_{k-1} \;-\; \sum_{l=1}^{p-1} \sum_{i=l+1}^{p} \gamma_{i,1}^s \Upsilon_{k-1}^{l,1} \;-\; \sum_{l=1}^{q-1} \sum_{i=l+1}^{q} \gamma_{i,2}^s \Upsilon_{k-1}^{l,2}. \quad (3.19)$$

Applying (3.16) for (3.18) and (3.19) $n$ times, where $1 \leq n \leq p-1$ and $k \geq n+1$, we get

$$\Phi_k = \sum_{l=1}^{n} (-1)^{l-1} a_{l,1} \Phi_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq p} \gamma_{j_n,1}^s \cdots \gamma_{j_1,1}^s \Phi_{k-n}^{j_0,1} +$$
$$\sum_{l=1}^{n} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq q} \gamma_{j_n,2}^s \cdots \gamma_{j_1,2}^s \Phi_{k-n}^{j_0,2} \quad (3.20)$$

and

$$\Upsilon_k = \sum_{l=1}^{n} (-1)^{l-1} a_{l,1} \Upsilon_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq p} \gamma_{j_n,1}^s \cdots \gamma_{j_1,1}^s \Upsilon_{k-n}^{j_0,1} +$$
$$\sum_{l=1}^{n} (-1)^{l-1} a_{l,2} \Upsilon_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq q} \gamma_{j_n,2}^s \cdots \gamma_{j_1,2}^s \Upsilon_{k-n}^{j_0,2}. \quad (3.21)$$

Then by (3.15) and the choosing $n = k-1$ we get

$$\begin{aligned} \Phi_k &= \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^{k-1} a_{k,2} \\ \Upsilon_k &= \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,1} \Upsilon_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Upsilon_{k-l} + (-1)^{k-1} a_{k,1} \end{aligned} \quad (3.22)$$

for $2 \leq k \leq p$. If $p < q$ we can apply (3.16) for (3.18) and (3.19) $n$ times, where $p \leq n \leq q-1$ and $k \geq n+1$, and we have

$$\Phi_k = \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{n} (-1)^{l-1} a_{l,2} \Phi_{k-l} +$$
$$(-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq q} \gamma_{j_n,2}^s \cdots \gamma_{j_1,2}^s \Phi_{k-n}^{j_0,2} \quad (3.23)$$

and

$$\Upsilon_k = \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Upsilon_{k-l} + \sum_{l=1}^{n} (-1)^{l-1} a_{l,2} \Upsilon_{k-l} + \\ (-1)^n \sum_{1 \le j_0 < \cdots < j_n \le q} \gamma^s_{j_n,2} \cdots \gamma^s_{j_1,2} \Upsilon^{j_0,2}_{k-n}. \quad (3.24)$$

By (3.15) and $k = n+1$ we have

$$\begin{aligned} \Phi_k &= \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^{k-1} a_{k,2} \\ \Upsilon_k &= \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Upsilon_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Upsilon_{k-l} \end{aligned} \quad (3.25)$$

for $p+1 \le k \le q$. By similar methods we get for $k \ge q+1$ that

$$\begin{aligned} \Phi_k &= \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \Phi_{k-l} \\ \Upsilon_k &= \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Upsilon_{k-l} + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \Upsilon_{k-l}. \end{aligned} \quad (3.26)$$

The convergence of the infinite series $\sum_{l=1}^\infty \Phi_l$ and $\sum_{l=1}^\infty \Upsilon_l$ depends on the roots of the characteristic polynomial of (3.26). More precisely, $\sum_{l=1}^\infty \Phi_l$ and $\sum_{l=1}^\infty \Upsilon_l$ are convergent if and only if the roots of the characteristic polynomial are strictly less than 1. The characteristic polynomial is

$$x^q = \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} x^{q-l} + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} x^{q-l}.$$

Since the roots of a polynomial depend continuously on the coefficients of the polynomial. Except the coefficient of $x^q$ the coefficients tend to zero as $s$ tends to infinity. Therefore the roots tend to zero as $s$ tends to infinity. So there exists a $\delta > 0$ such that $\sum_{l=1}^\infty \Phi_l$ and $\sum_{l=1}^\infty \Upsilon_l$ are convergent for $s \in (\delta, \infty)$. Let $\delta$ the infinum of $s$ such that $\sum_{l=1}^\infty \Phi_l$ and $\sum_{l=1}^\infty \Upsilon_l$ are convergent. Let

$$d(s) = \gamma^s_{0,1} + \gamma^s_{0,2} + \gamma^s_{0,1} \sum_{l=1}^{\infty} \Phi_l + \gamma^s_{0,2} \sum_{l=1}^{\infty} \Upsilon_l \text{ for } s \in (\delta, \infty). \quad (3.27)$$

Then there exists a unique $s^* \in (\delta, \infty)$ such that $d(s^*) = 1$. The sequence $s_k$ (see (3.13)) is monotone increasing and bounded by $s^*$, therefore it is convergent. It is easy to see that $\lim_{k\to\infty} s_k = \sup_k s_k = s^*$.

Let

$$\Phi = \sum_{k=1}^{\infty} \Phi_k \text{ and } \Upsilon = \sum_{k=1}^{\infty} \Upsilon_k.$$

Then by (3.26)

$$\Phi=\sum_{k=q+1}^{\infty}\Phi_k+\sum_{k=1}^{q}\Phi_k=$$
$$\sum_{k=q+1}^{\infty}\left(\sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l}+\sum_{l=1}^{q}(-1)^{l-1}a_{l,2}\Phi_{k-l}\right)+\sum_{k=1}^{q}\Phi_k=$$
$$\sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\sum_{k=q+1-l}^{\infty}\Phi_k+\sum_{l=1}^{q}(-1)^{l-1}a_{l,2}\sum_{k=q+1-l}^{\infty}\Phi_k+\sum_{k=1}^{q}\Phi_k=$$
$$\sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\left(\Phi-\sum_{k=1}^{q-l}\Phi_k\right)+\sum_{l=1}^{q}(-1)^{l-1}a_{l,2}\left(\Phi-\sum_{k=1}^{q-l}\Phi_k\right)+\sum_{k=1}^{q}\Phi_k.$$

Therefore

$$\Phi=\frac{\sum_{l=1}^{p}(-1)^{l}a_{l,1}\sum_{k=1}^{q-l}\Phi_k+\sum_{l=1}^{q}(-1)^{l}a_{l,2}\sum_{k=1}^{q-l}\Phi_k+\sum_{k=1}^{q}\Phi_k}{1+\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\sum_{l=1}^{q}(-1)^{l}a_{l,2}},\tag{3.28}$$

and similarly

$$\Upsilon=\frac{\sum_{l=1}^{p}(-1)^{l}a_{l,1}\sum_{k=1}^{q-l}\Upsilon_k+\sum_{l=1}^{q}(-1)^{l}a_{l,2}\sum_{k=1}^{q-l}\Upsilon_k+\sum_{k=1}^{q}\Upsilon_k}{1+\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\sum_{l=1}^{q}(-1)^{l}a_{l,2}}.\tag{3.29}$$

Applying (3.15), (3.22) and (3.25) we get

$$\sum_{k=1}^{q}\Phi_k=\Phi_1+\sum_{k=2}^{p}\Phi_k+\sum_{k=p+1}^{q}\Phi_k=$$
$$a_{1,2}+\sum_{k=2}^{p}\left(\sum_{l=1}^{k-1}(-1)^{l-1}a_{l,1}\Phi_{k-l}+\sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Phi_{k-l}+(-1)^{k-1}a_{k,2}\right)+$$
$$\sum_{k=p+1}^{q}\left(\sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l}+\sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Phi_{k-l}+(-1)^{k-1}a_{k,2}\right)=$$
$$\sum_{k=1}^{q}(-1)^{k-1}a_{k,2}+\sum_{l=1}^{p}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,1}\Phi_k+\sum_{l=1}^{q}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,2}\Phi_k,\tag{3.30}$$

and by similar arguments

$$\sum_{k=1}^{q}\Upsilon_k=\sum_{k=1}^{p}(-1)^{k-1}a_{k,1}+\sum_{l=1}^{p}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,1}\Upsilon_k+\sum_{l=1}^{q}\sum_{k=1}^{q-l}(-1)^{l-1}a_{l,2}\Upsilon_k.\tag{3.31}$$

Hence the numerator of (3.28) is $\sum_{k=1}^{q}(-1)^{k-1}a_{k,2}$ and the numerator of (3.29) is $\sum_{k=1}^{p}(-1)^{k-1}a_{k,1}$, which implies that

$$\begin{aligned}\Phi&=\frac{\sum_{k=1}^{q}(-1)^{k-1}a_{k,2}}{1+\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\sum_{l=1}^{q}(-1)^{l}a_{l,2}}\text{ and}\\ \Upsilon&=\frac{\sum_{k=1}^{p}(-1)^{k-1}a_{k,1}}{1+\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\sum_{l=1}^{q}(-1)^{l}a_{l,2}}.\end{aligned}\tag{3.32}$$

Then $d(s)=1$ (see (3.27)) is equivalent to

$$\gamma_{0,1}^s+\gamma_{0,2}^s+\sum_{l=1}^{p}(-1)^{l-1}a_{l,1}+\sum_{l=1}^{q}(-1)^{l-1}a_{l,2}+\gamma_{0,1}^s\sum_{l=1}^{p}(-1)^{l}a_{l,1}+\gamma_{0,2}^s\sum_{l=1}^{q}(-1)^{l}a_{l,2}=1.$$

Let us observe that

$$x^{p+1}-\sum_{m=0}^{p}(-1)^m\sum_{0\le j_0<\cdots<j_m\le p}\gamma_{j_0,1}^s\cdots\gamma_{j_m,1}^s x^{p-m}=\prod_{k=0}^{p}\left(x-\gamma_{k,1}^s\right)\text{ and}$$

$$x^{q+1}-\sum_{m=0}^{q}(-1)^m\sum_{0\le j_0<\cdots<j_m\le q}\gamma_{j_0,2}^s\cdots\gamma_{j_m,2}^s x^{q-m}=\prod_{k=0}^{q}\left(x-\gamma_{k,2}^s\right).$$

Then by $x=1$ we get that $d(s)=1$ is equivalent to

$$2-\prod_{k=0}^{p}\left(1-\gamma_{k,1}^s\right)-\prod_{k=0}^{q}\left(1-\gamma_{k,2}^s\right)=1$$

which is (3.14).

The proof will be complete if we show that (3.14) has unique solution. We have that the left hand side is equal to 2 if $s=0$ and the derivative is

$$\sum_{l=0}^{p}\gamma_{l,1}^s\log\gamma_{l,1}\prod_{\substack{k=0\\k\neq l}}^{p}\left(1-\gamma_{k,1}^s\right)+\sum_{l=0}^{q}\gamma_{l,2}^s\log\gamma_{l,2}\prod_{\substack{k=0\\k\neq l}}^{q}\left(1-\gamma_{k,2}^s\right)$$

which is negative for $s>0$. This completes the proof. □

Now we show that the unique solution of (3.14) is an upper bound for the Hausdorff dimension. To give a good cover of the attractor, we need to introduce another sequence of subsets of $\Sigma^*$. Let

$$\mathcal{C}_0=\{(0,1),(0,2)\}\tag{3.33}$$

and by induction let

$$\mathcal{C}_k=\bigcup_{j=0}^{p}\bigcup_{\substack{\underline{i}\in\mathcal{C}_{k-1}\\ \kappa_0\neq 1\vee j\le i_0}}\{(j,1)\underline{i}\}\;\bigcup\;\bigcup_{j=0}^{q}\bigcup_{\substack{\underline{i}\in\mathcal{C}_{k-1}\\ \kappa_0\neq 2\vee j\le i_0}}\{(j,2)\underline{i}\}.\tag{3.34}$$

**Lemma 3.5.** *Let $\widetilde{s}_k$ the unique solution of*

$$\sum_{\underline{i}\in\mathcal{C}_k}\gamma_{\underline{i}}^s=1,$$

*and let $\widetilde{s}=\sup_k\widetilde{s}_k$ then*

$$\dim_H\Lambda\le\min\{1,\widetilde{s}\}.$$

Note that the sequence $\widetilde{s}_k$ is bounded since $\mathcal{C}_k\subseteq(A_1\cup A_2)^{k+1}$.

*Proof.* Using that for every $(i,\kappa),(j,\kappa)\in A_\kappa$,

$$\varphi_{(i,\kappa)}\circ\varphi_{(j,\kappa)}\equiv\varphi_{(j,\kappa)}\circ\varphi_{(i,\kappa)},$$

and $\gamma_{j,\kappa},\gamma_{i,\kappa}\le\gamma_{0,\kappa}$ we have that the set of closed intervals

$$\left\{\varphi_{\underline{i}}([0,1])\right\}_{\underline{i}\in\mathcal{C}_k}$$

gives a cover of $\Lambda$ with diameter at most $\gamma_{\max}^k$, where $\gamma_{\max} = \max_{i,\kappa}\{\gamma_{i,\kappa}\}$. Then

$$\mathcal{H}^{\widetilde{s}}_{\gamma_{\max}^k}(\Lambda) \leq \sum_{\underline{i}\in\mathcal{C}_k} \left|\varphi_{\underline{i}}([0,1])\right|^{\widetilde{s}} = \sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{\widetilde{s}} \leq \sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{\widetilde{s}_k} = 1.$$

This proves the Lemma. □

*Proof of Theorem 1.1.* By the definition of $\mathcal{C}_k$ we have that for every $k \geq 1$

$$\mathcal{C}_k \subset \bigcup_{l=1}^{k} \mathcal{U}_k^l. \tag{3.35}$$

More precisely, every $\underline{i} \in \mathcal{C}_k$ can be decomposed as a juxtaposition $\underline{i} = \underline{j}_1 \cdots \underline{j}_r$, where each $\underline{j}_l \in \mathcal{U}_k$. By similar arguments as in the proof of Proposition 3.2, one can show that the system $\widetilde{\Psi}_k = \left\{\varphi_{\underline{i}}\right\}_{\underline{i}\in\mathcal{C}_k}$ satisfies transversality condition on $(\xi, 1-\xi)^{p+q}$. Since $\xi > 0$ was arbitrary by Theorem 2.2 we have

$$\dim_H \widetilde{\Lambda}_k = \min\{1, \widetilde{s}_k\} \text{ for } \mathcal{L}\text{-a.e. } \underline{c} \in (0,1)^{p+q}, \tag{3.36}$$

where $\widetilde{\Lambda}_k$ denotes the attractor of $\left\{\varphi_{\underline{i}}\right\}_{\underline{i}\in\mathcal{C}_k}$. Using (3.35) we have $\widetilde{\Lambda}_k \subseteq \Lambda_k \subseteq \Lambda$ which implies

$$\dim_H \widetilde{\Lambda}_k \leq \dim_H \Lambda_k \leq \dim_H \Lambda.$$

Therefore by Lemma 3.3 and Lemma 3.5 we have

$$\min\{1, \widetilde{s}_k\} \leq \min\{1, s_k\} \leq \min\{1, \widetilde{s}\}.$$

By Lemma 3.4, $s_k$ is convergent and $\lim_{k\to\infty} s_k = \sup_k s_k = s$. This implies that $\min\{1, s\} = \min\{1, \widetilde{s}\}$, moreover

$$\dim_H \Lambda = \min\{1, s\}.$$

To complete the proof we have to prove the measure claim. If $s > 1$ then there exists a $k \geq 2$ such that $s_k > 1$. Therefore, by Theorem 2.2 and Proposition 3.2, $\mathcal{L}(\Lambda) \geq \mathcal{L}(\Lambda_k) > 0$ for a.e. $\underline{c} \in (0,1)^{p+q} \cap \{\underline{c} : s > 1\}$. □

## 4. Box dimension of the generalized 4-corner set

In this section we show an application of the results for two dimensional, diagonally self-affine iterated function systems. Before we compute the box dimension of the generalized 4-corner set (see Figure 1), we state a general theorem on the box dimension of diagonally self-affine sets.

Let

$$f_i(x,y) = (\alpha_i x + t_i, \beta_i y + u_i) \tag{4.1}$$

for $i = 0, \dots, m$ such that

$$\begin{aligned} &0 < \alpha_i, \beta_i < 1 \\ &f_i([0,1]^2) \subseteq [0,1]^2 \text{ for } i = 0,\dots,m \\ &f_i((0,1)^2) \bigcap f_j((0,1)^2) = \emptyset \text{ for } i \neq j. \end{aligned} \tag{4.2}$$

Denote the attractor of $\Psi = \{f_i(x,y)\}_{i=0}^m$ by $\Lambda$ and define $\mathrm{proj}_x\Lambda$ (and $\mathrm{proj}_y\Lambda$) as the projection of $\Lambda$ onto the $x$-axis (and $y$-axis, respectively).

**Theorem 4.1.** *Let $f_i$ be in form (4.1) for $i = 0, \dots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2). Then the attractor $\Lambda$ of $\Psi$ satisfies*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\}$$

*where $d_\alpha$ and $d_\beta$ are the unique solutions of*

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha} \beta_i^{d_\alpha - s_\alpha} = 1 \text{ and } \sum_{i=0}^{m} \beta_i^{s_\beta} \alpha_i^{d_\beta - s_\beta} = 1,$$

*where $s_\alpha = \dim_B \mathrm{proj}_x \Lambda$ and $s_\beta = \dim_B \mathrm{proj}_y \Lambda$.*

Using this and [12, Theorem 2.1] we can compute the box dimension of the attractor at least for almost all translations such that (4.2) holds.

**Corollary 4.2.** *Let $f_i$ be in form (4.1) for $i = 0, \dots, m$ and let $\mathcal{T} \subset \mathbb{R}^{2m+2}$ be the set of translation vectors such that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2). Then the attractor $\Lambda$ of $\Psi$ satisfies*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\} \text{ for almost every translations in } \mathcal{T} \text{ with respect to } 2m+2\text{-dimensional Lebesgue measure}$$

*where $d_\alpha$ and $d_\beta$ are the unique solutions of*

$$\sum_{i=0}^{m} \alpha_i^{\min\{1, s_\alpha\}} \beta_i^{d_\alpha - \min\{1, s_\alpha\}} = 1 \text{ and } \sum_{i=0}^{m} \beta_i^{\min\left\{1, s_\beta\right\}} \alpha_i^{d_\beta - \min\left\{1, s_\beta\right\}} = 1,$$

*and $s_\alpha$, $s_\beta$ are the unique solutions of*

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha} = 1 \text{ and } \sum_{i=0}^{m} \beta_i^{s_\beta} = 1.$$

In the proof of Theorem 4.1 we follow the proof of Feng, Wang [8, Theorem 1] and Barański [1, Theorem B] with slight manipulations. For the convenience of the reader, we present here this lengthy calculation. The proof is broken into Lemma 4.3, Lemma 4.4 and Lemma 4.5.

Before we prove the theorem, let us introduce some notation. Let $\Sigma = \{0, \dots, m\}^{\mathbb{N}}$ and $\Sigma^* = \bigcup_{n=0}^{\infty} \{0, \dots, m\}^n$. Denote the right cut on $\Sigma^*$ by $\delta$. More precisely, let $\delta(\emptyset) = \emptyset$ and

$$\delta(i_0 \cdots i_k) = i_0 \cdots i_{k-1}.$$

For any $\underline{i} \in \Sigma^*$ let $f_{\underline{i}} = f_{i_0} \circ \cdots \circ f_{i_k}$ and $\alpha_{\underline{i}} = \alpha_{i_0} \cdots \alpha_{i_k}$, $\beta_{\underline{i}} = \beta_{i_0} \cdots \beta_{i_k}$. For every $0 < r < 1$ let

$$\Delta_r = \left\{\underline{i} \in \Sigma^* : \min\left\{\alpha_{\delta\underline{i}}, \beta_{\delta\underline{i}}\right\} \geq r, \min\left\{\alpha_{\underline{i}}, \beta_{\underline{i}}\right\} < r\right\}$$

and

$$\Delta_r^\alpha = \left\{\underline{i} \in \Delta_r : \alpha_{\underline{i}} \geq \beta_{\underline{i}}\right\} \text{ and } \Delta_r^\beta = \left\{\underline{i} \in \Delta_r : \alpha_{\underline{i}} < \beta_{\underline{i}}\right\}.$$

It is easy to see that $\Delta_r$ is a partition of $\Sigma$.

For every $\underline{i} \in \Delta_r^\alpha$ we set $\omega_\alpha(\underline{i}) = \left[\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right]$ and similarly, for every $\underline{i} \in \Delta_r^\beta$ we set $\omega_\beta(\underline{i}) = \left[\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right]$. For any $\underline{i} \in \Delta_r^\alpha$ we divide $f_{\underline{i}}([0,1]^2)$ into $\omega_\alpha(\underline{i})$ equal rectangles with height $\beta_{\underline{i}}$ and width $\alpha_{\underline{i}}/\omega_\alpha(\underline{i})$, denote the $k$th rectangle by $R_k^\alpha(\underline{i})$ for $k = 1, \dots, \omega_\alpha(\underline{i})$. Similarly, for $\underline{i} \in \Delta_r^\beta$ we divide $f_{\underline{i}}([0,1]^2)$ into $\omega_\beta(\underline{i})$ equal rectangles with width $\alpha_{\underline{i}}$ and height $\beta_{\underline{i}}/\omega_\beta(\underline{i})$ and denote the $k$th rectangle by $R_k^\beta(\underline{i})$ for $k = 1, \dots, \omega_\beta(\underline{i})$.

Let

$$\begin{aligned} C_r^\alpha &= \left\{ R_k^\alpha(\underline{i}) : \underline{i} \in \Delta_r^\alpha, 1 \le k \le \omega_\alpha(\underline{i}), R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset \right\} \\ C_r^\beta &= \left\{ R_k^\beta(\underline{i}) : \underline{i} \in \Delta_r^\beta, 1 \le k \le \omega_\beta(\underline{i}), R_k^\beta(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset \right\}, \end{aligned}$$

moreover

$$\eta_r^\alpha(\underline{i}) = \sharp \left\{ R_k^\alpha(\underline{i}) : 1 \le k \le \omega_\alpha(\underline{i}), R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset \right\} \text{ for } \underline{i} \in \Delta_r^\alpha \text{ and}$$
$$\eta_r^\alpha(\underline{i}) = \sharp \left\{ R_k^\beta(\underline{i}) : 1 \le k \le \omega_\beta(\underline{i}), R_k^\beta(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset \right\} \text{ for } \underline{i} \in \Delta_r^\beta.$$

**Lemma 4.3.** *Let $f_i$ be as in form (4.1) for $i = 0, \dots, m$ and let us suppose that $\Psi = \{f_i(x,y)\}_{i=0}^m$ satisfies (4.2). Moreover, let $\widetilde{N}_r = \sharp \left( C_r^\alpha \cup C_r^\beta \right)$. Then the attractor $\Lambda$ of $\Psi$ satisfies*

$$\overline{\dim}_B \Lambda = \limsup_{r \to 0+} \frac{\log \widetilde{N}_r}{-\log r} \text{ and } \underline{\dim}_B \Lambda = \liminf_{r \to 0+} \frac{\log \widetilde{N}_r}{-\log r}.$$

*Proof.* Let $N_r$ be the minimal number of squares with side length $r$ cover the attractor $\Lambda$.

By definition $C_r^\alpha \cup C_r^\beta$ covers $\Lambda$ and since for every $c \ge 1$ real number $\frac{1}{2}c \le [c] \le c$ we have that every rectangle in $C_r^\alpha \cup C_r^\beta$ has side length at most $2r$. Therefore

$$N_{2r} \le \widetilde{N}_r.$$

Let $\alpha_{\min} = \min_{i=0,\dots,m} \alpha_i$ and $\beta_{\min} = \min_{i=0,\dots,m} \beta_i$, moreover let $\rho = \min \{\alpha_{\min}, \beta_{\min}\}$.

Then every rectangle in $C_r^\alpha \cup C_r^\beta$ have side length at least $\rho r$. Therefore, by condition (4.2), every square with side length $\frac{\rho}{2} r$ can intersect at most 4 rectangles in $C_r^\alpha \cup C_r^\beta$, which implies that

$$4N_{\frac{\rho}{2}r} \ge \widetilde{N}_r.$$

One can finish the proof using the definition of the lower and upper box dimension. $\square$

For $\underline{i} \in \Delta_r^\alpha$ by some simple manipulation we get that

$$\begin{aligned} &\eta_r^\alpha(\underline{i}) = \sharp \left\{ R_k^\alpha(\underline{i}) : 1 \le k \le \omega_\alpha(\underline{i}), R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \neq \emptyset \right\} = \\ &\sharp \left\{ \left[ \frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})} \right] \times [0,1] : 1 \le k \le \omega_\alpha(\underline{i}), \left[ \frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})} \right] \times [0,1] \cap \Lambda \neq \emptyset \right\} = \\ &\qquad \sharp \left\{ \left[ \frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})} \right] : 1 \le k \le \omega_\alpha(\underline{i}), \left[ \frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})} \right] \cap \mathrm{proj}_x \Lambda \neq \emptyset \right\}. \end{aligned} \tag{4.3}$$

and by similar arguments for $\underline{i} \in \Delta_r^\beta$

$$\eta_r^\beta(\underline{i}) = \sharp \left\{ \left[ \frac{k-1}{\omega_\beta(\underline{i})}, \frac{k}{\omega_\beta(\underline{i})} \right] : 1 \le k \le \omega_\beta(\underline{i}), \left[ \frac{k-1}{\omega_\beta(\underline{i})}, \frac{k}{\omega_\beta(\underline{i})} \right] \cap \mathrm{proj}_y \Lambda \neq \emptyset \right\}. \tag{4.4}$$

Let us divide the unit interval into $n \in \mathbb{N}$ equal parts and denote $N_{\frac{1}{n}}(\mathrm{proj}_x \Lambda)$ (and $N_{\frac{1}{n}}(\mathrm{proj}_y \Lambda)$) the number of intervals with length $\frac{1}{n}$ intersect the set $\mathrm{proj}_x \Lambda$ (and $\mathrm{proj}_y \Lambda$, respectively). Since $\mathrm{proj}_x \Lambda$ and $\mathrm{proj}_y \Lambda$ are self-similar sets, the box dimensions exist, therefore for every $\varepsilon > 0$ exists a $c = c(\varepsilon) > 0$ such that for every integer $n \geq 1$

$$\begin{aligned} c^{-1}n^{s_\alpha-\varepsilon} &\leq N_{\frac{1}{n}}(\mathrm{proj}_x\Lambda) \leq cn^{s_\alpha+\varepsilon} \text{ and} \\ c^{-1}n^{s_\beta-\varepsilon} &\leq N_{\frac{1}{n}}(\mathrm{proj}_y\Lambda) \leq cn^{s_\beta+\varepsilon}, \end{aligned} \tag{4.5}$$

where $s_\alpha = \dim_B \mathrm{proj}_x\Lambda$ and $s_\beta = \dim_B \mathrm{proj}_y\Lambda$. Using (4.3) and (4.4) we have

$$\widetilde{N}_r = \sum_{\underline{i}\in\Delta_r^\alpha} \eta_r^\alpha(\underline{i}) + \sum_{\underline{i}\in\Delta_r^\beta} \eta_r^\beta(\underline{i}) \leq c\sum_{\underline{i}\in\Delta_r^\alpha} \omega_\alpha(\underline{i})^{s_\alpha+\varepsilon} + c\sum_{\underline{i}\in\Delta_r^\beta} \omega_\beta(\underline{i})^{s_\beta+\varepsilon} \leq$$

$$c\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon} + c\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon} \tag{4.6}$$

and similarly

$$\widetilde{N}_r \geq c^{-1}2^{-(s_\alpha-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon} + c^{-1}2^{-(s_\beta-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}. \tag{4.7}$$

Let $d_\alpha(t)$ and $d_\beta(t)$ be the unique solutions for $t \geq -\min\{s_\alpha, s_\beta\}$ of

$$\sum_{i=0}^m\left(\frac{\alpha_i}{\beta_i}\right)^{s_\alpha+t}\beta_i^{d_\alpha(t)} = 1 \text{ and } \sum_{i=0}^m\left(\frac{\beta_i}{\alpha_i}\right)^{s_\beta+t}\alpha_i^{d_\beta(t)} = 1.$$

We remark that $d_\alpha(0) = d_\alpha$ and $d_\beta(0) = d_\beta$.

**Lemma 4.4.** *Let $f_i$ be in form (4.1) for $i = 0, \dots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (4.2), then the attractor $\Lambda$ of $\Psi$ satisfies that $\overline{\dim}_B\Lambda \leq \max\{d_\alpha, d_\beta\}$.*

*Proof.* Let $\varepsilon > 0$ be arbitrary small. Then by (4.6)

$$\frac{\log\widetilde{N}_r}{-\log r} \leq \frac{\log c}{-\log r} + \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\right)}{-\log r} \leq \frac{\log c}{-\log r} +$$

$$\max\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\}\left(1 + \frac{\log\rho}{\log r}\right) + \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)}\right)}{-\log r}.$$

Since $\Delta_r$ is a partition, $\sum_{\underline{i}\in\Delta_r}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} = 1$ and $\sum_{\underline{i}\in\Delta_r}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)} = 1$ which implies that

$$\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)} \leq 2.$$

Therefore

$$\frac{\log\widetilde{N}_r}{-\log r} \leq \frac{\log c}{-\log r} + \max\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\}\left(1 + \frac{\log\rho}{\log r}\right) + \frac{\log 2}{-\log r}.$$

Taking limit superior as $r$ tends to 0 and by Lemma 4.3

$$\overline{\dim}_B\Lambda \leq \max\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\}$$

for every $\varepsilon > 0$. Finally, since $\varepsilon > 0$ was arbitrary, we proved the lemma. $\square$

**Lemma 4.5.** *Let $f_i$ be in form (4.1) for $i = 0, \dots, m$ and let us suppose that $\Psi = \{f_i(x,y)\}_{i=0}^m$ satisfies (4.2), then*

$$\underline{\dim}_B \Lambda \geq \max\{d_\alpha, d_\beta\}.$$

Before we prove the lower bound of the lower box dimension, we have to state another lemma about the dimension of the projections. To state this lemma we need a sublemma about the partitions of $\Sigma$. First let us introduce some notation. Let $\mathcal{G}$ be a partition of $\Sigma$ containing only cylinder sets, and denote $\lceil \mathcal{G} \rceil$ the length of the longest and denote $\lfloor \mathcal{G} \rfloor$ the length of the shortest cylinder set of $\mathcal{G}$. h

**Sublemma 4.6.** *Let $\mathcal{G}$ be a partition of $\Sigma = \{0, \dots, m\}^{\mathbb{N}}$ containing only cylinder sets and let $\gamma_i$, $i = 0, \dots, m$ be positive real numbers such that $\sum_{i=0}^m \gamma_i > 1$. Then*

$$\sum_{\underline{i} \in \mathcal{G}} \gamma_{\mathbf{i}} \geq \left( \sum_{i=0}^m \gamma_i \right)^{\lfloor \mathcal{G} \rfloor}.$$

*Proof.* We prove the statement of the sublemma by induction for the length of the longest cylinder set of $\mathcal{G}$.

For $\lceil \mathcal{G} \rceil = 1$ the statement holds trivially. Let us suppose that for every partitions with length of the longest cylinder set equal to $n$ the statement is true. Let $\mathcal{G}$ be a partition containing only cylinder sets with $\lceil \mathcal{G} \rceil = n + 1$.

If $\lceil \mathcal{G} \rceil = \lfloor \mathcal{G} \rfloor$ then the statement is true since

$$\sum_{\underline{i} \in \mathcal{G}} \gamma_{\underline{i}} = \left( \sum_{i=0}^m \gamma_i \right)^{\lfloor \mathcal{G} \rfloor}.$$

Therefore without loss of generality we may assume that $\lfloor \mathcal{G} \rfloor < \lceil \mathcal{G} \rceil$. Let $[i_0 \cdots i_n] \in \mathcal{G}$ be one of the longest cylinder sets of $\mathcal{G}$. Since $\mathcal{G}$ is a partition of $\Sigma$, $[i_0 \cdots i_{n-1} j] \in \mathcal{G}$ for every $j = 0, \dots, m$. Using this fact we can define a partition $\mathcal{G}_2$ such that for every $\underline{i} \in \mathcal{G}$ with length strictly less than $n+1$, $\underline{i} \in \mathcal{G}_2$ and for every $\underline{i} \in \mathcal{G}$ with length $n+1$, $\underline{i}|_n \in \mathcal{G}_2$. Then

$$\sum_{\underline{i} \in \mathcal{G}} \gamma_{\underline{i}} \geq \sum_{\underline{i} \in \mathcal{G}_2} \gamma_{\underline{i}} \geq \left( \sum_{i=0}^m \gamma_i \right)^{\lfloor \mathcal{G} \rfloor}.$$

In the last inequality we used the inductional assumption and $\lfloor \mathcal{G} \rfloor = \lfloor \mathcal{G}_2 \rfloor$ by the definition of $\mathcal{G}_2$. $\square$

**Lemma 4.7.** *Let $f_i$ be in form (4.1) for $i = 0, \dots, m$ and let us suppose that $\Psi = \{f_i(x,y)\}_{i=0}^m$ satisfies (4.2), then*

$$\sum_{i=0}^m \alpha_i^{s_\alpha} \beta_i^{s_\beta} \leq 1. \tag{4.8}$$

*Proof.* We begin the proof of the lemma by dividing the $[0,1]$ interval on the $x$ and $y$ axis into intervals with length $r$. Let $\varepsilon > 0$ be arbitrary small but fixed. Let us take the intervals which intersect $\mathrm{proj}_x \Lambda$ on the $x$ axis and $\mathrm{proj}_y \Lambda$ on the $y$ axis, moreover take the left and the right neighbor interval of those intervals. Then for every sufficiently small $r$ the number of intervals on the $x$ axis (and $y$ axis) is at most $3\left(\frac{1}{r}\right)^{s_\alpha + \varepsilon}$ (and $3\left(\frac{1}{r}\right)^{s_\beta + \varepsilon}$). Let us take the direct product of these intervals.

It is easy to see that the cover constructed in this way covers the approximate squares $C_r^\alpha \cup C_r^\beta$ and this implies that the area of $C_r^\alpha \cup C_r^\beta$ is less than or equal to the area of the squares constructed above. That is

$$\begin{aligned} 9\left(\frac{1}{r}\right)^{s_\alpha+\varepsilon}\left(\frac{1}{r}\right)^{s_\beta+\varepsilon} r^2 &\geq c^{-1}\sum_{\underline{i}\in\Delta_r^\alpha}\beta_{\underline{i}}\frac{\alpha_{\underline{i}}}{\omega_\alpha(\underline{i})}\omega_\alpha(\underline{i})^{s_\alpha-\varepsilon}+c^{-1}\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}\frac{\beta_{\underline{i}}}{\omega_\beta(\underline{i})}\omega_\beta(\underline{i})^{s_\beta-\varepsilon} \\ &\geq c^{-1}\sum_{\underline{i}\in\Delta_r^\alpha}\beta_{\underline{i}}^2\omega_\alpha(\underline{i})^{s_\alpha-\varepsilon}+c^{-1}\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}^2\omega_\beta(\underline{i})^{s_\beta-\varepsilon}, \end{aligned}$$

where $c$ is a constant depending only on $\varepsilon$ as in (4.5). By simple algebraic manipulations and using the definitions of $\omega_\alpha(\underline{i}), \omega_\beta(\underline{i})$ and $\Delta_r^\alpha, \Delta_r^\beta$ we have

$$\begin{aligned} \sum_{\underline{i}\in\Delta_r^\alpha}\beta_{\underline{i}}^2\omega_\alpha(\underline{i})^{s_\alpha-\varepsilon}r^{s_\alpha+s_\beta-2} &\geq c_1 r^\varepsilon\sum_{\underline{i}\in\Delta_r^\alpha}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}, \text{ and} \\ \sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\mathbf{i}}^2\omega_\beta(\underline{i})^{s_\beta-\varepsilon}r^{s_\alpha+s_\beta-2} &\geq c_1 r^\varepsilon\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}, \end{aligned}$$

where $c_1$ depends only on $\varepsilon$. Then there exists a constant $\widetilde{c}$ depending only on $\varepsilon$ such that for every sufficiently small $r$

$$\widetilde{c}r^{-3\varepsilon} \geq \sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}.$$

Since $\varepsilon$ was arbitrary we have that

$$0 \leq \liminf_{r\to0+}\frac{\log\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}}{\log r}. \tag{4.9}$$

Now let us suppose indirectly that $\sum_{i=0}^m \alpha_i^{s_\alpha}\beta_i^{s_\beta} > 1$. Then by using Sublemma 4.6 we have

$$\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta} \geq \left(\sum_{i=0}^m \alpha_i^{s_\alpha}\beta_i^{s_\beta}\right)^{\lfloor\Delta_r\rfloor}.$$

It is easy to see that $\lfloor\Delta_r\rfloor = \lceil\frac{\log r}{\log\rho}\rceil$, where $\rho = \min_i\{\alpha_i, \beta_i\}$. This implies that

$$\limsup_{r\to0+}\frac{\log\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}}{\log r} \leq \frac{\log\sum_{i=0}^m\alpha_i^{s_\alpha}\beta_i^{s_\beta}}{\log\rho} < 0,$$

which contradicts to (4.9). □

*Proof of Lemma 4.5.* By Lemma 4.7 we divide the proof into two parts. First let us assume that

$$\sum_{i=0}^m \alpha_i^{s_\alpha}\beta_i^{s_\beta} = 1. \tag{4.10}$$

Let us observe that in this case $d_\alpha = d_\beta = s_\alpha + s_\beta$. Then by inequality (4.5) we have

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}+\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\right)}{-\log r} \geq$$

$$s_\alpha + s_\beta + \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{s_\alpha+s_\beta}+\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{s_\alpha+s_\beta}\right)}{-\log r} \geq$$

$$s_\alpha + s_\beta - \varepsilon\frac{\lceil\Delta_r\rceil \log\max_i\left\{\frac{\alpha_i}{\beta_i},\frac{\beta_i}{\alpha_i}\right\}}{-\log r} + \frac{\log\left(\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}\right)}{-\log r}.$$

It is easy to see that $\lceil\Delta_r\rceil = \frac{\log r}{\log\max_i\{\alpha_i,\beta_i\}}$. Applying this fact and our assumption (4.10) we get for every $\varepsilon > 0$ that

$$\liminf_{r\to 0}\frac{\log \widetilde{N}_r}{-\log r} \geq s_\alpha + s_\beta - \varepsilon\frac{1}{-\log\max_i\{\alpha_i,\beta_i\}},$$

and this completes the proof in the first case.

In the second case let us assume that

$$\sum_{i=0}^{m}\alpha_i^{s_\alpha}\beta_i^{s_\beta} < 1. \tag{4.11}$$

Without loss of generality we assume that $d_\alpha \geq d_\beta$.

Then there exists a $\varepsilon^* > 0$ by (4.11) such that for every $0 < \varepsilon < \varepsilon^*$,

$$\sum_{i=0}^{m}\alpha_i^{s_\alpha-\varepsilon}\beta_i^{s_\beta-\varepsilon} < 1.$$

This implies that

$$d_\beta(-\varepsilon), d_\alpha(-\varepsilon) \leq s_\alpha + s_\beta - 2\varepsilon. \tag{4.12}$$

Then for every $\underline{i} \in \Delta_r^\beta$

$$\frac{\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}}{\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon}} = \left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+s_\beta-2\varepsilon-d_\alpha(-\varepsilon)}\alpha_{\underline{i}}^{d_\alpha(-\varepsilon)-d_\beta(-\varepsilon)} \leq \alpha_{\underline{i}}^{d_\alpha(-\varepsilon)-d_\beta(-\varepsilon)} \tag{4.13}$$

and for every $\underline{i} \in \Delta_r^\alpha$

$$\frac{\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon}}{\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}} \leq \beta_{\underline{i}}^{d_\beta(-\varepsilon)-d_\alpha(-\varepsilon)}. \tag{4.14}$$

Now we prove the Lemma in the case when $d_\alpha > d_\beta$. Then there exists a $\varepsilon^{**} > 0$ such that for every $0 < \varepsilon < \varepsilon^{**}$, $d_\alpha(-\varepsilon) > d_\beta(-\varepsilon)$. Then by (4.13)

$$\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\mathbf{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \leq \frac{1}{2}\sum_{\underline{i}\in\Delta_r^\beta}\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \leq \frac{1}{2}$$

holds for sufficiently small $r > 0$. Therefore

$$\sum_{\underline{i}\in\Delta_r^\alpha}\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \geq \frac{1}{2}. \tag{4.15}$$

Using (4.7)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log\left(c^{-1}2^{-(s_\alpha-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}+c^{-1}2^{-(s_\beta-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\right)}{-\log r} \geq$$
$$\frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r}+\frac{\log r^{-d_\alpha(-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)}}{-\log r},$$

and by (4.15)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq d_\alpha(-\varepsilon)+\frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r}+\frac{\log 2}{\log r}.$$

Taking limit inferior $r$ to 0 and Lemma 4.3 we get

$$\underline{\dim}_B\Lambda \geq d_\alpha(-\varepsilon).$$

Since $\varepsilon>0$ was arbitrary small we proved the Lemma in the case $d_\alpha>d_\beta$.

Now let us consider the case $d_\alpha=d_\beta$. The fact (4.12) and (4.13), (4.14) imply for every sufficiently small $\varepsilon>0$ that

$$\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \leq \sum_{\underline{i}\in\Delta_r^\beta}\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \text{ or}$$
$$\sum_{\underline{i}\in\Delta_r^\alpha}\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \leq \sum_{\underline{i}\in\Delta_r^\alpha}\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}.$$

Therefore

$$\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)}+\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)} \geq 1. \tag{4.16}$$

Using (4.7)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r}+\min\{d_\alpha(-\varepsilon),d_\beta(-\varepsilon)\}+$$
$$\frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)}+\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)}\right)}{-\log r}$$

and by (4.16)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r}+\min\{d_\alpha(-\varepsilon),d_\beta(-\varepsilon)\}.$$

Taking limit inferior $r$ to 0 and Lemma 4.3 we have

$$\underline{\dim}_B\Lambda \geq \min\{d_\alpha(-\varepsilon),d_\beta(-\varepsilon)\}.$$

Since $\varepsilon>0$ was arbitrary small and $d_\alpha=d_\beta$ this completes the proof of the lemma. □

*Proof of Theorem 4.1.* The proof is the combination of Lemma 4.4 and Lemma 4.5. □

Now we consider the generalized 4-corner set. Let $\Psi = \{f_0(\underline{x}), f_1(\underline{x}), f_2(\underline{x}), f_3(\underline{x})\}$ an iterated function system on the real plane such that

$$\begin{aligned} f_0(\underline{x}) &= \begin{pmatrix} \alpha_0 & 0 \\ 0 & \beta_0 \end{pmatrix} \underline{x} \\ f_1(\underline{x}) &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1-\beta_1 \end{pmatrix} \\ f_2(\underline{x}) &= \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} \underline{x} + \begin{pmatrix} 1-\alpha_2 \\ 0 \end{pmatrix} \\ f_3(\underline{x}) &= \begin{pmatrix} \alpha_3 & 0 \\ 0 & \beta_3 \end{pmatrix} \underline{x} + \begin{pmatrix} 1-\alpha_3 \\ 1-\beta_3 \end{pmatrix}. \end{aligned} \tag{4.17}$$

Assume that $\Psi$ satisfies (4.2), see Figure 1. This condition is equivalent to (4.18)-(4.23)

$$\alpha_0 + \alpha_2 < 1 \tag{4.18}$$
$$\alpha_1 + \alpha_3 < 1 \tag{4.19}$$
$$\beta_0 + \beta_1 < 1 \tag{4.20}$$
$$\beta_2 + \beta_3 < 1 \tag{4.21}$$
$$\alpha_0 + \alpha_3 < 1 \text{ or } \beta_0 + \beta_3 < 1 \tag{4.22}$$
$$\alpha_1 + \alpha_2 < 1 \text{ or } \beta_1 + \beta_2 < 1. \tag{4.23}$$

Let $\mathcal{R}$ be the set of possible parameters, that is

$$\mathcal{R} = \left\{(\underline{\alpha}, \underline{\beta}) \in (0,1)^8 : (\underline{\alpha}, \underline{\beta}) \text{ satisfies (4.18)-(4.23)}\right\}.$$

**Theorem 4.8.** *Let $\Psi$ as in (4.17) and suppose that satisfies (4.2). Let $\Lambda$ the attractor of $\Psi$. Then*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\} \text{ for } \mathcal{L}\text{-a.e.}(\underline{\alpha}, \underline{\beta}) \in \mathcal{R},$$

*where*

$$\sum_{i=0}^{3} \alpha_i^{\min\{1,s_\alpha\}} \beta_i^{d_\alpha - \min\{1,s_\alpha\}} = 1 \text{ and } \sum_{i=0}^{3} \beta_i^{\min\left\{1,s_\beta\right\}} \alpha_i^{d_\beta - \min\left\{1,s_\beta\right\}} = 1$$

*where $s_\alpha$ is the unique solution of*

$$\alpha_0^s + \alpha_1^s + \alpha_2^s + \alpha_3^s - \alpha_0^s\alpha_1^s - \alpha_2^s\alpha_3^s = 1, \tag{4.24}$$

*and similarly $s_\beta$ is the unique solution of*

$$\beta_0^s + \beta_1^s + \beta_2^s + \beta_3^s - \beta_0^s\beta_2^s - \beta_1^s\beta_3^s = 1. \tag{4.25}$$

*Proof.* The proof is an easy consequence of Theorem 1.1 and Theorem 4.1. □

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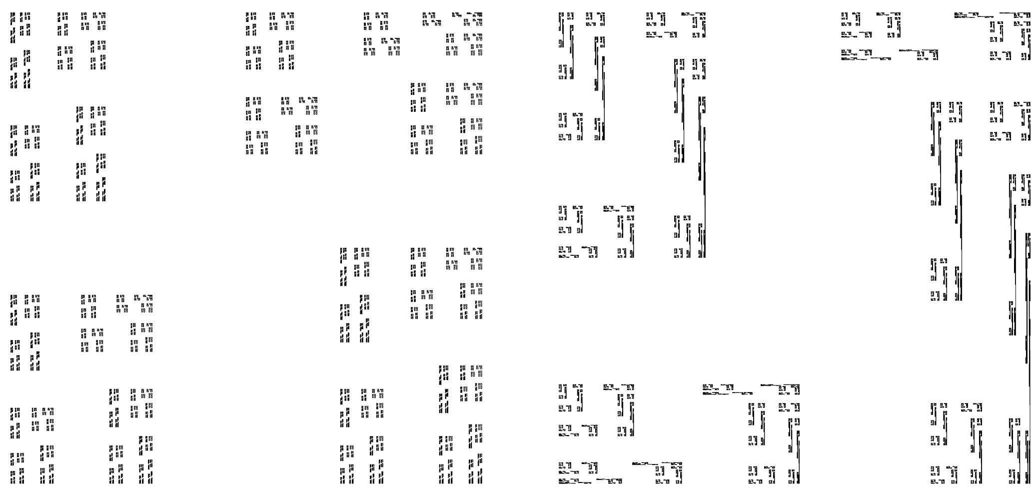

Figure 3. Generalized 4-corner sets with box dimension $\approx 1.39444$ and $\approx 1.40819$ for sufficiently small perturbation.

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