ON THE HAUSDORFF DIMENSION OF A FAMILY OF SELF-SIMILAR SETS WITH COMPLICATED OVERLAPS

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Abstract. We investigate the properties of the Hausdorff dimension of the attractor of the iterated function system (IFS) $\{\gamma x, \lambda x, \lambda x+1\}$. Since two maps have the same fixed point, there are very complicated overlaps, and it is not possible to directly apply known techniques. We give a formula for the Hausdorff dimension of the attractor for Lebesgue-almost all parameters $(\gamma, \lambda), \gamma < \lambda$. This result only holds for almost all parameters: we find a dense set of parameters (γ, λ) for which the Hausdorff dimension of the attractor is strictly smaller.

1. Introduction and Statements

Our paper is motivated by a question of Pablo Shmerkin at the conference in Greifswald in 2008. The question was the following:

Question. What is the Hausdorff dimension of the attractor, which is generated by the IFS $\{\frac{1}{4}x, \frac{1}{3}x, \frac{1}{3}x+\frac{2}{3}\}$?

Let us denote the Hausdorff dimension of a compact subset Λ of $\mathbb{R}$ by $\dim_H \Lambda$. For the definition and basic properties of Hausdorff dimension we refer the reader to [1] or [2]. Let us recall here the definition of the attractor.

Let $\{f_0, \dots, f_n\}$ be a family of continuous self-maps on the real line. We will in addition assume that each f_i is a contraction, that is, $|f_i(x) - f_i(y)| \leq r_i|x-y|$ for all x, y and for some $0 < r_i < 1$. Then there exists a unique, nonempty compact subset Λ of $\mathbb{R}$ which satisfies

$$\Lambda = \bigcup_{k=0}^{n} f_k(\Lambda).$$

We call this set Λ the attractor of the iterated function system (IFS) $\{f_0(x), f_1(x), \dots, f_n(x)\}$.

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Let us suppose that the functions of the IFS are similarities in the form $\{f_i(x) = \lambda_i x + d_i\}_{i=0}^n$, where $0 < |\lambda_i| < 1$ for every $i \in \{0, \dots, n\}$. We say that the attractor of the IFS or the IFS itself is self-similar. It is well known if a self-similar IFS satisfies the so called *open set condition* (OSC), i.e. there exists an open set U such that for every $i, j \in \{0, \dots, n\}$, $f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ if $i \neq j$, then the Hausdorff dimension of the attractor is the unique solution of

$$\sum_{i=0}^{n} |\lambda_i|^s = 1, \tag{1.1}$$

see for example [4]. Even if the OSC does not hold, the solution of equation (1.1) is called *similarity dimension* of the IFS. The similarity dimension is always an upper bound for the Hausdorff dimension of the attractor, see [1]. In the case when the IFS has overlapping structure, i.e. the open set condition does not hold, the Hausdorff dimension of the attractor Λ of IFS $\{f_i(x) = \lambda_i x + d_i\}_{i=0}^n$ is

$$\dim_H \Lambda = \min\{1, s\} \text{ for a.e. } \mathbf{d} \in \mathbb{R}^{n+1}, \tag{1.2}$$

where s is the unique solution of (1.1), see [11] and [3].

In this article, we consider the IFS $\{\gamma x, \lambda x, \lambda x + 1\}$, where we assume throughout the paper that $0 < \gamma < \lambda < 1$. Let us denote the attractor of this IFS by $\Lambda^{\gamma,\lambda}$. The problem of calculating the Hausdorff dimension of the attractor of this system is far from being simple.

The special property of our chosen class of IFS is that the first two maps have a common fixed point. This fact implies that the two maps commute, so we are going to observe an immense (increasing exponentially under iteration) amount of exact and partial overlaps in our system. Needless to say, the OSC does not hold.

Iterated function systems that do not satisfy OSC were first studied in [9], where the *transversality condition* method was first introduced. See [7], [8] for the most general treatment of this approach. Since this time, several other methods were proposed: *weak separation condition*, *finite type condition* and others (see, for example, [5], [6] and [14]). However, neither of those is going to work for overlaps as severe as our system has.

For this reason, we are forced to modify the transversality method, applying it only to some subsystems of the IFS (details will be presented in the following sections). The main result of this paper is as follows:

Theorem 1.1. *Let $\Lambda^{\gamma,\lambda}$ the attractor of the IFS $\{\gamma x, \lambda x, \lambda x+1\}$. Then for Lebesgue almost every $0<\gamma<\lambda<\frac{1}{2}$*

$$\dim_H \Lambda^{\gamma,\lambda} = \min\left\{1, s^{\gamma,\lambda}\right\}, \tag{1.3}$$

where $s^{\gamma,\lambda}$ is the unique solution of

$$2\lambda^s + \gamma^s - \lambda^s\gamma^s = 1. \tag{1.4}$$

Moreover $\mathcal{L}(\Lambda^{\gamma,\lambda})>0$ for Lebesgue almost every (γ,λ) such that $s^{\gamma,\lambda}>1$.

More precisely, the above statements are true for every fixed $0<\lambda<\frac{1}{2}$ and Lebesgue almost every $0<\gamma<\lambda$.

Note here that the assumption that $\lambda<\frac{1}{2}$ is not really restrictive: the attractor of our system contains the attractor of its subsystem $\{\lambda x, \lambda x+1\}$, which for $\lambda\geq\frac{1}{2}$ is an interval and, in particular, it has dimension 1.

Our result only holds for almost all parameters. And indeed, we can present a family of parameter values for which the result does not hold.

Proposition 1.2. *Let q and p integers and $q>p$, $(q,p)=1$. Let $\Lambda^{\lambda,q,p}$ be the attractor of $\left\{\lambda^{\frac{q}{p}}x, \lambda x, \lambda x+1\right\}$. Then*

$$\dim_H \Lambda^{\lambda,q,p} \leq \min\left\{1, s^{\lambda}_{p,q}\right\}, \tag{1.5}$$

where $s^{\lambda}_{p,q}$ is the unique solution of

$$2\lambda^s + \sum_{k=1}^{p-1}\lambda^{\left(\frac{q}{p}k+1\right)s} = 1.$$

Note that this family of exceptional parameter values is dense in $\{(\gamma,\lambda): 2\lambda+\gamma<1, \gamma<\lambda\}$, where the statement of Proposition 1.2 excludes the possibility that the assertion of Theorem 1.1 holds. This implies that the function $(\gamma,\lambda)\mapsto\dim_H\Lambda^{\gamma,\lambda}$ cannot be continuous.

2. Transversality methods

First let us introduce the *transversality condition* for self-similar IFS with one parameter. The definition corresponds to the definition in [12],[13] which was introduced for general IFS.

Let U be an open, bounded interval of $\mathbb{R}$ and Σ a finite set of symbols. Let $\Psi_t = \left\{\psi_i^t(x) = \lambda_i(t)x + d_i(t)\right\}_{i\in\Sigma}$, where $\lambda_i, d_i \in C^1(\overline{U})$ and $0<\alpha\leq\lambda_i(t)\leq\beta<1$ for every $i\in\Sigma$ and $t\in\overline{U}$ and for some $\alpha,\beta\in(0,1)$. Let Λ^t be the attractor of Ψ_t

and π_t be the natural projection from the symbolic space $\Sigma^{\mathbb{N}}$ to Λ^t. More precisely, let $\mathbf{i} = (i_0 i_1 \dots) \in \Sigma^{\mathbb{N}}$ and

$$\pi_t(\mathbf{i}) = \lim_{n\to\infty} \psi^t_{i_0} \circ \psi^t_{i_1} \circ \dots \circ \psi^t_{i_n}(0). \tag{2.1}$$

It is well-known that the limit exists and independent of the base point 0. Moreover, π_t is a surjective function from $\Sigma^{\mathbb{N}}$ onto Λ^t. Denote σ the left-shift operator on $\Sigma^{\mathbb{N}}$. More precisely, let $\sigma : (i_0 i_1 \dots) \mapsto (i_1 i_2 \dots)$. It is easy to see that

$$\pi_t(\mathbf{i}) = \psi^t_{i_0}(\pi_t(\sigma \mathbf{i})).$$

Definition 2.1. *We say that Ψ_t satisfies the* **transversality condition** *on an open, bounded interval $U \subset \mathbb{R}$, if there exists a constant $C > 0$ such that for every $\mathbf{i}, \mathbf{j} \in \Sigma^{\mathbb{N}}$ with $i_0 \neq j_0$ it holds that*

$$\mathcal{L}(t \in U : |\pi_t(\mathbf{i}) - \pi_t(\mathbf{j})| \leq r) \leq Cr \text{ for every } r > 0,$$

where $\mathcal{L}$ is the Lebesgue measure on the real line.

This definition is equivalent to the ones given in e.g. [12], [13]. Now let us recall the Theorem of K. Simon, B. Solomyak and M. Urbański [12, Theorem 3.1] in the self-similar case with one parameter.

Theorem 2.2 (Simon, Solomyak, Urbański)**.** *Suppose that Ψ_t satisfies the transversality condition on an open, bounded interval U. Then*

(1) $\dim_H \Lambda^t = \min\{s(t), 1\}$ *for Lebesgue-a.e.* $t \in U$,
(2) $\mathcal{L}(\Lambda^t) > 0$ *for Lebesgue-a.e.* $t \in U$ *such that* $s(t) > 1$,

where $s(t)$ is the similarity dimension of Ψ_t. More precisely, $s(t)$ satisfies the equation

$$\sum_{i\in\Sigma} \lambda_i(t)^{s(t)} = 1.$$

We can use the following Lemma to prove transversality which follows from [12, Lemma 7.3].

Lemma 2.3. *Let $U \subset \mathbb{R}$ be an open, bounded interval and $f_{\mathbf{i},\mathbf{j}}(t) = \pi_t(\mathbf{i}) - \pi_t(\mathbf{j})$. If for every $\mathbf{i}, \mathbf{j} \in \Sigma^{\mathbb{N}}$ with $i_0 \neq j_0$ and for every $t_0 \in U$*

$$f_{\mathbf{i},\mathbf{j}}(t_0) = 0 \Rightarrow \left|\frac{df_{\mathbf{i},\mathbf{j}}}{dt}(t_0)\right| > 0 \tag{2.2}$$

then there is transversality on any open interval V whose closure is contained in U.

3. Proofs

Let us fix a $\lambda \in (0, \frac{1}{2})$. Without loss of generality we can assume that $\gamma = c\lambda$, where $0 < c < 1$. Let $\psi_0^c(x) = c\lambda x$, $\psi_1^c(x) = \lambda x$ and $\psi_2^c(x) = \lambda x + 1$. We note that ψ_1^c, ψ_2^c do not depend on c. Let us denote $\Sigma^* = \bigcup_{n=1}^{\infty} \Sigma^n$ and for every $n \geq 1$ let $\psi_{\underline{i}}^c = \psi_{i_0}^c \circ \psi_{i_1}^c \circ \cdots \circ \psi_{i_n}^c$ where $\underline{i} \in \Sigma^n$.

We note that $\Psi^c = \{\psi_0^c, \psi_1^c, \psi_2^c\}$ does not satisfy the transversality condition, since for every $\mathbf{i}, \mathbf{j} \in \{0,1\}^{\mathbb{N}}$ and $0 < c < 1$ we have $\pi_c(\mathbf{i}) - \pi_c(\mathbf{j}) \equiv 0$. In order to prove Theorem 1.1, we are going to introduce well-chosen systems Ψ_n^c which do satisfy transversality. Moreover, we are going to show that the attractor of Ψ_n^c is contained in, and has dimension arbitrarily close to, the attractor $\Lambda^{c\lambda,\lambda}$ of IFS Ψ^c, so that we are able to conclude information on the dimension of $\Lambda^{c\lambda,\lambda}$ by studying these subsystems with transversality.

First of all, we have to consider some properties of the natural projection. Let $\mathbf{i} \in \Sigma^{\mathbb{N}}$ and denote by $\mathbf{i}(n)$ the first n elements of $\mathbf{i}$, moreover denote by $\sharp_i \mathbf{i}(n)$ the number of is in $\mathbf{i}(n)$. Similarly, let $\sharp_i \mathbf{i}(n,l)$ be the number of is between the nth and lth elements of $\mathbf{i}$. We note that $\mathbf{i}(0)$ is the empty word and $\sharp_i \mathbf{i}(-1,l) := \sharp_i \mathbf{i}(l)$. Then

$$\pi_c(\mathbf{i}) = \sum_{k=0}^{\infty} \delta_{i_k} c^{\sharp_0 \mathbf{i}(k)} \lambda^k, \tag{3.1}$$

where $\delta_i = 1$ if $i = 2$, else $\delta_i = 0$. We can write the natural projection in another form.

Let $\alpha_2^{\mathbf{i}}$ be the number of 2's in $\mathbf{i}$, more precisely, let $\alpha_2^{\mathbf{i}} = \lim_{n\to\infty} \sharp_0 \mathbf{i}(n)$ if $\mathbf{i}$ is an infinite length word of symbols and $\alpha_2^{\mathbf{i}} = \sharp_2 \mathbf{i}(|\mathbf{i}|)$ if $\mathbf{i}$ is finite. If $\mathbf{i}$ is infinite then $\alpha_2^{\mathbf{i}}$ can be equal to infinite. We will use the notation $n_l^{\mathbf{i}}$ for the place of the lth 2 in $\mathbf{i}$ (here $\mathbf{i}$ can be finite or infinite) for every $1 \leq l \leq \alpha_2^{\mathbf{i}}$. We define $n_0^{\mathbf{i}} := -1$ and if $\alpha_2^{\mathbf{i}}$ is finite, then let $n_{\alpha_2^{\mathbf{i}}+1}^{\mathbf{i}} := |\mathbf{i}|$. We note that if $\mathbf{i}$ does not contain any 2's then $\pi_c(\mathbf{i}) = 0$.

Observe that with this notation π_c can be written as

$$\pi_c(\mathbf{i}) = \sum_{k=1}^{\alpha_2^{\mathbf{i}}} c^{\sharp_0 \mathbf{i}(n_k^{\mathbf{i}})} \lambda^{n_k^{\mathbf{i}}},$$

we note that we define the empty sum as 0.

If $\alpha_2^{\mathbf{i}} \neq 0$, then let $P_{\mathbf{i}} = \left\{ \sharp_0 \mathbf{i}(n_l^{\mathbf{i}}) : l \geq 1 \right\}$ else let $P_{\mathbf{i}} = \emptyset$. Moreover let

$$m_k^{\mathbf{i}} = \sup\left\{ j : \sharp_0 \mathbf{i}(n_j^{\mathbf{i}}) = k \right\}, \quad r_k^{\mathbf{i}} = \inf\left\{ j : \sharp_0 \mathbf{i}(n_j^{\mathbf{i}}) = k \right\}. \tag{3.2}$$

for $k \in P_{\mathbf{i}}$. We note that $P_{\mathbf{i}}$ can be finite or infinite, and if $P_{\mathbf{i}}$ is finite then $m^{\mathbf{i}}_{\max P_{\mathbf{i}}}$ can be infinite. Therefore

$$\pi_c(\mathbf{i}) = \sum_{k \in P_{\mathbf{i}}} c^k d^{\mathbf{i}}_k(\lambda), \text{ where } d^{\mathbf{i}}_k(\lambda) = \sum_{l=r^{\mathbf{i}}_k}^{m^{\mathbf{i}}_k} \lambda^{n^{\mathbf{i}}_l}. \tag{3.3}$$

We call $n^{\mathbf{i}}_{m^{\mathbf{i}}_k}$ the degree and $n^{\mathbf{i}}_{r^{\mathbf{i}}_k}$ the lower degree of $d^{\mathbf{i}}_k(\lambda)$ denoted by deg and lowerdeg respectively.

Lemma 3.1. *For every* $\mathbf{i} \in \Sigma^{\mathbb{N}}$ *and* $\lambda \in (0, \frac{1}{2})$ *we have*

(1) $\mathrm{lowerdeg} d^{\mathbf{i}}_k(\lambda) \geq k$ *for every* $k \in P_{\mathbf{i}}$,
(2) $\deg d^{\mathbf{i}}_k(\lambda) + l - k + 1 \leq \mathrm{lowerdeg} d^{\mathbf{i}}_l(\lambda)$ *for every* $k, l \in P_{\mathbf{i}}$, $k < l$,
(3) $\lambda^{l-k} d^{\mathbf{i}}_k(\lambda) \geq d^{\mathbf{i}}_l(\lambda)$ *for every* $k, l \in P_{\mathbf{i}}$, $k < l$.

Proof. Since $\sharp_0 \mathbf{i}(k) \leq k$, (1) is obvious.

Let $k, l \in P_{\mathbf{i}}$, $k < l$. Since $\deg d^{\mathbf{i}}_k(\lambda) = n^{\mathbf{i}}_{m^{\mathbf{i}}_k}$ and $\mathrm{lowerdeg} d^{\mathbf{i}}_l(\lambda) = n^{\mathbf{i}}_{r^{\mathbf{i}}_l}$, between the $n^{\mathbf{i}}_{m^{\mathbf{i}}_k}$ and $n^{\mathbf{i}}_{r^{\mathbf{i}}_l}$ elements of $\mathbf{i}$ there have to be $l-k$ zeros. Therefore

$$l - k = \sharp_0 \mathbf{i}(n^{\mathbf{i}}_{m^{\mathbf{i}}_k}, n^{\mathbf{i}}_{r^{\mathbf{i}}_l}) \leq n^{\mathbf{i}}_{r^{\mathbf{i}}_l} - n^{\mathbf{i}}_{m^{\mathbf{i}}_k} - 1.$$

This completes the proof of (2). The property (3) is an easy consequence of (2) by using the fact $\lambda < \frac{1}{2}$. □

Now we are going to define the families Ψ^c_n of iterated function systems for which transversality holds. First of all, we define sets of finite length words of symbols Σ_i by induction. Let $\Sigma_1 = \{1, 2\}$ and for every $n \geq 1$ let

$$\Sigma_{n+1} = \bigcup_{\mathbf{i} \in \Sigma_n, i_0 \neq 1} \{0\mathbf{i}\} \cup \bigcup_{\mathbf{i} \in \Sigma_n} \{1\mathbf{i}, 2\mathbf{i}\}. \tag{3.4}$$

For example $\Sigma_2 = \{1, 2\}^2 \cup \{02\}$ and $\Sigma_3 = \{1, 2\}^3 \cup \{102, 202, 021, 022, 002\}$ etc. Obviously, for every $n \geq 1$, $\Sigma_n \subset \Sigma^n$.

Lemma 3.2. *For every* $n \geq 1$ *and for every* $\underline{i}, \underline{j} \in \Sigma_n$, $\underline{i} = \underline{j}$ *if and only if*

$$\alpha^{\underline{i}}_2 = \alpha^{\underline{j}}_2 \textit{ and } \forall 0 \leq l \leq \alpha^{\underline{i}}_2 = \alpha^{\underline{j}}_2,\ n^{\underline{i}}_l = n^{\underline{j}}_l \textit{ and } \sharp_0 \underline{i}(n^{\underline{i}}_l, n^{\underline{i}}_{l+1}) = \sharp_0 \underline{j}(n^{\underline{j}}_l, n^{\underline{j}}_{l+1}) \tag{3.5}$$

Proof. The implication $\left(\underline{i} = \underline{j}\right) \implies$ (3.5) is obvious for every $n \geq 1$. We are going to prove the other direction by induction.

For $n = 1$, (3.5) $\implies \left(\underline{i} = \underline{j}\right)$ is trivial. Let us suppose that Σ_n satisfies the statement and let $\underline{i}, \underline{j} \in \Sigma_{n+1}$ such that $\underline{i}, \underline{j}$ satisfy (3.5). Therefore $\underline{i} = 2\underline{i}'$ and

$\underline{j} = 2\underline{j}'$ or $\underline{i} = 0\underline{i}'$ or $1\underline{i}'$ and $\underline{j} = 0\underline{j}'$ or $1\underline{j}'$, where $\underline{i}', \underline{j}' \in \Sigma_n$. In the first case by using the inductional assumption we have $\underline{i} = \underline{j}$.

In the second case we will show that $\underline{i} = 0\underline{i}'$ if and only if $\underline{j} = 0\underline{j}'$. Let us suppose that $\underline{i} = 0\underline{i}'$ then

$$\sharp_0 \underline{j}(n_1^{\underline{j}}) = \sharp_0 \underline{i}(n_1^{\underline{i}}) = n_1^{\underline{i}} = n_1^{\underline{j}}.$$

In the middle equation we used (3.4). Therefore $\underline{j} = 0\underline{j}'$. The reversed direction is similar. By using the inductional assumption we have $\underline{i} = \underline{j}$. □

Lemma 3.3. *For every arbitrary small $\varepsilon > 0$ and for every $n \geq 2$ the system $\Psi_n^c = \left\{\psi_{\underline{i}}^c\right\}_{\underline{i}\in\Sigma_n}$ satisfies the transversality condition on $c \in (\varepsilon, 1-\varepsilon)$.*

Proof. We note that $\lambda \in (0, \frac{1}{2})$ is fixed. Let $\varepsilon > 0$ be arbitrary small but fixed. We are going to prove transversality of Ψ_n^c by using Lemma 2.3. More precisely, we are going to show that (2.2) holds on $U = (\frac{\varepsilon}{2}, 1)$.

Let us suppose that $c \in (\frac{\varepsilon}{2}, 1)$. Let $\widetilde{\mathbf{i}}, \widetilde{\mathbf{j}} \in \Sigma_n^{\mathbb{N}}$, such that $\widetilde{\mathbf{i}} = (\underline{i}_0\underline{i}_1\underline{i}_2\dots)$, $\widetilde{\mathbf{j}} = (\underline{j}_0\underline{j}_1\underline{j}_2\dots)$ and $\underline{i}_0 \neq \underline{j}_0$. We can define the natural projection of $\Psi_n^c = \left\{\psi_{\underline{i}}^c\right\}_{\underline{i}\in\Sigma_n}$ in similar way as in (2.1). Denote the natural projection of Ψ_n^c by $\widetilde{\pi}_c^n$.

Let us assume that $\widetilde{\pi}_{c_0}^n(\widetilde{\mathbf{i}}) = \widetilde{\pi}_{c_0}^n(\widetilde{\mathbf{j}})$ for a $c_0 \in (\frac{\varepsilon}{2}, 1)$. Let $\mathbf{i} = \widetilde{\mathbf{i}}$, $\mathbf{j} = \widetilde{\mathbf{j}}$ as elements of $\Sigma^{\mathbb{N}}$. Therefore $\widetilde{\pi}_c^n(\widetilde{\mathbf{i}}) = \pi_c(\mathbf{i})$ and $\widetilde{\pi}_c^n(\widetilde{\mathbf{j}}) = \pi_c(\mathbf{j})$.

If $\min P_{\mathbf{i}} = \min P_{\mathbf{j}}$ then there is the same number of zeros before the $n_1^{\mathbf{i}}$th element of $\mathbf{i}$ and before $n_1^{\mathbf{j}}$th element of $\mathbf{j}$. If $n_1^{\mathbf{i}} > n_1^{\mathbf{j}}$ then some simple algebra, using that $\lambda < \frac{1}{2}$, shows that $\pi_{c_0}(\mathbf{i}) < \pi_{c_0}(\mathbf{j})$, which is a contradiction. Likewise, we cannot have $n_1^{\mathbf{i}} < n_1^{\mathbf{j}}$, so necessarily $n_1^{\mathbf{i}} = n_1^{\mathbf{j}}$.

However, if $n_1^{\mathbf{i}} = n_1^{\mathbf{j}}$, then we have that

$$\pi_{c_0}(\mathbf{i}) - \pi_{c_0}(\mathbf{j}) = c^{\min P_{\mathbf{i}}} \lambda^{n_1^{\mathbf{i}}} \left(\pi_{c_0}(\sigma^{n_1^{\mathbf{j}}+1}\mathbf{i}) - \pi_{c_0}(\sigma^{n_1^{\mathbf{j}}+1}\mathbf{j})\right),$$

where σ is the left-shift operator of $\Sigma^{\mathbb{N}} = \{0,1,2\}^{\mathbb{N}}$.

Since we supposed that $\underline{i}_0 \neq \underline{j}_0 \in \Sigma_n$, by Lemma 3.2 and $c > \frac{\varepsilon}{2}$ we can assume without loss of generality that $\min P_{\mathbf{i}} > \min P_{\mathbf{j}}$. Let us denote $l_{\mathbf{j}} = \min P_{\mathbf{j}}$. Therefore

$$f_{\mathbf{j},\mathbf{i}}(c) = \widetilde{\pi}_c^n(\widetilde{\mathbf{j}}) - \widetilde{\pi}_c^n(\widetilde{\mathbf{i}}) = \pi_c(\mathbf{j}) - \pi_c(\mathbf{i}) = c^{l_{\mathbf{j}}} d_{l_{\mathbf{j}}}^{\mathbf{j}}(\lambda) + \sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}} c^k d_k^{\mathbf{j}}(\lambda) - \sum_{k\in P_{\mathbf{i}}} c^k d_k^{\mathbf{i}}(\lambda) =$$

$$c^{l_{\mathbf{j}}} d_{l_{\mathbf{j}}}^{\mathbf{j}}(\lambda) \left(1 + \sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}} c^{k-l_{\mathbf{j}}} \frac{d_k^{\mathbf{j}}(\lambda)}{d_{l_{\mathbf{j}}}^{\mathbf{j}}(\lambda)} - \sum_{k\in P_{\mathbf{i}}} c^{k-l_{\mathbf{j}}} \frac{d_k^{\mathbf{i}}(\lambda)}{d_{l_{\mathbf{j}}}^{\mathbf{j}}(\lambda)}\right).$$

Let us define

$$g_{\mathbf{j},\mathbf{i}}(c)=1+\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}c^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{j}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}-\sum_{k\in P_{\mathbf{i}}}c^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{i}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}.$$

Note that, since $c>\frac{\varepsilon}{2}>0$ one has $f_{\mathbf{j},\mathbf{i}}(c_0)=0$ if and only if $g_{\mathbf{j},\mathbf{i}}(c_0)=0$. On the other hand, it is easy to see that if $g_{\mathbf{j},\mathbf{i}}(c_0)=0$, then

$$\frac{df_{\mathbf{j},\mathbf{i}}}{dc}(c_0)=0\Leftrightarrow\frac{dg_{\mathbf{j},\mathbf{i}}}{dc}(c_0)=0.$$

Therefore it is enough to prove

$$\frac{dg_{\mathbf{j},\mathbf{i}}}{dc}(c)=0\Rightarrow g_{\mathbf{j},\mathbf{i}}(c)>0.$$

Let us suppose that $\frac{dg_{\mathbf{j},\mathbf{i}}}{dc}(c_0)=0$ for some $c_0\in(\frac{\varepsilon}{2},1)$. Then

$$0=c_0\frac{dg_{\mathbf{j},\mathbf{i}}}{dc}(c_0)=c_0\left(\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}(k-l_{\mathbf{j}})\,c_0^{k-l_{\mathbf{j}}-1}\frac{d^{\mathbf{j}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}-\sum_{k\in P_{\mathbf{i}}}(k-l_{\mathbf{j}})\,c_0^{k-l_{\mathbf{j}}-1}\frac{d^{\mathbf{i}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}\right)\leq$$

$$\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}(k-l_{\mathbf{j}})\,c_0^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{j}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}-\sum_{k\in P_{\mathbf{i}}}c_0^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{i}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}=$$

$$\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}(k-l_{\mathbf{j}}-1)\,c_0^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{j}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}+\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}c_0^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{j}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}-\sum_{k\in P_{\mathbf{i}}}c_0^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{i}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}.$$

By using (3) of Lemma 3.1 we have

$$0\leq\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}(k-l_{\mathbf{j}}-1)\,c_0^{k-l_{\mathbf{j}}}\lambda^{k-l_{\mathbf{j}}}+\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}c_0^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{j}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}-\sum_{k\in P_{\mathbf{i}}}c_0^{k-l_{\mathbf{j}}}\frac{d^{\mathbf{i}}_k(\lambda)}{d^{\mathbf{j}}_{l_{\mathbf{j}}}(\lambda)}$$

We can give an upper bound for the first term,

$$\sum_{k\in P_{\mathbf{j}}\setminus\{l_{\mathbf{j}}\}}(k-l_{\mathbf{j}}-1)\,c_0^{k-l_{\mathbf{j}}}\lambda^{k-l_{\mathbf{j}}}\leq\sum_{k=1}^{\infty}(k-1)c_0^k\lambda^k=\frac{(c_0\lambda)^2}{(1-c_0\lambda)^2}<1.$$

In the last inequality we used that $\lambda<\frac{1}{2}$. Therefore by the definition of $g_{\mathbf{j},\mathbf{i}}(c)$ we have that $0<g_{\mathbf{j},\mathbf{i}}(c_0)$.

Using Lemma 2.3 for $U=(\frac{\varepsilon}{2},1)$ we have that Ψ^c_n satisfies transversality on $V=(\varepsilon,1-\varepsilon)$. □

Let us denote the attractor of $\Psi^c_n=\left\{\psi^c_{\underline{i}}\right\}_{\underline{i}\in\Sigma_n}$ by $\Lambda^{c\lambda,\lambda}_{\Sigma_n}$.

Proposition 3.4. *For every* $0 < \lambda < \frac{1}{2}$ *and Lebesgue almost every* $0 < c < 1$

$$\lim_{n\to\infty} \dim_H \Lambda^{c\lambda,\lambda}_{\Sigma_n} = \dim_H \Lambda^{c\lambda,\lambda}.$$

Proof. First we note that by using the Lemma 3.3 and Theorem 2.2 we have for every $\varepsilon > 0$, $n \geq 1$ and almost every $\varepsilon < c < 1 - \varepsilon$

$$\dim_H \Lambda^{c\lambda,\lambda}_{\Sigma_n} = \min\left\{1, s^{c,\lambda}_n\right\} \tag{3.6}$$

where $s^{c,\lambda}_n$ is the unique solution of

$$\sum_{\underline{i}\in\Sigma_n} \left(c^{\sharp_0 \underline{i}} \lambda^n\right)^s = 1.$$

Since for every $\varepsilon > 0$ the dimension formula (3.6) holds for a.e. $c \in (\varepsilon, 1 - \varepsilon)$, it holds for a.e. $c \in (0, 1)$. We note that $s^{c,\lambda}_n$ is a bounded, monotone increasing sequence, therefore it is convergent. Let $s^*_{c,\lambda}$ be the limit of this sequence.

The lower bound is trivial since for every n, $\Lambda^{c\lambda,\lambda}_{\Sigma_n} \subseteq \Lambda^{c\lambda,\lambda}$. Therefore

$$\min\left\{s^*_{c,\lambda}, 1\right\} \leq \dim_H \Lambda^{c\lambda,\lambda}.$$

for every $0 < \lambda < \frac{1}{2}$ and Lebesgue almost every $0 < c < 1$.

Now we prove the upper bound. It is easy to see that the interval $\left[0, \frac{1}{1-\lambda}\right]$ is the convex hull of $\Lambda^{c\lambda,\lambda}$. By using the fact

$$\psi^c_0 \circ \psi^c_1(x) \equiv \psi^c_1 \circ \psi^c_0(x)$$

and $0 < c < 1$ we have that for every $\underline{i} \in \Sigma^n$ there exists $\underline{j} \in \Sigma_n$ such that

$$\psi^c_{\underline{i}}\left(\left[0, \frac{1}{1-\lambda}\right]\right) \subseteq \psi^c_{\underline{j}}\left(\left[0, \frac{1}{1-\lambda}\right]\right).$$

We note how to find such $\underline{j} \in \Sigma_n$. The positions of the 2's in $\underline{j}$ are the same as in $\underline{i}$. Before the first appearance of 2 and between any two consecutive appearances of 2, we keep the same number of 0's as in $\underline{i}$, but change the order so that all 1's come before all 0's. After the last occurrence of a 2 (or everywhere if there are no 2's in $\underline{i}$), we replace all 0's by 1's.

For each $n \in \mathbb{N}$, we consider the covering of $\Lambda^{c\lambda,\lambda}$ given by $\left\{\psi^c_{\underline{i}}\left(\left[0, \frac{1}{1-\lambda}\right]\right)\right\}_{\underline{i}\in\Sigma_n}$, and note that the diameters of sets in the cover are at most λ^n. Therefore by using the definition of Hausdorff measure (see [1]) we have

$$\mathcal{H}^s_{\lambda^n}(\Lambda^{c\lambda,\lambda}) \leq \sum_{\underline{i}\in\Sigma_n} \left(\frac{c^{\sharp_0 \underline{i}} \lambda^n}{1-\lambda}\right)^s,$$

where $\mathcal{H}^s_\delta(\Lambda) = \inf\left\{\sum_i |U_i|^s : \Lambda \subset \bigcup_i U_i, |U_i| < \delta\right\}$. Let $\varepsilon > 0$ be arbitrary small. Then

$$\mathcal{H}^{s^*_{c,\lambda}+\varepsilon}_{\lambda^n}(\Lambda^{c\lambda,\lambda}) \leq \sum_{\underline{i}\in\Sigma_n} \left(\frac{c^{\sharp_0 \underline{i}}\lambda^n}{1-\lambda}\right)^{s^*_{c,\lambda}+\varepsilon} \leq \lambda^{n\varepsilon}\left(\frac{1}{1-\lambda}\right)^{s^*_{c,\lambda}+\varepsilon}$$

which tends to 0 as $n \to \infty$. Therefore by the definition of Hausdorff dimension (see [1])

$$\dim_H \Lambda^{c\lambda,\lambda} \leq s^*_{c,\lambda} + \varepsilon$$

where $\varepsilon > 0$ was arbitrary. Thus the proposition is proved. □

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\widetilde{\Sigma}_n$ the following set of symbols

$$\widetilde{\Sigma}_n = \{1, 2, 02, 002, \dots, \overbrace{0\dots0}^{n-1}2\}. \tag{3.7}$$

Let $\Lambda^{c\lambda,\lambda}_{\widetilde{\Sigma}_n}$ the attractor of $\left\{\psi^c_{\underline{i}}\right\}_{\underline{i}\in\widetilde{\Sigma}_n}$. Notice that for every $\underline{i} \in \Sigma_n$ can be decomposed as a juxtaposition $\underline{i} = \underline{j}_1 \cdots \underline{j}_k$, where each $\underline{j}_r$ is in $\widetilde{\Sigma}_n$. Therefore for every $0 < \lambda < \frac{1}{2}$ and Lebesgue almost every $0 < c < 1$, $\Lambda^{c\lambda,\lambda}_{\Sigma_n} \subseteq \Lambda^{c\lambda,\lambda}_{\widetilde{\Sigma}_n}$, and therefore by using Proposition 3.4,

$$\dim_H \Lambda^{c\lambda,\lambda} = \lim_{n\to\infty} \dim_H \Lambda^{c\lambda,\lambda}_{\Sigma_n} \leq \lim_{n\to\infty} \dim_H \Lambda^{c\lambda,\lambda}_{\widetilde{\Sigma}_n}.$$

The lower bound is trivial, therefore

$$\dim_H \Lambda^{c\lambda,\lambda} = \lim_{n\to\infty} \dim_H \Lambda^{c\lambda,\lambda}_{\widetilde{\Sigma}_n}. \tag{3.8}$$

We use the fact that for every $n \geq 2$, $\left\{\psi^c_{\underline{i}}\right\}_{\underline{i}\in\widetilde{\Sigma}_n}$ satisfies the transversality condition on $(\varepsilon, 1-\varepsilon)$ for all $\varepsilon > 0$. As the proof of this claim is very similar to the proof of Lemma 3.3, we omit it. By Theorem 2.2 and by similar arguments as in the beginning of the proof of Proposition 3.4, we have for every $n \geq 1$ and almost every $0 < c < 1$

$$\dim_H \Lambda^{c\lambda,\lambda}_{\widetilde{\Sigma}_n} = \min\left\{1, \widetilde{s}^{c,\lambda}_n\right\}$$

where $\widetilde{s}^{c,\lambda}_n$ is the unique solution of

$$2\lambda^s + \sum_{k=1}^{n-1}\left(c^k\lambda^{k+1}\right)^s = 1.$$

It is easy to see by (3.8) that $s^{c\lambda,\lambda} = \lim_{n\to\infty} \widetilde{s}_n^{c,\lambda}$ is the unique solution of

$$2\lambda^s + \sum_{k=1}^{\infty}\left(c^k\lambda^{k+1}\right)^s = 1. \tag{3.9}$$

We note that the function $f_1(s) = 2\lambda^s + \frac{\lambda^s\gamma^s}{1-\gamma^s}$ is strictly monotone increasing for every $\gamma, \lambda \in (0,1)$, moreover $\lim_{s\to 0+} f_1(s) = +\infty$ and $\lim_{s\to\infty} f_1(s) = 0$. Therefore the equation (3.9) has a unique solution, which also satisfies the following equation:

$$2\lambda^s + (c\lambda)^s - (c\lambda^2)^s = 1. \tag{3.10}$$

By similar arguments one can prove that (3.10) has unique solution as well, which was the first statement of Theorem 1.1.

Now we prove the measure claim of Theorem 1.1. If $s^{c\lambda,\lambda} > 1$, then $\widetilde{s}_n^{c,\lambda} > 1$ for large enough n, so that, by transversality, $\widetilde{\Lambda}_{\widetilde{\Sigma}_n}^{c\lambda,\lambda}$ has positive Lebesgue measure for almost every c. Since $\widetilde{\Lambda}_{\widetilde{\Sigma}_n}^{c\lambda,\lambda} \subset \Lambda^{c\lambda,\lambda}$, this completes the proof of Theorem 1.1. □

Finally, we are proving Proposition 1.2.

Proof of Proposition 1.2. Let q, p integers such that $(q,p) = 1$ and $q > p$ and let $\left\{\psi_0(x) = \lambda^{\frac{q}{p}}x, \psi_1(x) = \lambda x, \psi_2(x) = \lambda x + 1\right\}$. It is easy to see that

$$\psi_{01}(x) \equiv \psi_{10}(x), \qquad \psi_{\underbrace{0\dots0}_{p}}(x) \equiv \psi_{\underbrace{1\dots1}_{q}}(x).$$

Therefore if we have $\mathbf{i} \in \Sigma^{\mathbb{N}} = \{0,1,2\}^{\mathbb{N}}$ then we can choose $\mathbf{j} \in \widetilde{\Sigma}_p^{\mathbb{N}}$ ($\widetilde{\Sigma}_p$ is defined as in (3.7)) such that

$$\pi(\mathbf{i}) = \pi(\mathbf{j}).$$

Since, whenever there are at least p consecutive zeros in $\mathbf{i}$, we can replace each block of p consecutive zeros by blocks of q consecutive ones, and then we can rearrange the zeros and ones between two consecutive twos, by moving the ones to the front. Therefore

$$\dim_H \Lambda^{\lambda,q,p} = \dim_H \Lambda_{\widetilde{\Sigma}_p}^{\lambda,q,p}$$

where $\Lambda_{\widetilde{\Sigma}_p}^{\lambda,q,p}$ is the attractor of IFS $\left\{\psi_{\underline{i}}\right\}_{\underline{i}\in\widetilde{\Sigma}_p}$. Since the Hausdorff dimension of a self-similar set is always at most the minimum of the similarity dimension (see (1.1)) and the dimension of ambient space, we have

$$\dim_H \Lambda_{\widetilde{\Sigma}_p}^{\lambda,q,p} \le \min\left\{1, s_{q,p}^{\lambda}\right\}$$

where $s_{q,p}^{\lambda}$ is the unique solution of

$$2\lambda^s + \sum_{k=1}^{p-1} \lambda^{\left(\frac{q}{p}k+1\right)s} = 1,$$

which was to be proved. □

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