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Dimension Theory of Non-conformal Attractors and Overlapping Self-similar Sets

PhD Thesis

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Introduction

In this thesis we investigate some properties of self-similar sets and self-affine sets. Especially, we focus on the dimension theory of fractals generated by iterated function systems (IFS).

More precisely, let $\Phi = \{f_1, \dots, f_n\}$ be a set of contracting functions (that is, $\|D_{\underline{x}} f\| < 1$) of $\mathbb{R}^d$ mapping an open bounded set U into itself. Then it is well known (see [H]), that there exists a unique, non-empty, compact subset Λ of $\mathbb{R}^d$ such that

$$\Lambda = \bigcap_{k=1}^{\infty} \bigcup_{i_1,\dots,i_k=1}^{n} f_{i_1} \circ \cdots f_{i_k}(U) \text{ and } \Lambda = \bigcup_{i=1}^{n} f_i(\Lambda).$$

We call the set Λ as the attractor of the iterated function system Φ.

One of the important properties of these sets is the dimension. In this thesis we mainly focus on the so-called *Minkowski dimension* (or box dimension) and *Hausdorff dimension*. We denote the Hausdorff dimension (and respectively the box dimension) of the set Λ by $\dim_H \Lambda$ ($\dim_B \Lambda$). Moreover, let us denote the s-dimensional Hausdorff measure by $\mathcal{H}^s$. For the definition and basic properties of the Hausdorff and box dimension and the Hausdorff measure we refer to [Fa1, Fa2].

The simplest case is when the functions are contracting similarities

$$\Phi = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^{n}$$

on the real line. In that case we call the attractor of Φ *self-similar set*. Then the non-trivial upper on the Hausdorff and box dimension of the attractor is the similarity dimension which is defined as the unique solution of

$$\sum_{i=1}^{n} \lambda_i^s = 1.$$

The dimension theory of self-similar sets is quite well understood in the cases when some separation conditions hold. Hutchinson proved that whenever the cylinders $\{f_i(\Lambda)\}_{i=1}^{n}$ are well separated, more precisely, the *open set* *condition* (OSC) holds (there exists an open, bounded subset U of $\mathbb{R}$ such that $f_i(U) \subset U$ for every i and $f_i(U) \cap f_j(U) = \emptyset$ if $i \neq j$) then the similarity dimension is equal to the Hausdorff dimension, see [H]. The box dimension is equal to the Hausdorff dimension independently of separation conditions, see [Fa5].

However, in case of heavy overlaps in between the cylinders we know very little about the structure of attractor Λ. To study such kind of Iterated Function Systems there are two known methods:

- Instead of an individual IFS we consider a one-parameter family of IFS and we use the so-called *transversality condition* introduced by Pollicott Simon [PoSi] (see Section 1.2). See [PeSo1], [PeSo2] for the most general treatment of this method. In this thesis we use this approach.

- In some very particular cases we can apply the so-called *Weak Separation Condition* [Ze], [LNR], [NW1] or some variants of it. With this method we can handle IFS like $\left\{f_i(x) = \frac{1}{N}x + t_i\right\}_{i=1}^{m}$, where $N, t_i \in \mathbb{Z}$.

In particular, when some of the maps of the IFS have common fixed points then non of the known methods can be applied directly. One of the most important novelties of this thesis is to handle the cases of non-distinct fixed points.

The simplest situation when two maps share the same fixed point was considered in [B3]. More precisely, in [B3] we considered the IFS $\{\gamma x, \lambda x, \lambda x + 1\}$ and its attractor Λ on the real line, where $\gamma < \lambda$. Let $I = [0, \frac{1}{1-\lambda}]$ be the convex hull of the attractor Λ. See Figure 1 for the image of I by the functions of this IFS. The problem of calculating the dimension was raised by Pablo Shmerkin at the conference in Greifswald in 2008. The novelty of the result obtained in [B3] about the dimension of Λ was to tackle the difficulty which comes from the fact that the first two maps have the same fixed point.

In *Chapter 1* we study two types of self-similar iterated function systems with non-distinct fixed points. In both of the cases we assume that *the images of the convex hull of the attractor are overlapping only for the functions which share the same fixed point*. In the first case we suppose that every fixed point belongs to at most two functions. For an example of such type of IFS see Figure 2. Our assumption in the second case is that there are exactly two different fixed points but a fixed point belongs to arbitrary many functions. For an example see Figure 3.

For both of the cases we calculate the Hausdorff and box dimension for almost every contracting parameters. Moreover, for the case in Figure 2 we calculate that the proper dimensional Hausdorff measure of the attractor is zero. For precise details see Section 1.1. Chapter 1 is based on [B1] and [B2].

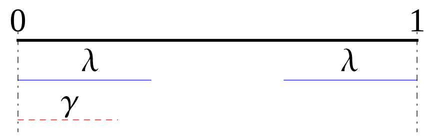

Figure 1: The simplest example of IFS with some of the functions share the same fixed point, considered in [B3].

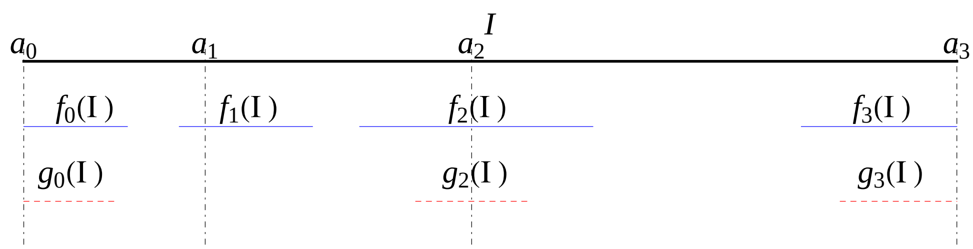

Figure 2: Images of the convex hull of the attractor of IFS $\{f_0, g_0, f_1, f_2, g_2, f_3, g_3\}$, where $a_0 = \mathrm{Fix}(f_0) = \mathrm{Fix}(g_0)$, $a_1 = \mathrm{Fix}(f_1)$, $a_2 = \mathrm{Fix}(f_2) = \mathrm{Fix}(g_2)$ and $a_3 = \mathrm{Fix}(f_3) = \mathrm{Fix}(g_3)$

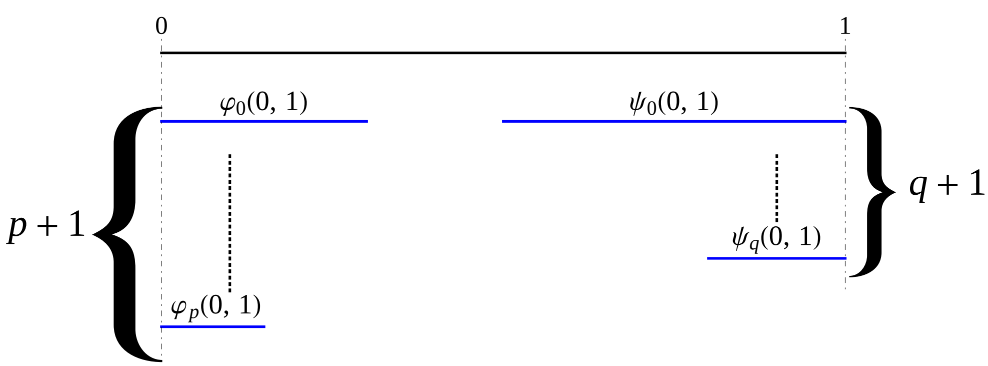

Figure 3: Images of the convex hull of the attractor of IFS $\{\phi_i\}_{i=0}^{p} \cup \{\psi_j\}_{j=0}^{q}$ where $\mathrm{Fix}(\phi_i) = 0$ and $\mathrm{Fix}(\psi_j) = 1$ for every i, j.

In the last two decades considerable attention has been paid to the dimension theory of non-conformal sets. We call a set Λ *conformal* if it is an attractor of an IFS containing $C^{1+\alpha}$ conformal homeomorphisms, where we call a function conformal if its derivative is a similarity transformation at every point. The dimension theory of conformal attractors is very closely related to the dimension theory of self-similar sets.

The dimension theory of non-conformal IFS is very difficult and there are only very few results. The most important tool of this field is *the sub-additive pressure*, which was defined by K. Falconer [Fa4] and L. Barreira [Barr]. (For the precise definition of sub-additive pressure, see Section 2.1.) Unfortunately, we know very little about sub-additive pressure itself.

The simplest non-conformal situation is the case of self-affine sets. A set $\Lambda \subset \mathbb{R}^d$ is called self-affine if it is an attractor of an IFS containing contracting affine maps $\{f_i(x) = A_i x + a_i\}_{i=1}^m$, where A_i are $d \times d$ real matrices. The dimension theory of self-affine sets is far from well understood even in the diagonal case. That is, when all A_i are diagonal matrices.

To study the dimension of a self-affine attractor we consider the k-th approximation of the attractor with the so called k-th cylinders which are naturally defined by the k fold application of the functions of the IFS. To measure the contribution of such a k cylinder to the covering sum which appears in the definition of the Hausdorff measure for each of these k-th cylinders we consider the singular value function. These are non-negative valued functions defined in a neighborhood of the attractor. The dimension of the attractor is related to the exponential growth rate of the sum of the values of these exponentially many singular value functions in the self affine case. Precisely, the Falconer Theorem (see [Fa6]) states that the Hausdorff- and box dimension of a self-affine attractor coincide for almost every translation parameters and equal to the singularity dimension, whenever the norm of all the affine maps of IFS is smaller than 1/3. This bound was improved to 1/2 by Solomyak in [So1]. To verify this it was essential that the exponential growth rate is the same wherever we evaluate these singular value functions, since the singular value functions are constant in the self-affine case.

Falconer [Fa4] and Barreira [Barr] considered the situation when the IFS is no longer self-affine. They introduced a technical condition named 1-bunched property, which implies that the cylinder sets in each iteration are convex. In this case, it turns out that the exponential growth rate of the sum of the value of the singular value functions does not depend on wherever they are evaluated. We express this phenomenon as the "insensitivity property holds". This is a very important property of the sub-additive pressure and in general we do not know if it holds or not.

The main goal of *Chapter 2* is to verify this property in a special case when the 1-bunched property does not hold but the IFS consists of maps with lower triangular derivative matrices. This result is a generalization of the result of K. Simon and A. Manning [MS2]. They proved the same assertion on the real plane.

Even if the 1-bunched condition is not satisfied, Zhang [Zh] found that the zero of the sub-additive pressure is an upper bound for the Hausdorff dimension. As an application, we supply two examples of such IFS for which we are able to calculate the Hausdorff dimension using that the insensibility property holds.

The main theorem of the chapter can be also considered as a generalization of a recent paper by K. Falconer and J. Miao [FM]. They gave a formula to estimate the Hausdorff dimension of self-affine fractals generated by upper-triangular matrices. We will show a formula to estimate the sub-additive pressure in the non-conformal case and we will prove that the sub-additive pressure depends only on the diagonal elements of the derivative matrices in the case when the derivative matrices are triangular. Chapter 2 is based on [B4] and take a part of the author's Master Thesis.

In *Chapter 3* we focus on a special family of self-affine sets which is called the generalized four corner set $\Lambda(\underline{\alpha}, \underline{\beta})$ on the real plane. The generalized 4-corner set is the attractor of the self-affine iterated function system (IFS) of Figure 4. (The precise definition will be given in Section 3.1.) The parameters $\underline{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ and $\underline{\beta} = (\beta_0, \beta_1, \beta_2, \beta_3)$ are chosen such that the rectangles R_0, R_1, R_2, R_3 on Figure 4 are disjoint. One of the main goals of the chapter is to determine the box dimension of this set for Lebesgue typical parameters.

We will prove that for Lebesgue-typical parameters $\underline{\alpha}, \underline{\beta}$ the Hausdorff dimension and even the box dimension of the generalized 4-corner set is strictly smaller than the singularity dimension. The reason of this phenomena is the very special relative geometric position of the rectangles which generate the generalized 4-corner set. The speciality of the maps is that the fixed points are the corners of the unit square, so they do not move when we change the parameters $\underline{\alpha}, \underline{\beta}$. Therefore the orthogonal projection to the x-axis (and to the y-axis respectively) is an attractor of a special iterated function system of four similarities where the similarities derived from the maps having fixed points with same coordinate y (and with same coordinate x) have common fixed points. Applying the results of Chapter 1 we are able to handle this difficulty. Chapter 3 is based on [B1].

In *Chapter 4* we study the dimension theory of the slices of the Sierpiński gasket. In particular, we describe the multifractal analysis of the size of the slices which correspond to a countable dense set of angles. We recall that the

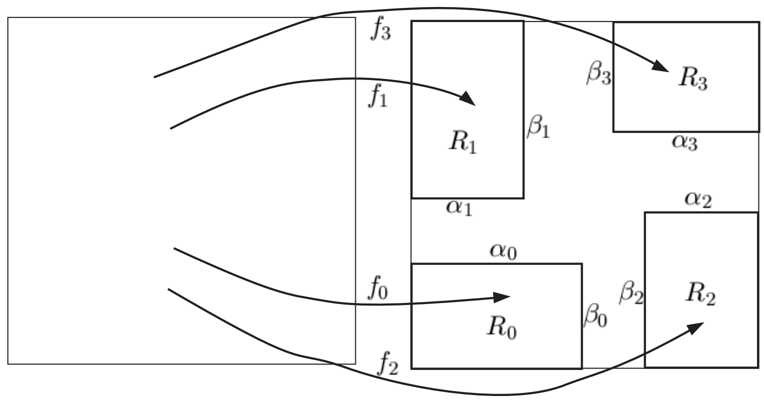

Figure 4: Maps of the generalized 4-corner set.

Sierpiński gasket is the attractor of the IFS $\left\{\frac{1}{2}\underline{x}, \frac{1}{2}\underline{x}+(\frac{1}{2},0), \frac{1}{2}\underline{x}+(\frac{1}{4},\frac{\sqrt{3}}{4})\right\}$ on the real plane. Liu, Xi and Zhao showed a formula for the box and Hausdorff dimension of the intersections of the Sierpiński carpet with Lebesgue-typical planar lines of rational slopes and conjectured that this value is strictly less than the dimension of the Sierpiński carpet minus one (for precise details see [LXZ]). Manning and Simon verified the conjecture in [MS1] and proved a dimension conservation phenomena for the carpet (see [MS1, Theorem 9] and [MS1, Proposition 4]).

One of the main goals of this chapter is to prove that both of the theorems are valid for the Sierpiński gasket (for precise details see Section 4.1). Moreover, respectively to the natural self-similar measure, we prove that the dimension of the typical slices is strictly greater than $\frac{\log 3}{\log 2}-1$ for rational slopes, were $\frac{\log 3}{\log 2}$ is the Hausdorff dimension of the Sierpiński gasket. We recall the definition of the *self-similar measure*. Let $\{f_1,\dots,f_n\}$ be an IFS (not necessarily self-similar) and let $(p_1,\dots,p_n)$ be a probability vector. Then there exists a unique Borel regular probability measure μ such that

$$\mu=\sum_{i=1}^{n}p_i\mu\circ f_i^{-1},$$

see [H]. We call the measure μ self-similar if the corresponding IFS is self-similar. If the self-similar IFS satisfies the OSC then the proper dimensional Hausdorff measure restricted and normalized to the attractor is also a self-similar measure and we call it as the *natural self-similar measure*.

In [Fur], Furstenberg introduced and proved a dimension conservation formula for homogeneous fractals (for example homotheticly self-similar sets). Denote proj_θ the θ-angle projection from the real plane into the y-axis, then for any self-similar set Λ there exists a $\delta \geq 0$ such that

$$\delta + \dim_H \left\{ x \in \mathrm{proj}_\theta \Lambda : \dim_H \mathrm{proj}_\theta^{-1}(x) \cap \Lambda \geq \delta \right\} = \dim_H \Lambda.$$

We describe the multifractal analysis of the slices, we will give a formula for the function

$$\Gamma : \delta \mapsto \dim_H \left\{ x \in \mathrm{proj}_\theta \Lambda : \dim_H \mathrm{proj}_\theta^{-1}(x) \cap \Lambda \geq \delta \right\}$$

in the case when Λ is the Sierpiński gasket and $\tan\theta$ is rational.

Chapter 4 is based on [BFS] which is a joint work with Andrew Ferguson and Károly Simon.

Finally, in *Chapter 5* we investigate some properties of the invariant measure of iterated function systems with random perturbations.

For an IFS $\{f_i\}_{i=1}^n$ the natural coding of the elements of its attractor Λ by the elements of $\Sigma = \{1, \ldots, n\}^{\mathbb{N}}$ is called the natural projection π and then $\pi : \Sigma \mapsto \Lambda$. Let $\mu = (p_1, \ldots, p_n)^{\mathbb{N}}$ be a Bernoulli measure on the space Σ. Let $h = -\sum_{i=1}^n p_i \log p_i$ be the entropy of the left-shift operator with respect to the Bernoulli measure μ. Denote by ν the push-down measure of μ, that is $\nu = \mu \circ \pi^{-1}$. It was proved in [BNS], for non-linear, contracting on average, iterated function systems (and later extended in [FST]) that

$$\dim_H(\nu) \leq \frac{h}{|\chi|},$$

where $\dim_{\mathrm{H}}(\nu)$ is the Hausdorff dimension of the measure ν and χ is the Lyapunov exponent of the IFS associated to the Bernoulli measure μ.

One can expect that, at least "typically", the measure ν is absolutely continuous when $h/|\chi| > 1$. Essentially the only known approach to this is transversality. For example, in the linear case with uniform contraction ratios, see [PeSc] and [PeSo2]. In the linear case for non-uniform contraction ratios, see [N] and [NW2]. In the non-linear case, see for example [SSU2]. We note that there is another direction in the study of iterated function systems with overlaps, which is concerned with concrete, but not-typical systems, often of arithmetic nature, for which there is a dimension drop, see for example [LNR].

In the last chapter, we are interested in studying absolute continuity with L^2 density. We will study a modification of the problem, namely we consider a random perturbation of the functions. The linear case was studied by Peres, Simon and Solomyak in [PSS1]. They proved absolute continuity for random linear IFS, with non-uniform contraction ratios and also L^2 and continuous density in the uniform case. We would like to extend this result by proving L^2 density with non-uniform contraction ratios and in non-linear case.

Let Y_ε be uniformly distributed in $[1-\varepsilon, 1+\varepsilon]$ and let $f_i \in C^{1+\alpha}$ be contractions with fixed points a_i. We consider the iterated function system $\{Y_\varepsilon f_i + a_i(1-Y_\varepsilon)\}_{i=1}^n$, were each of the maps are chosen with probability p_i. We will prove that the invariant density is in L^2 and the L^2-norm does not grow faster than $1/\sqrt{\varepsilon}$, as ε vanishes.

Throughout the chapter we will use the method of [Per]. The proof relies on defining a piecewise hyperbolic dynamical system on the cube, with an SRB-measure with the property that its projection is the density of the iterated function system. Chapter 5 is based on [BP] which is a joint work with Tomas Persson.

Chapter 1

Hausdorff dimension of self-similar sets with heavy overlaps

1.1 Definitions and Statements

Throughout the chapter we study two families of self-similar iterated function systems. Firstly, we assume that exactly two different fixed points belong to the functions of the examined IFS. Precisely,

Principal Assumptions of Case A:

A1. Let $\mathcal{R}$ be a finite set of linear, real functions such that for every $\varphi \in \mathcal{R}$, $\mathrm{Fix}(\varphi) \in \{0, 1\}$ and $\varphi([0,1]) \subseteq [0,1]$.

A2. For arbitrary $\varphi_i, \varphi_j \in \mathcal{R}$ suppose either $\varphi_i([0,1]) \cap \varphi_j([0,1]) = \emptyset$ or $\mathrm{Fix}(\varphi_i) = \mathrm{Fix}(\varphi_j)$.

Theorem 1.1.1. *Let* $\mathcal{R} = \{\phi_{i,1}(x) = \gamma_{i,1}x\}_{i=0}^{p} \cup \{\phi_{i,2}(x) = \gamma_{i,2}x + (1-\gamma_{i,2})\}_{i=0}^{q}$ *such that* $0 < \gamma_{i,1} < \gamma_{0,1} < 1$ *for* $i = 1, \ldots, p$ *and* $0 < \gamma_{j,2} < \gamma_{0,2} < 1$ *for* $j = 1, \ldots, q$ *(see Figure 3 page 3), then*

$$\dim_B \Lambda = \dim_H \Lambda = \min\{1, s\}, \tag{1.1.1}$$

where s *is the unique solution of*

$$\prod_{i=0}^{p}(1-\gamma_{i,1}^{s}) + \prod_{i=0}^{q}(1-\gamma_{i,2}^{s}) = 1 \tag{1.1.2}$$

for Lebesgue almost every $(\underline{\gamma}_1, \underline{\gamma}_2) \in (0, \gamma_{0,1})^p \times (0, \gamma_{0,2})^q$*, where* $\underline{\gamma}_1 = (\gamma_{1,1}, \ldots, \gamma_{p,1})$ *and respectively* $\underline{\gamma}_2 = (\gamma_{1,2}, \ldots, \gamma_{q,2})$.

Moreover $\mathcal{L}(\Lambda) > 0$ *for Lebesgue almost every* $(\underline{\gamma}_1, \underline{\gamma}_2)$ *if* $s > 1$.

Note that whenever $\gamma_{0,1} + \gamma_{0,2} \geq 1$ the attractor of $\mathcal{R}$ is an interval which implies immediately Theorem 1.1.1. In this way without loss of generality we may assume that $\gamma_{0,1}+\gamma_{0,2} < 1$, which is equivalent to $\varphi_{0,1}([0,1]) \cap \varphi_{0,2}([0,1]) = \emptyset$. Then the IFS $\mathcal{R}$ satisfies obviously the assumptions (A1) and (A2). The proof of Theorem 1.1.1 is based on [B1].

On the other hand, we study the case when every fixed point belongs to at most two functions (see Figure 2, page 3). Precisely,

Principal Assumptions of Case B:

B1. $\mathcal{S} = \mathcal{F} \cup \mathcal{G}$

B2. $\mathcal{F} = \{f_i(x) = \lambda_i x + a_i(1-\lambda_i)\}_{i=0}^{N-1}$ where $0 < \lambda_i < 1$ and the fixed points satisfy: $a_0 < a_1 < \cdots < a_{N-1}$.

B3. Let $I = [a_0, a_{N-1}]$ (the convex hull of the attractor). We require that $f_{i-1}(I) < f_i(I)$ that is

$$f_{i-1}(a_{N-1}) < f_i(a_0) \text{ for every } i = 1, \ldots, N-1. \tag{1.1.3}$$

B4. $\mathcal{G} = \{g_i(x) = \beta_i x + a_i(1-\beta_i)\}_{i \in \mathcal{J}}$, where $\mathcal{J} \subseteq \{0, \ldots, N-1\}$ and $0 < \beta_i < \lambda_i$ for every $i \in \mathcal{J}$.

Observe that for every $i \in \mathcal{J}$, $\mathrm{Fix}(f_i) = \mathrm{Fix}(g_i) = a_i$.

Denote $\underline{\beta} \in (0,1)^{\sharp\mathcal{J}}$ the vector of contraction ratios of $\mathcal{G}$ and $\underline{\lambda} \in (0,1)^N$ the vector of contraction ratios of $\mathcal{F}$. Moreover, let $\underline{a} \in \mathbb{R}^N$ be the vector of fixed points and denote the attractor of $\mathcal{S}$ by Ω. For the simplicity we write $\mathcal{I} = \{0, \ldots, N-1\}$.

Theorem 1.1.2. *Let* $\mathcal{S}$ *be as in (B1)-(B4) then the attractor* Ω *of* $\mathcal{S}$ *satisfies that*

$$\dim_B \Omega = \dim_H \Omega = \min\{1, s\}, \tag{1.1.4}$$

where s *is the unique solution of*

$$\sum_{i=0}^{N-1} \lambda_i^s + \sum_{i\in\mathcal{J}} \beta_i^s - \sum_{i\in\mathcal{J}} \lambda_i^s \beta_i^s = 1, \tag{1.1.5}$$

for Lebesgue almost every $\underline{\beta}$ *in*

$$\left\{\underline{\beta} : 0 < \beta_i < \min\left\{\lambda_i, \frac{2}{(1+\sqrt{2})(\alpha_i^2\lambda_{\max}+2)}\right\}\right\}, \tag{1.1.6}$$

where $\lambda_{\max} = \max_i \{\lambda_i\}$ *and*

$$\alpha_i = \frac{\max\{a_{N-1} - a_i, a_i - a_0\}}{\min\{f_{i+1}(a_0) - a_i, a_i - f_{i-1}(a_{n-1})\}} \text{ for every } i \in \mathcal{I}.$$

Moreover $\mathcal{L}(\Omega) > 0$ *for Lebesgue almost every* $\underline{\beta}$ *such that* $\underline{\beta}$ *satisfies (1.1.6) and* $s > 1$.

In the proof of Theorem 1.1.2 we are going to show that s is always an upper bound for the Hausdorff and Box dimension. Moreover we will prove that the s dimensional Hausdorff measure of the attractor is zero.

Theorem 1.1.3. *Assume that* $\mathcal{S}$ *satisfies (B1)-(B4) and let* s *be the unique solution of (1.1.5) then*

$$\mathcal{H}^s(\Omega) = 0.$$

To prove Theorem 1.1.1 and Theorem 1.1.2, we are going to use the so-called transversality method. Note, that our original system does not satisfy the transversality condition (see later the precise arguments), but some well-chosen subsystems of the sufficiently high iterations do so. To verify this we use two methods of checking the transversality condition. One of them was introduced by Simon, Solomyak and Urbański [SSU1], [SSU2] and the other one is due to [PeSo1], [PeSo2]. For the convenience of the reader in Section 1.2 we summarize these methods.

There is a big difference between the structure of the two families of IFS, the chosen subsystems and the proofs of the transversality conditions are significantly different. Therefore we study them in two different sections. In Section 1.3 we prove Theorem 1.1.1 and in Section 1.4, Theorem 1.1.2. In both of the cases we construct the appropriate natural projections, the subsystems.

In Section 1.5 we prove Theorem 1.1.3. The method of the proof is similar to that of [PSS2, Theorem 1.1] obtained by a modification of the Brandt, Graf method [BG].

The results of the chapter are based on [B1] and [B2].

1.2 Transversality methods

First let us introduce the *transversality condition* for self-similar IFS on the real line with d dimensional parameter-space. The definition corresponds to the definition in [SSU1],[SSU2] which was introduced for much more general IFS.

Let U be an open, bounded subset of $\mathbb{R}^d$ with smooth boundary and $\mathcal{I}$ a finite set of symbols. Let $\Psi_{\underline{t}} = \left\{\psi_i^{\underline{t}}(x) = \lambda_i(\underline{t})x + d_i(\underline{t})\right\}_{i \in \mathcal{I}}$, where $\lambda_i, d_i \in C^1(\overline{U})$ and $0 < \alpha \leq \lambda_i(\underline{t}) \leq \beta < 1$ for every $i \in \mathcal{I}$ and $\underline{t} \in \overline{U}$ and for some $\alpha, \beta \in (0,1)$. Let $\Lambda^{\underline{t}}$ be the attractor of $\Psi_{\underline{t}}$ and $\pi_{\underline{t}}$ is the natural projection from the symbolic space $\Sigma = \mathcal{I}^{\mathbb{N}}$ to $\Lambda^{\underline{t}}$. More precisely, for $\mathbf{i} = (i_0 i_1 \dots) \in \Sigma$ we write

$$\pi_{\underline{t}}(\mathbf{i}) = \lim_{n \to \infty} \psi_{i_0}^{\underline{t}} \circ \psi_{i_1}^{\underline{t}} \circ \dots \circ \psi_{i_n}^{\underline{t}}(0). \tag{1.2.1}$$

It is well-known that the limit exists and independent of the base point 0. Moreover, $\pi_{\underline{t}}$ is a continuous, surjective function from Σ onto $\Lambda^{\underline{t}}$. Denote σ the left-shift operator on Σ. That is $\sigma : (i_0 i_1 \dots) \mapsto (i_1 i_2 \dots)$. It is easy to see that

$$\pi_{\underline{t}}(\mathbf{i}) = \psi_{i_0}^{\underline{t}}(\pi_{\underline{t}}(\sigma \mathbf{i})).$$

Definition 1.2.1. We say that $\Psi_{\underline{t}}$ satisfies the **transversality condition** on an open, bounded set $U \subset \mathbb{R}^d$, if for any $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$ there exists a constant $C = C(i_0, j_0)$ such that

$$\mathcal{L}_d(\underline{t} \in U : |\pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})| \leq r) \leq Cr \text{ for every } r > 0,$$

where $\mathcal{L}_d$ is the d dimensional Lebesgue measure.

In short, we say that there is transversality if the transversality condition holds. This definition is equivalent to the ones given in e.g. [SSU1], [SSU2]. As a special case of [SSU1, Theorem 3.1] we obtain:

Theorem 1.2.1 (Simon, Solomyak, Urbański)**.** *Suppose that $\Psi_{\underline{t}}$ satisfies the transversality condition on an open, bounded set $U \subset \mathbb{R}^d$. Then*

1. $\dim_H \Lambda^{\underline{t}} = \min\{s(\underline{t}), 1\}$ *for Lebesgue-a.e.* $\underline{t} \in U$,
2. $\mathcal{L}_1(\Lambda^{\underline{t}}) > 0$ *for Lebesgue-a.e.* $\underline{t} \in U$ *such that* $s(\underline{t}) > 1$,

where $s(\underline{t})$ is the similarity dimension of $\Psi_{\underline{t}}$. More precisely, $s(\underline{t})$ satisfies the equation

$$\sum_{i \in \mathcal{I}} \lambda_i(\underline{t})^{s(\underline{t})} = 1. \tag{1.2.2}$$

We can use the following Lemma to prove transversality which follows from [SSU1, Lemma 7.3].

Lemma 1.2.2. *Let $U \subset \mathbb{R}^d$ be an open, bounded set with smooth boundary and $f_{\mathbf{i},\mathbf{j}}(\underline{t}) = \pi_{\underline{t}}(\mathbf{i}) - \pi_{\underline{t}}(\mathbf{j})$. If for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$ and for every $\underline{t}_0 \in U$*

$$f_{\mathbf{i},\mathbf{j}}(\underline{t}_0) = 0 \Rightarrow \|\mathrm{grad}_{\underline{t}} f_{\mathbf{i},\mathbf{j}}\big|_{\underline{t}=\underline{t}_0}\| > 0 \qquad (1.2.3)$$

then there is transversality on any open subset V whose closure is contained in U.

There is another Lemma which is useful to prove transversality by controlling the double roots of infinite series. The proof of the Lemma below depends on the so-called $(*)$-functions which were introduced by Solomyak [So2] and further developed by Peres and Solomyak [PeSo1] and [PeSo2]. Although, the following Lemma was not proved explicitly in [PeSo2] but one can easily see that a simple modification of the proofs [PeSo2, Lemma 5.1], [PeSo2, Corollary 5.2] yields:

Lemma 1.2.3. *Let the function $g : [0, 1) \mapsto \mathbb{R}$ be given in the following form:*

$$g(x) = 1 + \sum_{k=1}^{\infty} a_k x^k.$$

Let us suppose that $a_1 \in (-d, d)$ and for every $k \geq 2$, $a_k \in (-b, b)$, where $d, b > 0$. Then

$$g(x_0) = 0 \Rightarrow g'(x_0) < 0 \text{ for every } x_0 \in \left(0, \frac{1}{1+\sqrt{b}}\right).$$

1.3 Proof of Theorem 1.1.1

1.3.1 Natural projection

Let p, q be positive integers and let

$$\varphi_{i,1}(x) = \gamma_{i,1} x \text{ for } i = 0, \ldots, p$$
$$\varphi_{i,2}(x) = \gamma_{i,2} x + (1 - \gamma_{i,2}) \text{ for } i = 0, \ldots, q.$$

Then our main assumptions (A1), (A2) are equivalent to $0 < \gamma_{i,1} < \gamma_{0,1} < 1$ for every $i = 1, \ldots, p$ and $0 < \gamma_{i,2} < \gamma_{0,2} < 1$ for every $i = 1, \ldots, q$, moreover,

$$\gamma_{0,1} + \gamma_{0,2} < 1.$$

Therefore, without loss of generality we can assume that

$$\gamma_{i,1} = c_{i,1}\gamma_{0,1}$$
$$\gamma_{i,2} = c_{i,2}\gamma_{0,2},$$

where $0 < c_{i,1}, c_{j,2} < 1$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Then $\mathcal{R}$ can be written in the form

$$\mathcal{R} = \{\gamma_{0,1}x, \gamma_{0,2}x + (1-\gamma_{0,2})\} \bigcup \{c_{i,1}\gamma_{0,1}x\}_{i=1}^{p} \bigcup \{c_{i,2}\gamma_{0,2}x + (1 - c_{i,2}\gamma_{0,2})\}_{i=1}^{q}.$$

Let us introduce the vectors of parameters, namely, $\underline{c}_1 = (c_{1,1}, \ldots, c_{p,1}) \in (0,1)^p$ and $\underline{c}_2 = (c_{1,2}, \ldots, c_{q,2}) \in (0,1)^q$, moreover $\underline{c} = (\underline{c}_1, \underline{c}_2)$.

Denote the set of symbols of the functions with fixed point 0 by A_1, and similarly, denote the set of symbols of the functions with fixed point 1 by A_2. So

$$A_1 = \{(0,1), \ldots, (p,1)\} \text{ and } A_2 = \{(0,2), \ldots, (q,2)\}.$$

Let Σ be the symbolic space generated by $A_1 \cup A_2$ and Σ^* the set of finite words. That is, $\Sigma = (A_1 \cup A_2)^{\mathbb{N}}$ and $\Sigma^* = \bigcup_{n=0}^{\infty} (A_1 \cup A_2)^n$. For any $\underline{i} = ((i_0, \kappa_0)(i_1, \kappa_1) \cdots (i_n, \kappa_n)) \in \Sigma^*$ we use the notation

$$\varphi_{\underline{i}} = \varphi_{i_0,\kappa_0} \circ \varphi_{i_1,\kappa_1} \circ \cdots \circ \varphi_{i_n,\kappa_n} \text{ and } \gamma_{\underline{i}} = \gamma_{i_0,\kappa_0} \cdots \gamma_{i_n,\kappa_n}.$$

For an $\mathbf{i} \in \Sigma$ we write $\mathbf{i}(k)$ as the first k elements of $\mathbf{i}$. In particular, $\mathbf{i}(k) = ((i_0, \kappa_0) \cdots (i_{k-1}, \kappa_{k-1}))$ and $\mathbf{i}(0) = \emptyset$. For $j = 1, 2$ and $i = 0, \ldots, p$ or q, we define $\sharp_{i,j}\mathbf{i}(k)$ as the number of (i,j) in $\mathbf{i}(k)$. Moreover, for $j = 1, 2$ we define $\sharp_j \mathbf{i}(k)$ as the number of symbols from A_j in $\mathbf{i}(k)$. Clearly, $\sharp_1\mathbf{i}(k) = \sum_{i=0}^{p} \sharp_{i,1}\mathbf{i}(k)$ and respectively $\sharp_2\mathbf{i}(k) = \sum_{i=0}^{q} \sharp_{i,2}\mathbf{i}(k)$. Using the notations above and the definition of the natural projection (1.2.1),

$$\pi_{\underline{c}}(\mathbf{i}) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{q} \delta_{(i_k,\kappa_k)}^{(l,2)} (1-\gamma_{l,2}) \right) \gamma_{0,1}^{\sharp_1\mathbf{i}(k)} \gamma_{0,2}^{\sharp_2\mathbf{i}(k)} \prod_{i=1}^{p} c_{i,1}^{\sharp_{(i,1)}\mathbf{i}(k)} \prod_{i=1}^{q} c_{i,2}^{\sharp_{(i,2)}\mathbf{i}(k)}, \tag{1.3.1}$$

where

$$\delta_j^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}.$$

The set of k's satisfying $(i_k, \kappa_k) \in A_2$ gives us non-zero elements in the infinite sum above. Hence it is useful to define $\beta_i^{\mathbf{i}}$ as the number of $(i,2)$ in $\mathbf{i}$ and $\beta^{\mathbf{i}}$ the number of symbols from A_2 in $\mathbf{i}$. Clearly, $\beta_i^{\mathbf{i}} = \lim_{k\to\infty} \sharp_{(i,2)}\mathbf{i}(k)$ and

$\beta^{\mathbf{i}} = \sum_{l=0}^{q} \beta_l^{\mathbf{i}}$. Moreover, let $m_k^{\mathbf{i}}$ be the position of the kth symbol from A_2 in $\mathbf{i}$. Applying the notation, $\sharp_2 \mathbf{i}(m_k^{\mathbf{i}}) = k-1$ and

$$\pi_{\underline{c}}(\mathbf{i}) = \sum_{k=1}^{\beta^{\mathbf{i}}} \left(\sum_{l=0}^{q} \delta^{(l,2)}_{(i_{m_k^{\mathbf{i}}}, \kappa_{m_k^{\mathbf{i}}})} (1-\gamma_{l,2}) \right) \gamma_{0,2}^{k-1} \gamma_{0,1}^{\sharp_1 \mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{p} c_{l,1}^{\sharp_{(l,1)} \mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)} \mathbf{i}(m_k^{\mathbf{i}})}. \tag{1.3.2}$$

For every $i = 1, \dots, p$ we write (1.3.2) as the power series of $c_{i,1}$. So we collect all the different exponents of $c_{i,1}$ into the set $P_{\mathbf{i}}^i$. It is easy to see that if $\beta^{\mathbf{i}} = 0$ then $P_{\mathbf{i}}^i = \emptyset$, otherwise

$$P_{\mathbf{i}}^i = \left\{ m \geq 0 : \exists k \geq 1, \sharp_{(i,1)} \mathbf{i}(m_k^{\mathbf{i}}) = m \right\} \text{ for } i = 1, \dots, p.$$

Then we can write the natural projection in the following form

$$\pi_{\underline{c}}(\mathbf{i}) = \sum_{m \in P_{\mathbf{i}}^i} h_i^m(\mathbf{i}) c_{(i,1)}^m. \tag{1.3.3}$$

For every $m \in P_{\mathbf{i}}^i$ the coefficient $h_i^m(\mathbf{i})$ of $c_{i,1}^m$ is the sum of those elements of (1.3.2) divided by $c_{i,1}^m$ which's indexes k satisfy $\sharp_{(i,1)} \mathbf{i}(m_k^{\mathbf{i}}) = m$. Precisely,

$$h_i^m(\mathbf{i}) = \sum_{k=\underline{s}_m^i(\mathbf{i})}^{\overline{s}_m^i(\mathbf{i})} \left(\sum_{l=0}^{q} \delta^{(l,2)}_{(i_{m_k^{\mathbf{i}}}, \kappa_{m_k^{\mathbf{i}}})} (1-\gamma_{l,2}) \right) \gamma_{0,2}^{k-1} \gamma_{0,1}^{\sharp_1 \mathbf{i}(m_k^{\mathbf{i}})} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)} \mathbf{i}(m_k^{\mathbf{i}})} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)} \mathbf{i}(m_k^{\mathbf{i}})}. \tag{1.3.4}$$

where

$$\overline{s}_m^i(\mathbf{i}) = \sup \left\{ k : \sharp_{(i,1)} \mathbf{i}(m_k^{\mathbf{i}}) = m \right\} \quad \text{and} \quad \underline{s}_m^i(\mathbf{i}) = \inf \left\{ k : \sharp_{(i,1)} \mathbf{i}(m_k^{\mathbf{i}}) = m \right\}.$$

Lemma 1.3.1. *Let $\mathbf{i} \in \Sigma$ then for every $i = 1, \dots, p$ and every $m \in P_{\mathbf{i}}^i$*

$$h_i^m(\mathbf{i}) \leq \gamma_{0,2}^{\underline{s}_m^i(\mathbf{i})-1} \gamma_{0,1}^{\sharp_1 \mathbf{i}\left(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})}\right)} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)} \mathbf{i}\left(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})}\right)}.$$

Moreover, if $0 \in P_{\mathbf{i}}^i$ then

$$h_i^0(\mathbf{i}) \geq \gamma_{0,1}^{m_1^{\mathbf{i}}-1} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)} \mathbf{i}(m_1^{\mathbf{i}})} (1-\gamma_{0,2}).$$

Proof. Let $\mathbf{i} \in \Sigma$ and for $m \in P_{\mathbf{i}}^{i}$ let $\underline{i}_m = ((i_{m_{\underline{s}_m^i(\mathbf{i})}}, \kappa_{m_{\underline{s}_m^i(\mathbf{i})}}) \cdots (i_{m_{\overline{s}_m^i(\mathbf{i})}}, \kappa_{m_{\overline{s}_m^i(\mathbf{i})}}))$. By the definition of $\overline{s}_m^i(\mathbf{i})$ and $\underline{s}_m^i(\mathbf{i})$, the segment $\underline{i}_m$ of $\mathbf{i}$ corresponds to the coefficient $h_i^m(\mathbf{i})$. By (1.3.4)

$$h_i^m(\mathbf{i}) = \gamma_{0,2}^{\underline{s}_m^i(\mathbf{i})-1} \gamma_{0,1}^{\sharp_1 \mathbf{i}\left(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})}\right)} \prod_{\substack{l=1 \\ l \neq i}}^{p} c_{l,1}^{\sharp_{(l,1)} \mathbf{i}\left(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})}\right)} \prod_{l=1}^{q} c_{l,2}^{\sharp_{(l,2)} \mathbf{i}\left(m^{\mathbf{i}}_{\underline{s}_m^i(\mathbf{i})}\right)} \varphi_{\underline{i}_m}(0).$$

By the definition, $\kappa_{m_{\underline{s}_m^i(\mathbf{i})}} = 2$ which implies that

$$1 - \gamma_{0,2} \leq \varphi_{\underline{i}_m}(0) \leq 1,$$

for every $m \in P_{\mathbf{i}}^{i}$.

If $0 \in P_{\mathbf{i}}^{i}$ then before the first $(i, 1)$ there has to be at least one symbol from A_2. Therefore $\underline{s}_0^{\mathbf{i}} = 1$. Moreover, before the place of the first symbol from A_2 the number of symbols from A_1 is $m_1^{\mathbf{i}} - 1$. This proves the assertion of the Lemma. $\square$

1.3.2 Proof of the transversality condition

For every $\mathbf{i}, \mathbf{j} \in A_\kappa^{\mathbb{N}}$ $(\kappa = 1, 2)$ $\pi_{\underline{c}}(\mathbf{i}) \equiv \pi_{\underline{c}}(\mathbf{j})$ as functions of $\underline{c}$. This implies the IFS $\mathcal{R}$ does not satisfy the transversality condition. The goal of this section is to introduce a sequence of iterated function systems which satisfy the transversality and are suitable to approximate the Hausdorff dimension of the attractor of $\mathcal{R}$.

Since $\varphi_{i_0,\kappa} \circ \varphi_{i_1,\kappa} = \varphi_{i_1,\kappa} \circ \varphi_{i_0,\kappa}$ holds for every $(i_0, \kappa), (i_1, \kappa) \in A_\kappa$ which is in the way of transversality. To eliminate this problem we choose a sequence of subsets of Σ^* such that we order the symbols in each word by the first coordinate.

Define

$$\begin{aligned} &\mathcal{P}_0 = \{(0,1); (0,2)\} \text{ and} \\ &\mathcal{P}_1 = \{(1,2)(0,1); \dots; (q,2)(0,1); (1,1)(0,2); \dots; (p,1)(0,2)\} \end{aligned} \tag{1.3.5}$$

and by induction for $k \geq 2$

$$\mathcal{P}_k = \left(\bigcup_{j=1}^{p} \bigcup_{\substack{\underline{i} \in \mathcal{P}_{k-1} \\ \kappa_0 \neq 1 \vee j \leq i_0}} \{(j,1)\underline{i}\} \right) \bigcup \left(\bigcup_{j=1}^{q} \bigcup_{\substack{\underline{i} \in \mathcal{P}_{k-1} \\ \kappa_0 \neq 2 \vee j \leq i_0}} \{(j,2)\underline{i}\} \right). \tag{1.3.6}$$

and

$$\mathcal{U}_k = \bigcup_{l=0}^{k} \mathcal{P}_l. \tag{1.3.7}$$

Denote $\Sigma_k = \mathcal{U}_k^{\mathbb{N}}$ and the sequence of IFS's

$$\Psi_k = \{\varphi_{\underline{i}}\}_{\underline{i}\in\mathcal{U}_k}. \tag{1.3.8}$$

Proposition 1.3.2. *Let $\xi > 0$ be arbitrary small, then the system Ψ_k satisfies the transversality condition on $\underline{c} \in (\xi, 1-\xi)^{p+q}$ for every $k \geq 1$.*

Proof. Suppose that $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$ and let $\mathbf{i}', \mathbf{j}' \in \Sigma_k = \mathcal{U}_k^{\mathbb{N}}$ such that $\underline{i}_0 \neq \underline{j}_0 \in \mathcal{U}_k$. Denote $\mathbf{i}'$ (and $\mathbf{j}'$) as the element of Σ by $\mathbf{i}$ (and $\mathbf{j}$ respectively). To prove transversality by Lemma 1.2.2 it is enough to show that

$$\pi_{\underline{c}}(\mathbf{i}) = \pi_{\underline{c}}(\mathbf{j}) \implies \mathrm{grad}_{\underline{c}}\left(\pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j})\right) \neq 0. \tag{1.3.9}$$

Suppose that $\pi_{\underline{c}}(\mathbf{i}) = \pi_{\underline{c}}(\mathbf{j})$. Since $\gamma_{0,1} + \gamma_{0,2} < 1$, the first element of $\mathbf{i}$, (i_0, κ_0), and the first element of $\mathbf{j}$, (j_0, τ_0), have to satisfy that $\kappa_0 = \tau_0$. Then $\mathbf{i}, \mathbf{j}$ can be written in the form

$$\mathbf{i} = \overbrace{(0,\kappa)\cdots(0,\kappa)}^{r_0}\overbrace{(1,\kappa)\cdots(1,\kappa)}^{r_1}\cdots\overbrace{(s,\kappa)\cdots(s,\kappa)}^{r_s}(l_1, 3-\kappa)\cdots$$
$$\mathbf{j} = \overbrace{(0,\kappa)\cdots(0,\kappa)}^{t_0}\overbrace{(1,\kappa)\cdots(1,\kappa)}^{t_1}\cdots\overbrace{(s,\kappa)\cdots(s,\kappa)}^{t_s}(l_2, 3-\kappa)\cdots,$$

where $r_i, t_i \geq 0$ for $i = 1, \dots, s$, $s = p$ if $\kappa = 1$ and $s = q$ otherwise.

If $r_i \leq t_i$ for every $i = 0, \dots, s$ and there exists an $1 \leq i \leq s$ such that $r_i < t_i$ then by $\gamma_{0,1} + \gamma_{0,2} < 1$, $\pi_{\underline{c}}(\mathbf{i}) \neq \pi_{\underline{c}}(\mathbf{j})$, which is a contradiction. Therefore there are two possibilities, there exist $i \neq j$ such that $r_i > t_i$ and $r_j < t_j$ or $r_i = t_i$ for every $i = 0, \dots, s$. In the last case

$$0 = \pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j}) = \gamma_{0,\kappa}^{\sum_{i=0}^{s} r_i} \prod_{i=1}^{s} c_{i,\kappa}^{r_i} \left(\pi_{\underline{c}}(\sigma^{\sum_{i=0}^{s} r_i}\mathbf{i}) - \pi_{\underline{c}}(\sigma^{\sum_{i=0}^{s} r_i}\mathbf{j})\right).$$

Since $c_{i,\kappa} > \xi/2$ for every $\kappa = 1, 2$ and $i = 1, \dots, p$ or q and moreover $\underline{i}_0 \neq \underline{j}_0$ without loss of generality we can assume the first case.

Firstly, let us suppose that $\kappa = 1$ then $\mathbf{i}$ and $\mathbf{j}$ are in the form

$$\mathbf{i} = \overbrace{(0,1)\cdots(0,1)}^{r_0}\overbrace{(1,1)\cdots(1,1)}^{r_1}\cdots\overbrace{(s,1)\cdots(s,1)}^{r_p}(l_1, 2)\cdots$$
$$\mathbf{j} = \overbrace{(0,1)\cdots(0,1)}^{t_0}\overbrace{(1,1)\cdots(1,1)}^{t_1}\cdots\overbrace{(s,1)\cdots(s,1)}^{t_p}(l_2, 2)\cdots,$$

and there exists $1 \le j \le p$ such that $r_j < t_j$. There exists also an $0 \le i \le p$ such that $r_i > t_i$ and $i \neq j$, but we prove transversality derivation in $c_{j,1}$.

Let

$$\mathbf{i}^* = \overbrace{(0,1)\cdots(0,1)}^{r_0}\cdots\overbrace{(j-1,1)\cdots(j-1,1)}^{r_{j-1}}\overbrace{(j+1,1)\cdots(j+1,1)}^{r_{j+1}}\cdots(l_1,2)\cdots$$

and

$$\mathbf{j}^* = \overbrace{(0,1)\cdots(0,1)}^{t_0}\cdots\overbrace{(j,1)\cdots(j,1)}^{t_j-r_j}\cdots(l_2,2)\cdots.$$

Then

$$\pi_{\underline{c}}(\mathbf{i}) - \pi_{\underline{c}}(\mathbf{j}) = \gamma_{j,1}^{r_j} c_{j,1}^{r_j}\left(\pi_{\underline{c}}(\mathbf{i}^*) - \pi_{\underline{c}}(\mathbf{j}^*)\right).$$

Let $a(\underline{c}) = \pi_{\underline{c}}(\mathbf{i}^*) - \pi_{\underline{c}}(\mathbf{j}^*)$. Since $c_{j,1} > \xi/2$ to prove transversality it is enough to show that

$$a(\underline{c}) = 0 \implies \frac{\partial a}{\partial c_{j,1}}(\underline{c}) \neq 0$$

for every $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$. But instead of showing that we prove

$$\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = 0 \implies a(\underline{c}) > 0 \tag{1.3.10}$$

for every $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$. By (1.3.3) we have

$$a(\underline{c}) = h_j^0(\mathbf{i}^*) + \sum_{m \in P_{\mathbf{i}^*}^j \setminus\{0\}} h_j^m(\mathbf{i}^*) c_{j,1}^m - \sum_{m \in P_{\mathbf{j}^*}^j} h_j^m(\mathbf{j}^*) c_{j,1}^m.$$

Let $\underline{c} \in (\xi/2, 1-\xi/2)^{p+q}$ such that $\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = 0$ then

$$0 = c_{j,1}\frac{\partial a}{\partial c_{j,1}}(\underline{c}) = h_j^0(\mathbf{i}^*)\left(\sum_{m \in P_{\mathbf{i}^*}^j \setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)} m c_{j,1}^m - \sum_{m \in P_{\mathbf{j}^*}^j} \frac{h_j^m(\mathbf{j}^*)}{h_j^0(\mathbf{i}^*)} m c_{j,1}^m\right) \le$$

$$h_j^0(\mathbf{i}^*)\left(\sum_{m \in P_{\mathbf{i}^*}^j \setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}(m-1) c_{j,1}^m + \sum_{m \in P_{\mathbf{i}^*}^j \setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)} c_{j,1}^m - \sum_{m \in P_{\mathbf{j}^*}^j} \frac{h_j^m(\mathbf{j}^*)}{h_j^0(\mathbf{i}^*)} c_{j,1}^m\right).$$

It is enough to prove that

$$\sum_{m \in P_{\mathbf{i}^*}^j \setminus\{0\}} \frac{h_j^m(\mathbf{i}^*)}{h_j^0(\mathbf{i}^*)}(m-1) c_{j,1}^m < 1.$$

By Lemma 1.3.1 we have

$$\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{h^m_j(\mathbf{i}^*)}{h^0_j(\mathbf{i}^*)}(m-1)c^m_{j,1} \le$$

$$\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0\}} \frac{\gamma_{0,2}^{\underline{s}^j_m(\mathbf{i}^*)-1}\gamma_{0,1}^{\sharp_1\mathbf{i}^*\left(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)}\right)}\prod_{\substack{l=1\\ l\neq j}}^p c_{l,1}^{\sharp_{(l,1)}\mathbf{i}^*\left(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)}\right)}}{\gamma_{0,1}^{m_1^{\mathbf{i}^*}-1}\prod_{\substack{l=1\\ l\neq j}}^p c_{l,1}^{\sharp_{(l,1)}\mathbf{i}(m_1^{\mathbf{i}^*})}(1-\gamma_{0,2})}(m-1)c^m_{j,1}. \quad (1.3.11)$$

Since $\mathbf{i}^*$ does not contain $(j,1)$ before the first element from A_2, $\underline{s}^j_0(\mathbf{i}^*)=1$ and $\sharp_1\mathbf{i}^*\left(m^{\mathbf{i}^*}_{\underline{s}^j_m(\mathbf{i}^*)}\right) \ge m_1^{\mathbf{i}^*}+m-1$ for every $m\in P^j_{\mathbf{i}^*}\setminus\{0\}$.

Let $q_1=\min P^j_{\mathbf{i}^*}\setminus\{0\}$ and $q_2=\min P^j_{\mathbf{i}^*}\setminus\{0,q_1\}$. We define the minimum of the empty set as infinity. Then $\underline{s}^j_{q_1}(\mathbf{i}^*)\ge 2$ and $\underline{s}^j_{q_2}(\mathbf{i}^*)\ge 3$. This implies that the right hand side of (1.3.11) is less than or equal to

$$\frac{\gamma_{0,1}^{q_1}\gamma_{0,2}}{1-\gamma_{0,2}}(q_1-1)c^{q_1}_{j,1}+\frac{\gamma_{0,1}^{q_2}\gamma_{0,2}^2}{1-\gamma_{0,2}}(q_2-1)c^{q_2}_{j,1}+\frac{\gamma_{0,2}^3}{1-\gamma_{0,2}}\sum_{m\in P^j_{\mathbf{i}^*}\setminus\{0,q_1,q_2\}}\gamma_{0,1}^m(m-1)c^m_{j,1}. \quad (1.3.12)$$

Using that $(n-1)\gamma_{0,1}^n \le \frac{-\gamma_{0,1}}{e\ln\gamma_{0,1}}$ for every $n\in\mathbb{N}$, we get that (1.3.12) is less than or equal to

$$\frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}}+\frac{\gamma_{0,2}^3}{1-\gamma_{0,2}}\sum_{m=3}^{\infty}(m-1)\gamma_{0,1}^m = \frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}}+\frac{\gamma_{0,2}^3}{1-\gamma_{0,2}}\frac{\gamma_{0,1}^3(2-\gamma_{0,1})}{(1-\gamma_{0,1})^2}.$$

Using the assumption $\gamma_{0,1}+\gamma_{0,2}<1$ by some algebraic manipulation we get that

$$\frac{-\gamma_{0,1}(\gamma_{0,2}+\gamma_{0,2}^2)}{(1-\gamma_{0,2})e\ln\gamma_{0,1}}+\frac{\gamma_{0,2}^3}{1-\gamma_{0,2}}\frac{\gamma_{0,1}^3(2-\gamma_{0,1})}{(1-\gamma_{0,1})^2}<1,$$

which implies (1.3.10).

To prove transversality in the second case when $\kappa=2$ we introduce the function $\eta(x)=-x+1$. Let us observe that $\eta\circ\eta(x)=x$. Let

$$\widetilde{\varphi}_{i,1}(x) := \eta\circ\varphi_{i,1}\circ\eta(x)=\gamma_{i,1}x+(1-\gamma_{i,1}) \text{ for } i=0,\dots,p, \text{ and}$$
$$\widetilde{\varphi}_{i,2}(x) := \eta\circ\varphi_{i,2}\circ\eta(x)=\gamma_{i,2}x \text{ for } i=0,\dots,q.$$

The IFS $\widetilde{\mathcal{R}} = \{\widetilde{\varphi}_{i,1}\}_{i=0}^{p} \cup \{\widetilde{\varphi}_{i,2}\}_{i=0}^{q}$ and $\mathcal{R}$ are equivalent. More precisely, let $\widetilde{\pi}_{\underline{c}}$ be the natural projection of $\widetilde{\mathcal{R}}$ then $\widetilde{\pi}_{\underline{c}}(\mathbf{i}) = -\pi_{\underline{c}}(\mathbf{i}) + 1$ for every $\mathbf{i} \in \Sigma$. Using this fact one can prove transversality in the case $\kappa = 2$ as in $\kappa = 1$.

The proof can be finished applying Lemma 1.2.2. $\square$

1.3.3 Hausdorff dimension

In the first part of the section we calculate the Hausdorff dimension of the attractor of Ψ_k (see (1.3.8)) and in the second part we will prove that the limit will correspond with the dimension of the attractor of $\mathcal{R}$.

Let for $k \geq 0$

$$d_k(s) = \sum_{\underline{i} \in \mathcal{U}_k} \gamma_{\underline{i}}^s.$$

By the definition of $\mathcal{U}_k$ (see (1.3.7)) for $k \geq 1$

$$d_k(s) = \gamma_{0,1}^s + \gamma_{0,2}^s + \gamma_{0,1}^s \sum_{l=1}^{k} \Phi_l + \gamma_{0,2}^s \sum_{l=1}^{k} \Upsilon_l$$

where

$$\Phi_k = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k, h_k) = (0,1)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,1}^s}$$

and

$$\Upsilon_k = \sum_{\substack{\underline{i} \in \mathcal{P}_k \\ (i_k, h_k) = (0,2)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,2}^s}.$$

Lemma 1.3.3. *Let us denote the attractor of Ψ_k by Λ_k. Then*

$$\dim_H \Lambda_k = \min\{1, s_k\} \text{ for } \mathcal{L}\text{ebesgue-a.e. } \underline{c} \in (0,1)^{p+q}$$

where s_k is the unique solution of $d_k(s) = 1$.

Proof. By Proposition 1.3.2, Ψ_k satisfies the transversality condition on $\underline{c} \in (\xi, 1-\xi)^{p+q}$ for every arbitrary small $\xi > 0$. Since $d_k(s)$ is the sum of the contraction ratios of the functions in the IFS Ψ_k to the power s, Theorem 1.2.1 implies that the Hausdorff dimension of Λ_k is equal to $\min\{1, s_k\}$ where s_k is the unique solution of

$$d_k(s) = 1 \tag{1.3.13}$$

for Lebesgue almost every $\underline{c} \in (\xi, 1-\xi)^{p+q}$. Since $\xi > 0$ was arbitrary the lemma is proved. $\square$

Lemma 1.3.4. *Let s_k be the unique solution of $d_k(s) = 1$. Then the limit $\lim_{k\to\infty} s_k = s$ exists and s is the unique solution of*

$$\prod_{i=0}^{p}(1-\gamma_{i,1}^s) + \prod_{i=0}^{q}(1-\gamma_{i,2}^s) = 1. \tag{1.3.14}$$

The proof of Formula (1.3.14) is a sequence of tedious algebraic manipulations carried out in the following pages.

Proof of Lemma 1.3.4. Without loss of generality we can assume that $p \le q$. Let

$$\Phi_k^{i,\kappa} = \sum_{\substack{\underline{i}\in\mathcal{P}_k \\ (i_k,\kappa_k)=(0,1) \\ (i_1,\kappa_1)=(i,\kappa)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,1}^s}, \qquad \Upsilon_k^{i,\kappa} = \sum_{\substack{\underline{i}\in\mathcal{P}_k \\ (i_k,\kappa_k)=(0,2) \\ (i_1,\kappa_1)=(i,\kappa)}} \frac{\gamma_{\underline{i}}^s}{\gamma_{0,2}^s},$$

then $\Phi_k = \sum_{i=1}^{p}\Phi_k^{i,1} + \sum_{i=1}^{q}\Phi_k^{i,2}$ and $\Upsilon_k = \sum_{i=1}^{p}\Upsilon_k^{i,1} + \sum_{i=1}^{q}\Upsilon_k^{i,2}$. By the definition of $\mathcal{P}_k$ (see (1.3.5), (1.3.6)) we have

$$\begin{aligned} &\Phi_1^{i,1} = 0 \text{ for } i = 1,\dots,p, \\ &\Phi_1^{i,2} = \gamma_{i,2}^s \text{ for } i = 1,\dots,q, \\ &\Upsilon_1^{i,1} = \gamma_{i,1}^s \text{ for } i = 1,\dots,p, \\ &\Upsilon_1^{i,2} = 0 \text{ for } i = 1,\dots,q, \end{aligned} \tag{1.3.15}$$

moreover for $k \ge 2$

$$\begin{aligned} \Phi_k^{i,\kappa} &= \gamma_{i,\kappa}^s \left(\Phi_{k-1} - \sum_{l=1}^{i-1}\Phi_{k-1}^{l,\kappa}\right) \\ \Upsilon_k^{i,\kappa} &= \gamma_{i,\kappa}^s \left(\Upsilon_{k-1} - \sum_{l=1}^{i-1}\Upsilon_{k-1}^{l,\kappa}\right). \end{aligned} \tag{1.3.16}$$

Denote

$$\begin{aligned} a_{k,1} &= \sum_{1\le j_0<\dots<j_{k-1}\le p} \gamma_{j_0,1}^s \cdots \gamma_{j_{k-1},1}^s \text{ for } i = 1,\dots,p, \\ a_{k,2} &= \sum_{1\le j_0<\dots<j_{k-1}\le q} \gamma_{j_0,2}^s \cdots \gamma_{j_{k-1},2}^s \text{ for } i = 1,\dots,q. \end{aligned} \tag{1.3.17}$$

Applying (1.3.16) we have for $k \geq 2$

$$\begin{aligned}
\Phi_k = &\sum_{i=1}^{p} \Phi_k^{i,1} + \sum_{i=1}^{q} \Phi_k^{i,2} = \\
&\sum_{i=1}^{p} \gamma_{i,1}^s \left(\Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,1} \right) + \sum_{i=1}^{q} \gamma_{i,2}^s \left(\Phi_{k-1} - \sum_{l=1}^{i-1} \Phi_{k-1}^{l,2} \right) = \\
&a_{1,1}\Phi_{k-1} + a_{1,2}\Phi_{k-1} - \sum_{l=1}^{p-1} \sum_{i=l+1}^{p} \gamma_{i,1}^s \Phi_{k-1}^{l,1} - \sum_{l=1}^{q-1} \sum_{i=l+1}^{q} \gamma_{i,2}^s \Phi_{k-1}^{l,2}, \quad (1.3.18)
\end{aligned}$$

and similarly

$$\Upsilon_k = a_{1,1}\Upsilon_{k-1} + a_{1,2}\Upsilon_{k-1} - \sum_{l=1}^{p-1} \sum_{i=l+1}^{p} \gamma_{i,1}^s \Upsilon_{k-1}^{l,1} - \sum_{l=1}^{q-1} \sum_{i=l+1}^{q} \gamma_{i,2}^s \Upsilon_{k-1}^{l,2}. \quad (1.3.19)$$

Applying (1.3.16) for (1.3.18) and (1.3.19) n times, where $1 \leq n \leq p-1$ and $k \geq n+1$, we get

$$\begin{aligned}
\Phi_k = &\sum_{l=1}^{n} (-1)^{l-1} a_{l,1} \Phi_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq p} \gamma_{j_n,1}^s \cdots \gamma_{j_1,1}^s \Phi_{k-n}^{j_0,1} + \\
&\sum_{l=1}^{n} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq q} \gamma_{j_n,2}^s \cdots \gamma_{j_1,2}^s \Phi_{k-n}^{j_0,2} \quad (1.3.20)
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_k = &\sum_{l=1}^{n} (-1)^{l-1} a_{l,1} \Upsilon_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq p} \gamma_{j_n,1}^s \cdots \gamma_{j_1,1}^s \Upsilon_{k-n}^{j_0,1} + \\
&\sum_{l=1}^{n} (-1)^{l-1} a_{l,2} \Upsilon_{k-l} + (-1)^n \sum_{1 \leq j_0 < \cdots < j_n \leq q} \gamma_{j_n,2}^s \cdots \gamma_{j_1,2}^s \Upsilon_{k-n}^{j_0,2}. \quad (1.3.21)
\end{aligned}$$

Then by (1.3.15) and the choice $n = k-1$ we get

$$\begin{aligned}
\Phi_k &= \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^{k-1} a_{k,2} \\
\Upsilon_k &= \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,1} \Upsilon_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Upsilon_{k-l} + (-1)^{k-1} a_{k,1}
\end{aligned} \quad (1.3.22)$$

for $2 \leq k \leq p$. If $p < q$ we can apply (1.3.16) for (1.3.18) and (1.3.19) n times, where $p \leq n \leq q-1$ and $k \geq n+1$, and we have

$$\Phi_k = \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l} + \sum_{l=1}^{n}(-1)^{l-1}a_{l,2}\Phi_{k-l} + \\ (-1)^n \sum_{1\leq j_0<\cdots<j_n\leq q} \gamma^s_{j_n,2}\cdots\gamma^s_{j_1,2}\Phi^{j_0,2}_{k-n} \quad (1.3.23)$$

and

$$\Upsilon_k = \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + \sum_{l=1}^{n}(-1)^{l-1}a_{l,2}\Upsilon_{k-l} + \\ (-1)^n \sum_{1\leq j_0<\cdots<j_n\leq q} \gamma^s_{j_n,2}\cdots\gamma^s_{j_1,2}\Upsilon^{j_0,2}_{k-n}. \quad (1.3.24)$$

By (1.3.15) and $k = n+1$ we have

$$\begin{aligned} \Phi_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l} + \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Phi_{k-l} + (-1)^{k-1}a_{k,2} \\ \Upsilon_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + \sum_{l=1}^{k-1}(-1)^{l-1}a_{l,2}\Upsilon_{k-l} \end{aligned} \quad (1.3.25)$$

for $p+1 \leq k \leq q$. By similar methods we get for $k \geq q+1$ that

$$\begin{aligned} \Phi_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Phi_{k-l} + \sum_{l=1}^{q}(-1)^{l-1}a_{l,2}\Phi_{k-l} \\ \Upsilon_k &= \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}\Upsilon_{k-l} + \sum_{l=1}^{q}(-1)^{l-1}a_{l,2}\Upsilon_{k-l}. \end{aligned} \quad (1.3.26)$$

The convergence of the infinite series $\sum_{l=1}^{\infty}\Phi_l$ and $\sum_{l=1}^{\infty}\Upsilon_l$ depends on the roots of the characteristic polynomial of (1.3.26). More precisely, $\sum_{l=1}^{\infty}\Phi_l$ and $\sum_{l=1}^{\infty}\Upsilon_l$ are convergent if and only if the roots of the characteristic polynomial are strictly less than 1. The characteristic polynomial is

$$x^q = \sum_{l=1}^{p}(-1)^{l-1}a_{l,1}x^{q-l} + \sum_{l=1}^{q}(-1)^{l-1}a_{l,2}x^{q-l}.$$

Since the roots of a polynomial depend continuously on the coefficients of the polynomial. Except the coefficient of x^q the coefficients tend to zero as s tends to infinity. Therefore the roots tend to zero as s tends to infinity. So there exists a $\delta > 0$ such that $\sum_{l=1}^{\infty} \Phi_l$ and $\sum_{l=1}^{\infty} \Upsilon_l$ are convergent for $s \in (\delta, \infty)$. Let δ the infinum of s such that $\sum_{l=1}^{\infty} \Phi_l$ and $\sum_{l=1}^{\infty} \Upsilon_l$ are convergent. Let

$$d(s) = \gamma_{0,1}^s + \gamma_{0,2}^s + \gamma_{0,1}^s \sum_{l=1}^{\infty} \Phi_l + \gamma_{0,2}^s \sum_{l=1}^{\infty} \Upsilon_l \text{ for } s \in (\delta, \infty). \tag{1.3.27}$$

Then there exists a unique $s^* \in (\delta, \infty)$ such that $d(s^*) = 1$. The sequence s_k (see (1.3.13)) is monotone increasing and bounded by s^*, therefore it is convergent. It is easy to see that $\lim_{k\to\infty} s_k = \sup_k s_k = s^*$.

Let

$$\Phi = \sum_{k=1}^{\infty} \Phi_k \text{ and } \Upsilon = \sum_{k=1}^{\infty} \Upsilon_k.$$

Then by (1.3.26)

$$\begin{aligned}
\Phi = &\sum_{k=q+1}^{\infty} \Phi_k + \sum_{k=1}^{q} \Phi_k = \\
&\sum_{k=q+1}^{\infty} \left(\sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \Phi_{k-l} \right) + \sum_{k=1}^{q} \Phi_k = \\
&\sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \sum_{k=q+1-l}^{\infty} \Phi_k + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \sum_{k=q+1-l}^{\infty} \Phi_k + \sum_{k=1}^{q} \Phi_k = \\
&\sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \left(\Phi - \sum_{k=1}^{q-l} \Phi_k \right) + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} \left(\Phi - \sum_{k=1}^{q-l} \Phi_k \right) + \sum_{k=1}^{q} \Phi_k.
\end{aligned}$$

Therefore

$$\Phi = \frac{\sum_{l=1}^{p} (-1)^l a_{l,1} \sum_{k=1}^{q-l} \Phi_k + \sum_{l=1}^{q} (-1)^l a_{l,2} \sum_{k=1}^{q-l} \Phi_k + \sum_{k=1}^{q} \Phi_k}{1 + \sum_{l=1}^{p} (-1)^l a_{l,1} + \sum_{l=1}^{q} (-1)^l a_{l,2}}, \tag{1.3.28}$$

and similarly

$$\Upsilon = \frac{\sum_{l=1}^{p} (-1)^l a_{l,1} \sum_{k=1}^{q-l} \Upsilon_k + \sum_{l=1}^{q} (-1)^l a_{l,2} \sum_{k=1}^{q-l} \Upsilon_k + \sum_{k=1}^{q} \Upsilon_k}{1 + \sum_{l=1}^{p} (-1)^l a_{l,1} + \sum_{l=1}^{q} (-1)^l a_{l,2}}. \tag{1.3.29}$$

Applying (1.3.15), (1.3.22) and (1.3.25) we get

$$\sum_{k=1}^{q} \Phi_k = \Phi_1 + \sum_{k=2}^{p} \Phi_k + \sum_{k=p+1}^{q} \Phi_k =$$

$$a_{1,2} + \sum_{k=2}^{p} \left(\sum_{l=1}^{k-1} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^{k-1} a_{k,2} \right) +$$

$$\sum_{k=p+1}^{q} \left(\sum_{l=1}^{p} (-1)^{l-1} a_{l,1} \Phi_{k-l} + \sum_{l=1}^{k-1} (-1)^{l-1} a_{l,2} \Phi_{k-l} + (-1)^{k-1} a_{k,2} \right) =$$

$$\sum_{k=1}^{q} (-1)^{k-1} a_{k,2} + \sum_{l=1}^{p} \sum_{k=1}^{q-l} (-1)^{l-1} a_{l,1} \Phi_k + \sum_{l=1}^{q} \sum_{k=1}^{q-l} (-1)^{l-1} a_{l,2} \Phi_k, \quad (1.3.30)$$

and by similar arguments

$$\sum_{k=1}^{q} \Upsilon_k = \sum_{k=1}^{p} (-1)^{k-1} a_{k,1} + \sum_{l=1}^{p} \sum_{k=1}^{q-l} (-1)^{l-1} a_{l,1} \Upsilon_k + \sum_{l=1}^{q} \sum_{k=1}^{q-l} (-1)^{l-1} a_{l,2} \Upsilon_k. \quad (1.3.31)$$

Hence the numerator of (1.3.28) is $\sum_{k=1}^{q} (-1)^{k-1} a_{k,2}$ and the numerator of (1.3.29) is $\sum_{k=1}^{p} (-1)^{k-1} a_{k,1}$, which implies that

$$\begin{aligned} \Phi &= \frac{\sum_{k=1}^{q} (-1)^{k-1} a_{k,2}}{1 + \sum_{l=1}^{p} (-1)^l a_{l,1} + \sum_{l=1}^{q} (-1)^l a_{l,2}} \text{ and} \\ \Upsilon &= \frac{\sum_{k=1}^{p} (-1)^{k-1} a_{k,1}}{1 + \sum_{l=1}^{p} (-1)^l a_{l,1} + \sum_{l=1}^{q} (-1)^l a_{l,2}}. \end{aligned} \quad (1.3.32)$$

Then $d(s) = 1$ (see (1.3.27)) is equivalent to

$$\gamma_{0,1}^s + \gamma_{0,2}^s + \sum_{l=1}^{p} (-1)^{l-1} a_{l,1} + \sum_{l=1}^{q} (-1)^{l-1} a_{l,2} + \gamma_{0,1}^s \sum_{l=1}^{p} (-1)^l a_{l,1} + \gamma_{0,2}^s \sum_{l=1}^{q} (-1)^l a_{l,2} = 1.$$

Let us observe that

$$x^{p+1} - \sum_{m=0}^{p} (-1)^m \sum_{0 \le j_0 < \cdots < j_m \le p} \gamma_{j_0,1}^s \cdots \gamma_{j_m,1}^s x^{p-m} = \prod_{k=0}^{p} \left(x - \gamma_{k,1}^s \right) \text{ and}$$

$$x^{q+1} - \sum_{m=0}^{q} (-1)^m \sum_{0 \le j_0 < \cdots < j_m \le q} \gamma_{j_0,2}^s \cdots \gamma_{j_m,2}^s x^{q-m} = \prod_{k=0}^{q} \left(x - \gamma_{k,2}^s \right).$$

Then by $x = 1$ we get that $d(s) = 1$ is equivalent to

$$2 - \prod_{k=0}^{p} \left(1 - \gamma_{k,1}^s \right) - \prod_{k=0}^{q} \left(1 - \gamma_{k,2}^s \right) = 1$$

which is (1.3.14).

The proof will be complete if we show that (1.3.14) has unique solution. We have that the left hand side is equal to 2 if $s = 0$ and the derivative is

$$\sum_{l=0}^{p} \gamma_{l,1}^{s} \log \gamma_{l,1} \prod_{\substack{k=0 \\ k\neq l}}^{p} \left(1-\gamma_{k,1}^{s}\right) + \sum_{l=0}^{q} \gamma_{l,2}^{s} \log \gamma_{l,2} \prod_{\substack{k=0 \\ k\neq l}}^{q} \left(1-\gamma_{k,2}^{s}\right)$$

which is negative for $s > 0$. This completes the proof. $\square$

Now we show that the unique solution of (1.3.14) is an upper bound for the Hausdorff dimension. To give a good cover of the attractor, we need to introduce another sequence of subsets of Σ^*. Let

$$\mathcal{C}_0 = \{(0,1),(0,2)\} \tag{1.3.33}$$

and by induction let

$$\mathcal{C}_k = \bigcup_{j=0}^{p} \bigcup_{\substack{\underline{i}\in\mathcal{C}_{k-1} \\ \kappa_0\neq 1 \vee j\leq i_0}} \{(j,1)\underline{i}\} \bigcup \bigcup_{j=0}^{q} \bigcup_{\substack{\underline{i}\in\mathcal{C}_{k-1} \\ \kappa_0\neq 2 \vee j\leq i_0}} \{(j,2)\underline{i}\}. \tag{1.3.34}$$

Lemma 1.3.5. *Let $\widetilde{s}_k$ the unique solution of*

$$\sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{s} = 1,$$

and let $\widetilde{s} = \sup_k \widetilde{s}_k$ then

$$\dim_H \Lambda \leq \min\{1, \widetilde{s}\}.$$

Note that the sequence $\widetilde{s}_k$ is bounded since $\mathcal{C}_k \subseteq (A_1 \cup A_2)^{k+1}$.

Proof. Using that for every $(i,\kappa),(j,\kappa) \in A_\kappa$,

$$\varphi_{(i,\kappa)} \circ \varphi_{(j,\kappa)} \equiv \varphi_{(j,\kappa)} \circ \varphi_{(i,\kappa)},$$

and $\gamma_{j,\kappa}, \gamma_{i,\kappa} \leq \gamma_{0,\kappa}$ we have that the set of closed intervals

$$\{\varphi_{\underline{i}}([0,1])\}_{\underline{i}\in\mathcal{C}_k}$$

gives a cover of Λ with diameter at most $\gamma_{\max}^k$, where $\gamma_{\max} = \max_{i,\kappa}\{\gamma_{i,\kappa}\}$. Then

$$\mathcal{H}_{\gamma_{\max}^k}^{\widetilde{s}}(\Lambda) \leq \sum_{\underline{i}\in\mathcal{C}_k} |\varphi_{\underline{i}}([0,1])|^{\widetilde{s}} = \sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{\widetilde{s}} \leq \sum_{\underline{i}\in\mathcal{C}_k} \gamma_{\underline{i}}^{\widetilde{s}_k} = 1.$$

This proves the Lemma. $\square$

Proof of Theorem 1.1.1. By the definition of $\mathcal{C}_k$ we have that for every $k \geq 1$

$$\mathcal{C}_k \subset \bigcup_{l=1}^{k} \mathcal{U}_k^l. \tag{1.3.35}$$

More precisely, every $\underline{i} \in \mathcal{C}_k$ can be decomposed as a juxtaposition $\underline{i} = \underline{j}_1 \cdots \underline{j}_r$, where each $\underline{j}_l \in \mathcal{U}_k$. By similar arguments as in the proof of Proposition 1.3.2, one can show that the system $\widetilde{\Psi}_k = \{\varphi_{\underline{i}}\}_{\underline{i} \in \mathcal{C}_k}$ satisfies transversality condition on $(\xi, 1-\xi)^{p+q}$. Since $\xi > 0$ was arbitrary by Theorem 1.2.1 we have

$$\dim_H \widetilde{\Lambda}_k = \min\{1, \widetilde{s}_k\} \text{ for } \mathcal{L}\text{-a.e. } \underline{c} \in (0,1)^{p+q}, \tag{1.3.36}$$

where $\widetilde{\Lambda}_k$ denotes the attractor of $\{\varphi_{\underline{i}}\}_{\underline{i} \in \mathcal{C}_k}$. Using (1.3.35) we have $\widetilde{\Lambda}_k \subseteq \Lambda_k \subseteq \Lambda$ which implies

$$\dim_H \widetilde{\Lambda}_k \leq \dim_H \Lambda_k \leq \dim_H \Lambda.$$

Therefore by Lemma 1.3.3 and Lemma 1.3.5 we have

$$\min\{1, \widetilde{s}_k\} \leq \min\{1, s_k\} \leq \min\{1, \widetilde{s}\}.$$

By Lemma 1.3.4, s_k is convergent and $\lim_{k\to\infty} s_k = \sup_k s_k = s$. This implies that $\min\{1, s\} = \min\{1, \widetilde{s}\}$, moreover

$$\dim_H \Lambda = \min\{1, s\}.$$

To complete the proof we have to prove the measure claim. If $s > 1$ then there exists a $k \geq 2$ such that $s_k > 1$. Therefore, by Theorem 1.2.1 and Proposition 1.3.2, $\mathcal{L}(\Lambda) \geq \mathcal{L}(\Lambda_k) > 0$ for a.e. $\underline{c} \in (0,1)^{p+q} \cap \{\underline{c} : s > 1\}$. $\square$

1.4 Proof of Theorem 1.1.2

1.4.1 Natural Projection

Because of the special nature of the IFS $\mathcal{S} = \mathcal{F} \cup \mathcal{G}$ under consideration, it is reasonable to modify the way as the elements of $\mathcal{S}$ are labeled. Namely, we label the functions of $\mathcal{S}$ by pairs of integers like (i, κ), where $\kappa = 1$ if the function is from $\mathcal{F}$ and $\kappa = 2$ when the function is from $\mathcal{G}$. In both cases $i \in \{0, \ldots, N-1\}$, where we recall that N was defined in our Principal Assumptions as the cardinality of $\mathcal{F}$. From now on we write in the rest of the chapter, $\mathcal{I} = \{(0,1), (1,1), \ldots, (N-1,1)\}$ for $N \geq 2$. According to this new notation the contraction ratio and the fixed point of the functions from $\mathcal{F}$ are $0 < \lambda_{(i,1)} < 1$, and $a_{(i,1)} \in \mathbb{R}$, $(i,1) \in \mathcal{I}$. That is

$$f_{(i,1)}(x) = \lambda_{(i,1)} x + a_{(i,1)}(1 - \lambda_{(i,1)}), \quad (i,1) \in \mathcal{I}. \tag{1.4.1}$$

Let $\mathcal{J} \subseteq \{(0,2), \ldots, (N-1,2)\}$ and denote $\mathcal{N} = \{i : (i,2) \in \mathcal{J}\}$. Like above, the contraction ratio and the fixed point of the functions from $\mathcal{G}$ are $0 < \lambda_{(i,2)} < 1$ and $a_{(i,2)} \in \mathbb{R}$, $(i,2) \in \mathcal{J}$. That is

$$f_{(i,2)}(x) = \lambda_{(i,2)} x + a_{(i,2)}(1 - \lambda_{(i,2)}) \text{ for } i \in \mathcal{N}. \tag{1.4.2}$$

So

$$\mathcal{F} = \left\{ f_{(i,1)} \right\}_{i=0}^{N-1} \text{ and } \mathcal{G} = \left\{ f_{(i,2)} \right\}_{i \in \mathcal{N}}.$$

According to our principal assumptions (B1)-(B4) we have the following relations:

$$a_i := a_{(i,1)} = a_{(i,2)} \text{ and } 0 < \lambda_{(i,2)} < \lambda_{(i,1)} < 1 \text{ for every } i \in \mathcal{N}.$$

Moreover, by definition $a_0 < a_1 < \cdots < a_{N-1}$ and

$$f_{(i-1,1)}(a_{N-1}) < f_{(i,1)}(a_0), \tag{1.4.3}$$

see (1.1.3). For simplicity denote $\underline{\lambda}_1$ the vector of contraction ratios of $\mathcal{F}$ and similarly $\underline{\lambda}_2$ the vector of contraction ratios of $\mathcal{G}$. We denote the attractor of $\mathcal{S}$ by $\Omega(\underline{\lambda}, \underline{a})$, where $\underline{\lambda} = \underline{\lambda}_1 \times \underline{\lambda}_2$ and the vector of the distinct fixed points of the functions of $\mathcal{S}$ is $\underline{a} = (a_0, \ldots, a_{N-1})$. As usual we write

$$\underline{\gamma}^{\underline{k}} := \prod_{i=1}^{m} \gamma_i^{k_i}, \quad \underline{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m, \quad \underline{\gamma} \in \mathbb{R}^m. \tag{1.4.4}$$

The symbolic space is

$$\Sigma := (\mathcal{I} \cup \mathcal{J})^{\mathbb{N}}.$$

The natural projection $\pi_{\underline{\lambda},\underline{a}}$ from the symbolic space Σ to the attractor Ω is defined exactly as in (1.2.1).

We remind that for an $\mathbf{i} = ((i_0, \kappa_0)(i_1, \kappa_1)(i_2, \kappa_2) \cdots) \in \Sigma$ we write $\mathbf{i}(k)$ for the sequence of the first k elements of $\mathbf{i}$ and we denote the number of $(i, \kappa) \in \mathcal{I} \cup \mathcal{J}$ in $\mathbf{i}(k)$ by $\sharp_{(i,\kappa)} \mathbf{i}(k)$. We form the vector $\sharp \mathbf{i}(k) \in \{0, \ldots, k\}^{\sharp\mathcal{I} + \sharp\mathcal{J}}$ as

$$\sharp \mathbf{i}(k) := \left(\sharp_{(0,1)} \mathbf{i}(k), \sharp_{(1,1)} \mathbf{i}(k), \ldots, \sharp_{(N-1,1)} \mathbf{i}(k), \sharp_{(\min \mathcal{J},2)} \mathbf{i}(k), \ldots, \sharp_{(\max \mathcal{J},2)} \mathbf{i}(k) \right).$$

Using the notation introduced in (1.4.4), clearly,

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) = \sum_{k=0}^{\infty} a_{i_k} (1 - \lambda_{(i_k,\kappa_k)}) \underline{\lambda}^{\sharp \mathbf{i}(k)}. \tag{1.4.5}$$

Equivalently,

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) = a_{i_0} + \sum_{k=0}^{\infty} \left(a_{i_{k+1}} - a_{i_k} \right) \underline{\lambda}^{\sharp \mathbf{i}(k+1)}. \tag{1.4.6}$$

In this way only those elements of the sum above have non-zero contribution for which $a_{i_{k+1}} \neq a_{i_k}$. Now we partition the elements of $\mathbf{i}$ into blocks to rewrite the natural projection. Let $p_l^{\mathbf{i}}$ be the $l+1$-th element of the set $\{k : i_{k-1} \neq i_k\}$ where $\mathbf{i} = ((i_0, \kappa_0)(i_1, \kappa_1)\dots)$. For $l = 0$, let the 0-th block of $\mathbf{i}$ be $b_0^{\mathbf{i}} = \Big((i_0, \kappa_0)\dots(i_{p_0^{\mathbf{i}}-1}, \kappa_{p_0^{\mathbf{i}}-1})\Big)$, and for $l \geq 1$ the l-th block of $\mathbf{i}$ is $b_l^{\mathbf{i}} = \Big((i_{p_{l-1}^{\mathbf{i}}}, \kappa_{p_{l-1}^{\mathbf{i}}})\dots(i_{p_l^{\mathbf{i}}-1}, \kappa_{p_l^{\mathbf{i}}-1})\Big)$. Therefore all functions which correspond to any symbols in a block share the same fixed point.

We write $k_l^{\mathbf{i}}$ for the length of the l-th block $b_l^{\mathbf{i}}$. Obviously, the length of the first l blocks is $p_l^{\mathbf{i}} = \sum_{j=0}^{l} k_j^{\mathbf{i}}$.

In this way the decomposition of $\mathbf{i}$ into blocks is as follows:

$$\mathbf{i} = (\underbrace{(i_0, \kappa_0)\cdots(i_{k_0^{\mathbf{i}}-1}, \kappa_{k_0^{\mathbf{i}}-1})}_{b_0^{\mathbf{i}}}\cdots\underbrace{(i_{p_l^{\mathbf{i}}}, \kappa_{p_l^{\mathbf{i}}})\cdots(i_{p_l^{\mathbf{i}}+k_{l+1}^{\mathbf{i}}-1}, \kappa_{p_l^{\mathbf{i}}+k_{l+1}^{\mathbf{i}}-1})}_{b_{l+1}^{\mathbf{i}}}\cdots)$$

or simply $\mathbf{i} = b_0^{\mathbf{i}} b_1^{\mathbf{i}} b_2^{\mathbf{i}} \dots$. Let $a_{b_l^{\mathbf{i}}}$ be the common fixed point of all the functions $f_{(i,\kappa)}$, where $(i, \kappa) \in b_l^{\mathbf{i}}$. That is

$$a_{b_l^{\mathbf{i}}} := a_{i_{p_{l-1}^{\mathbf{i}}}} = a_{i_{p_{l-1}^{\mathbf{i}}+1}} = \dots = a_{i_{p_{l-1}^{\mathbf{i}}+k_l^{\mathbf{i}}-1}}.$$

For a block $b = ((i_u, \kappa_u), \dots, (i_v, \kappa_v))$ we define

$$f_b := f_{(i_u,\kappa_u)} \circ \dots \circ f_{(i_v,\kappa_v)}. \tag{1.4.7}$$

By the two notations above we have

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) = \lim_{l\to\infty} f_{b_0^{\mathbf{i}}} \circ \dots \circ f_{b_l^{\mathbf{i}}}(0) = a_{b_0^{\mathbf{i}}} + \sum_l (a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}}) \underline{\lambda}^{\sharp \mathbf{i}(p_l^{\mathbf{i}})}. \tag{1.4.8}$$

We define both the empty sum, and for every $0 < \alpha < 1$, α^{∞} as 0. Let us assume about the first element (i_0, κ_0) of $\mathbf{i}$ that $i_0 \in \mathcal{N}$. To find the exponent of $\lambda_{i_0,2}$ we introduce a set $Q^{\mathbf{i}}$ as follows: First for every $l \geq 0$ we assign an integer $m(l)$ which is the total number of the appearances of $(i_0, 2)$ in the union of the first l blocks. Observe we always assign the same $m(l)$ to more than one consecutive l. Among these, the smallest one is called $r_m^{\mathbf{i}}$ and the biggest one is $o_m^{\mathbf{i}} \geq 1 + r_m^{\mathbf{i}}$ The collection of the distinct integers $m(l)$ assigned in this way to some $l \geq 0$ is the set $Q^{\mathbf{i}}$. That is

$$Q^{\mathbf{i}} = \left\{ m \geq 0 : \exists l \geq 0,\ m = \sharp_{(i_0,2)} \mathbf{i}(p_l^{\mathbf{i}}) \right\}. \tag{1.4.9}$$

and

$$o_m^{\mathbf{i}} = \sup\left\{l : \sharp_{(i_0,2)}\mathbf{i}(p_l^{\mathbf{i}}) = m\right\},\ r_m^{\mathbf{i}} = \inf\left\{l : \sharp_{(i_0,2)}\mathbf{i}(p_l^{\mathbf{i}}) = m\right\}. \qquad (1.4.10)$$

It is possible that $o_m^{\mathbf{i}} = \infty$. Now we partition the sum in (1.4.8) according to the exponent of $(i_0, 2)$:

$$\begin{aligned}
\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) &= a_{b_0^{\mathbf{i}}} + \sum_{l=0}^{\infty}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})} \\
&= a_{b_0^{\mathbf{i}}} + \sum_{m\in Q^{\mathbf{i}}}\sum_{l=r_m^{\mathbf{i}}}^{o_m^{\mathbf{i}}}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})} \\
&= a_{b_0^{\mathbf{i}}} + \sum_{m\in Q^{\mathbf{i}}} d_{\mathbf{i}}^m \lambda_{(i_0,2)}^m, \qquad (1.4.11)
\end{aligned}$$

where

$$d_{\mathbf{i}}^m = \sum_{l=r_m^{\mathbf{i}}}^{o_m^{\mathbf{i}}}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\frac{\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}}{\lambda_{(i_0,2)}^{\sharp_{(i_0,2)}\mathbf{i}(p_l^{\mathbf{i}})}} = \sum_{l=r_m^{\mathbf{i}}}^{o_m^{\mathbf{i}}}(a_{b_{l+1}^{\mathbf{i}}} - a_{b_l^{\mathbf{i}}})\frac{\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}. \qquad (1.4.12)$$

Note that for $l = r_m^{\mathbf{i}}, \ldots, o_m^{\mathbf{i}}$ the ratio $\frac{\underline{\lambda}^{\sharp\mathbf{i}(p_l^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}$ is independent of $\lambda_{(i_0,2)}$, by the definition of m.

Lemma 1.4.1. *Let* $\mathbf{i} = ((i_0,\kappa_0)(i_1,\kappa_1)\cdots) \in \Sigma$ *such that* $i_0 \in \mathcal{N}$. *Then for every* $m \in Q^{\mathbf{i}}$ *we have*

$$|d_{\mathbf{i}}^m| \leq \frac{\underline{\lambda}^{\sharp\mathbf{i}(p_{r_m^{\mathbf{i}}}^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}\max\left\{a_{N-1} - a_{i_0}, a_{i_0} - a_0\right\}. \qquad (1.4.13)$$

Moreover if $0 \in Q^{\mathbf{i}}$ *then*

$$|d_{\mathbf{i}}^0| \geq \lambda_{(i_0,1)}^{k_0^{\mathbf{i}}}\min\left\{f_{(i_0+1,1)}(a_0) - a_{i_0}, a_{i_0} - f_{(i_0-1,1)}(a_{N-1})\right\}. \qquad (1.4.14)$$

Proof. The statement of the lemma follows easily from the following observation:

$$d_{\mathbf{i}}^m = \frac{\underline{\lambda}^{\sharp\mathbf{i}(p_{r_m^{\mathbf{i}}}^{\mathbf{i}})}}{\lambda_{(i_0,2)}^m}\left(f_{\underline{i}}(a_{i_0}) - a_{i_0}\right), \qquad (1.4.15)$$

where $\underline{i} := (b_{r_m^{\mathbf{i}}+1}^{\mathbf{i}}\cdots b_{o_m^{\mathbf{i}}}^{\mathbf{i}})$ and using the notation of (1.4.7) we define

$$f_{\underline{i}} = f_{b_{r_m^{\mathbf{i}}+1}^{\mathbf{i}}} \circ \cdots \circ f_{b_{o_m^{\mathbf{i}}}^{\mathbf{i}}}.$$

To verify (1.4.15) we fix an $\mathbf{i} = ((i_0, \kappa_0)(i_1, \kappa_1)\cdots) \in \Sigma$ and $m \in Q^{\mathbf{i}}$. Using that $a_{b^{\mathbf{i}}_{o^{\mathbf{i}}_m+1}} = a_{b^{\mathbf{i}}_{r^{\mathbf{i}}_m}} = a_{i_0}$ by definition we have

$$f_{\underline{i}}(a_{i_0}) = a_{b^{\mathbf{i}}_{r^{\mathbf{i}}_m+1}} + \sum_{l=r^{\mathbf{i}}_m+1}^{o^{\mathbf{i}}_m-1} (a_{b^{\mathbf{i}}_{l+1}} - a_{b^{\mathbf{i}}_l}) \underline{\lambda}^{\sharp\mathbf{i}(p^{\mathbf{i}}_l) - \sharp\mathbf{i}(p^{\mathbf{i}}_{r^{\mathbf{i}}_m})} + (a_{b^{\mathbf{i}}_{o^{\mathbf{i}}_m+1}} - a_{b^{\mathbf{i}}_{o^{\mathbf{i}}_m}}) \underline{\lambda}^{\sharp\mathbf{i}(p^{\mathbf{i}}_{o^{\mathbf{i}}_m}) - \sharp\mathbf{i}(p^{\mathbf{i}}_{r^{\mathbf{i}}_m})}$$

and

$$d^m_{\mathbf{i}} = \sum_{l=r^{\mathbf{i}}_m}^{o^{\mathbf{i}}_m} (a_{b^{\mathbf{i}}_{l+1}} - a_{b^{\mathbf{i}}_l}) \frac{\underline{\lambda}^{\sharp\mathbf{i}(p^{\mathbf{i}}_l)}}{\lambda^m_{(i_0,2)}} = \frac{\underline{\lambda}^{\sharp\mathbf{i}(p^{\mathbf{i}}_{r^{\mathbf{i}}_m})}}{\lambda^m_{(i_0,2)}} \left(f_{\underline{i}}(a_{i_0}) - a_{b^{\mathbf{i}}_{r^{\mathbf{i}}_m}} \right).$$

Which completes the proof of (1.4.15). Therefore

$$|d^m_{\mathbf{i}}| \leq \frac{\underline{\lambda}^{\sharp\mathbf{i}(p^{\mathbf{i}}_{r^{\mathbf{i}}_m})}}{\lambda^m_{(i_0,2)}} \max\left\{a_{N-1} - a_{i_0}, a_{i_0} - a_0\right\}.$$

Now let us suppose that $0 \in Q^{\mathbf{i}}$ then $r^{\mathbf{i}}_0 = 0$. Moreover $b^{\mathbf{i}}_0$ contains only $(i_0, 1)$. Then by $|b^{\mathbf{i}}_0| = k^{\mathbf{i}}_0$ we have

$$d^m_{\mathbf{i}} = \lambda^{k^{\mathbf{i}}_0}_{(i_0,1)} \left(f_{\underline{i}'}(a_{i_0}) - a_{i_0} \right),$$

where $\underline{i}' = (b^{\mathbf{i}}_1 \cdots b^{\mathbf{i}}_{o^{\mathbf{i}}_0})$. By definition, $b^{\mathbf{i}}_1$ does not contain elements from $\{(i_0,1),(i_0,2)\}$. Then by (1.4.3) and $\lambda_{(i,2)} < \lambda_{(i,1)}$ we have

$$|f_{\underline{i}'}(a_{i_0}) - a_{i_0}| \geq \min\left\{ f_{(i_0+1,1)}(a_0) - a_{i_0}, a_{i_0} - f_{(i_0-1,1)}(a_{N-1}) \right\}$$

which completes the proof. □

1.4.2 Proof of the transversality condition

Similarly to the case of IFS $\mathcal{R}$, the IFS $\mathcal{S}$ does not satisfy either the transversality condition, because for every $i \in \mathcal{N}$ and every $\mathbf{i}, \mathbf{j} \in \{(i,1),(i,2)\}^{\mathbb{N}}$ with $(i,\kappa_0) \neq (i,\tau_0)$ we have $\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) \equiv \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}) \equiv a_i$. In this section we prove transversality for a series of suitable subsystems, but with substantially different method compare to Section 1.3.2. For $k \geq 2$ let

$$\mathcal{U}_k = \mathcal{I} \bigcup \left(\bigcup_{l=0}^{k-2} \bigcup_{\underline{i} \in \mathcal{J}^l} \bigcup_{u \in \mathcal{N}} \bigcup_{v=0, u \neq v}^{N-1} \{\underline{i}(u,2)(v,1)\} \right). \tag{1.4.16}$$

For a $k \geq 2$ we define

$$\Psi_k = \{f_{\underline{i}}\}_{\underline{i} \in \mathcal{U}_k}. \tag{1.4.17}$$

We prove in Lemma 1.4.2 below that for every $k \geq 2$ the IFS Ψ_k satisfies transversality on a certain parameter domain $R_{\underline{\varepsilon}}$. Using this, in Proposition 1.4.4, we verify that the transversality holds on a domain which approximates the parameter domain that appears in Theorem 1.1.2. First we introduce the corresponding notation. Let us denote the attractor of Ψ_k by $\Omega_k^{\underline{\lambda}}$ and the natural projection from $\Sigma_k := \mathcal{U}_k^{\mathbb{N}}$ onto $\Omega_k^{\underline{\lambda}}$ by $\pi_k^{\underline{\lambda}}$. Denote the elements of Σ_k by $\mathbf{i}' = (\underline{i}_0 \underline{i}_1 \cdots)$.

Lemma 1.4.2. *Let $0 < \varepsilon_i < \lambda_{(i,1)}$ for every $i = 0, \ldots, N-1$. Then for every $k \geq 2$ and every $\mathbf{i}' = (\underline{i}_0 \underline{i}_1 \cdots), \mathbf{j}' = (\underline{j}_0 \underline{j}_1 \cdots) \in \Sigma_k$ such that $\underline{i}_0 \neq \underline{j}_0 \in \mathcal{U}_k$,*

$$\pi_k^{\widetilde{\underline{\lambda}}}(\mathbf{i}') = \pi_k^{\widetilde{\underline{\lambda}}}(\mathbf{j}') \implies \left| \frac{\partial}{\partial \lambda_{(i,2)}} \left(\pi_k^{\underline{\lambda}}(\mathbf{i}') - \pi_k^{\underline{\lambda}}(\mathbf{j}') \right) \Big|_{\underline{\lambda} = \widetilde{\underline{\lambda}}} \right| > 0, \tag{1.4.18}$$

for some i and for every

$$\widetilde{\underline{\lambda}}_2 \in R_{\underline{\varepsilon}} = \prod_{i \in \mathcal{N}} \left(\varepsilon_i, \min \left\{ \lambda_{(i,1)}, \frac{1}{1 + \sqrt{\lambda_{\max} \alpha_i \left(1 + \frac{\alpha_i}{\varepsilon_i}\right)}} \right\} \right), \tag{1.4.19}$$

if it exists, where $\lambda_{\max} = \max_{i=0,\ldots,N-1} \left\{ \lambda_{(i,1)} \right\}$ and

$$\alpha_i = \frac{\max\left\{a_{N-1} - a_i, a_i - a_0\right\}}{\min\left\{f_{(i+1,1)}(a_0) - a_i, a_i - f_{(i-1,1)}(a_{N-1})\right\}}.$$

To prove Lemma 1.4.2 we need the following Sublemma:

Sublemma 1.4.3. *Let $\underline{i}, \underline{j}$ finite length word of symbols such that*

$$\underline{i} = \overbrace{(i,1) \cdots (i,1)}^{k_1} (l_1, \kappa_1)$$
$$\underline{j} = \overbrace{(i,2) \cdots (i,2)}^{k_2} (l_2, \kappa_2)$$

where $l_1, l_2 \neq i$. If $f_{\underline{i}}([a_0, a_{N-1}]) \cap f_{\underline{j}}([a_0, a_{N-1}]) \neq \emptyset$ then

$$\frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}} \leq \alpha_i.$$

Proof. Since for every $(i,2) \in \mathcal{J}$, $\lambda_{(i,2)} < \lambda_{(i,1)}$, we have that $f_{\underline{i}}([a_0,a_{N-1}]) \cap f_{\underline{j}}([a_0,a_{N-1}]) \neq \emptyset$ implies

$$\begin{aligned}\lambda_{(i,1)}^{k_1}\lambda_{(l_1,\kappa_1)}a_0 + \lambda_{(i,1)}^{k_1}a_{l_1}(1-\lambda_{(l_1,\kappa_1)}) + a_i(1-\lambda_{(i,1)}^{k_1}) \leq \\ \lambda_{(i,2)}^{k_2}\lambda_{(l_2,\kappa_2)}a_{N-1} + \lambda_{(i,2)}^{k_2}a_{l_2}(1-\lambda_{(l_2,\kappa_2)}) + a_i(1-\lambda_{(i,2)}^{k_2}),\end{aligned}$$

$$\begin{aligned}\lambda_{(i,2)}^{k_2}\lambda_{(l_2,\kappa_2)}a_0 + \lambda_{(i,2)}^{k_2}a_{l_2}(1-\lambda_{(l_2,\kappa_2)}) + a_i(1-\lambda_{(i,2)}^{k_2}) \leq \\ \lambda_{(i,1)}^{k_1}\lambda_{(l_1,\kappa_1)}a_{N-1} + \lambda_{(i,1)}^{k_1}a_{l_1}(1-\lambda_{(l_1,\kappa_1)}) + a_i(1-\lambda_{(i,1)}^{k_1}).\end{aligned}$$

Using the fact that $\mathcal{F}$ satisfies (1.4.3), we have $l_1, l_2 > i$ or $l_1, l_2 < i$. One can finish the proof by some obvious algebraic manipulations. □

Proof of Lemma 1.4.2. Let $0 < \varepsilon_i < \lambda_{(i,1)}$ and suppose that $\varepsilon_i < \lambda_{(i,2)}$ for every $i \in \mathcal{N}$. Let $\mathbf{i}', \mathbf{j}' \in \Sigma_k$ such that $\underline{i}_0 \neq \underline{j}_0$ and $\pi_k^{\lambda}(\mathbf{i}') = \pi_k^{\lambda}(\mathbf{j}')$. Divide $\underline{i}_0$ and $\underline{j}_0$ into blocks such that $\underline{i}_0 = (b_0^{i_0} \cdots b_l^{i_0})$ and $\underline{j}_0 = (b_0^{j_0} \cdots b_q^{j_0})$. By definition, a block consists of such pairs which share the same first component. If u is the common first element in the case of the block $b_0^{i_0}$ and v for $b_0^{j_0}$ then applying (1.4.3) we obtain that $u = v$. That is the first elements of all of the pairs that are contained either in $b_0^{i_0}$ or in $b_0^{j_0}$ are the same. First let us assume that both of $\underline{i}_0$ and $\underline{j}_0$ begin with $(i,2)$. Then by the definition of $\mathcal{U}_k$ (see (1.4.16)), $b_0^{i_0}, b_0^{j_0}$ contain only $(i,2)$. Since $\mathcal{S}$ satisfies (1.4.3) we have that $|b_0^{i_0}| = |b_0^{j_0}| = n$. This implies that

$$0 = \pi_k^{\lambda}(\mathbf{i}') - \pi_k^{\lambda}(\mathbf{j}') = \lambda_{(i,2)}^{n}\left(\pi_k^{\lambda}(\mathbf{i}'^*) - \pi_k^{\lambda}(\mathbf{j}'^*)\right)$$

where the first element of $\mathbf{i}'^*$ is $(b_1^{i_0} \cdots b_l^{i_0}) \in \Sigma_k$ and the first element of $\mathbf{j}'^*$ is $(b_1^{j_0} \cdots b_q^{j_0}) \in \Sigma_k$. Since $\lambda_{(i,2)} > \varepsilon_i$, without loss of generality we can assume that $\underline{i}_0 = (i,1)$ and $b_0^{j_0}$ contains only $(i,2)$ for an $i \in \mathcal{N}$. Let us write $\mathbf{i}, \mathbf{j}$ for the elements of $\Sigma = (\mathcal{I} \cup \mathcal{J})^{\mathbb{N}}$ that correspond to $\mathbf{i}', \mathbf{j}'$ respectively. Then $\pi_k^{\lambda}(\mathbf{i}') \equiv \pi_{\underline{\lambda},\underline{a}}(\mathbf{i})$ and $\pi_k^{\lambda}(\mathbf{j}') \equiv \pi_{\underline{\lambda},\underline{a}}(\mathbf{j})$.

If $\sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}}) \geq \sharp_{(i,2)}\mathbf{j}(k_0^{\mathbf{j}})$ then by (1.4.3), $\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) \neq \pi_{\underline{\lambda},\underline{a}}(\mathbf{j})$ therefore without loss of generality we assume that $\sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}}) < \sharp_{(i,2)}\mathbf{j}(k_0^{\mathbf{j}})$. Then

$$\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}) - \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}) = \lambda_{(i,2)}^{\sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}})}\left(\pi_{\underline{\lambda},\underline{a}}(\mathbf{i}^*) - \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}^*)\right),$$

where

$$\mathbf{i}^* = (\overbrace{(i,1)\cdots(i,1)}^{\sharp_{(i,1)}\mathbf{i}(k_0^{\mathbf{i}})} b_1^{\mathbf{i}} \cdots) \text{ and } \mathbf{j}^* = (\overbrace{(i,2)\cdots(i,2)}^{\sharp_{(i,2)}\mathbf{j}(k_0^{\mathbf{j}}) - \sharp_{(i,2)}\mathbf{i}(k_0^{\mathbf{i}})} b_1^{\mathbf{j}} b_2^{\mathbf{j}} \cdots).$$

Since $\lambda_{(i,2)} > \varepsilon_i > 0$ it is enough to prove that

$$f(\underline{\lambda}) = 0 \implies \|\mathrm{grad} f(\underline{\lambda})\| > 0, \tag{1.4.20}$$

where $f(\underline{\lambda}) = \pi_{\underline{\lambda},\underline{a}}(\mathbf{i}^*) - \pi_{\underline{\lambda},\underline{a}}(\mathbf{j}^*)$. Let $m = \min Q^{\mathbf{j}^*}$ then by (1.4.11) we have

$$f(\underline{\lambda}) = d^0_{\mathbf{i}^*}\left(1 + \sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d^k_{\mathbf{i}^*}}{d^0_{\mathbf{i}^*}}\lambda^k_{(i,2)} - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d^k_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}}\lambda^k_{(i,2)}\right) =$$
$$d^0_{\mathbf{i}^*}\left(1 + \sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d^k_{\mathbf{i}^*}}{d^0_{\mathbf{i}^*}}\lambda^k_{(i,2)} - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d^k_{\mathbf{j}^*}\lambda^m_{(i,2)}}{d^0_{\mathbf{i}^*}\lambda_{(i,2)}}\lambda^{k-m+1}_{(i,2)}\right).$$

Now we give upper bound for the absolute value of the coefficients. It is easy to see by Lemma 1.4.1 and Sublemma 1.4.3 that

$$\begin{aligned} \left|\frac{d^k_{\mathbf{i}^*}}{d^0_{\mathbf{i}^*}}\right| &\leq \lambda_{\max}\alpha_i && \text{for every } k \in Q_i^{\mathbf{i}^*}\setminus\{0\} \\ \left|\frac{d^m_{\mathbf{j}^*}\lambda^m_{(i,2)}}{d^0_{\mathbf{i}^*}\lambda_{(i,2)}}\right| &\leq \frac{\alpha_i^2}{\varepsilon_i} && \text{and} \\ \left|\frac{d^k_{\mathbf{j}^*}\lambda^m_{(i,2)}}{d^0_{\mathbf{i}^*}\lambda_{(i,2)}}\right| &\leq \lambda_{\max}\frac{\alpha_i^2}{\varepsilon_i} && \text{for every } k \in Q_i^{\mathbf{j}^*}\setminus\{m\}\,. \end{aligned}$$

Therefore absolute value of the coefficient of $\lambda_{(i,2)}$ is at most $\lambda_{\max}\alpha_i + \frac{\alpha_i^2}{\varepsilon_i}$ and the absolute value of the coefficient of $\lambda^k_{(i,2)}$ for $k \geq 2$ is at most $\lambda_{\max}\alpha_i + \lambda_{\max}\frac{\alpha_i^2}{\varepsilon_i}$. If $f(\widetilde{\underline{\lambda}}) = 0$ then

$$\frac{\partial f}{\partial\lambda_{(i,2)}}(\underline{\lambda}) = d^0_{\mathbf{i}^*}\left(\sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d^k_{\mathbf{i}^*}}{d^0_{\mathbf{i}^*}}k\lambda^{k-1}_{(i,2)} - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d^k_{\mathbf{j}^*}\lambda^m_{(i,2)}}{d^0_{\mathbf{i}^*}\lambda_{(i,2)}}(k-m+1)\lambda^{k-m}_{(i,2)}\right.$$
$$\left. - \sum_{k\in Q^{\mathbf{j}^*}} (m-1)\frac{d^k_{\mathbf{j}^*}\lambda^{m-2}_{(i,2)}}{d^0_{\mathbf{i}^*}}\lambda^{k-m+1}_{(i,2)}\right),$$

and by Lemma 1.2.3 we obtain that for $\lambda_{(i,2)} \in \left(\varepsilon_i, \frac{1}{1+\sqrt{\lambda_{\max}\alpha_i\left(1+\frac{\alpha_i}{\varepsilon_i}\right)}}\right)$ the following inequality holds:

$$\sum_{k\in Q^{\mathbf{i}^*}\setminus\{0\}} \frac{d^k_{\mathbf{i}^*}}{d^0_{\mathbf{i}^*}}k\lambda^{k-1}_{(i,2)} - \sum_{k\in Q^{\mathbf{j}^*}} \frac{d^k_{\mathbf{j}^*}\lambda^m_{(i,2)}}{d^0_{\mathbf{i}^*}\lambda_{(i,2)}}(k-m+1)\lambda^{k-m}_{(i,2)} < 0. \tag{1.4.21}$$

On the other hand, (1.4.15) yields that for suitable $\underline{i}', \underline{j}'$ we have

$$\frac{d^m_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}} = \frac{\frac{\underline{\lambda}^{\sharp\mathbf{j}(p^{\mathbf{j}}_{r^{\mathbf{j}}_m})}}{\lambda^m_{(i,2)}}\left(f_{\underline{j}'}(a_i) - a_i\right)}{\lambda^{k_0^{\mathbf{i}^*}}_{(i,1)}\left(f_{\underline{i}'}(a_i) - a_i\right)}.$$

Let i'_0 and j'_0 be the first element of the first component of $\underline{i}'$, $\underline{j}'$. Then by (1.4.3), $i'_0, j'_0 > i$ or $i'_0, j'_0 < i$ which implies that $\frac{d^m_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}} > 0$. Therefore by Lemma 1.4.1 we have for $\lambda_{(i,2)} < \frac{1}{1+\lambda_{\max}\alpha_i}$ that

$$\sum_{k \in Q^{\mathbf{j}^*}} (m-1)\frac{d^k_{\mathbf{j}^*}\lambda^{m-2}_{(i,2)}}{d^0_{\mathbf{i}^*}}\lambda^{k-m+1}_{(i,2)} = (m-1)\frac{d^m_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}}\lambda^{m-1}_{(i,2)}\left(1 + \sum_{k \in Q^{\mathbf{j}^*}\setminus\{m\}} \frac{d^k_{\mathbf{j}^*}}{d^m_{\mathbf{j}^*}}\lambda^{k-m}_{(i,2)}\right) \geq$$

$$(m-1)\frac{d^m_{\mathbf{j}^*}}{d^0_{\mathbf{i}^*}}\lambda^{m-1}_{(i,2)}\left(1 - \sum_{k=1}^{\infty} \lambda_{\max}\alpha_i\lambda^k_{(i,2)}\right) \geq 0. \quad (1.4.22)$$

Observe that $\frac{1}{1+\sqrt{\lambda_{\max}\alpha_i\left(1+\frac{\alpha_i}{\varepsilon_i}\right)}} < \frac{1}{1+\lambda_{\max}\alpha_i}$ holds for every $0 < \varepsilon_i < 1$. Using this (1.4.21) and (1.4.22) we have

$$f(\underline{\widetilde{\lambda}}) = 0 \implies \frac{\partial f}{\partial \lambda_{(i,2)}}(\underline{\widetilde{\lambda}}) < 0$$

which was to be proved. $\square$

Proposition 1.4.4. *For every $k \geq 2$, the IFS Ψ_k satisfies the transversality condition on*

$$\underline{\lambda}_2 \in \mathcal{T}_N(\xi) = \prod_{i \in \mathcal{N}} (\xi, \min\left\{\lambda_{(i,1)}, \frac{2}{(1+\sqrt{2})(\alpha_i^2\lambda_{\max}+2)}\right\} - \xi) \qquad (1.4.23)$$

where $\xi > 0$ is arbitrary small and

$$\alpha_i = \frac{\max\left\{a_{N-1} - a_i, a_i - a_0\right\}}{\min\left\{f_{i+1}(a_0) - a_i, a_i - f_{i-1}(a_{N-1})\right\}} \textit{ for } i \in \mathcal{N}.$$

Proof. Let

$$g_i(x) = \frac{1}{1+\sqrt{\lambda_{\max}\alpha_i\left(1+\frac{\alpha_i}{x}\right)}}.$$

We can extend g_i onto $[0,\infty)$ as $g_i(0) = 0$, which is a fixed point of g_i. It is easy to see by simple calculations that g_i is strictly monotone increasing and has a unique positive fixed point ε_i^*.

Hence, we can cover the rectangle $\prod_{i\in\mathcal{N}}(0, \min\left\{\lambda_{(i,1)}, \varepsilon_i^*\right\})$ by countable many rectangles in the type $R_{\underline{\varepsilon}}$, see (1.4.19).

It follows from Lemma 1.4.2 that for every $k \geq 2$ and $\mathbf{i}', \mathbf{j}' \in \Sigma_k$ with $\underline{i}_0 \neq \underline{j}_0$ the function $\pi_k^{\underline{\lambda}}(\mathbf{i}') - \pi_k^{\underline{\lambda}}(\mathbf{j}')$ satisfies (1.2.3) on the rectangle $\prod_{i\in\mathcal{N}}(0, \min\left\{\lambda_{(i,1)}, \varepsilon_i^*\right\})$.

Now we are going to prove that

$$\frac{2}{(\sqrt{2}+1)(\alpha_i^2\lambda_{\max}+2)} \leq \varepsilon_i^*. \tag{1.4.24}$$

To verify this, observe that

$$\varepsilon_i^* = \frac{2}{\sqrt{(\alpha_i^2\lambda_{\max}+2)^2 + 4(\alpha_i\lambda_{\max}-1)} + \alpha_i^2\lambda_{\max}+2}.$$

If the second term under the square root is non-positive, that is if $\alpha_i\lambda_{\max} \leq 1$ then clearly (1.4.24) holds. Otherwise, $\alpha_i\lambda_{\max} > 1$. Then $\alpha_i > 1$. A simple calculation yields: $4(\alpha_i\lambda_{\max}-1) \leq (\alpha_i^2\lambda_{\max}+2)^2$ which follows that (1.4.24) holds. To complete the proof we apply Lemma 1.2.2 for the rectangle on the right hand side of (1.4.23) with $\xi = 0$. $\square$

1.4.3 Hausdorff dimension

Before we prove the theorems we have to introduce a sequence of functions. For every $k \geq 2$ we introduce the function $h_{\underline{\lambda},k}(s)$ which is defined as the sum of the s-powers of the contraction ratios of the IFS Ψ_k. That is

$$h_{\underline{\lambda},k}(s) = \sum_{i=0}^{N-1} \lambda_{(i,1)}^s + \sum_{l=0}^{k-2}\left(\sum_{i\in\mathcal{N}} \lambda_{(i,2)}^s\right)^l \sum_{i\in\mathcal{N}} \sum_{j=0,j\neq i}^{N-1} \lambda_{(i,2)}^s \lambda_{(j,1)}^s. \tag{1.4.25}$$

Let $s_k(\underline{\lambda})$ be the unique solution of $h_{\underline{\lambda},k}(s) = 1$. Therefore $\dim_H \Omega_k^{\underline{\lambda}} \leq \min\{1, s_k(\underline{\lambda})\}$, where $\Omega_k^{\underline{\lambda}}$ is the attractor of Ψ_k.

Since the sequence $s_k(\underline{\lambda})$ is monotone increasing and bounded, it is convergent. It is easy to see by some algebraic manipulation that the limit of $s_k(\underline{\lambda})$ is the unique solution of

$$\sum_{i=0}^{N-1} \lambda_{(i,1)}^s + \sum_{i\in\mathcal{N}} \lambda_{(i,2)}^s \left(1 - \lambda_{(i,1)}^s\right) = 1.$$

This equation corresponds to (1.1.5).

Moreover, we need to introduce a sequence of subsets of Σ^*. Let

$$\mathcal{C}_1 = \mathcal{I} = \{(0,1), \dots, (N-1,1)\} \tag{1.4.26}$$

and by induction let

$$\mathcal{C}_{k+1} = \bigcup_{j=0}^{N-1} \bigcup_{\underline{i} \in \mathcal{C}_k} \{(j,1)\underline{i}\} \cup \bigcup_{j \in \mathcal{N}} \bigcup_{\substack{\underline{i} \in \mathcal{C}_k \\ (i_0,\kappa_0) \neq (j,1)}} \{(j,2)\underline{i}\}. \tag{1.4.27}$$

Then we can look at the elements of $\mathcal{C}_k$ either as certain sequences of length k of symbols from $\mathcal{I} \cup \mathcal{J}$ or juxtapositions of at most k elements of $\mathcal{U}_k$.

Lemma 1.4.5. *Let $\widetilde{s}_k(\underline{\lambda})$ be the unique solution of*

$$\sum_{\underline{i} \in \mathcal{C}_k} \lambda_{\underline{i}}^s = 1,$$

and let $\widetilde{s}(\underline{\lambda}) = \sup_k \widetilde{s}_k(\underline{\lambda})$ then

$$\dim_H \Omega_{\underline{\lambda},\underline{a}} \leq \min\{1, \widetilde{s}(\underline{\lambda})\}.$$

Moreover,

$$\mathcal{H}^{\widetilde{s}(\underline{\lambda})}(\Omega_{\underline{\lambda},\underline{a}}) \leq (a_{N-1} - a_0)^{\widetilde{s}(\underline{\lambda})}.$$

Note that $\widetilde{s}_k(\underline{\lambda})$ is bounded since $\mathcal{C}_k \subset (\mathcal{I} \cup \mathcal{J})^k$.

Proof. Using that for every $i \in \mathcal{N}$

$$f_{(i,1)} \circ f_{(i,2)} \equiv f_{(i,2)} \circ f_{(i,1)},$$

and $0 < \lambda_{(i,2)} < \lambda_{(i,1)} < 1$ we have that the set of closed intervals

$$\{f_{\underline{i}}([a_0, a_{N-1}])\}_{\underline{i} \in \mathcal{C}_k}$$

gives a cover of $\Omega_{\underline{\lambda},\underline{a}}$ with diameter at most $\lambda_{\max}^k$. Then

$$\mathcal{H}^{\widetilde{s}(\underline{\lambda})}_{\lambda_{\max}^k}(\Omega_{\underline{\lambda},\underline{a}}) \leq \sum_{\underline{i} \in \mathcal{C}_k} |f_{\underline{i}}([a_0, a_{N-1}])|^{\widetilde{s}(\underline{\lambda})} = (a_{N-1} - a_0)^{\widetilde{s}(\underline{\lambda})} \sum_{\underline{i} \in \mathcal{C}_k} f'_{\underline{i}}(0)^{\widetilde{s}(\underline{\lambda})} \leq$$

$$(a_{N-1} - a_0)^{\widetilde{s}(\underline{\lambda})} \underbrace{\sum_{\underline{i} \in \mathcal{C}_k} f'_{\underline{i}}(0)^{\widetilde{s}_k(\underline{\lambda})}}_{1} = (a_{N-1} - a_0)^{\widetilde{s}(\underline{\lambda})}.$$

This proves the upper bound of the dimension and the measure claim of the Lemma. $\square$

Proof of Theorem 1.1.2. Let $\xi > 0$. By the definition of $\mathcal{C}_k$ we have that for every $k \geq 1$

$$\mathcal{C}_k \subset \bigcup_{l=1}^{k} \mathcal{U}_k^l. \tag{1.4.28}$$

As it was mentioned above, every $\underline{i} \in \mathcal{C}_k$ can be decomposed as a juxtaposition $\underline{i} = \underline{j}_1 \cdots \underline{j}_r$, where each $\underline{j}_l$ is in $\mathcal{U}_k$ and $1 \leq r \leq k$. By using this fact and Proposition 1.4.4 we have that the system $\widetilde{\Psi}_k = \{f_{\underline{i}}\}_{\underline{i} \in \mathcal{C}_k}$ satisfies transversality on $\mathcal{T}_N(\xi)$. By Theorem 1.2.1 we have

$$\dim_H \widetilde{\Omega}_k^{\underline{\lambda}} = \min\{1, \widetilde{s}_k(\underline{\lambda})\} \text{ for } \mathcal{L}\text{-a.e. } \underline{\lambda}_2 \in \mathcal{T}_N(\xi), \tag{1.4.29}$$

where $\widetilde{\Omega}_k^{\underline{\lambda}}$ denotes the attractor of $\{f_{\underline{i}}\}_{\underline{i} \in \mathcal{C}_k}$. Using (1.4.28)

$$\dim_H \widetilde{\Omega}_k^{\underline{\lambda}} \leq \dim_H \Omega_k^{\underline{\lambda}}.$$

Moreover by Proposition 1.4.4 and Theorem 1.2.1 we have

$$\dim_H \Omega_k^{\underline{\lambda}} = \min\{1, s_k(\underline{\lambda})\} \text{ for } \mathcal{L}\text{-a.e. } \underline{\lambda}_2 \in \mathcal{T}_N(\xi).$$

Since $\widetilde{\Omega}_k^{\underline{\lambda}}, \Omega_k^{\underline{\lambda}} \subseteq \Omega_{\underline{\lambda},\underline{a}}$ for every $k \geq 2$ by Lemma 1.4.5 we have

$$\min\{1, \widetilde{s}_k(\underline{\lambda})\} \leq \min\{1, s_k(\underline{\lambda})\} \leq \min\{1, \widetilde{s}(\underline{\lambda})\}.$$

Since $s_k(\underline{\lambda})$ is strictly monotone increasing $\lim_{k\to\infty} s_k(\underline{\lambda}) = \sup_k s_k(\underline{\lambda})$. This implies that $\min\{1, s(\underline{\lambda})\} = \min\{1, \widetilde{s}(\underline{\lambda})\}$, moreover

$$\dim_H \Omega_{\underline{\lambda},\underline{a}} = \min\{1, s(\underline{\lambda})\}.$$

To complete the proof of the last assertion of Theorem 1.1.2 first observe that whenever $s(\underline{\lambda}) > 1$ then there exists a $k \geq 2$ such that $s_k(\underline{\lambda}) > 1$. Therefore, by Theorem 1.2.1 and Proposition 1.4.4, $\mathcal{L}(\Omega_{\underline{\lambda},\underline{a}}) \geq \mathcal{L}\left(\Omega_k^{\underline{\lambda}}\right) > 0$ for a.e. $\underline{\lambda}_2 \in \mathcal{T}_N(\xi) \cap \{\underline{\lambda}_2 : s(\underline{\lambda}) > 1\}$. Since ξ was arbitrary, this completes the proof. $\square$

1.4.4 Example

To visualize the behavior of the vector of contracting ratios we consider an easy example, where the functions of $\mathcal{F}$ are uniformly distributed with uniform contracting ratio, that is

$$\mathcal{F} = \{f_i(x) = \lambda x + i(1-\lambda)\}_{i=0}^{N-1},$$

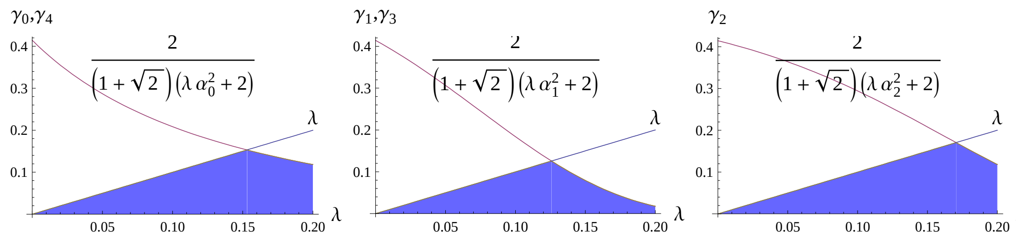

Figure 1.1: Transversality region for $N = 5$ fixed points

where $0 < \lambda < \frac{1}{N}$. Let us add to the system the following N functions:

$$\mathcal{G} = \{g_i(x) = \gamma_i x + i(1-\gamma_i)\}_{i=0}^{N-1}.$$

Note that the fixed point of both f_i and g_i is i, $i = 0, \ldots, N-1$. It is easy to see that for every $i = 1, \ldots, N-2$

$$\alpha_i = \alpha_{N-1-i} = \frac{\max\{N-1-i, i\}}{\min\{1-(i+1)\lambda, 1-(N-i)\lambda\}} \text{ and } \alpha_0 = \alpha_{N-1} = \frac{N-1}{1-\lambda},$$

where α_i is as in Theorem 1.1.2. To satisfy the assumptions of Theorem 1.1.2 it is enough to require that

$$0 < \gamma_i < \min\left\{\lambda, \frac{2}{(1+\sqrt{2})(\alpha_i^2\lambda+2)}\right\} \tag{1.4.30}$$

holds for $i = 0, \ldots, N-1$. For example, when $N = 5$ then we can choose γ_i from the appropriate shaded region of Figure 1.1. In general, first we observe that

$$\alpha_i \leq \alpha_1 = \alpha_{N-2} = \frac{N-2}{1-(N-1)\lambda},$$

holds for every $i = 0, \ldots, N-1$. So by (1.4.30) the assumptions of Theorem 1.1.2 hold if we assume that

$$0 < \gamma_i < \min\left\{\lambda, \frac{2}{(1+\sqrt{2})\left(\left(\frac{N-2}{1-(N-1)\lambda}\right)^2\lambda+2\right)}\right\}, \quad 0 \leq i \leq N-1. \tag{1.4.31}$$

We know that $0 < \lambda$ must be smaller than $1/N$. By (1.4.31) we obtain that whenever $\lambda < 0.4764/N$ holds then the assumptions of Theorem 1.1.2 are satisfied for $\gamma_i < \lambda$.

1.5 Proof of Theorem 1.1.3

To prove Theorem 1.1.3 we use the method of Bandt and Graf [BG]. More precisely, we use it in the way as it was used by Peres, Simon and Solomyak, [PSS2] with some modifications.

Without loss of generality we may assume that $s(\underline{\lambda}) \leq 1$. (Otherwise $\mathcal{H}^s(\Omega) = 0$ holds obviously.) Let us denote the local inverse of the left-shift operator σ on $\Sigma = (\mathcal{I} \cup \mathcal{J})^{\mathbb{N}}$ by $\sigma^{-1}_{(i,\kappa)}$. More precisely, for every $\mathbf{i} \in \Sigma$ let $\sigma^{-1}_{(i,\kappa)}\mathbf{i} = (i, \kappa)\mathbf{i}$. Denote $\sigma^{-1}_{\underline{i}} := \sigma^{-1}_{(i_0,\kappa_0)} \circ \cdots \circ \sigma^{-1}_{(i_n,\kappa_n)}$ for an $\underline{i} \in \Sigma^*$. Let

$$\widehat{\Sigma} = \bigcup_{k=0}^{\infty} \bigcup_{\underline{i} \in (\mathcal{I} \cup \mathcal{J})^k} \left\{ \sigma^{-1}_{\underline{i}} \mathcal{J}^{\mathbb{N}} \right\},$$

which is the subset of Σ such that every $\mathbf{i} \in \widehat{\Sigma}$ contains only finitely many symbols of $\mathcal{I}$. Then

$$\Omega_{\underline{\lambda},\underline{a}} = \pi_{\underline{\lambda},\underline{a}} \left(\widehat{\Sigma} \right) \bigcup \pi_{\underline{\lambda},\underline{a}} \left(\Sigma \backslash \widehat{\Sigma} \right).$$

Let

$$\mathcal{U}_\infty = \mathcal{I} \bigcup \left(\bigcup_{l=0}^{\infty} \bigcup_{\underline{i} \in \mathcal{J}^l} \bigcup_{i \in \mathcal{N}} \bigcup_{j=0, j \neq i}^{N-1} \{ \underline{i}(i,2)(j,1) \} \right).$$

Cf. to (1.4.16) the definition of $\mathcal{U}_k$.

Lemma 1.5.1.

$$\pi_{\underline{\lambda},\underline{a}} \left(\Sigma \backslash \widehat{\Sigma} \right) \subseteq \pi_{\underline{\lambda},\underline{a}} \left(\mathcal{U}_\infty^{\mathbb{N}} \right).$$

Proof. For every $\mathbf{i} \in \Sigma \backslash \widehat{\Sigma}$ there are at most two possibilities, it contains finitely or infinitely many blocks. If $\mathbf{i}$ contains an infinite length block (which is equivalent to $\mathbf{i}$ contains finitely many blocks) then every element in the last block can be changed to a suitable $i \in \mathcal{I}$ without the modification of the value of the natural projection.

The fact $f_{(i,1)} \circ f_{(i,2)} \equiv f_{(i,2)} \circ f_{(i,1)}$ completes the proof. □

Since Hausdorff dimension of $\pi_{\underline{\lambda},\underline{a}} \left(\widehat{\Sigma} \right)$ is equal to the Hausdorff dimension of the attractor of $\mathcal{G}$, which is the unique solution of $\sum_{i \in \mathcal{N}} \lambda^s_{(i,2)} = 1$, we have

$$\mathcal{H}^{s(\underline{\lambda})}(\Omega_{\underline{\lambda},\underline{a}}) = \mathcal{H}^{s(\underline{\lambda})} \left(\pi_{\underline{\lambda},\underline{a}} \left(\mathcal{U}_\infty^{\mathbb{N}} \right) \right). \tag{1.5.1}$$

We say that $\bar{i}$ and $\bar{j}$ elements of $\mathcal{U}_\infty^*$ (the set of finite length symbols of $\mathcal{U}_\infty$) are *incomparable* if there are no $\overline{\eta} \in \widetilde{\Sigma}_\infty^*$ such that $\bar{i} = \bar{j}\overline{\eta}$ or $\bar{j} = \bar{i}\overline{\eta}$ holds.

We define an outer measure. Let

$$\mu^s(K) = \inf\left\{ \sum_{k\in I} |U_k|^s : \text{ open, } K \subseteq \bigcup_{k\in I} U_k \right\}.$$

Lemma 1.5.2. *For measurable $K \subseteq \pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)$, $\mathcal{H}^{s(\underline{\lambda})}(K)$ coincides with the outer measure $\mu^{s(\underline{\lambda})}(K)$. Moreover,*

$$\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right) \cap f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\right) = 0$$

for every $\underline{i}, \underline{j} \in \mathcal{U}_\infty^*$ such that $\underline{i}$ and $\underline{j}$ are incomparable.

The proof of this lemma coincides with the proof of [BG, Proposition 3].

Proof of Theorem 1.1.3. Without loss of generality we can assume that for every $i \in \mathcal{N}$ the quotient $\frac{\log \lambda_{(i,2)}}{\log \lambda_{(i,1)}}$ is irrational. Otherwise $\dim_H \Omega_{\underline{\lambda},\underline{a}} < s(\underline{\lambda})$ trivially.

Let $\underline{i} = (i,1)\cdots(i,1)(j,\kappa_1)$ and $\underline{j} = (i,2)\cdots(i,2)(j,\kappa_2)$ such that $\sharp_{(i,1)}(\underline{i}) = k_1$, $\sharp_{(i,2)}(\underline{j}) = k_2$ and $j \neq i$. Then

$$f_{\underline{i}}^{-1} \circ f_{\underline{j}}(x) = \frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}} x + \left(1 - \frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}}\right)\left(a_j(1 - \frac{1}{\lambda_{(i,1)}}) + \frac{a_i}{\lambda_{(i,1)}}\right).$$

Therefore for every $\delta > 0$ there exists $\underline{i}, \underline{j} \in \mathcal{U}_\infty^*$ incomparable words such that

$$\sup_{x\in[a_{(0,1)},a_{(n-1,1)}]} \left\{ \left| x - f_{\underline{i}}^{-1} \circ f_{\underline{j}}(x) \right| \right\} < \delta. \tag{1.5.2}$$

Indirectly, let us suppose that $\mathcal{H}^{s(\underline{\lambda})}(\Omega_{\underline{\lambda},\underline{a}}) > 0$ and let $\xi \in (1, \frac{3}{2})$. Since $\Omega_{\underline{\lambda},\underline{a}}$ is compact, there exists $U_1, \ldots, U_l$ finite cover of $\Omega_{\underline{\lambda},\underline{a}}$ such that

$$\sum_{m=1}^{l} |U_l|^{s(\underline{\lambda})} < \xi \mathcal{H}^{s(\underline{\lambda})}(\Omega_{\underline{\lambda},\underline{a}}) = \xi \mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_{\infty,n}^{\mathbb{N}}\right)\right) \tag{1.5.3}$$

by (1.5.1). Let

$$\delta = \inf\left\{ |a - x| : a \in \Omega_{\underline{\lambda},\underline{a}}, x \notin \bigcup_{m=1}^{l} U_m \right\} \leq$$

$$\inf\left\{ |a - x| : a \in \pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right), x \notin \bigcup_{m=1}^{l} U_m \right\}. \tag{1.5.4}$$

Let $\underline{i}, \underline{j} \in \mathcal{U}_\infty^*$ such that

$$\sup_{x\in[a_0,a_{N-1}]}\left\{\left|x - f_{\underline{i}}^{-1}\circ f_{\underline{j}}(x)\right|\right\} < \delta$$

and $\frac{\lambda_{(i,2)}^{k_2}}{\lambda_{(i,1)}^{k_1}} > 2-\xi$. Therefore by (1.5.4) we have

$$f_{\underline{i}}^{-1}\circ f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right) \subseteq \bigcup_{m=1}^{l} U_m$$

and

$$f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\bigcup f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right) \subseteq f_{\underline{i}}\left(\bigcup_{m=1}^{l} U_m\right).$$

So, we have by Lemma 1.5.2 that

$$\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\right) + \mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\right) = \\ \mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\bigcup f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\right)$$

which is less than or equal to

$$\sum_{m=1}^{l}\left|f_{\underline{i}}(U_m)\right|^{s(\underline{\lambda})} = \lambda_{(i,1)}^{k_1 s(\underline{\lambda})}\sum_{m=1}^{l}\left|U_m\right|^{s(\underline{\lambda})} < \lambda_{(i,1)}^{k_1 s(\underline{\lambda})}\xi\mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)$$

In the last inequality we have used (1.5.3) and (1.5.1).

However, by the definition of Hausdorff measure,

$$\mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{i}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\right) + \mathcal{H}^{s(\underline{\lambda})}\left(f_{\underline{j}}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right)\right) = \\ \lambda_{(i,1)}^{k_1 s(\underline{\lambda})}\mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right) + \lambda_{(i,2)}^{k_2 s(\underline{\lambda})}\mathcal{H}^{s(\underline{\lambda})}\left(\pi_{\underline{\lambda},\underline{a}}\left(\mathcal{U}_\infty^{\mathbb{N}}\right)\right).$$

Since we assumed that $\mathcal{H}^{s(\underline{\lambda})}\left(\Omega_{\underline{\lambda},\underline{a}}\right) > 0$ and by Lemma 1.4.5, $\mathcal{H}^{s(\underline{\lambda})}\left(\Omega_{\underline{\lambda},\underline{a}}\right)$ is finite, by (1.5.1) we have $2-\xi < \xi - 1$ which is a contradiction. $\square$

Chapter 2

Sub-additive pressure of Iterated Function Systems with triangular maps

2.1 Definitions and Statements

Let $M \subset \mathbb{R}^n$ be a non-empty, open and bounded set, and let $F_i : M \mapsto M$ contractive maps for every $i = 1, ..., l$. For an $\mathbf{i} = i_1 i_2 ... i_k,\ i_j \in \{1, ..., l\}$, we write $F_{\mathbf{i}}(\underline{x}) = F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_n}(\underline{x})$. Our principal assumption about the maps F_i, $i = 1, ..., l$ is that

$$F_i(x_1, ..., x_n) = \left(f_i^1(x_1), f_i^2(x_1, x_2), ..., f_i^n(x_1, ..., x_n)\right), \tag{2.1.1}$$

and $F_i(x_1, ..., x_n) \in C^{1+\varepsilon}(\overline{M})$ for every $i = 1, ..., l$. Moreover we require that $D_{\underline{x}} F_i$ is a regular (non-singular matrix) for every $\underline{x} \in \overline{M}$ and every $i \in \{1, ..., l\}$. Denote the elements of $D_{\underline{x}} F_{\mathbf{i}}$ by $x_{ij}(\mathbf{i}, \underline{x})$.

Proposition 2.1.1. *There exists a real constant* $0 < C < \infty$ *such that*

$$C^{-1} < \frac{|x_{ii}(\mathbf{i}, \underline{x})|}{|x_{ii}(\mathbf{i}, \underline{y})|} < C \tag{2.1.2}$$

for every $\underline{x}, \underline{y} \in \overline{M}$ *and for every* $\mathbf{i} \in \{1, ..., l\}^*$.

Proof. Let $G_i^{(m)} : \mathbb{R}^m \mapsto \mathbb{R}^m$ for every integer m between 1 and n, be the restriction of F_i to the first m component, i.e.:

$$G_i^{(m)}(x_1, ..., x_m) := \left(f_i^1(x_1), f_i^2(x_1, x_2), ..., f_i^m(x_1, ..., x_m)\right).$$

From [Pes1, Page 198; Propostion 20.1 (3)] it follows that for every $\underline{x}, \underline{y} \in \overline{M}$, for every $\mathbf{i} \in \{1, ..., l\}^*$ finite sequence, and for $1 \le m \le n$ there exists a real $0 < C_m < \infty$ constant that

$$C_m^{-1} < \frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})} < C_m.$$

Since for every m, the matrix $D_{\underline{x}} G_{\mathbf{i}}^{(m)}$ is a lower triangular matrix, the Jacobian is the following

$$\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x}) = \left| x_{11}(\mathbf{i}, \underline{x}) \cdots x_{mm}(\mathbf{i}, \underline{x}) \right|.$$

Therefore for every integer $1 \le m < n$ and for every $\underline{x}, \underline{y} \in M$

$$\frac{C_m^{-1}}{C_{m+1}} < \frac{\frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})}}{\frac{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{y})}} < \frac{C_m}{C_{m+1}^{-1}}$$

and

$$\frac{\frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})}}{\frac{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{y})}} = \frac{\left| x_{m+1m+1}\left(\mathbf{i}, \underline{y}\right) \right|}{\left| x_{m+1m+1}\left(\mathbf{i}, \underline{x}\right) \right|}.$$

Then $C := \max_{1 \le m < n-1} \left\{ \frac{C_m}{C_{m+1}^{-1}}, C_1 \right\}$ choice completes the proof of the proposition. □

The singular values of a linear contraction T are the positive square roots of the eigenvalues of TT^*, where T^* is the transpose of T. Let $\alpha_k(D_{\underline{x}} F_{\mathbf{i}})$ be the k-th greatest singular value of the matrix $D_{\underline{x}} F_{\mathbf{i}}$. The singular value function ϕ^s is defined for $0 \le s \le n$ as

$$\phi^s(D_{\underline{x}} F_{\mathbf{i}}) := \alpha_1(D_{\underline{x}} F_{\mathbf{i}}) ... \alpha_{k-1}(D_{\underline{x}} F_{\mathbf{i}}) \alpha_k(D_{\underline{x}} F_{\mathbf{i}})^{s-k+1} \tag{2.1.3}$$

where $k - 1 < s \le k$ and k is a positive integer. We define the maximum and the minimum of the singular value function as

$$\overline{\phi}^s(\mathbf{i}) := \max_{\underline{x} \in \overline{M}} \phi^s(D_{\underline{x}} F_{\mathbf{i}}) \ , \ \underline{\phi}^s(\mathbf{i}) := \min_{\underline{x} \in \overline{M}} \phi^s(D_{\underline{x}} F_{\mathbf{i}}).$$

We define the sub-additive pressure after K. Falconer [Fa4] and L. Barreira [Barr]:

$$P(s) := \lim_{k \to \infty} \frac{1}{k} \log \sum_{|\mathbf{i}|=k} \overline{\phi}^s(\mathbf{i}) \tag{2.1.4}$$

and define the lower pressure:

$$\underline{P}(s) := \liminf_{k\to\infty} \frac{1}{k} \log \sum_{|\mathbf{i}|=k} \underline{\phi}^s(\mathbf{i}). \tag{2.1.5}$$

Theorem 2.1.2. *Let $0 \le s \le n$. If $F_1, ..., F_l$ are contractive maps in form (2.1.1) and $F_i \in C^{1+\varepsilon}$ for every $1 \le i \le l$ then*

$$P(s) = \underline{P}(s).$$

The proof of Theorem 2.1.2 is based on [B4] which uses the technique of [FM]. The result of the chapter was part of author's Master Thesis.

2.2 Proof of Theorem 2.1.2

The m-dimensional exterior algebra Φ^m is a vector space spanned by formal elements $v_1 \wedge ... \wedge v_m$ with $v_i \in \mathbb{R}^n$ such that $v_1 \wedge ... \wedge v_m = 0$ if $v_i = v_j$ for some $i \ne j$, and such that interchanging two different elements reverses the sign, i.e. $v_1 \wedge ...v_i...v_j... \wedge v_m = -v_1 \wedge ...v_j...v_i... \wedge v_m$, if $i \ne j$. Then Φ^m has dimension $\binom{n}{m}$ with basis $\{e_{j_1} \wedge ... \wedge e_{j_m} : 1 \le j_1 < ... < j_m \le n\}$ where $e_1, ...e_n$ are a given set of orthonormal vectors in $\mathbb{R}^n$.

Let us define a scalar product on Φ^m in the following way. Let

$$< v_1 \wedge \cdots \wedge v_m, u_1 \wedge \cdots \wedge u_m >_{\Phi^m} = \det\left((< v_i, u_j >)_{i,j=1...m}\right),$$

where $< .,. >$ is the usual scalar product on $\mathbb{R}^n$. One can extend $< .,. >_{\Phi^m}$ to every element of Φ^m the natural way. Then Φ^m becomes a Hilbert-space. Let us define the norm $\|.\|$ on Φ^m by $< .,. >_{\Phi^m}$ the usual way. Then it is easy to see that $\|v_1 \wedge ... \wedge v_m\|$ is equal to the absolute m-dimensional volume of the parallelepiped spanned by $v_1, ...v_m$, for every $v_1 \wedge ... \wedge v_m$, see [K, p. 44].

We may also define an other norm $\|.\|_\infty$ on Φ^m by

$$\left\| \sum_{1\le i_1<...<i_m\le m} \lambda_{i_1...i_m}(e_{i_1} \wedge ... \wedge e_{i_m}) \right\|_\infty := \max\left|\lambda_{i_1...i_m}\right|.$$

If $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is linear then there is an induced linear mapping $\widetilde{T} : \Phi^m \mapsto \Phi^m$ given by

$$\widetilde{T}(v_1 \wedge ... \wedge v_m) := (Tv_1) \wedge ... \wedge (Tv_m).$$

The norms on Φ^m induce norms on the space of linear mappings $\mathfrak{L}(\Phi^m, \Phi^m)$ in the usual way by

$$\left\|\widetilde{T}\right\| = \sup_{w\in\Phi^m, w\neq 0} \frac{\left\|\widetilde{T}w\right\|}{\|w\|}.$$

Then with respect to the norm $\|.\|$

$$\left\|\widetilde{T}\right\| = \phi^m(T) \tag{2.2.1}$$

and with respect to the $\|.\|_\infty$

$$\left\|\widetilde{T}\right\|_\infty = \max\left\{\left|T^{(m)}\right| : T^{(m)} \text{ is an } m\times m \text{ minor of } T\ \right\}, \tag{2.2.2}$$

where $T^{(m)} = T\binom{r_1,\dots r_m}{s_1,\dots,s_m}$ is the determinant of that $m \times m$ minor of $n \times n$ matrix T which is determined by the elements of T in the rows $1 \leq r_1 < \dots < r_m \leq n$ and columns $1 \leq s_1 < \dots < s_m \leq n$. The space of linear mappings $\mathfrak{L}(\Phi^m, \Phi^m)$ is of finite dimension $\binom{n}{m}^2$. Since any two norms on a finite dimensional normed space are equivalent, there are constants $0 < c_1 < c_2 < \infty$ depending only on n and m such that

$$c_1 \left\|\widetilde{T}\right\|_\infty \leq \left\|\widetilde{T}\right\| \leq c_2 \left\|\widetilde{T}\right\|_\infty. \tag{2.2.3}$$

Now we notice several lemmas relating to minors of matrices. We will need some well-known lemmas.

Lemma 2.2.1. *Let $x_i \geq 0$, $i = 1, \dots, m$ and $p \in \mathbb{R}^+$.*

1. *If $p > 1$, then $(x_1^p + \dots + x_m^p) \leq (x_1 + \dots + x_m)^p \leq m^{p-1}(x_1^p + \dots + x_m^p)$*
2. *If $0 < p \leq 1$, then $m^{p-1}(x_1^p+\dots+x_m^p) \leq (x_1+\dots+x_m)^p \leq (x_1^p+\dots+x_m^p)$.*

Lemma 2.2.2. *Let a_n be a sequence of real numbers such that $a_{n+m} \leq a_n + a_m$. Then there exists $\lim_{n\to\infty} \frac{a_n}{n}$ and it equals to $\inf_n \frac{a_n}{n}$.*

We first look at the expansion of $m\times m$ minors of the product of k matrices $A = A_1 A_2 \cdots A_k$, where for $i = 1, \dots, k$

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i & \dots & a_{1n}^i \\ a_{21}^i & a_{22}^i & \dots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^i & a_{n2}^i & \dots & a_{nn}^i \end{bmatrix}.$$

Lemma 2.2.3. *For $1 \leq m \leq n$, the $m \times m$ minors of $A = A_1 \cdots A_k$ have formal expansions in terms of the entries of the A_i of the form*

$$A\binom{r_1, ...r_m}{s_1, ..., s_m} = \sum_{c_1,...,c_k} \pm a^1_{1(c_1)} \cdots a^1_{m(c_1)} a^2_{1(c_2)} \cdots a^2_{m(c_2)} \cdots a^k_{1(c_k)} \cdots a^k_{m(c_k)}$$

such that for each $i = 1, ..., k$, the $a^i_{1(c_i)} \cdots a^i_{m(c_i)}$ are distinct entries a^i_{rs} of A_i. In particular, for each i, $1(c_i), ..., m(c_i)$ denote pairs (r, s) corresponding to entries in m different rows and columns of the ith matrix A_i, and the sum is over all such entry combinations $(c_1, ..., c_k)$ with appropriate sign $\pm$.

The proof of this Lemma can be found in [FM, Lemmma 2.2]. Now we consider lower triangular matrices. For $i = 1, ..., k$, let

$$U_i = \begin{bmatrix} u^i_1 & 0 & ... & 0 \\ u^i_{21} & u^i_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u^i_{n1} & u^i_{n2} & ... & u^i_n \end{bmatrix}.$$

We consider the product

$$U = U_1 \cdots U_k = \begin{bmatrix} u_1 & 0 & ... & 0 \\ u_{21} & u_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & ... & u_n \end{bmatrix}.$$

We note that

$$u_{rs} = \sum_{r \geq r_1 \geq ... \geq r_{k-1} \geq s} u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s} \quad 1 \leq r \leq s \leq n \tag{2.2.4}$$

since all other products are 0.

Lemma 2.2.4. *With the notation as above, let $U_1, ..., U_k$ be lower triangular matrices and $U = U_1 \cdots U_k$. Then*

1. *If $r < s$, $u_{rs} = 0$*
2. *If $r = s$, $u_{rs} \equiv u_r = u^1_r \cdots u^k_r$*
3. *If $r > s$, then the sum (2.2.4) for u_{rs} has at most $k^{r-s} \leq k^{n-1}$ non-zero terms. Moreover, each non-zero summand $u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}$ has at most $n-1$ non-diagonal terms in the product, i.e. terms with $r \neq r_1$ or $r_i \neq r_{i+1}$ or $r_{k-1} \neq s$.*

The proof can also be found in [FM, Lemma 2.3] for upper-triangular matrices. Now we extend the estimate of Lemma 2.2.4 to minors.

Lemma 2.2.5. *Let $U_1, ..., U_k$ and U be lower triangular matrices as above. Then each $m \times m$ minor of U has an expansion of the form*

$$U\begin{pmatrix} r_1, ...r_m \\ s_1, ..., s_m \end{pmatrix} = \sum_{c_1,...,c_k} \pm u^1_{1(c_1)} u^2_{1(c_2)} \cdots u^k_{1(c_k)} \cdots u^1_{m(c_1)} u^2_{m(c_2)} \cdots u^k_{m(c_k)}$$

where $1(c_i), ..., m(c_i)$ are as in Lemma 2.2.3 and

1. *there are at most $m!k^{m(n-1)}$ terms in the sum which are non-zero,*

2. *each summand contains at most $(n-1)^m$ non-diagonal elements in the product.*

The proof is equivalent to the proof of [FM, Lemma 2.4]. Before we prove Theorem 2.1.2, we define two sums.

$$H(s,r) = \max_{\substack{j_1,...,j_{m-1} \\ j'_1,...,j'_m}} \sum_{|\mathbf{i}|=r} (d_{j_1j_1}(\mathbf{i}) \cdots d_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s} (d_{j'_1j'_1}(\mathbf{i}) \cdots d_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \tag{2.2.5}$$

where $m-1 < s \leq m$ and $d_{jj}(\mathbf{i}) = \inf_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$. Moreover

$$T(s,r) = \max_{\substack{j_1,...,j_{m-1} \\ j'_1,...,j'_m}} \sum_{|\mathbf{i}|=r} (t_{j_1j_1}(\mathbf{i}) \cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s} (t_{j'_1j'_1}(\mathbf{i}) \cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \tag{2.2.6}$$

where $m-1 < s \leq m$ and $t_{jj}(\mathbf{i}) = \sup_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$. It is easy to see from Proposition 2.1.1 and the definition of the two sums that

$$H(s,r) \leq T(s,r) \leq C^s H(s,r). \tag{2.2.7}$$

Lemma 2.2.6. *For every positive integers r, z, $T(s, r+z) \leq T(s,r)T(s,z)$. Moreover $\lim_{r\to\infty} \frac{\log T(s,r)}{r}$ exists and equal with $\inf_r \frac{\log T(s,r)}{r}$.*

Proof of Lemma 2.2.6. From the definition of $T(s,r)$ it follows

$$T(s,r+z) = \max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r+z} (t_{j_1j_1}(\mathbf{i})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s}(t_{j'_1j'_1}(\mathbf{i})\cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \leq$$

$$\leq \max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}} (\sum_{|\mathbf{i}|=r}\sum_{|\mathbf{h}|=z}((t_{j_1j_1}(\mathbf{i})t_{j_1j_1}(\mathbf{h})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i})t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}\times$$

$$\times (t_{j'_1j'_1}(\mathbf{i})t_{j'_1j'_1}(\mathbf{h})\cdots t_{j'_mj'_m}(\mathbf{i})t_{j'_mj'_m}(\mathbf{h}))^{s-m+1}) =$$

$$= \max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}} (\sum_{|\mathbf{i}|=r}(t_{j_1j_1}(\mathbf{i})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s}(t_{j'_1j'_1}(\mathbf{i})\cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1}\times$$

$$\times \sum_{|\mathbf{h}|=z}(t_{j_1j_1}(\mathbf{h})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}(t_{j'_1j'_1}(\mathbf{h})\cdots t_{j'_mj'_m}(\mathbf{h}))^{s-m+1})) \leq$$

$$\leq T(s,r)T(s,z).$$

The existence of the limit follows from Lemma 2.2.2. □

The proof of Theorem 2.1.2 follows the method of the proof of [FM, Theorem 2.5], but our theorem is not a consequence of it. The most important difference is that the functions in [FM] are affine maps. So the derivatives in our case are not constant matrices. Moreover, in the proof of [FM, Theorem 2.5], the singular value functions and the minors of the derivative matrices were compared. During the proof of Theorem 2.1.2 we will do this as well, however, we have to introduce in the proof a new IFS, which will be the r-th iteration of the original IFS, to take separation between the growth rate of the non-zero and the non-diagonal terms of the minors of the derivative matrices.

To control the consequences of the phenomenon of not constant matrices, we have to state the following lemma.

Lemma 2.2.7. *Let X be a compact subset of $\mathbb{R}^n$ and let $\{f_i\}$ be finitely many continuous, real valued functions. Then*

$$\sup_{\underline{x}\in X}\max_i f_i(\underline{x}) = \max_i \sup_{\underline{x}\in X} f_i(\underline{x}).$$

Proof of Lemma 2.2.7. Since X is compact, we have $\underline{x}_i \in X$ such that $f_i(\underline{x}_i) = \sup_{\underline{x}} f_i(\underline{x})$. Therefore

$$\sup_{\underline{x}}\max_i f_i(\underline{x}) \leq \max_i\sup_{\underline{x}} f_i(\underline{x}) = \max_i f_i(\underline{x}_i) = \max_{i,j} f_i(\underline{x}_j) = \max_j\max_i f_i(\underline{x}_j)$$
$$\leq \sup_{\underline{x}}\max_i f_i(\underline{x}),$$

which was to be proved. □

Proof of Theorem 2.1.2. Let

$$\{G_h\}_{h=1}^{l^r} = \{F_{i_1 \ldots i_r}\}_{i_1=1,\ldots,i_r=1}^{l,\ldots,l}. \tag{2.2.8}$$

In this case an index h is equivalent to a $\mathbf{i} \in \{1, ..., l\}^r$ finite sequence, length r. Let us define

$$\overline{\phi'}^s(\mathbf{h}) = \sup_{\underline{x}} \phi^s(D_{\underline{x}} G_{\mathbf{h}}),$$
$$\underline{\phi'}^s(\mathbf{h}) = \inf_{\underline{x}} \phi^s(D_{\underline{x}} G_{\mathbf{h}})$$

for $\mathbf{h} \in \{1, ..., l^r\}^*$, corresponding to IFS $\{G_h\}_{h=1}^{l^r}$, see (2.1.3).

It is easy to see that

$$\sum_{|\mathbf{i}|=kr} \phi^s(D_{\underline{x}} F_{\mathbf{i}}) = \sum_{|\mathbf{h}|=k} \phi^s(D_{\underline{x}} G_{\mathbf{h}}), \tag{2.2.9}$$

where $\mathbf{i} \in \{1, ..., l\}^{kr}$ and $\mathbf{h} \in \{1, ..., l^r\}^k$. The elements of $D_{\underline{x}} G_h$, denoted by $y_{ij}(h, \underline{x})$, are equal to $x_{ij}(\mathbf{i}, \underline{x})$ for an appropriate finite sequence $\mathbf{i}$ with length r. It is very simple to see that

$$\phi^s(D_{\underline{x}} G_{\mathbf{h}}) = (\phi^{m-1}(D_{\underline{x}} G_{\mathbf{h}}))^{m-s} (\phi^m(D_{\underline{x}} G_{\mathbf{h}}))^{s-m+1},$$

where $m-1 < s \le m$. By using relations (2.2.1), (2.2.2) and (2.2.3) it follows that

$$\phi^m(D_{\underline{x}} G_{\mathbf{h}}) \ge c_2 \max \left\{ \left| D_{\underline{x}} G_{\mathbf{h}}^{(m)} \right| : D_{\underline{x}} G_{\mathbf{h}}^{(m)} \text{ is an } m \times m \text{ minor of } D_{\underline{x}} G_{\mathbf{h}} \right\}.$$

The maximum $m \times m$ minor of $D_{\underline{x}} G_{\mathbf{h}}$ is at least the largest product of m distinct diagonal elements of $D_{\underline{x}} G_{\mathbf{h}}$, since such products are themselves minors of triangular matrices. Therefore

$$\underline{\phi'}^s(\mathbf{h}) \ge$$
$$c_2^s \left(\inf_{\underline{x}} \left| y_{j_1 j_1}(\mathbf{h}, \underline{x}) \cdots y_{j_{m-1} j_{m-1}}(\mathbf{h}, \underline{x}) \right| \right)^{m-s} \left(\inf_{\underline{x}} \left| y_{j_1' j_1'}(\mathbf{h}, \underline{x}) \cdots y_{j_m' j_m'}(\mathbf{h}, \underline{x}) \right| \right)^{s-m+1}$$

for every $j_1, ..., j_{m-1}, j_1', ..., j_m'$.

By the chain rule $D_{\underline{x}} G_{\mathbf{h}} = D_{G_{h_2 \ldots h_k}(\underline{x})} G_{h_1} D_{G_{h_3 \ldots h_k}(\underline{x})} G_{h_2} \cdots D_{\underline{x}} G_{h_k}$, $y_{jj}(\mathbf{h}, \underline{x}) = y_{jj}(h_1, G_{h_2 \ldots h_k}(\underline{x}))\, y_{jj}(h_2, G_{h_3 \ldots h_k}(\underline{x})) \cdots y_{jj}(h_k, \underline{x})$. It follows with the notation $\inf_{\underline{x}} |y_{jj}(h, \underline{x})| = d'_{jj}(h)$ that

$$\inf_{\underline{x}} \left| y_{j_1 j_1}(\mathbf{h}, \underline{x}) \cdots y_{j_{m-1} j_{m-1}}(\mathbf{h}, \underline{x}) \right|^{m-s} \inf_{\underline{x}} \left| y_{j_1' j_1'}(\mathbf{h}, \underline{x}) \cdots y_{j_m' j_m'}(\mathbf{h}, \underline{x}) \right|^{s-m+1} \ge$$
$$\ge (d'_{j_1 j_1}(h_1) \cdots d'_{j_1 j_1}(h_k) d'_{j_2 j_2}(h_1) \cdots d'_{j_{m-1} j_{m-1}}(h_1) \cdots d'_{j_{m-1} j_{m-1}}(h_k))^{m-s} \times$$
$$\times (d'_{j_1' j_1'}(h_1) \cdots d'_{j_1' j_1'}(h_k) d'_{j_2' j_2'}(h_1) \cdots d'_{j_m' j_m'}(h_1) \cdots d'_{j_m' j_m'}(h_k))^{s-m+1}.$$

The next inequality follows from the rearrangement of the product

$$\sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \geq$$
$$c_2^s \sum_{|\mathbf{h}|=k} (d'_{j_1j_1}(h_1)\cdots d'_{j_{m-1}j_{m-1}}(h_1))^{m-s}(d'_{j'_1j'_1}(h_1)\cdots d'_{j'_mj'_m}(h_1))^{s-m+1}\cdots$$
$$\cdots(d'_{j_1j_1}(h_k)\cdots d'_{j_{m-1}j_{m-1}}(h_k))^{m-s}(d'_{j'_1j'_1}(h_k)\cdots d'_{j'_mj'_m}(h_k))^{s-m+1} =$$
$$c_2^s((d'_{j_1j_1}(1)\cdots d'_{j_{m-1}j_{m-1}}(1))^{m-s}(d'_{j'_1j'_1}(1)\cdots d'_{j'_mj'_m}(1))^{s-m+1}+\cdots$$
$$\cdots+(d'_{j_1j_1}(l^r)\cdots d'_{j_{m-1}j_{m-1}}(l^r))^{m-s}(d'_{j'_1j'_1}(l^r)\cdots d'_{j'_mj'_m}(l^r))^{s-m+1})^k.$$

The inequality above is true for every $j_1,...,j_{m-1},j'_1,...,j'_m$, therefore we obtain the maximum. From the definition of $\{G_h\}_{h=1}^{l^r}$ and $H(s,r)$, see (2.2.5) and (2.2.8), it follows

$$\sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \geq c_2^s H(s,r)^k. \tag{2.2.10}$$

By using relations (2.2.1), (2.2.2) and (2.2.3) it follows similarly that

$$\phi^m(D_{\underline{x}}G_{\mathbf{h}}) \leq c_1 \max\left\{\left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right| : D_{\underline{x}}G_{\mathbf{h}}^{(m)} \text{ is an } m\times m \text{ minor of } D_{\underline{x}}G_{\mathbf{h}}\right\}.$$

Therefore

$$\sum_{|\mathbf{h}|=k} \overline{\phi'}^s(\mathbf{i}) \leq$$
$$c_1^2 \sum_{|\mathbf{h}|=k}\left(\sup_{\underline{x}} \max_{m-1\times m-1 \text{ minor}} \left|D_{\underline{x}}G_{\mathbf{h}}^{(m-1)}\right|\right)^{m-s}\left(\sup_{\underline{x}} \max_{m\times m \text{ minor}} \left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right|\right)^{s-m+1}.$$

By Lemma 2.2.7, the order of the supremum and the maximum can be changed in this situation and we can estimate the sum with

$$C \max_{\left\{\substack{r_1,...,r_{m-1}\\ s_1,...,s_{m-1}}\right\}} \max_{\left\{\substack{r'_1,...,r'_m\\ s'_1,...,s'_m}\right\}} \sum_{|\mathbf{h}|=k}\left(\sup_{\underline{x}}\left|D_{\underline{x}}G_{\mathbf{h}}^{(m-1)}\right|\right)^{m-s}\left(\sup_{\underline{x}}\left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right|\right)^{s-m+1}$$

where $r_1,...,r_{m-1}$ are the rows and $s_1,...,s_{m-1}$ are the columns of the $(m-1)\times(m-1)$ minor, and $r'_1,...,r'_m$ are the rows and $s'_1,...,s'_m$ are the columns of $m \times m$ minor, moreover $C = c_1^2 \binom{n}{m}^2 \binom{n}{m-1}^2$. By the chain rule $D_{\underline{x}} G_{\mathbf{h}} = D_{G_{h_2 \dots h_k}(\underline{x})} G_{h_1} D_{G_{h_3 \dots h_k}(\underline{x})} G_{h_2} \dots D_{\underline{x}} G_{h_k}$, we obtain

$$
\begin{aligned}
& D_{\underline{x}} G_{\mathbf{h}} \begin{pmatrix} r_1, \dots, r_m \\ s_1, \dots, s_m \end{pmatrix} = \\
& \sum_{c_1, \dots, c_k} \pm y_{1(c_1)}(h_1, G_{h_2 \dots h_k}(\underline{x})) \dots y_{1(c_k)}(h_k, \underline{x}) \dots y_{m(c_1)}(h_1, G_{h_2 \dots h_k}(\underline{x})) \times \\
& \times y_{m(c_2)}(h_2, G_{h_3 \dots h_k}(\underline{x})) \dots y_{m(c_k)}(h_k, \underline{x}).
\end{aligned}
\tag{2.2.11}
$$

Therefore

$$
\begin{aligned}
& \sup_{\underline{x}} \left| D_{\underline{x}} G_{\mathbf{h}}^{(m)} \right| \le \\
& \quad \sum_{c_1, \dots, c_k} \sup_{\underline{x}} \left| y_{1(c_1)}(h_1, \underline{x}) \right| \dots \sup_{\underline{x}} \left| y_{1(c_k)}(h_k, \underline{x}) \right| \dots \sup_{\underline{x}} \left| y_{m(c_1)}(h_1, \underline{x}) \right| \times \\
& \qquad \times \sup_{\underline{x}} \left| y_{m(c_2)}(h_2, \underline{x}) \right| \dots \sup_{\underline{x}} \left| y_{m(c_k)}(h_k, \underline{x}) \right|.
\end{aligned}
\tag{2.2.12}
$$

Denote by $t'_{kl}(h) := \sup_{\underline{x}} |y_{kl}(h, \underline{x})|$ the supremum. It follows from the inequality (2.2.12) and the Lemma 2.2.1

$$
\begin{aligned}
& \sum_{|\mathbf{h}|=k} \sup_{\underline{x}} \left| D_{\underline{x}} G_{\mathbf{h}}^{(m-1)} \right|^{m-s} \sup_{\underline{x}} \left| D_{\underline{x}} G_{\mathbf{h}}^{(m)} \right|^{s-m+1} \le \\
& \sum_{\substack{c_1, \dots, c_k \\ c'_1, \dots, c'_k}} ((t'_{1(c_1)}(1) \dots t'_{m-1(c_1)}(1))^{m-s} (t'_{1(c'_1)}(1) \dots t'_{m(c'_1)}(1))^{s-m+1} + \\
& \dots + (t'_{1(c_1)}(l^r) \dots t'_{m-1(c_1)}(l^r))^{m-s} (t'_{1(c'_1)}(l^r) \dots t'_{m(c'_1)}(l^r))^{s-m+1}) \times \\
& \dots \times ((t'_{1(c_k)}(1) \dots t'_{m-1(c_k)}(1))^{m-s} (t'_{1(c'_k)}(1) \dots t'_{m(c'_k)}(1))^{s-m+1} + \\
& \dots + (t'_{1(c_k)}(l^r) \dots t'_{m-1(c_k)}(l^r))^{m-s} (t'_{1(c'_k)}(l^r) \dots t'_{m(c'_k)}(l^r))^{s-m+1}).
\end{aligned}
\tag{2.2.13}
$$

Lemma 2.2.5 implies that each non-zero term of the sum above has at most $2(n-1)^m = b$ of the indices $1(c_1), \dots, m-1(c_1), \dots, 1(c_k), \dots, m-1(c_k)$, $1(c'_1), \dots, m(c'_1), \dots, 1(c'_k), \dots, m(c'_k)$ that are non-diagonal terms. Thus, for each set of indices $(c_1, \dots, c_k, c'_1, \dots, c'_k)$, we have at least $k-b$ of these indices such that $1(c_r), \dots, m-1(c_r), 1(c'_r), \dots, m(c'_r)$ are all diagonal entries.

For such c_r and c'_r

$$((t'_{1(c_r)}(1)...t'_{m-1(c_r)}(1))^{m-s}(t'_{1(c'_r)}(1)...t'_{m(c'_r)}(1))^{s-m+1}+...$$
$$...+(t'_{1(c_r)}(l^r)...t'_{m-1(c_1)}(l))^{m-s}(t'_{1(c'_r)}(l^r)...t'_{m(c'_r)}(l^r))^{s-m+1})\leq$$
$$\max_{\{j_1,...,j_{m-1}\},\{j'_1,...,j'_m\}}((t'_{j_1j_1}(1)...t'_{j_{m-1}j_{m-1}}(1))^{m-s}(t'_{j'_1}(1)...t'_{j'_m}(1))^{s-m+1}+...$$
$$...+(t'_{j_1j_1}(l^r)...t'_{j_{m-1}j_{m-1}}(l^r))^{m-s}(t'_{j'_1}(l^r)...t'_{j'_mj'_m}(l^r))^{s-m+1})=T(s,r).$$

The last equality follows from the definition of $\{G_h\}_{h=1}^{l^r}$ and $T(s,r)$. Hence from (2.2.13)

$$\begin{aligned}&\sum_{|\mathbf{h}|=k}\sup_{\underline{x}}\left|D_{\underline{x}}G_{\mathbf{h}}^{(m-1)}\right|^{m-s}\sup_{\underline{x}}\left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right|^{s-m+1}\leq\\&\sum_{\substack{c_1,...,c_k\\c'_1,...,c'_k}}\left(T(s,r)^{k-b}(l^r)^b\right)\leq c''k^ql^{rb}T(s,r)^{k-b},\end{aligned}\tag{2.2.14}$$

where, using Lemma 2.2.5, $c''=m!(m-1)!$ and $q=(2m-1)(n-1)$.

By using (2.2.7), (2.2.9), (2.2.10) and (2.2.14)

$$\sum_{|\mathbf{i}|=kr}\overline{\phi}^s(\mathbf{i})=\sum_{|\mathbf{h}|=k}\overline{\phi'}^s(\mathbf{h})\leq c''k^ql^{rb}T(s,r)^{k-b}\leq c''(C^s)^kk^ql^{rb}T(s,r)^{-b}H(s,r)^k\leq$$
$$c'''(C^s)^kk^ql^{rb}T(s,r)^{-b}\sum_{|\mathbf{h}|=k}\underline{\phi}'^s(\mathbf{h})=c'''k^ql^{rb}T(s,r)^{-b}\sum_{|\mathbf{i}|=kr}\underline{\phi}^s(\mathbf{i}).\tag{2.2.15}$$

We take the logarithm of both sides of the inequality and we divide by kr, then

$$\begin{aligned}&\frac{\log\sum_{|\mathbf{i}|=kr}\overline{\phi}^s(\mathbf{i})}{kr}\leq\\&\frac{\log c'''}{kr}+\frac{q\log k}{kr}+\frac{rb\log l}{kr}+\frac{(kb)\log(C^s)}{kr}+\frac{-b\log T(s,r)}{kr}+\frac{\log\sum_{|\mathbf{i}|=kr}\underline{\phi}^s(\mathbf{i})}{kr}\end{aligned}\tag{2.2.16}$$

is true for every positive k,r integer. We take limit inferior of both sides. The limit exists in the left-hand side of the inequality and in the right-hand side the limit of every term exists and equals zero except the last term. Therefore

$$P(s)\leq\underline{P}(s)$$

While the opposite relation is trivial this completes the proof. □

The next corollary is a consequence of the previous proof.

Corollary 2.2.8. *For $0 \le s \le n$. If $F_1, ..., F_l$ contractive maps in form (2.1.1) and $F_i \in C^{1+\varepsilon}$ for every $1 \le i \le l$ then*

$$P(s) = \lim_{r\to\infty} \frac{1}{r} \log(\max_{\substack{j_1,\dots,j_{m-1} \\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} \left(|x_{j_1 j_1}(\mathbf{i}, \underline{x})| \dots |x_{j_{m-1} j_{m-1}}(\mathbf{i}, \underline{x})| \right)^{m-s} \times$$

$$\times \left(|x_{j'_1 j'_1}(\mathbf{i}, \underline{x})| \dots |x_{j'_m j'_m}(\mathbf{i}, \underline{x})| \right)^{s-m+1}) \quad (2.2.17)$$

for every $\underline{x} \in M$.

Proof. It follows from inequality (2.2.7) that the $\lim_{r\to\infty} \frac{\log H(s,r)}{r}$ exists and

$$\lim_{r\to\infty} \frac{\log H(s,r)}{r} = \lim_{r\to\infty} \frac{\log T(s,r)}{r}.$$

It is clear by (2.2.15) that $\lim_{r\to\infty} \frac{\log T(s,r)}{r} = P(s)$. Because of the definition $H(s,r), T(s,r)$, this is exactly what we want to prove. □

2.3 Some applications

In this section we compute the Hausdorff dimension of some non-conformal IFS by using Corollary 2.2.8. It follows from [Zh] that the Hausdorff dimension is less than or equal to s_0 where $P(s_0) = 0$. We will show some examples where the root is exactly the dimension.

2.3.1 Example 1

The easiest example is the non-linear modified Sierpiński triangle, see Figure 2.1. Let

$$T = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and $T_i\underline{x} = T\underline{x} + \underline{v}_i$ for $i = 1, 2, 3$, where $v_1 = \binom{0}{0}, v_2 = \binom{\frac{2}{3}}{0}, v_3 = \binom{\frac{1}{3}}{\frac{2}{3}}$. We call the attractor of this IFS as modified Sierpiński gasket. Clearly, the Hausdorff and box dimension is $\frac{\ln 3}{\ln 3} = 1$.

Let $f_i : [0,1] \mapsto [0,1]$ be functions for $i = 1, 2, 3$ in $C^{1+\varepsilon}$ such that

$$F_i(x, y) = (\frac{x}{3} + v_i, y/3 + f_i(x) + w_i)$$

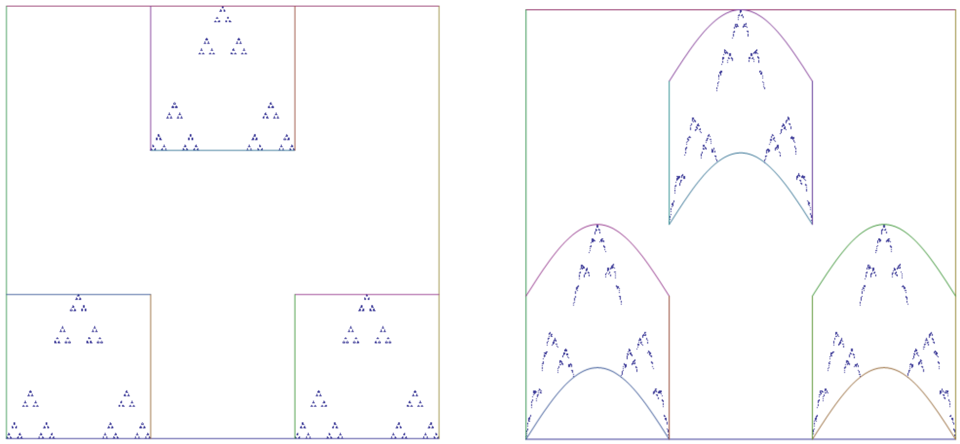

Figure 2.1: The image of the modified and the non-linear modified Sierpinski-triangular for $f_i(x) = \sin(\pi x)/6$ for every i.

are contractions where $(v_1, w_1) = (0,0), (v_2, w_2) = \left(\frac{2}{3}, 0\right), (v_3, w_3) = \left(\frac{1}{3}, \frac{1}{2}\right)$. We can consider the attractor as a non-linear Sierpiński triangle.

We prove that the Hausdorff dimension of the non-linear modified Sierpiński gasket is equal to 1, with the assumption that for $i = 1, 2, 3$, $f_i \in C^{1+\varepsilon}$ and

$$(f_i'(x))^2 + \mid f_i'(x) \mid \sqrt{(f_i'(x))^2 + \frac{4}{9}} < \frac{16}{9}.$$

We need this assumption to provide that the $\{F_1, F_2, F_3\}$ is contracting.

From the definition in this case it is easy to see that $x_{11}(\mathbf{i}, \underline{x}) = x_{22}(\mathbf{i}, \underline{x}) = \frac{1}{3}^{|\mathbf{i}|}$. We can suppose that $1 \le s < 2$. Then by using Corollary 2.2.8

$$P(s) = \lim_{r \to \infty} \frac{1}{r} \log \left(\max_{\substack{j_1, \\ j_1', j_2'}} \sum_{|\mathbf{i}|=r} \left(|x_{j_1 j_1}(\mathbf{i}, \underline{x})|\right)^{2-s} \times \left(\left|x_{j_1' j_1'}(\mathbf{i}, \underline{x})\right| \left|x_{j_2' j_2'}(\mathbf{i}, \underline{x})\right| \right)^{s-2+1} \right) =$$

$$\lim_{r \to \infty} \frac{1}{r} \log \left(\sum_{|\mathbf{i}|=r} \left(\frac{1}{3}^{|\mathbf{i}|} \right)^{2-s} \left(\frac{1}{3}^{|\mathbf{i}|} \frac{1}{3}^{|\mathbf{i}|} \right)^{s-1} \right) = \lim_{r \to \infty} \frac{1}{r} \log \left(3^r \frac{1}{3}^{sr} \right) = \log 3 - s \log 3.$$

It is easy to see that $P(s) = 0$ if and only if $s = 1$, which is the upper bound of the Hausdorff dimension of the modified non-linear attractor, this follows from [Zh]. To get a lower bound it is enough to project it onto the x axis and we get the $[0,1]$ interval.

2.3.2 Example 2

The next example is a non-linear perturbation of a self-affine IFS, where the attractors of the original and the perturbed IFS are both graphs of real functions mapping $[0,1]$ into itself, see Figure 2.2. Let $c_1, c_2 \in (0,1)$. Consider the following self-affine IFS

$$g_0(\underline{x}) = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \underline{x}, \quad g_1(\underline{x}) = \begin{bmatrix} 1-c_1 & 0 \\ 0 & 1-c_2 \end{bmatrix} \underline{x} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

It is easy to see that the attractor of this IFS has Hausdorff dimension 1 since it is a graph of a strictly monotone function. We perturb this IFS as follows, let $\{\widetilde{g}_0, \widetilde{g}_1\}$ be the following

$$\widetilde{g}_0(x,y) = \begin{bmatrix} c_1 x \\ c_2 y + f_0(x) \end{bmatrix}, \quad \widetilde{g}_1(x,y) = \begin{bmatrix} (1-c_1)x + c_1 \\ (1-c_2)y + c_2 + f_1(x) \end{bmatrix}.$$

where $f_0, f_1 \in C^{1+\varepsilon}$ and f_i are periodic with period 1. Moreover we suppose that $\widetilde{g}_0, \widetilde{g}_1$ are contractions, namely the following inequalities hold

$$\begin{aligned} & c_1^2 + (f_0'(x))^2 + c_2^2 + \sqrt{(c_1^2 + (f_0'(x))^2 + c_2^2)^2 - 4c_1^2 c_2^2} < 2 \\ & (1-c_1)^2 + (f_1'(x))^2 + (1-c_2)^2 \\ & + \sqrt{((1-c_1)^2 + (f_1'(x))^2 + (1-c_2)^2)^2 - 4(1-c_1)^2(1-c_2)^2} < 2. \end{aligned}$$

In this case the Hausdorff dimension of the modified attractor is greater than or equal to 1 since the projection to the x axis is the $[0,1]$ interval. To get an upper bound we have to use the sub-additive pressure and Corollary 2.2.8. For every $\mathbf{i} \in \{0,1\}^*$ we have $x_{11}(\mathbf{i}, \underline{x}) = c_1^{\sharp_0 \mathbf{i}}(1-c_1)^{\sharp_1 \mathbf{i}}$ and $x_{22}(\mathbf{i}, \underline{x}) = c_2^{\sharp_0 \mathbf{i}}(1-c_2)^{\sharp_1 \mathbf{i}}$ where $\sharp_j \mathbf{i}$ is the number of js in $\mathbf{i}$. Then

$$\begin{aligned} & \max_j \sum_{|\mathbf{i}|=r} x_{jj}(\mathbf{i}, \underline{x})^{2-s} \left(x_{11}(\mathbf{i}, \underline{x})\, x_{22}(\mathbf{i}, \underline{x}) \right)^{s-2+1} = \\ & \max_j \sum_{|\mathbf{i}|=r} c_j^{(2-s)\sharp_0 \mathbf{i}} (1-c_j)^{(2-s)\sharp_1 \mathbf{i}} c_1^{(s-1)\sharp_0 \mathbf{i}} (1-c_1)^{(s-1)\sharp_1 \mathbf{i}} c_2^{(s-1)\sharp_0 \mathbf{i}} (1-c_2)^{(s-1)\sharp_1 \mathbf{i}} = \\ & \max \left\{ (c_1 c_2^{s-1} + (1-c_1)(1-c_2)^{s-1})^r, (c_2 c_1^{s-1} + (1-c_2)(1-c_1)^{s-1})^r \right\}. \end{aligned}$$

Therefore by formula (2.2.17) we have $P(1) = 0$, and by [Zh] 1 is an upper bound for Hausdorff dimension, so the Hausdorff dimension is exactly 1.

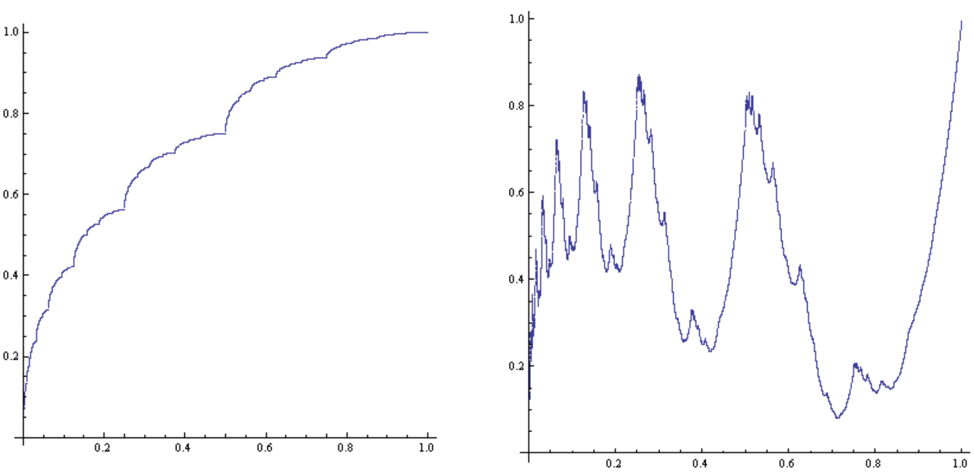

Figure 2.2: The images of the attractors in case $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{4}$, $f_0(x) = (1 - c_2)\sin(\pi x)$, $f_1(x) = -c_2 \sin(\pi x)$

Chapter 3

Box Dimension of the generalized 4-corner set

3.1 Definitions and Statements

In this chapter we consider the generalized 4-corner set $\Lambda(\underline{\alpha}, \underline{\beta})$ which is the attractor of the self-affine iterated function system (IFS) of Figure 4 on page 6. Precisely, let $\Psi = \{f_0(\underline{x}), f_1(\underline{x}), f_2(\underline{x}), f_3(\underline{x})\}$ be an iterated function system on the real plane and $\Lambda(\underline{\alpha}, \underline{\beta})$ its attractor, where

$$\begin{aligned} f_0(\underline{x}) &= \begin{pmatrix} \alpha_0 & 0 \\ 0 & \beta_0 \end{pmatrix} \underline{x}, \\ f_1(\underline{x}) &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1-\beta_1 \end{pmatrix}, \\ f_2(\underline{x}) &= \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} \underline{x} + \begin{pmatrix} 1-\alpha_2 \\ 0 \end{pmatrix}, \\ f_3(\underline{x}) &= \begin{pmatrix} \alpha_3 & 0 \\ 0 & \beta_3 \end{pmatrix} \underline{x} + \begin{pmatrix} 1-\alpha_3 \\ 1-\beta_3 \end{pmatrix}. \end{aligned} \tag{3.1.1}$$

Before we compute the box dimension of the generalized 4-corner set, we state a general theorem on the box dimension of diagonally self-affine sets.

Let

$$f_i(x, y) = (\alpha_i x + t_i, \beta_i y + u_i) \tag{3.1.2}$$

for $i = 0, \ldots, m$ such that

$$\begin{aligned} &0 < \alpha_i, \beta_i < 1 \\ &f_i([0,1]^2) \subseteq [0,1]^2 \text{ for } i = 0, \ldots, m \\ &f_i((0,1)^2) \bigcap f_j((0,1)^2) = \emptyset \text{ for } i \neq j. \end{aligned} \tag{3.1.3}$$

Denote the attractor of $\Psi = \{f_i(x,y)\}_{i=0}^m$ by Λ and define $\mathrm{proj}_x\Lambda$ (and $\mathrm{proj}_y\Lambda$) as the projection of Λ onto the x-axis (and y-axis, respectively).

Theorem 3.1.1. *Let f_i be in form (3.1.2) for $i = 0,\dots,m$ and let us suppose that $\Psi = \{f_i(x,y)\}_{i=0}^m$ satisfies (3.1.3). Then the attractor Λ of Ψ satisfies*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\}$$

where d_α and d_β are the unique solutions of

$$\sum_{i=0}^m \alpha_i^{s_\alpha}\beta_i^{d_\alpha - s_\alpha} = 1 \text{ and } \sum_{i=0}^m \beta_i^{s_\beta}\alpha_i^{d_\beta - s_\beta} = 1,$$

where $s_\alpha = \dim_B \mathrm{proj}_x\Lambda$ and $s_\beta = \dim_B \mathrm{proj}_y\Lambda$.

Using this and [SS, Theorem 2.1] we can compute the box dimension of the attractor at least for almost all translations such that (3.1.3) holds.

Corollary 3.1.2. *Let f_i be in form (3.1.2) for $i = 0,\dots,m$ and let $\mathcal{T} \subset \mathbb{R}^{2m+2}$ be the set of translation vectors such that $\Psi = \{f_i(x,y)\}_{i=0}^m$ satisfies (3.1.3). Then the attractor Λ of Ψ satisfies*

$$\dim_B \Lambda = \max\{d_\alpha, d_\beta\} \text{ for almost every translations in } \mathcal{T} \text{ with respect to } 2m+2\text{-dimensional Lebesgue measure}$$

where d_α and d_β are the unique solutions of

$$\sum_{i=0}^m \alpha_i^{\min\{1,s_\alpha\}}\beta_i^{d_\alpha - \min\{1,s_\alpha\}} = 1 \text{ and } \sum_{i=0}^m \beta_i^{\min\left\{1,s_\beta\right\}}\alpha_i^{d_\beta - \min\left\{1,s_\beta\right\}} = 1,$$

and s_α, s_β are the unique solutions of

$$\sum_{i=0}^m \alpha_i^{s_\alpha} = 1 \text{ and } \sum_{i=0}^m \beta_i^{s_\beta} = 1.$$

Now, using the main theorem of this chapter and the earlier result of Chapter 1 we are able to calculate the box dimension of the generalized 4-corner set for almost every parameters.

Theorem 3.1.3. *Let $\Lambda(\underline{\alpha},\underline{\beta})$ be the attractor of the self-affine IFS of Figure 4. Then*

$$\dim_B \Lambda(\underline{\alpha},\underline{\beta}) = \max\{d_\alpha, d_\beta\}, \text{ for Lebesgue almost every } (\underline{\alpha},\underline{\beta}) \text{ such that}$$
$$\max\{\alpha_i + \alpha_{i+2}, \beta_i + \beta_{i+2}\} < 1 \text{ and } \min\{\alpha_i + \alpha_{3-i}, \beta_i + \beta_{3-i}\} < 1 \text{ for } i = 0,1 \tag{3.1.4}$$

where d_α and d_β are defined in two steps. First we define two numbers s_α, s_β as the unique solutions of the equations

$$\alpha_0^{s_\alpha} + \alpha_1^{s_\alpha} + \alpha_2^{s_\alpha} + \alpha_3^{s_\alpha} - \alpha_0^{s_\alpha}\alpha_1^{s_\alpha} - \alpha_2^{s_\alpha}\alpha_3^{s_\alpha} = 1$$
$$\beta_0^{s_\beta} + \beta_1^{s_\beta} + \beta_2^{s_\beta} + \beta_3^{s_\beta} - \beta_0^{s_\beta}\beta_2^{s_\beta} - \beta_1^{s_\beta}\beta_3^{s_\beta} = 1.$$

Then we can define d_α and d_β as the unique real numbers such that

$$\sum_{i=0}^{3} \alpha_i^{\min\{1,s_\alpha\}} \beta_i^{d_\alpha - \min\{1,s_\alpha\}} = 1, \quad \sum_{i=0}^{3} \beta_i^{\min\left\{1,s_\beta\right\}} \alpha_i^{d_\beta - \min\left\{1,s_\beta\right\}} = 1. \quad (3.1.5)$$

Proof. The proof is an easy consequence of Theorem 1.1.1 and Theorem 3.1.1. □

The proof of Theorem 3.1.1 is based on [B1] which follows the method of Feng, Wang [FW, Theorem 1] and Barański [Bara, Theorem B] with slight modifications. The proof of Theorem 3.1.1 is decomposed into three lemmas, Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.3.

3.2 Proof of Theorem 3.1.1

Let us introduce some notation. Let $\Sigma = \{0, \ldots, m\}^{\mathbb{N}}$ and $\Sigma^* = \bigcup_{n=0}^{\infty} \{0, \ldots, m\}^n$. Denote the right cut on Σ^* by δ. More precisely, let $\delta(\emptyset) = \emptyset$ and

$$\delta(i_0 \cdots i_k) = i_0 \cdots i_{k-1}.$$

For any $\underline{i} \in \Sigma^*$ let $f_{\underline{i}} = f_{i_0} \circ \cdots \circ f_{i_k}$ and $\alpha_{\underline{i}} = \alpha_{i_0} \cdots \alpha_{i_k}$, $\beta_{\underline{i}} = \beta_{i_0} \cdots \beta_{i_k}$. For every $0 < r < 1$ let

$$\Delta_r = \{\underline{i} \in \Sigma^* : \min\{\alpha_{\delta\underline{i}}, \beta_{\delta\underline{i}}\} \geq r, \min\{\alpha_{\underline{i}}, \beta_{\underline{i}}\} < r\}$$

and

$$\Delta_r^\alpha = \{\underline{i} \in \Delta_r : \alpha_{\underline{i}} \geq \beta_{\underline{i}}\} \text{ and } \Delta_r^\beta = \{\underline{i} \in \Delta_r : \alpha_{\underline{i}} < \beta_{\underline{i}}\}.$$

It is easy to see that Δ_r is a partition of Σ.

For every $\underline{i} \in \Delta_r^\alpha$ we set $\omega_\alpha(\underline{i}) = \left[\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right]$ and similarly, for every $\underline{i} \in \Delta_r^\beta$ we set $\omega_\beta(\underline{i}) = \left[\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right]$. For any $\underline{i} \in \Delta_r^\alpha$ we divide $f_{\underline{i}}([0,1]^2)$ into $\omega_\alpha(\underline{i})$ equal rectangles with height $\beta_{\underline{i}}$ and width $\alpha_{\underline{i}}/\omega_\alpha(\underline{i})$, denote the kth rectangle by $R_k^\alpha(\underline{i})$ for $k = 1, \ldots, \omega_\alpha(\underline{i})$. Similarly, for $\underline{i} \in \Delta_r^\beta$ we divide $f_{\underline{i}}([0,1]^2)$ into $\omega_\beta(\underline{i})$ equal rectangles with width $\alpha_{\underline{i}}$ and height $\beta_{\underline{i}}/\omega_\beta(\underline{i})$ and denote the kth rectangle by $R_k^\beta(\underline{i})$ for $k = 1, \ldots, \omega_\beta(\underline{i})$.

Let

$$C_r^\alpha = \{R_k^\alpha(\underline{i}) : \underline{i} \in \Delta_r^\alpha, 1 \le k \le \omega_\alpha(\underline{i}), R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \ne \emptyset\}$$
$$C_r^\beta = \left\{R_k^\beta(\underline{i}) : \underline{i} \in \Delta_r^\beta, 1 \le k \le \omega_\beta(\underline{i}), R_k^\beta(\underline{i}) \cap f_{\underline{i}}(\Lambda) \ne \emptyset\right\},$$

moreover

$$\eta_r^\alpha(\underline{i}) = \sharp\{R_k^\alpha(\underline{i}) : 1 \le k \le \omega_\alpha(\underline{i}), R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \ne \emptyset\} \text{ for } \underline{i} \in \Delta_r^\alpha \text{ and}$$
$$\eta_r^\alpha(\underline{i}) = \sharp\left\{R_k^\beta(\underline{i}) : 1 \le k \le \omega_\beta(\underline{i}), R_k^\beta(\underline{i}) \cap f_{\underline{i}}(\Lambda) \ne \emptyset\right\} \text{ for } \underline{i} \in \Delta_r^\beta.$$

Lemma 3.2.1. *Let f_i be as in form (3.1.2) for $i = 0, \dots, m$ and let us suppose that $\Psi = \{f_i(x,y)\}_{i=0}^m$ satisfies (3.1.3). Moreover, let $\widetilde{N}_r = \sharp\left(C_r^\alpha \cup C_r^\beta\right)$. Then the attractor Λ of Ψ satisfies*

$$\overline{\dim}_B\Lambda = \limsup_{r\to 0+} \frac{\log \widetilde{N}_r}{-\log r} \text{ and } \underline{\dim}_B\Lambda = \liminf_{r\to 0+} \frac{\log \widetilde{N}_r}{-\log r}.$$

Proof. Denote the minimal number of squares with side length r covering the attractor Λ by N_r.

By definition $C_r^\alpha \cup C_r^\beta$ covers Λ and since for every $c \ge 1$ real number $\frac{1}{2}c \le [c] \le c$ we have that every rectangle in $C_r^\alpha \cup C_r^\beta$ has side length at most $2r$. Therefore

$$N_{2r} \le \widetilde{N}_r.$$

Let $\alpha_{\min} = \min_{i=0,\dots,m} \alpha_i$ and $\beta_{\min} = \min_{i=0,\dots,m} \beta_i$, moreover let $\rho = \min\{\alpha_{\min}, \beta_{\min}\}$.

Then every rectangle in $C_r^\alpha \cup C_r^\beta$ have side length at least ρr. Therefore, by condition (3.1.3), every square with side length $\frac{\rho}{2}r$ can intersect at most 4 rectangles in $C_r^\alpha \cup C_r^\beta$, which implies that

$$4N_{\frac{\rho}{2}r} \ge \widetilde{N}_r.$$

One can finish the proof using the definition of the lower and upper box dimension. $\square$

For $\underline{i} \in \Delta_r^\alpha$ by some simple manipulation we get that

$$\eta_r^\alpha(\underline{i}) = \sharp\{R_k^\alpha(\underline{i}) : 1 \le k \le \omega_\alpha(\underline{i}), R_k^\alpha(\underline{i}) \cap f_{\underline{i}}(\Lambda) \ne \emptyset\} =$$
$$\sharp\left\{\left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] \times [0,1] : 1 \le k \le \omega_\alpha(\underline{i}), \left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] \times [0,1] \cap \Lambda \ne \emptyset\right\} =$$
$$\sharp\left\{\left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] : 1 \le k \le \omega_\alpha(\underline{i}), \left[\frac{k-1}{\omega_\alpha(\underline{i})}, \frac{k}{\omega_\alpha(\underline{i})}\right] \cap \mathrm{proj}_x\Lambda \ne \emptyset\right\}. \quad (3.2.1)$$

and by similar arguments for $\underline{i} \in \Delta_r^\beta$

$$\eta_r^\beta(\underline{i}) = \sharp\left\{\left[\frac{k-1}{\omega_\beta(\underline{i})}, \frac{k}{\omega_\beta(\underline{i})}\right] : 1 \leq k \leq \omega_\beta(\underline{i}), \left[\frac{k-1}{\omega_\beta(\underline{i})}, \frac{k}{\omega_\beta(\underline{i})}\right] \cap \mathrm{proj}_y\Lambda \neq \emptyset\right\}. \tag{3.2.2}$$

Let us divide the unit interval into $n \in \mathbb{N}$ equal parts and denote $N_{\frac{1}{n}}(\mathrm{proj}_x\Lambda)$ (and $N_{\frac{1}{n}}(\mathrm{proj}_y\Lambda)$) the number of intervals with length $\frac{1}{n}$ intersect the set $\mathrm{proj}_x\Lambda$ (and $\mathrm{proj}_y\Lambda$, respectively). Since $\mathrm{proj}_x\Lambda$ and $\mathrm{proj}_y\Lambda$ are self-similar sets, the box dimensions exist, therefore for every $\varepsilon > 0$ exists a $c = c(\varepsilon) > 0$ such that for every integer $n \geq 1$

$$\begin{aligned} c^{-1}n^{s_\alpha-\varepsilon} \leq N_{\frac{1}{n}}(\mathrm{proj}_x\Lambda) \leq cn^{s_\alpha+\varepsilon} \text{ and} \\ c^{-1}n^{s_\beta-\varepsilon} \leq N_{\frac{1}{n}}(\mathrm{proj}_y\Lambda) \leq cn^{s_\beta+\varepsilon}, \end{aligned} \tag{3.2.3}$$

where $s_\alpha = \dim_B \mathrm{proj}_x\Lambda$ and $s_\beta = \dim_B \mathrm{proj}_y\Lambda$. Using (3.2.1) and (3.2.2) we have

$$\begin{aligned} \widetilde{N}_r = \sum_{\underline{i}\in\Delta_r^\alpha} \eta_r^\alpha(\underline{i}) + \sum_{\underline{i}\in\Delta_r^\beta} \eta_r^\beta(\underline{i}) \leq c\sum_{\underline{i}\in\Delta_r^\alpha} \omega_\alpha(\underline{i})^{s_\alpha+\varepsilon} + c\sum_{\underline{i}\in\Delta_r^\beta} \omega_\beta(\underline{i})^{s_\beta+\varepsilon} \leq \\ c\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon} + c\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon} \end{aligned} \tag{3.2.4}$$

and similarly

$$\widetilde{N}_r \geq c^{-1}2^{-(s_\alpha-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon} + c^{-1}2^{-(s_\beta-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}. \tag{3.2.5}$$

Let $d_\alpha(t)$ and $d_\beta(t)$ be the unique solutions for $t \geq -\min\{s_\alpha, s_\beta\}$ of

$$\sum_{i=0}^m \left(\frac{\alpha_i}{\beta_i}\right)^{s_\alpha+t}\beta_i^{d_\alpha(t)} = 1 \text{ and } \sum_{i=0}^m \left(\frac{\beta_i}{\alpha_i}\right)^{s_\beta+t}\alpha_i^{d_\beta(t)} = 1.$$

We remark that $d_\alpha(0) = d_\alpha$ and $d_\beta(0) = d_\beta$.

Lemma 3.2.2. *Let f_i be in form (3.1.2) for $i = 0, \ldots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^m$ satisfies (3.1.3), then the attractor Λ of Ψ satisfies that $\overline{\dim}_B\Lambda \leq \max\{d_\alpha, d_\beta\}$.*

Proof. Let $\varepsilon > 0$ be arbitrary small. Then by (3.2.4)

$$\frac{\log \widetilde{N}_r}{-\log r} \leq \frac{\log c}{-\log r} + \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\right)}{-\log r} \leq \frac{\log c}{-\log r} +$$

$$\max\left\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\right\}\left(1+\frac{\log\rho}{\log r}\right) + \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)}\right)}{-\log r}.$$

Since Δ_r is a partition, $\sum_{\underline{i}\in\Delta_r}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} = 1$ and $\sum_{\underline{i}\in\Delta_r}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)} = 1$ which implies that

$$\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+\varepsilon}\beta_{\underline{i}}^{d_\alpha(\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta+\varepsilon}\alpha_{\underline{i}}^{d_\beta(\varepsilon)} \leq 2.$$

Therefore

$$\frac{\log \widetilde{N}_r}{-\log r} \leq \frac{\log c}{-\log r} + \max\left\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\right\}\left(1+\frac{\log\rho}{\log r}\right) + \frac{\log 2}{-\log r}.$$

Taking limit superior as r tends to 0 and by Lemma 3.2.1

$$\overline{\dim}_B\Lambda \leq \max\left\{d_\alpha(\varepsilon), d_\beta(\varepsilon)\right\}$$

for every $\varepsilon > 0$. Finally, since $\varepsilon > 0$ was arbitrary, we proved the lemma. □

Lemma 3.2.3. *Let f_i be in form (3.1.2) for $i = 0, \ldots, m$ and let us suppose that $\Psi = \{f_i(x,y)\}_{i=0}^m$ satisfies (3.1.3), then*

$$\underline{\dim}_B\Lambda \geq \max\left\{d_\alpha, d_\beta\right\}.$$

Before we prove the lower bound of the lower box dimension, we have to state another lemma about the dimension of the projections. To state this lemma we need a sublemma about the partitions of Σ. First let us introduce some notation. Let $\mathcal{G}$ be a partition of Σ containing only cylinder sets, and denote $\lceil\mathcal{G}\rceil$ the length of the longest and denote $\lfloor\mathcal{G}\rfloor$ the length of the shortest cylinder set of $\mathcal{G}$. h

Sublemma 3.2.4. *Let $\mathcal{G}$ be a partition of $\Sigma = \{0, \ldots, m\}^{\mathbb{N}}$ containing only cylinder sets and let γ_i, $i = 0, \ldots, m$ be positive real numbers such that $\sum_{i=0}^m \gamma_i > 1$. Then*

$$\sum_{\underline{i}\in\mathcal{G}}\gamma_{\mathbf{i}} \geq \left(\sum_{i=0}^m \gamma_i\right)^{\lfloor\mathcal{G}\rfloor}.$$

Proof. We prove the statement of the sublemma by induction for the length of the longest cylinder set of $\mathcal{G}$.

For $\lceil \mathcal{G} \rceil = 1$ the statement holds trivially. Let us suppose that the statement of the sublemma is true for every partition in which the length of the longest cylinder set is equal to n . Let $\mathcal{G}$ be a partition containing only cylinder sets with $\lceil \mathcal{G} \rceil = n + 1$.

If $\lceil \mathcal{G} \rceil = \lfloor \mathcal{G} \rfloor$ then the statement is true since

$$\sum_{\underline{i} \in \mathcal{G}} \gamma_{\underline{i}} = \left(\sum_{i=0}^{m} \gamma_i \right)^{\lfloor \mathcal{G} \rfloor} .$$

Therefore without loss of generality we may assume that $\lfloor \mathcal{G} \rfloor < \lceil \mathcal{G} \rceil$. Let $[i_0 \cdots i_n] \in \mathcal{G}$ be one of the longest cylinder sets of $\mathcal{G}$. Since $\mathcal{G}$ is a partition of Σ, $[i_0 \cdots i_{n-1} j] \in \mathcal{G}$ for every $j = 0, \ldots, m$. Using this fact we can define a partition $\mathcal{G}_2$ such that for every $\underline{i} \in \mathcal{G}$ with length strictly less than $n+1$, $\underline{i} \in \mathcal{G}_2$ and for every $\underline{i} \in \mathcal{G}$ with length $n+1$, $\underline{i}|_n \in \mathcal{G}_2$. Then

$$\sum_{\underline{i} \in \mathcal{G}} \gamma_{\underline{i}} \geq \sum_{\underline{i} \in \mathcal{G}_2} \gamma_{\underline{i}} \geq \left(\sum_{i=0}^{m} \gamma_i \right)^{\lfloor \mathcal{G} \rfloor} .$$

In the last inequality we used the inductional assumption and $\lfloor \mathcal{G} \rfloor = \lfloor \mathcal{G}_2 \rfloor$ by the definition of $\mathcal{G}_2$. $\square$

Lemma 3.2.5. *Let f_i be in form (3.1.2) for $i = 0, \ldots, m$ and let us suppose that $\Psi = \{f_i(x, y)\}_{i=0}^{m}$ satisfies (3.1.3), then*

$$\sum_{i=0}^{m} \alpha_i^{s_\alpha} \beta_i^{s_\beta} \leq 1. \tag{3.2.6}$$

Proof. We begin the proof of the lemma by dividing the $[0, 1]$ interval on the x and y axis into r long intervals. Let $\varepsilon > 0$ be arbitrary small but fixed. Let us take the intervals which intersect $\mathrm{proj}_x \Lambda$ on the x axis and $\mathrm{proj}_y \Lambda$ on the y axis, moreover take the left and the right neighbor interval of those intervals. Then for every sufficiently small r the number of intervals on the x axis (and y axis) is at most $3 \left(\frac{1}{r} \right)^{s_\alpha + \varepsilon}$ (and $3 \left(\frac{1}{r} \right)^{s_\beta + \varepsilon}$). Let us take the direct product of these intervals. It is easy to see that the cover constructed in this way covers the approximate squares $C_r^\alpha \cup C_r^\beta$ and this implies that the area of $C_r^\alpha \cup C_r^\beta$ is less than or equal to the area of the squares constructed above.

That is

$$9\left(\frac{1}{r}\right)^{s_\alpha+\varepsilon}\left(\frac{1}{r}\right)^{s_\beta+\varepsilon} r^2 \geq c^{-1}\sum_{\underline{i}\in\Delta_r^\alpha}\beta_{\underline{i}}\frac{\alpha_{\underline{i}}}{\omega_\alpha(\underline{i})}\omega_\alpha(\underline{i})^{s_\alpha-\varepsilon}+c^{-1}\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}\frac{\beta_{\underline{i}}}{\omega_\beta(\underline{i})}\omega_\beta(\underline{i})^{s_\beta-\varepsilon}$$
$$\geq c^{-1}\sum_{\underline{i}\in\Delta_r^\alpha}\beta_{\underline{i}}^2\omega_\alpha(\underline{i})^{s_\alpha-\varepsilon}+c^{-1}\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}^2\omega_\beta(\underline{i})^{s_\beta-\varepsilon},$$

where c is a constant depending only on ε as in (3.2.3). By simple algebraic manipulations and using the definitions of $\omega_\alpha(\underline{i}), \omega_\beta(\underline{i})$ and $\Delta_r^\alpha, \Delta_r^\beta$ we have

$$\sum_{\underline{i}\in\Delta_r^\alpha}\beta_{\underline{i}}^2\omega_\alpha(\underline{i})^{s_\alpha-\varepsilon}r^{s_\alpha+s_\beta-2} \geq c_1 r^\varepsilon\sum_{\underline{i}\in\Delta_r^\alpha}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}, \text{ and}$$
$$\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\mathbf{i}}^2\omega_\beta(\underline{i})^{s_\beta-\varepsilon}r^{s_\alpha+s_\beta-2} \geq c_1 r^\varepsilon\sum_{\underline{i}\in\Delta_r^\beta}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta},$$

where c_1 depends only on ε. Then there exists a constant $\widetilde{c}$ depending only on ε such that for every sufficiently small r

$$\widetilde{c}r^{-3\varepsilon} \geq \sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}.$$

Since ε was arbitrary we have that

$$0 \leq \liminf_{r\to0+}\frac{\log\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}}{\log r}. \tag{3.2.7}$$

Now we argue by contradiction. Let us suppose that $\sum_{i=0}^m \alpha_i^{s_\alpha}\beta_i^{s_\beta} > 1$. Then by using Sublemma 3.2.4 we have

$$\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta} \geq \left(\sum_{i=0}^m \alpha_i^{s_\alpha}\beta_i^{s_\beta}\right)^{\lfloor\Delta_r\rfloor}.$$

It is easy to see that $\lfloor\Delta_r\rfloor = \lceil\frac{\log r}{\log\rho}\rceil$, where $\rho = \min_i\{\alpha_i, \beta_i\}$. This implies that

$$\limsup_{r\to0+}\frac{\log\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}}{\log r} \leq \frac{\log\sum_{i=0}^m\alpha_i^{s_\alpha}\beta_i^{s_\beta}}{\log\rho} < 0,$$

which contradicts (3.2.7). □

Proof of Lemma 3.2.3. By Lemma 3.2.5 we divide the proof into two parts. First let us assume that

$$\sum_{i=0}^m \alpha_i^{s_\alpha}\beta_i^{s_\beta} = 1. \tag{3.2.8}$$

Let us observe that in this case $d_\alpha = d_\beta = s_\alpha + s_\beta$. Then by inequality (3.2.3) we have

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\right)}{-\log r} \geq$$
$$s_\alpha + s_\beta + \frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{s_\alpha+s_\beta} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{s_\alpha+s_\beta}\right)}{-\log r} \geq$$
$$s_\alpha + s_\beta - \varepsilon\frac{\lceil\Delta_r\rceil \log\max_i\left\{\frac{\alpha_i}{\beta_i}, \frac{\beta_i}{\alpha_i}\right\}}{-\log r} + \frac{\log\left(\sum_{\underline{i}\in\Delta_r}\alpha_{\underline{i}}^{s_\alpha}\beta_{\underline{i}}^{s_\beta}\right)}{-\log r}.$$

It is easy to see that $\lceil\Delta_r\rceil = \frac{\log r}{\log\max_i\{\alpha_i,\beta_i\}}$. Applying this fact and our assumption (3.2.8) we get for every $\varepsilon > 0$ that

$$\liminf_{r\to 0}\frac{\log \widetilde{N}_r}{-\log r} \geq s_\alpha + s_\beta - \varepsilon\frac{1}{-\log\max_i\{\alpha_i,\beta_i\}},$$

and this completes the proof in the first case.

In the second case let us assume that

$$\sum_{i=0}^{m}\alpha_i^{s_\alpha}\beta_i^{s_\beta} < 1. \tag{3.2.9}$$

Without loss of generality we may suppose that $d_\alpha \geq d_\beta$.

Then there exists an $\varepsilon^* > 0$ by (3.2.9) such that for every $0 < \varepsilon < \varepsilon^*$,

$$\sum_{i=0}^{m}\alpha_i^{s_\alpha-\varepsilon}\beta_i^{s_\beta-\varepsilon} < 1.$$

This implies that

$$d_\beta(-\varepsilon), d_\alpha(-\varepsilon) \leq s_\alpha + s_\beta - 2\varepsilon. \tag{3.2.10}$$

Then for every $\underline{i}\in\Delta_r^\beta$

$$\frac{\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}}{\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon}} = \left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha+s_\beta-2\varepsilon-d_\alpha(-\varepsilon)}\alpha_{\underline{i}}^{d_\alpha(-\varepsilon)-d_\beta(-\varepsilon)} \leq \alpha_{\underline{i}}^{d_\alpha(-\varepsilon)-d_\beta(-\varepsilon)} \tag{3.2.11}$$

and for every $\underline{i}\in\Delta_r^\alpha$

$$\frac{\beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon}}{\alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}} \leq \beta_{\underline{i}}^{d_\beta(-\varepsilon)-d_\alpha(-\varepsilon)}. \tag{3.2.12}$$

Now we prove the lemma in the case when $d_\alpha > d_\beta$. Then there exists a $\varepsilon^{**} > 0$ such that for every $0 < \varepsilon < \varepsilon^{**}$, $d_\alpha(-\varepsilon) > d_\beta(-\varepsilon)$. Then by (3.2.11)

$$\sum_{\underline{i}\in\Delta_r^\beta} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\mathbf{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \leq \frac{1}{2}\sum_{\underline{i}\in\Delta_r^\beta} \beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \leq \frac{1}{2}$$

holds for sufficiently small $r > 0$. Therefore

$$\sum_{\underline{i}\in\Delta_r^\alpha} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \geq \frac{1}{2}. \tag{3.2.13}$$

Using (3.2.5)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log\left(c^{-1}2^{-(s_\alpha-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon} + c^{-1}2^{-(s_\beta-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\right)}{-\log r} \geq$$
$$\frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r} + \frac{\log r^{-d_\alpha(-\varepsilon)}\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)}}{-\log r},$$

and by (3.2.13)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq d_\alpha(-\varepsilon) + \frac{\log(c^{-1}2^{-(\max\{s_\alpha,s_\beta\}-\varepsilon)})}{-\log r} + \frac{\log 2}{\log r}.$$

Taking liminf as r goes to 0 implies by Lemma 3.2.1 that

$$\underline{\dim}_B\Lambda \geq d_\alpha(-\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary small we proved the lemma in the case $d_\alpha > d_\beta$.

Now let us consider the case $d_\alpha = d_\beta$. The fact (3.2.10) and (3.2.11), (3.2.12) imply for every sufficiently small $\varepsilon > 0$ that

$$\sum_{\underline{i}\in\Delta_r^\beta} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon} \leq \sum_{\underline{i}\in\Delta_r^\beta} \beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \text{ or}$$
$$\sum_{\underline{i}\in\Delta_r^\alpha} \beta_{\underline{i}}^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)-s_\beta+\varepsilon} \leq \sum_{\underline{i}\in\Delta_r^\alpha} \alpha_{\underline{i}}^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)-s_\alpha+\varepsilon}.$$

Therefore

$$\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)} \geq 1. \tag{3.2.14}$$

Using (3.2.5)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log(c^{-1}2^{-(\max\left\{s_\alpha, s_\beta\right\}-\varepsilon)})}{-\log r} + \min\left\{d_\alpha(-\varepsilon), d_\beta(-\varepsilon)\right\} +$$
$$\frac{\log\left(\sum_{\underline{i}\in\Delta_r^\alpha}\left(\frac{\alpha_{\underline{i}}}{\beta_{\underline{i}}}\right)^{s_\alpha-\varepsilon}\beta_{\underline{i}}^{d_\alpha(-\varepsilon)} + \sum_{\underline{i}\in\Delta_r^\beta}\left(\frac{\beta_{\underline{i}}}{\alpha_{\underline{i}}}\right)^{s_\beta-\varepsilon}\alpha_{\underline{i}}^{d_\beta(-\varepsilon)}\right)}{-\log r}$$

and by (3.2.14)

$$\frac{\log \widetilde{N}_r}{-\log r} \geq \frac{\log(c^{-1}2^{-(\max\left\{s_\alpha, s_\beta\right\}-\varepsilon)})}{-\log r} + \min\left\{d_\alpha(-\varepsilon), d_\beta(-\varepsilon)\right\}.$$

Taking liminf as r goes to 0 implies by Lemma 3.2.1 that

$$\underline{\dim}_B\Lambda \geq \min\left\{d_\alpha(-\varepsilon), d_\beta(-\varepsilon)\right\}.$$

Since $\varepsilon > 0$ was arbitrary small and $d_\alpha = d_\beta$ this completes the proof of the lemma. □

Proof of Theorem 3.1.1. The proof is the combination of Lemma 3.2.2 and Lemma 3.2.3. □

Chapter 4

Dimension Theory of the intersections of the Sierpiński Gasket and lines with rational slope

4.1 Definitions and Statements

Denote by $\Delta \subset \mathbb{R}^2$ the usual Sierpiński gasket, that is, Δ is the unique non-empty compact set satisfying

$$\Delta = S_0(\Delta) \cup S_1(\Delta) \cup S_2(\Delta),$$

where

$$S_0(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right),\ S_1(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right),\ S_2(x,y) = \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right). \tag{4.1.1}$$

It is well known that $\dim_H \Delta = \dim_B \Delta = \frac{\log 3}{\log 2} = s$.

We denote by proj_θ the projection onto the line through the origin making angle θ with the x-axis. For $a \in \text{proj}_\theta(\Delta)$ we let

$$L_{\theta,a} = \{(x,y) : \text{proj}_\theta(x,y) = a\} = \{(x, a + x\tan\theta) : x \in \mathbb{R}\}.$$

The main subject of this chapter is to analyze the dimension theory of the slices $E_{\theta,a} = L_{\theta,a} \cap \Delta$. Since Δ is rotation and reflection invariant, without loss of generality we may assume that $\theta \in [0, \frac{\pi}{3})$.

Denote by ν the natural self-similar measure of Δ. That is, $\nu = \frac{\mathcal{H}^s|_\Delta}{\mathcal{H}^s(\Delta)}$, where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure. In this case, ν satisfies that

$$\nu = \sum_{i=0}^{2} \frac{1}{3} \nu \circ S_i^{-1}.$$

Denote by ν_θ the projection of ν by angle θ. That is, $\nu_\theta = \nu \circ \mathrm{proj}_\theta^{-1}$. Similarly, let Δ_θ be the projection of Δ.

For typical line segments, we have a special case of a theorem of Marstrand (see [Mar1] or [Mat, Theorem 10.11]).

Proposition 4.1.1 (Marstrand)**.** *For Lebesgue almost every $\theta \in [0, \frac{\pi}{3})$ and ν_θ-almost all $a \in \Delta_\theta$*

$$\dim_B E_{\theta,a} = \dim_H E_{\theta,a} = s - 1.$$

Let us define the (upper and lower) local dimension of a measure η at the point x by

$$\underline{d}_\eta(x) = \liminf_{r \to 0} \frac{\log \eta(B_r(x))}{\log r}, \ \overline{d}_\eta(x) = \limsup_{r \to 0} \frac{\log \eta(B_r(x))}{\log r}.$$

In the first result of this chapter, Proposition 4.1.2, we will show that a dimension conservation principle holds, connecting the local dimension of the projected natural measure and the box dimension of the slices. Manning and Simon proved such dimension conservation phenomena for the Sierpiński carpet, (see [MS1, Proposition 4]).

Proposition 4.1.2. *For every $\theta \in (0, \frac{\pi}{3})$ and $a \in \Delta_\theta$*

$$\underline{d}_{\nu_\theta}(a) + \overline{\dim}_B E_{\theta,a} = s, \tag{4.1.2}$$

$$\overline{d}_{\nu_\theta}(a) + \underline{\dim}_B E_{\theta,a} = s. \tag{4.1.3}$$

Feng and Hu proved in [FH, Theorem 2.12] that every self-similar measure is exact dimensional. That is, the lower and upper local-dimension coincide and this common value is almost everywhere constant. Moreover, Young proved in [You] that this constant is the Hausdorff dimension of the measure. In other words, if η is self-similar then

for η-almost all x, $\underline{d}_\eta(x) = \overline{d}_\eta(x) = d_\eta(x) = \dim_H \eta = \inf \{\dim_H A : \eta(A) = 1\}$.

Using the above results we easily deduce.

Corollary 4.1.3. *For every $\theta \in (0, \frac{\pi}{3})$ and ν_θ-almost every $a \in \Delta_\theta$ we have*

$$\dim_B E_{\theta,a} = s - \dim_H \nu_\theta \geq s - 1.$$

Furthermore, in Theorem 4.1.4 we prove that whenever $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ for positive integers p, q, the direction θ is exceptional in Marstrand's Theorem. More precisely, the dimension of Lebesgue almost all slices is a constant strictly smaller than $s-1$ but the dimension for almost all slices with respect to the projected measure is another constant strictly greater than $s-1$.

Theorem 4.1.4. *Let $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Then there exist constants $\alpha(\theta), \beta(\theta)$ depending only on θ such that*

1. *for Lebesgue almost all $a \in \Delta_\theta$*

$$\alpha(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} < s - 1,$$

2. *for ν_θ-almost all $a \in \Delta_\theta$*

$$\beta(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} > s - 1.$$

A simple calculation reveals that the tangent of the set of angles in this theorem is equal to $\mathbb{Q}' = \left\{0 < \sqrt{3}\frac{m}{n} < \sqrt{3} : \text{ if } m \text{ is odd then } n \text{ is odd } \right\}$.

In [Fur], Furstenberg introduced and proved a dimension conservation formula [Fur, Definition 1.1] for homogeneous fractals (for example self-similar sets with IFS containing only homothetic similarities). As a consequence of Theorem 4.1.4(2) and Corollary 4.1.3 we state the special case of Furstenberg dimension conservation formula for the Sierpiński gasket and rational slopes. By [Fur, Theorem 6.2], the formula is valid for arbitrary angles.

Furstenberg in [Fur, Theorem 6.2] stated the result as an inequality but combining the result as stated with the Marstrand Slicing Theorem (see [Mar2] or [Fa3, Theorem 5.8]) we see that

Lemma 4.1.5 (Marstrand Slicing Theorem)**.** *Let F be any subset of $\mathbb{R}^2$, and let E be a subset of the y-axis. If $\dim_H(F \cap L_{\theta,a}) \geq t$ for all $a \in E$, then $\dim_H F \geq t + \dim_H E$.*

Corollary 4.1.6 (Furstenberg)**.** *Let us fix $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Then the proj_θ satisfies the dimension conservation formula [Fur, Definition 1.1] by $\beta(\theta)$. Precisely,*

$$\beta(\theta) + \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \beta(\theta)\} = s. \tag{4.1.4}$$

Proof.

$$\begin{aligned}\dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \beta(\theta)\} \\ \geq \dim_H \{a \in \Delta_\theta : \dim_B E_{\theta,a} = \dim_H E_{\theta,a} = \beta(\theta)\} \\ \geq \dim_H \nu_\theta = s - \beta(\theta).\end{aligned}$$

The other direction follows from Lemma 4.1.5. ☐

One can prove by similar argument that

$$\beta(\theta) + \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \beta(\theta)\} = s.$$

The other main goal of the chapter is to analyze the behavior of the function $\Gamma : \delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \delta\}$ in the case when $\tan\theta = \frac{\sqrt{3}p}{2q+p}$, where $p, q \in \mathbb{N}$ and $(p, q) = 1$. For the analysis we use two matrices generated naturally by the projection and the IFS $\{S_0, S_1, S_2\}$. For the simplicity, we illustrate these matrices for the right-angle gasket.

More precisely, for technical reasons, we elect to prove our statements for the so-called right-angle Sierpiński gasket Λ which is the attractor of iterated function system

$$\Phi = \left\{F_0(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right),\ F_1(x,y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right),\ F_2(x,y) = \left(\frac{x}{2}, \frac{y}{2} + \frac{1}{2}\right)\right\}, \tag{4.1.5}$$

and intersections with rational slope lines. There is a linear transformation T

$$T = \begin{pmatrix} 1 & -\frac{\sqrt{3}}{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{pmatrix} \tag{4.1.6}$$

which maps the Sierpiński gasket into the right-angle Sierpińsi gasket. Since an invertible linear transformation does not change the dimension of any set we state our results for the usual Sierpiński gasket and for appropriate slopes. For the transformation see Figure 4.1.

Denote the angle θ projection of Λ to the y-axis by Λ_θ. Then $\Lambda_\theta = [-\tan\theta, 1]$. Moreover, let us consider the projected IFS of Φ. Namely, let

$$\phi = \left\{f_0(t) = \frac{t}{2}, f_1(t) = \frac{t}{2} + \frac{1}{2}, f_2(t) = \frac{t}{2} - \frac{p}{2q}\right\}.$$

By straightforward calculations and [NW1, Theorem 2.7.] we see that ϕ satisfies the finite type condition and therefore, the weak separation property.

Let us divide Λ_θ into $p+q$ equal intervals such that $I_k = \left[1 - \frac{k}{q}, 1 - \frac{k-1}{q}\right]$ for $k = 1, \ldots, p+q$. Moreover, let us divide I_k for every k into two equal

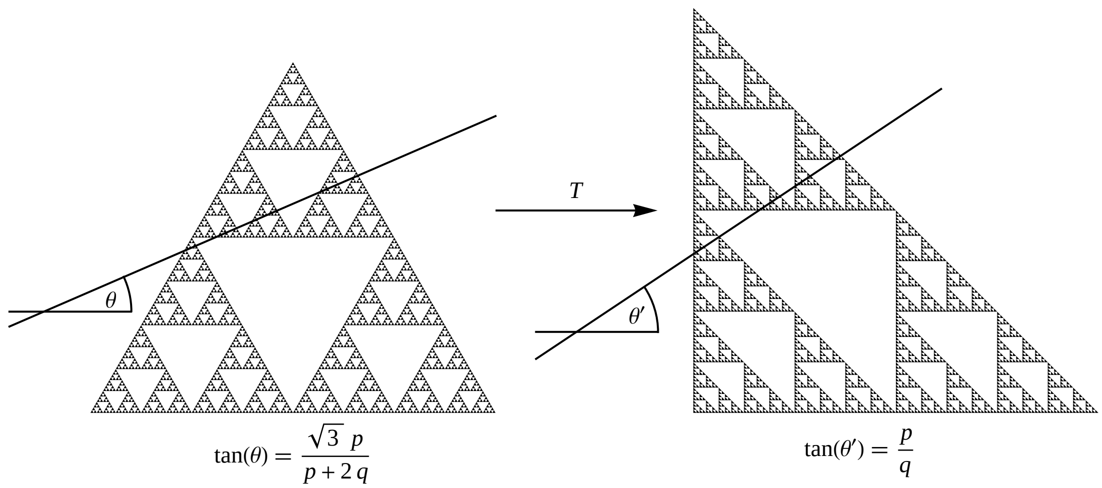

Figure 4.1: The transformation between the usual and right-angle Sierpiński gasket.

parts. Namely, let $I_k^0 = \left[1 - \frac{k}{q}, 1 - \frac{2k-1}{2q}\right]$ and $I_k^1 = \left[1 - \frac{2k-1}{2q}, 1 - \frac{k-1}{q}\right]$. Let us define the $(p+q) \times (p+q)$ matrices A_0, A_1 in the following way:

$$(A_n)_{i,j} = \sharp \{k \in \{0, 1, 2\} : f_k(I_j) = I_i^n\}. \tag{4.1.7}$$

For example, see the case $\frac{p}{q} = \frac{2}{3}$ of the construction in Figure 4.2 and the matrices are

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We note that by some simple calculations the matrices A_0, A_1 can be written in the form

$$\begin{aligned} &(A_n)_{i,j} = 1 \text{ if and only if } 2i + 1 - n \equiv j \mod p+q \text{ or} \\ &\quad 2q + p \geq 2i + n - 1 \geq q + 1 \text{ and } 2i + 1 - n - q \equiv j \mod p+q \end{aligned} \tag{4.1.8}$$

for $n = 0, 1$ and $1 \leq i, j \leq p + q$. Using these matrices we are able to explicitly express the quantities $\alpha(\theta), \beta(\theta)$.

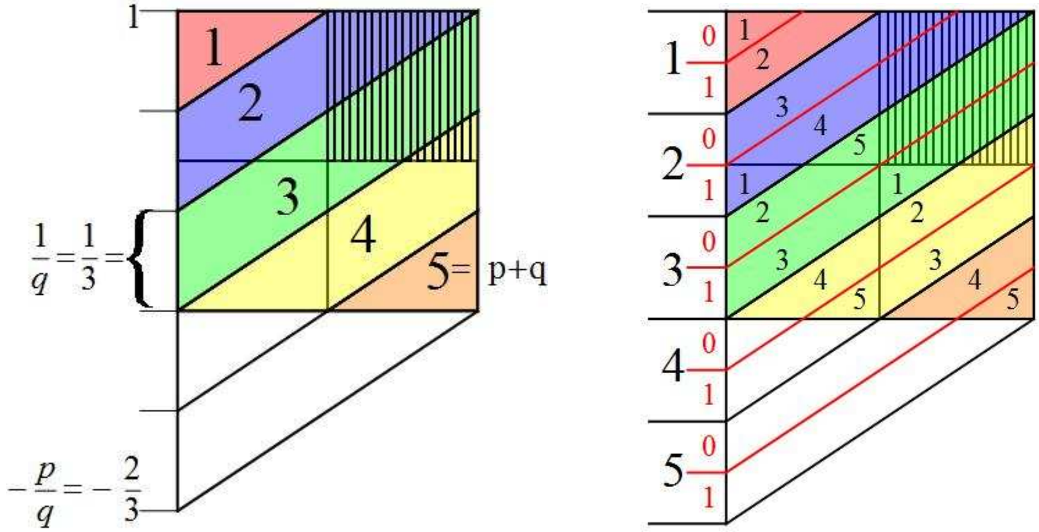

Figure 4.2: Graph of the projection and construction of matrices A_0, A_1 in the case $\frac{p}{q} = \frac{2}{3}$.

Proposition 4.1.7. *Let $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Moreover, let $\alpha(\theta)$ and $\beta(\theta)$ be as in Theorem 4.1.4. Then*

$$\alpha(\theta) = \frac{1}{\log 2} \lim_{n\to\infty} \frac{1}{n} \sum_{\xi_1,\ldots,\xi_n=0}^{1} \frac{1}{2^n} \log \underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{e},$$

$$\beta(\theta) = \frac{1}{\log 2} \lim_{n\to\infty} \frac{1}{n} \sum_{\xi_1,\ldots,\xi_n=0}^{1} \frac{1}{3^n} \underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{p} \log \left(\underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{p} \right),$$

where $\underline{e} = (1, \cdots, 1)$ and $\underline{p}$ is the unique probability vector such that $(A_0 + A_1)\, \underline{p} = 3\, \underline{p}$.

The proof of Proposition 4.1.7 will follow from the proof of Theorem 4.1.4. In order to obtain further information on the nature of the function $\Gamma : \ \delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \delta\}$ we will employ the theory of multifractal analysis for products of non-negative matrices [Fe1, Fe2, FL2]. Let $P(t)$ denote the pressure function which is defined as

$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\xi_1,\ldots,\xi_n=0}^{1} \left(\underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{e} \right)^t \tag{4.1.9}$$

and let us define

$$b_{\min} = \lim_{t\to-\infty} \frac{P(t)}{t},\ b_{\max} = \lim_{t\to\infty} \frac{P(t)}{t}.$$

Proposition 4.1.8. *Let $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Then*

1. $\dim_H \{a \in \Delta_\theta : \dim_B E_{\theta,a} = \alpha\} = \inf_t \left\{-\alpha t + \frac{P(t)}{\log 2}\right\}$ *for* $b_{\min} \leq\ \alpha \leq\ b_{\max}$.
2. $\dim_H \{a \in \Delta_\theta : d_{\nu_\theta}(a) = \alpha\} = \inf_t \left\{-(s-\alpha)t + \frac{P(t)}{\log 2}\right\}$ *for* $s - b_{\max} \leq \alpha \leq s - b_{\min}$.

Both of the functions are concave and continuous.

Proof. Proposition 4.1.8(2) follows immediately from [FL1, Theorem 1.1], [FL1, Theorem 1.2]. Proposition 4.1.8(1) follows from combining the dimension conservation principle Proposition 4.1.2 with Proposition 4.1.8(2). □

We note that Proposition 4.1.8(1) follows also from the results of [Fe2] and we will present a short alternative proof later by using it.

Theorem 4.1.9. *Let $p, q \in \mathbb{N}$ and let us suppose that $\tan\theta = \frac{\sqrt{3}p}{2q+p}$ and $\theta \in (0, \frac{\pi}{3})$. Then*

1. $\Gamma(\delta) = \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} \geq \delta\} = \inf_{t>0} \left\{-\delta t + \frac{P(t)}{\log 2}\right\}$ *if* $b_{\max} \geq\ \delta >\ \alpha(\theta)$ *and* $\Gamma(\delta) = 1$ *if* $\delta \leq \alpha(\theta)$. *The function* Γ *is decreasing and continuous.*
2. $\chi(\delta) = \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \delta\} = \inf_{t>0} \left\{-\delta t + \frac{P(t)}{\log 2}\right\}$ *for every* $b_{\max} \geq\ \delta \geq\ \alpha(\theta)$. *The function* χ *is decreasing and continuous.*

For an example of the function $\delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \delta\}$ with $\tan\theta = \frac{\sqrt{3}}{3}$ in the usual Sierpiński gasket case, see Figure 4.3.

The chapter is based on [BFS] which is a joint work with Andrew Ferguson and Károly Simon.

The organization of the chapter is the following: We prove Proposition 4.1.2 in Section 4.2, Theorem 4.1.4 in Section 4.3 and Theorem 4.1.9 in Section 4.4.

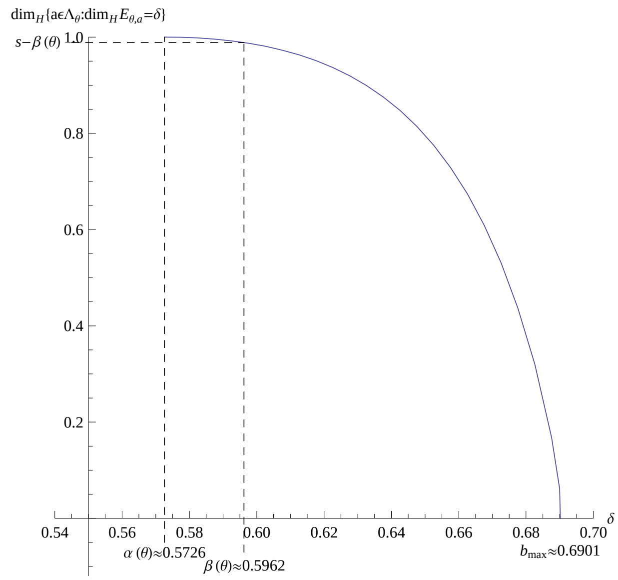

Figure 4.3: The graph of the function $\delta \mapsto \dim_H \{a \in \Delta_\theta : \dim_H E_{\theta,a} = \delta\}$ of the case $\frac{p}{q} = 1$.

4.2 Proof of Proposition 4.1.2

In this section we modify the method of [MS1, Proposition 4].

First, let us introduce some general notation. Let S_0, S_1, S_2 be as in (4.1.1), moreover let $\Sigma = \{0,1,2\}^{\mathbb{N}}$ and $\Sigma^* = \bigcup_{n=0}^{\infty} \{0,1,2\}^n$. Write $\sigma : \Sigma \mapsto \Sigma$ for the left shift operator. Moreover, let $\Pi : \Sigma \mapsto \Delta$ be the natural projection. That is, for every $\mathbf{i} = (i_1 i_2 \cdots) \in \Sigma$

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}(0).$$

Let μ be the equally distributed Bernoulli measure on Σ. That is, for every $\underline{i} \in \Sigma^*$ the measure of $[\underline{i}] = \{\mathbf{i} : \mathbf{i} = \underline{i}\omega\}$ is $\mu([\underline{i}]) = 3^{-|\underline{i}|}$, where $|\underline{i}|$ denotes the length of $\underline{i}$. Then $\nu = \Pi^*\mu = \mu \circ \Pi^{-1}$.

For simplicity we denote by $\Delta_{i_1\cdots i_n} = S_{i_1} \circ \cdots \circ S_{i_n}(\Delta)$. Let us call the n'th level "good sets" of $a \in \Delta_\theta$ the set of $(i_1 \cdots i_n)$ such that $\Delta_{i_1\cdots i_n}$ intersects the set $E_{\theta,a}$. More precisely,

$$G_n(\theta, a) = \{(i_1 \cdots i_n) : \Delta_{i_1\cdots i_n} \cap E_{\theta,a} \neq \emptyset\}. \tag{4.2.1}$$

Lemma 4.2.1. *For every $\theta \in [0, \frac{\pi}{3})$ and $a \in \Delta_\theta$*

$$\underline{\dim}_B E_{\theta,a} = \liminf_{n\to\infty} \frac{\log \sharp G_n(\theta, a)}{n \log 2} \text{ and } \overline{\dim}_B E_{\theta,a} = \limsup_{n\to\infty} \frac{\log \sharp G_n(\theta, a)}{n \log 2}.$$

Proof. Let us denote the minimal number of intervals with length r covering the set $E_{\theta,a}$ by $N_r(\theta, a)$. It is easy to see that

$$N_{2^{-n}}(\theta, a) \le \sharp G_n(\theta, a). \tag{4.2.2}$$

On the other hand, for a minimal cover of $E_{\theta,a}$ with intervals of side length 2^{-n}, for every interval there exists an $\underline{i}$ in $G_n(\theta, a)$ and for every "good" $\Delta_{\underline{i}}$ there exists an interval in the minimal cover such that $\Delta_{\underline{i}}$ intersects the interval. Moreover, for every interval with side length 2^{-n} there are at most $\left\lceil \frac{4\sqrt{3}(2+\pi)}{3} \right\rceil$ cylinders in $G_n(\theta, a)$ which intersects it. Therefore

$$\sharp G_n(\theta, a) \le \left\lceil \frac{4\sqrt{3}(2+\pi)}{3} \right\rceil N_{2^{-n}}(\theta, a). \tag{4.2.3}$$

The equations (4.2.2) and (4.2.3) imply the statement of the lemma. $\square$

Proof of Proposition 4.1.2. Let $\theta \in (0, \frac{\pi}{3})$ and $a \in \Delta_\theta$. Consider the $C(\theta)2^{-n}$ neighbourhood of a, where $C(\theta) = \frac{1}{2} \min\left\{\tan\theta, \cos(\theta + \frac{\pi}{6})\right\}$. Then

$$\nu_\theta(B_{C(\theta)2^{-n}}(a)) = \nu(B_{\cos\theta C(\theta)2^{-n}}(L_{\theta,a})) \ge \nu\left(\bigcup_{\underline{i}\in G_{n-c(\theta)}} \Delta_{\underline{i}}\right) = 3^{-n+c(\theta)} \sharp G_{n-c(\theta)}(\theta, a),$$

where $c(\theta) = \frac{\log(\cos\theta C(\theta))}{\log 2}$. Taking logarithm and dividing by $-n\log 2$ we have

$$\frac{\log \nu_\theta(B_{C(\theta)2^{-n}}(a))}{-n\log 2} \le \frac{(n - c(\theta))\log 3}{n \log 2} + \frac{\log \sharp G_{n-c(\theta)}(\theta, a)}{-n\log 2}.$$

Taking limit inferior and limit superior and using Lemma 4.2.1 we get

$$\begin{aligned} \underline{d}_{\nu_\theta}(a) + \overline{\dim}_B E_{\theta,a} &\le s, \\ \overline{d}_{\nu_\theta}(a) + \underline{\dim}_B E_{\theta,a} &\le s. \end{aligned} \tag{4.2.4}$$

For the reverse inequality we have to introduce the so called "bad" sets which do not intersect $E_{\theta,a}$ but intersect its neighbourhood. That is,

$$R_n(\theta, a) = \left\{(i_1 \cdots i_n) : \Delta_{i_1\cdots i_n} \cap E_{\theta,a} = \emptyset \text{ and } \Delta_{i_1\cdots i_n} \cap B_{\cos\theta C(\theta)2^{-n}}(L_{\theta,a}) \ne \emptyset\right\}.$$

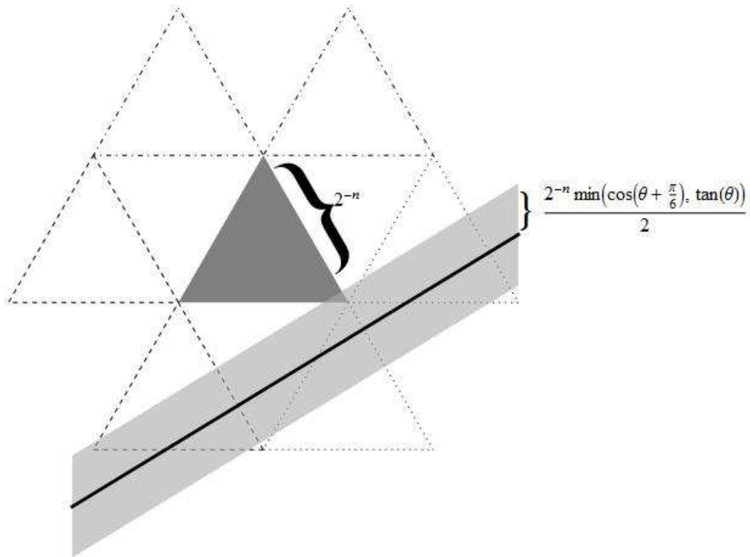

Figure 4.4: A “bad” set of the Sierpiński gasket

Then

$$\nu_\theta(B_{C(\theta)2^{-n}}(a)) = \nu(B_{\cos\theta C(\theta)2^{-n}}(L_{\theta,a})) \le 3^{-n}\left(\sharp R_n(\theta,a) + \sharp G_n(\theta,a)\right).$$

It is enough to prove that $\sharp R_n(\theta,a)$ is less than or equal to $\sharp G_n(\theta,a)$ up to a multiplicative constant.

Let $\Delta_{\underline{i}}$ be an arbitrary n'th level cylinder set of Δ. It is easy to see that if $\Delta_{\underline{i}}$ is not one of the corners of Δ then every corner of $\Delta_{\underline{i}}$ connects to another n'th level cylinder set, see Figure 4.4. We note that the constant $C(\theta)$ is chosen in the way that if the $\cos\theta C(\theta)2^{-n}$ neighbourhood of the line $L_{\theta,a}$ intersects a cylinder but not the line itself intersects it (that is it is a “bad” set) then the line intersects the closest neighbour of the cylinder. Therefore, for every $\underline{i} \in R_n(\theta,a)$ there exists at least one $\underline{j} \in G_n(\theta,a)$ such that $\Delta_{\underline{i}}$ and $\Delta_{\underline{j}}$ are connected to each other (by the choice of $C(\theta)$). Moreover, a cylinder set can be connected to at most 6 other cylinder sets. Therefore, $R_n(\theta,a) \le 6G_n(\theta,a)$.

Applying this, we have

$$\nu_\theta(B_{C(\theta)2^{-n}}(a)) \le 3^{-n}7\sharp G_n(\theta,a).$$

Taking logarithms, dividing by $-n\log 2$ and taking limit inferior and limit superior we get by Lemma 4.2.1

$$\begin{aligned} \underline{d}_{\nu_\theta}(a) + \overline{\dim}_B E_{\theta,a} &\ge s, \\ \overline{d}_{\nu_\theta}(a) + \underline{\dim}_B E_{\theta,a} &\ge s. \end{aligned} \tag{4.2.5}$$

The inequalities (4.2.4) and (4.2.5) imply the statements. $\square$

We note that Proposition 4.1.2 holds in the case when Δ is transformed in an invertible linear way, as well.

4.3 Proof of Theorem 4.1.4

Throughout the section we use the method of [MS1, Theorem 9] with a slight modification. We follow the way of the proof but the construction of the matrices are strictly different.

In the rest of the chapter we will focus on the right-angle Sierpiński gasket Λ and for rational slopes. We prove the statements in that case. For precise details of the right-angle Sierpiński gasket and the transformation between the right-angle and the usual one, see Section 4.1.

For the rest of the chapter we assume that $\theta \in (0, \frac{\pi}{2})$ such that $\tan\theta = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and the greatest common divisor is 1. (This is equivalent with the choice $\theta \in (0, \frac{\pi}{3})$ for Δ.)

Lemma 4.3.1. *Let θ and $a \in \Lambda_\theta$ be such that $\tan\theta = \frac{p}{q}$ and*

$$a = 1 - \frac{k-1}{q} - \frac{1}{q}\sum_{i=1}^{\infty} \frac{\xi_i}{2^i}$$

then

$$\underline{\dim}_B E_{\theta,a} = \liminf_{n\to\infty} \frac{\log \underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log 2} \text{ and } \overline{\dim}_B E_{\theta,a} = \limsup_{n\to\infty} \frac{\log \underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log 2},$$

where $\underline{e}_k$ is the k'th element of the natural basis of $\mathbb{R}^{p+q}$ and $\underline{e} = \sum_{k=1}^{p+q} \underline{e}_k$.

Proof. By the definition of the matrices A_0, A_1 it is easy to see that for every $n \geq 1$ and $\xi_1, \dots, \xi_n \in \{0, 1\}$ we have

$$(A_{\xi_1}\cdots A_{\xi_n})_{i,j} = \sharp\left\{\underline{i} \in \{0,1\}^n : f_{\underline{i}}(I_j) = I_i^{\xi_1,\dots,\xi_n}\right\},$$

where $I_i^{\xi_1,\dots,\xi_n}$ denotes the interval $[1-\frac{i-1}{q}-\frac{1}{q}\sum_{l=1}^{n}\frac{\xi_l}{2^l}-\frac{1}{q2^n}, 1-\frac{i-1}{q}-\frac{1}{q}\sum_{l=1}^{n}\frac{\xi_l}{2^l}]$. Therefore

$$\underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e} = \sharp\left\{\underline{i} \in \{0,1\}^n : \text{there exists a } 1 \leq j \leq p+q \text{ such that } f_{\underline{i}}(I_j) = I_k^{\xi_1,\dots,\xi_n}\right\}.$$

For every $I_k^{\xi_1,\dots,\xi_n}$ and every $(i_1,\dots,i_n)$ if there exists a $1\le j\le p+q$ such that $f_{i_1,\dots,i_n}(I_j)=I_k^{\xi_1,\dots,\xi_n}$ then $I_k^{\xi_1,\dots,\xi_n}\subseteq \mathrm{proj}_\theta\Lambda_{i_1,\dots,i_n}$. This implies that for every $a\in I_k^{\xi_1,\dots,\xi_n}$

$$\underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e}\le \sharp G_n(\theta,a).$$

On the other hand for every $a\in \mathrm{proj}_\theta\Lambda$ if $a\in \mathrm{int}(I_k^{\xi_1,\dots,\xi_n})$ then for every $(i_1,\dots,i_n)\in G_n(\theta,a)$ there exists a $1\le j\le p+q$ such that $f_{i_1,\dots,i_n}(I_j)=I_k^{\xi_1,\dots,\xi_n}$. If $a\in\partial(I_k^{\xi_1,\dots,\xi_n})$ then for every $(i_1,\dots,i_n)\in G_n(\theta,a)$ there exists a $(i'_1,\dots,i'_n)\in G_n(\theta,a)$ and a $1\le j\le p+q$ such that $f_{i'_1,\dots,i'_n}(I_j)=I_k^{\xi_1,\dots,\xi_n}$ as well as $\Lambda_{i_1,\dots,i_n}$ and $\Lambda_{i'_1,\dots,i'_n}$ are connected or equal. Since for every cylinder set can be connected to at most three other cylinder sets, for any $a\in I_k^{\xi_1,\dots,\xi_n}$

$$\sharp G_n(\theta,a)\le 3\underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{e}.$$

The proof is complete by Lemma 4.2.1.

□

One of the main properties of the matrices A_0, A_1 is stated in the following proposition.

Proposition 4.3.2. *Let p,q be integers such that the greatest common divisor is 1, and let A_0 and A_1 be defined as in (4.1.7) (or equivalently as in (4.1.8)). Then there exists an $n_0\ge 1$ and a finite sequence $(\xi_1,\dots,\xi_{n_0})\in\{0,1\}^{n_0}$ such that every element of $A_{\xi_1}\cdots A_{\xi_{n_0}}$ is strictly positive.*

Moreover, for every $n\ge 1$

$$\sharp\left\{(\xi_1,\dots,\xi_n)\in\{0,1\}^n : \exists 1\le i,j\le p+q \text{ such that } (A_{\xi_1,\dots,\xi_n})_{i,j}=0\right\}\le \sum_{l=0}^{(p+q-1)(p+q)-1}\binom{n}{l}2^l. \quad (4.3.1)$$

We note that $\binom{n}{m}=0$ whenever $n<m$.

The most of the proof of Proposition 4.3.2 is divided into the following three lemmas.

Lemma 4.3.3. *Let p,q be integers such that the greatest common divisor is 1, and let A_0 and A_1 be defined as in (4.1.7). Then there are at least one and at most two 1 in each column and in each row of A_n. Moreover, the sum of each column of A_0+A_1 is three.*

The proof is straightforward from the definition.

Lemma 4.3.4. *Let p, q be integers such that the greatest common divisor is 1, and let A_0 and A_1 be defined as in (4.1.7) and in (4.1.8). Then for every $1 \leq m \leq p+q$ distinct columns $1 \leq j_1, \ldots, j_m \leq p+q$ and every $n = 0, 1$ there exist m distinct rows $1 \leq i_1, \ldots, i_m \leq p+q$ such that $(A_n)_{i_k, j_k} = 1$ for every $k = 1, \ldots, m$. Note that $i_1, \ldots, i_m$ may depend on n.*

Proof. If $p+q$ is odd then for any j_k there exists a unique i_k such that $2i_k - 1 + n \equiv j_k \mod p+q$ and, by (4.1.8), $(A_n)_{i_k, j_k} = 1$. Moreover, if $j_k \neq j_{k'}$ then $i_k \neq i_{k'}$. This implies the statement of the lemma.

Now, let us assume that $p+q$ is even. Further, assume that there are two non-zero elements j_1, j_2 in the row i_1. Then

$$2i_1 - 1 + n \equiv j_1 \mod p+q \text{ and } 2i_1 - 1 + n - q \equiv j_2 \mod p+q.$$

It is easy to see that every element of the column j_2 is 0 except (i_1, j_2). Moreover, there exists $1 \leq i'_1 \leq p+q$ such that $2i'_1 - 1 + n \equiv j_1 \mod p+q$. In this case, every element of the row i'_1 is 0 except (i'_1, j_1). Otherwise, if there would be $j_3 \neq j_1$ such that $2i'_1 - 1 + n - q \equiv j_3 \mod p+q$ then $j_3 \equiv j_1 - q \equiv j_2 \mod p+q$, but every element of the column j_2 is zero except (i_1, j_2), which is a contradiction. Therefore, for A_n, $n = 0, 1$ and for every m distinct columns $j_1, \ldots, j_m$ there are at least m distinct rows $i_1, \ldots, i_m$ such that $(A_n)_{i_k, j_k} = 1$. □

Lemma 4.3.5. *Let p, q be integers such that the greatest common divisor is 1, and let A_0 and A_1 be defined as in (4.1.7) and in (4.1.8). Then for every $1 \leq m < p+q$ distinct columns $1 \leq j_1, \ldots, j_m \leq p+q$ there exists an $n \in \{0, 1\}$ and at least $m+1$ distinct rows $1 \leq i_1, \ldots, i_{m+1} \leq p+q$ such that $(A_n)_{i_k, j_k} = 1$ for $k = 1, \ldots, m$ and there exists a $j \in \{j_1, \ldots, j_m\}$ such that $(A_n)_{i_{m+1}, j} = 1$.*

Proof. We argue by contradiction. Let us fix the m distinct columns $1 \leq j_1, \ldots, j_m \leq p+q$. By Lemma 4.3.3 in every column there are at least one and at most two "1" elements and by Lemma 4.3.4 there are at least m different rows $1 \leq i_1, \ldots, i_m \leq p+q$ in A_0 and at least m different rows $1 \leq s_1, \ldots, s_m \leq p+q$ in A_1 such that $(A_0)_{i_k, j_k} = 1$ and $(A_1)_{s_k, j_k} = 1$. To get a contradiction we assume that

$$\forall i \notin \{i_1, \ldots, i_m\}, \forall s \notin \{s_1, \ldots, s_m\}, \forall k: \quad (A_0)_{i, j_k} = 0, \ (A_1)_{s, j_k} = 0. \quad (\mathbf{A1})$$

By Lemma 4.3.3, in the matrix $A_0 + A_1$ in every column there are exactly 3 non-zero elements. Therefore we can assume without loss of generality that there is an $0 \leq l \leq m$ such that in A_0 the columns $j_1, \ldots, j_l$ and in A_1 the columns $j_{l+1}, \ldots, j_m$ contain two non-zero elements. Namely, there are l

distinct rows $1 \leq i'_1, \dots, i'_l \leq$ and $m-l$ distinct rows $1 \leq s'_{l+1}, \dots, s'_m \leq p+q$ such that $(A_0)_{i'_k, j_k} = 1$ for $k = 1, \dots, l$ and $(A_1)_{s'_k, j_k} = 1$ for $k = l+1, \dots, m$. Moreover, by our assumption (**A1**) and Lemma 4.3.4, for every i'_k there exists a i_{t_k} such that $l+1 \leq t_k \leq m$ and $i'_k = i_{t_k}$, similarly for every s'_k there exists a s_{t_k} such that $1 \leq t_k \leq l$ and $s'_k = s_{t_k}$.

Let us define now a directed graph $G(V, E)$ such that the vertices are $V = \{j_1, \dots, j_m\}$ and there is an edge $j_k \to j_n$ if and only if $s'_k = s_n$ or $i'_k = i_n$. It is easy to see that

$$j_k \to j_n \Longleftrightarrow \begin{cases} j_n - q \equiv j_k \mod p+q & \text{if } p+q \text{ is odd} \\ j_k - q \equiv j_n \mod p+q & \text{if } p+q \text{ is even.} \end{cases} \tag{4.3.2}$$

Since from every vertex of G there is an edge pointing out, there is a circle $j_{n_1} \to j_{n_2} \to \cdots \to j_{n_t} \to j_{n_1}$, where $1 \leq t \leq m$. By (4.3.2) we have

$$j_{n_1} \equiv j_{n_2} - q \equiv \cdots \equiv j_{n_t} - (t-1)q \equiv j_{n_1} - tq \mod p+q \text{ if } p+q \text{ is odd or}$$
$$j_{n_1} \equiv j_{n_t} - q \equiv \cdots \equiv j_{n_2} - (t-1)q \equiv j_{n_1} - tq \mod p+q \text{ if } p+q \text{ is even.}$$

Then $tq \equiv 0 \mod p+q$. Since $(q, p+q) = 1$, then $t \equiv 0 \mod p+q$. Therefore $p+q \leq t \leq m < p+q$ which is a contradiction. □

Proof of Proposition 4.3.2. First, we prove the existence of such a sequence. It is easy to see by Lemma 4.3.4 that for every matrix B with non-negative elements and $n = 0, 1$, if the l'th column of B contains m non-zero elements then the l'th column of the matrix $A_n B$ contains at least m non-zero elements. Moreover, by Lemma 4.3.5, for every column l of B there exists an $n \in \{0, 1\}$ such that if it contains m non-zero elements then the l'th column of $A_n B$ contains at least $m+1$ non-zero elements.

Therefore, there exists an at most $n = (p+q)(p+q-1)+1$ length sequence $\{\xi_k\}_{k=1}^n$of $0, 1$ such that every element of the matrix $A_{\xi_n} \cdots A_{\xi_1}$ is non-zero.

For the second statement, let us observe that for any non-negative matrix B and any column $1 \leq j \leq p+q$ there is at most one matrix A_n such that the number of non-zero elements of the l'th column of $A_n B$ is equal to the number of non-zero elements in the l'th column of B. Therefore, if for a finite word $(\xi_1, \dots, \xi_n)$ and the matrix $A_{\xi_n} \cdots, A_{\xi_1}$ there is at least one zero element then the word $(\xi_1, \dots, \xi_n)$ may contain at most $(p+q-1)(p+q)-1$ arbitrary elements, but in the other places there have to be the matrix, which does not grow the number of non-zero elements in the columns. This implies the inequality. □

It is natural to introduce the dyadic symbolic space. Let $\Xi = \{0, 1\}^{\mathbb{N}}$ and Ξ^* be the set of dyadic finite length words. Define the natural projection

$\pi : \Xi \mapsto [0,1]$ by

$$\pi(\mathbf{i}) = \sum_{k=1}^{\infty} \frac{i_k}{2^k}.$$

Moreover, let σ be the left shift operator on Ξ.

For any θ with $\tan\theta \in \mathbb{Q}$ and $a \in \Lambda_\theta$ let us define $\Gamma_a = \left\{a + \frac{i}{q} \in \Lambda_\theta : i \in \mathbb{Z}\right\}$ and $F_{\theta,a} = \bigcup_{b\in\Gamma_a} E_{\theta,b}$.

Proposition 4.3.6. *Let $p, q \in \mathbb{N}$ be such that $(p,q) = 1$ and let $\theta \in (0, \frac{\pi}{2})$ be such that $\tan\theta = \frac{p}{q}$. Then for Lebesgue-almost every $a \in \Lambda_\theta$*

$$\dim_B E_{\theta,a} = \alpha(\theta),$$

where

$$\alpha(\theta) = \frac{1}{\log 2} \lim_{n\to\infty} \frac{1}{n} \log \underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{e}, \text{ for } \mathbb{P}\text{-a.a. } (\xi_1, \xi_2, \dots) \in \Xi, \tag{4.3.3}$$

where $\mathbb{P}$ is the equidistributed Bernoulli measure on Ξ. Similarly,

$$\alpha(\theta) = \frac{1}{\log 2} \lim_{n\to\infty} \frac{1}{n} \sum_{\xi_1,\dots,\xi_n} \frac{1}{2^n} \log \underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{e}. \tag{4.3.4}$$

Proof. Since A_0, A_1 are non-negative matrices, we have for any $(\xi_1, \dots, \xi_n) \in \Xi^*$ and $1 \le k \le n$

$$\underline{e} A_{\xi_1} \cdots A_{\xi_n} \underline{e} \le \underline{e} A_{\xi_1} \cdots A_{\xi_k} \underline{e}\ \underline{e} A_{\xi_{k+1}} \cdots A_{\xi_n} \underline{e}.$$

Let $\mathbb{P} = \left\{\frac{1}{2}, \frac{1}{2}\right\}^{\mathbb{N}}$ be the equidistributed Bernoulli measure on Ξ. Then by the sub-additive ergodic theorem (see [Wa, p. 231]) we have for $\mathbb{P}$-almost all $\underline{\xi} \in \Xi$ the limit (4.3.3) exists and constant. The equation (4.3.4) follows also from the sub-additive ergodic theorem.

It is easy to see that the measure $\sum_{k=1}^{p+q} \frac{1}{p+q} \mathbb{P} \circ \pi^{-1} \circ h_k|_{I_k}$ is equivalent with the Lebesgue measure on Λ_θ, where $h_k(x) = -qx + q - k$, so that $h_k(I_k) = [0,1]$. This and Lemma 4.3.1 implies that for Lebesgue almost every $a \in \Lambda_\theta$

$$\max_{b\in\Gamma_a} \dim_B E_{\theta,b} = \dim_B F_{\theta,a} = \alpha(\theta). \tag{4.3.5}$$

Let $(\xi_1, \dots, \xi_{n_0}) \in \{0,1\}^{n_0}$ be as in Proposition 4.3.2. Then for every $1 \le k \le p+q$ and every finite length word $(\zeta_1, \dots, \zeta_n) \in \{0,1\}^*$ and Lebesgue-almost every $a \in I_k^{\zeta_1,\dots,\zeta_n\xi_1\dots\xi_{n_0}}$ we have

$$\dim_B E_{\theta,a} = \dim_B F_{\theta,a'} = \alpha(\theta),$$

where $a' = 2^{n+n_0}\left(a - 1 + \frac{k-1}{q}\right) + \frac{1}{q}\sum_{i=1}^{n} 2^{n+n_0-i}\zeta_i + \frac{1}{q}\sum_{i=1}^{n_0} 2^{n_0-i}\xi_i + 1 - \frac{k-1}{q}$.
The statement of the proposition follows from the fact that the set
$\bigcup_{k=1}^{p+q}\bigcup_{n=0}^{\infty}\bigcup_{(\zeta_1,\dots,\zeta_n)\in\{0,1\}^n} I_k^{\zeta_1,\dots,\zeta_n\xi_1\dots\xi_{n_0}}$ has full Lebesgue measure in Λ_θ. $\square$

Lemma 4.3.7. *The function $\alpha(\theta) < s-1$ for every θ such that $\tan\theta \in \mathbb{Q}^+$.*

The proof of Lemma 4.3.7 coincides with the proof of [MS1, Theorem 9], (see [MS1, Subsection 3.4, Subsection 3.5]), therefore we omit it.

Finally, we have to state a proposition about the coincidence of the Hausdorff and box dimension for "typical" points before we prove Theorem 4.1.4.

Proposition 4.3.8. *Let $p, q \in \mathbb{N}$ be such that $(p,q) = 1$ and let $\theta \in (0, \frac{\pi}{2})$ be such that $\tan\theta = \frac{p}{q}$. Let η be a left shift invariant measure on Ξ such that*

$$\eta\left(\bigcup_{n=0}^{\infty}\bigcup_{(\zeta_1,\dots,\zeta_n)\in\{0,1\}^n}[\zeta_1,\dots,\zeta_n\xi_1\dots\xi_{n_0}]\right) = 1, \tag{4.3.6}$$

where $(\xi_1,\dots,\xi_{n_0})$ is as in Proposition 4.3.2. Let $\eta = \sum_{k=1}^{p+q}\eta_k$ be an arbitrary positive decomposition of η. (That is, $\eta_k([\zeta_1,\dots,\zeta_n]) > 0$ for any $1 \le k \le p+q$ and any cylinder set.) Then for λ-almost every $a \in \Lambda_\theta$

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a},$$

where

$$\lambda = \sum_{k=1}^{p+q}\eta_k \circ \pi^{-1} \circ h_k\big|_{I_k}.$$

The proof follows the proof of [LXZ, Theorem 1.1(3)] and [MS1, Proposition 8.]. The following lemma appears in a paper of Kenyon and Peres [KP, Proposition 2.6], the proof is attributed to Ledrappier. We state the lemma only for our special case.

Lemma 4.3.9 (Ledrappier)**.** *Let T_2 be the endomorphism $T_2(x) = 2x \mod 1$ on the one-dimensional torus S^1. Assume that $F \subset S^1 \times S^1 = \mathbb{T}^2$ is compact and invariant under $T_2 \times T_2$ and ν a T_2-invariant probability measure on S^1. Then for ν-a.e. x*

$$\dim_H \mathrm{proj}^{-1}(x) = \dim_B \mathrm{proj}^{-1}(x),$$

where $\mathrm{proj} : F \mapsto S^1$ *is the projection to the second coordinate.*

Proof of Proposition 4.3.8. It is easy to see that

$$F_{\theta,a} = \Lambda \cap \{(x,y) : px - qy \equiv -qa \mod 1\}.$$

Let $P : (x,y) \mapsto (x, (px - qy) \mod 1)$ be a map of $\mathbb{T}^2$ into itself. Then

$$\underline{\dim}_B P(F_{\theta,a}) = \underline{\dim}_B F_{\theta,a},\ \overline{\dim}_B P(F_{\theta,a}) = \overline{\dim}_B F_{\theta,a} \text{ and } \dim_H P(F_{\theta,a}) = \dim_H F_{\theta,a}.$$

and $P(\Lambda) \subset \mathbb{T}^2$ is compact and $T_2 \times T_2$-invariant. Moreover, let $Q(a) = -qa \mod 1$ be the mapping Λ_θ into S^1. Since η is left shift invariant then $\lambda \circ Q^{-1} = \eta \circ \pi^{-1}$ is T_2 invariant. Since

$$\mathrm{proj}^{-1}(-qa \mod 1) = P(F_{\theta,a})$$

by Lemma 4.3.9 we have for λ-almost all $a \in \Lambda_\theta$ that

$$\dim_H F_{\theta,a} = \dim_B F_{\theta,a}. \tag{4.3.7}$$

Let $(\xi_1, \ldots, \xi_{n_0}) \in \{0,1\}^{n_0}$ be as in Proposition 4.3.2. Then by the assumptions, for every $1 \le k \le p+q$ and every finite length word $(\zeta_1, \ldots, \zeta_n) \in \{0,1\}^*$ the measure $\lambda(I_k^{\zeta_1,\ldots,\zeta_n\xi_1\ldots\xi_{n_0}}) > 0$ and for λ-almost every $a \in I_k^{\zeta_1,\ldots,\zeta_n\xi_1\ldots\xi_{n_0}}$ the equation (4.3.7) holds. Moreover, the fact that the matrix $A_{\xi_1} \cdots A_{\xi_{n_0}}$ has strictly positive coefficients implies that

$$\dim_B E_{\theta,a} = \dim_B F_{\theta,a'} = \dim_H F_{\theta,a'} = \dim_H E_{\theta,a},$$

where $a' = 2^{n+n_0}\left(a - 1 + \frac{k-1}{q}\right) + \frac{1}{q}\sum_{i=1}^{n} 2^{n+n_0-i}\zeta_i + \frac{1}{q}\sum_{i=1}^{n_0} 2^{n_0-i}\xi_i + 1 - \frac{k-1}{q}$. The proof is completed by applying the assumption (4.3.6). $\square$

Proof of Theorem 4.1.4. Theorem 4.1.4(1) is an easy consequence of Proposition 4.3.6, Lemma 4.3.7 and Proposition 4.3.8.

The equalities of Theorem 4.1.4(2) follow from Corollary 4.1.3 and Proposition 4.3.8. It is enough to prove that $\beta(\theta) > s - 1$. To prove this fact, we use the method of [R].

Define a probability measure η on Ξ as

$$\eta([\xi_1, \ldots, \xi_n]) := \frac{1}{3^n}\underline{e}A_{\xi_1} \cdots A_{\xi_n}\underline{p},$$

where $\underline{p}$ is the unique probability vector such that $\frac{1}{3}(A_0 + A_1)\underline{p} = \underline{p}$. Then it is easy to see that η is left shift invariant. Moreover, by Perron-Frobenius Theorem, the measure η is mixing (that is, for any cylinder sets A, B of Ξ,

$\lim_{n\to\infty} \eta(\sigma^{-n}A\cap B) = \eta(A)\eta(B)$) and therefore, an ergodic probability measure. Decompose $\eta = \sum_{k=1}^{p+q} \eta_k$ as

$$\eta_k([\xi_1,\dots,\xi_n]) = \frac{1}{3^n}\underline{e}_k A_{\xi_1}\cdots A_{\xi_n}\underline{p}$$

for every cylinder set $[\xi_1,\dots,\xi_n]$. Let us recall that ν_θ is the projection of the natural self-similar measure on Λ. Observe that $\nu_\theta|_{I_k}\circ h_k = \eta_k\circ\pi^{-1}$ and define $\widetilde{\nu}_\theta(.) = \sum_{k=1}^{p+q} \nu_\theta|_{I_k}\circ h_k = \eta\circ\pi^{-1}$. Then $\widetilde{\nu}_\theta$ is T_2 invariant and mixing probability measure satisfying the assumptions of Proposition 4.3.8.

By the Volume lemma [PU, Theorem 10.4.1] and [PU, Theorem 10.4.2] we have

$$\dim_H \widetilde{\nu}_\theta = \lim_{n\to\infty} -\frac{1}{n\log 2}\sum_{\xi_1,\dots,\xi_n=0}^{1} \frac{1}{3^n}\underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{p}\log\left(\frac{1}{3^n}\underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{p}\right) \tag{4.3.8}$$

On the other hand, since $\nu_\theta|_{I_k}\circ h_k \ll \widetilde{\nu}_\theta$ for every $1\le k\le p+q$ which implies that $\dim_H \nu_\theta|_{I_k} = \dim_H \nu_\theta|_{I_k}\circ h_k \le \dim_H \widetilde{\nu}_\theta$. However,

$$\dim_H \widetilde{\nu}_\theta = \inf_{1\le k\le p+q} \dim_H \nu_\theta|_{I_k}\circ h_k = \inf_{1\le k\le p+q} \dim_H \nu_\theta|_{I_k} = \dim_H \nu_\theta.$$

By Lemma 4.3.7 there exists a $\delta > 0$ such that for sufficiently large n there exists a sequence $(\xi_1,\dots,\xi_n)$ that

$$\underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{p} < 2^{-(n+\delta n)}.$$

This implies that the limit in (4.3.8) is strictly less than 1. The proof can be finished by Corollary 4.1.3. $\square$

Proof of Proposition 4.1.7. The statement of the proposition follows from Proposition 4.3.6 and the proof of Theorem 4.1.4(2). $\square$

4.4 Proof of Theorem 4.1.9

In this section we would like to apply the results of [Fe1], [Fe2] and [FL2]. Let

$$\widetilde{\Lambda}_\theta = \left\{ a = 1 - \frac{k-1}{q} - \frac{1}{q}\sum_{i=1}^{\infty}\frac{\xi_i}{2^i} \in \Lambda_\theta : \exists k\ge 1,\ A_{\xi_1}\cdots A_{\xi_k} > 0 \right\}.$$

By Proposition 4.3.2 we have

$$\overline{\dim}_B \Lambda_\theta\backslash\widetilde{\Lambda}_\theta = 0. \tag{4.4.1}$$

Moreover, we can reformulate Lemma 4.3.1.

Lemma 4.4.1. *Let θ and $a \in \widetilde{\Lambda}_\theta$ be such that $\tan\theta = \frac{p}{q}$ and*

$$a = 1 - \frac{k-1}{q} - \frac{1}{q}\sum_{i=1}^{\infty}\frac{\xi_i}{2^i}$$

then

$$\underline{\dim}_B E_{\theta,a} = \liminf_{n\to\infty}\frac{\log \underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log 2} \text{ and } \overline{\dim}_B E_{\theta,a} = \limsup_{n\to\infty}\frac{\log \underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log 2}.$$

Proof of Proposition 4.1.8(1). As a consequence of Lemma 4.4.1 and (4.4.1) we have

$$\dim_H\{a \in \Lambda_\theta : \dim_B E_{\theta,a} = \alpha\} =$$
$$\dim_H\left\{1 - \frac{k-1}{q} - \frac{1}{q}\sum_{i=1}^{\infty}\frac{\xi_i}{2^i} \in \widetilde{\Lambda}_\theta : \lim_{n\to\infty}\frac{\log \underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n} = \alpha\log 2\right\} =$$
$$\dim_H\left\{(\xi_1,\xi_2,\dots) \in \Xi : \lim_{n\to\infty}\frac{\log \underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n} = \alpha\log 2\right\}.$$

By Proposition 4.3.2, one can finish the proof using [Fe2, Theorem 1.1]. □

By [Fe2, Lemma 2.2] and [FL2, Theorem 3.3] we can state a lemma for the pressure function.

Lemma 4.4.2. *Let $P(t)$ be defined as in (4.1.9). Then $P(t)$ is monotone increasing, convex and continuous for $t \in \mathbb{R}$. Moreover, for $t > 0$ the pressure is differentiable.*

Lemma 4.4.3. *For every $0 \le \delta \le \alpha(\theta)$,*

$$\dim_H\{a \in \Lambda_\theta : \dim_H E_{\theta,a} \ge \delta\} = 1.$$

Proof. For every $0 \le \delta \le \alpha(\theta)$ we have

$$\dim_H\{a \in \Lambda_\theta : \dim_H E_{\theta,a} \ge \delta\} \ge$$
$$\dim_H\{a \in \Lambda_\theta : \dim_H E_{\theta,a} = \dim_B E_{\theta,a} = \alpha(\theta)\} = 1.$$

The last equation follows from Theorem 4.1.4(1). The upper bound is trivial. □

Lemma 4.4.4. *Let $P(t)$ be defined as in (4.1.9). Then*

$$\lim_{t\to 0+} P'(t) = \alpha(\theta)\log 2.$$

Proof. First, we prove $\lim_{t\to 0+} P'(t) \geq \alpha(\theta)\log 2$. Suppose by way of contradiction that that there is a $t' > 0$ such that $P'(t') = \alpha(\theta)\log 2$ and for every $0 < t < t'$, $P'(t) < \alpha(\theta)\log 2$. Then

$$1 = \dim_H \{a \in \Lambda_\theta : \dim_B E_{\theta,a} = \alpha(\theta)\} = \inf_t \left\{-\alpha(\theta)t + \frac{P(t)}{\log 2}\right\} = -\alpha(\theta)t' + \frac{P(t')}{\log 2}.$$

Therefore $P(0) = \log 2$ and $P(t') = \log 2\alpha(\theta)t' + \log 2$ contradicting the assumption that $P'(t) < \alpha(\theta)\log 2$.

We now prove the other inequality, $\lim_{t\to 0+} P'(t) \leq \alpha(\theta)\log 2$, by contradiction, as well. Suppose that there is a $\lim_{t\to 0+} P'(t) > \delta > \alpha(\theta)$ then by Theorem 4.1.8(1) there is a $t^- \leq 0$

$$\dim_H \{a \in \Lambda_\theta : \dim_B E_{\theta,a} = \delta\} = \inf_t \left\{-\delta t + \frac{P(t)}{\log 2}\right\} = -\delta t^- + \frac{P(t^-)}{\log 2} >$$
$$-\alpha(\theta)t^- + \frac{P(t^-)}{\log 2} \geq \inf_t \left\{-\alpha(\theta)t + \frac{P(t)}{\log 2}\right\} = \dim_H \{a \in \Lambda_\theta : \dim_B E_{\theta,a} = \alpha(\theta)\} = 1,$$

which is a contradiction. (The last equality follows from Theorem 4.1.4(1).) □

Before we prove the case when $\alpha(\theta) < \delta \leq b_{\max}$ we need the so-called Gibbs measure.

Lemma 4.4.5. *For every $t > 0$ there is a unique ergodic, left shift invariant Gibbs measure μ_t on Ξ such that there exists a $C > 0$ that for any $(\xi_1, \ldots, \xi_k) \in \Xi^*$*

$$C^{-1} \leq \frac{\mu_t((\xi_1, \ldots, \xi_k))}{\left(\underline{e}A_{\xi_1}\cdots A_{\xi_k}\underline{e}\right)^t e^{-kP(t)}} \leq C.$$

Moreover,

$$\dim_H \mu_t = \frac{-tP'(t) + P(t)}{\log 2} \tag{4.4.2}$$

and

$$\lim_{n\to\infty} \frac{\log \underline{e}A_{\xi_1}\cdots A_{\xi_n}\underline{e}}{n\log 2} = \frac{P'(t)}{\log 2} \text{ for } \mu_t\text{-a.a. } (\xi_1, \xi_2, \ldots). \tag{4.4.3}$$

The proof of the lemma follows from [FL2, Theorem 3.2] and [FL2, Proof of Theorem 1.3].

Lemma 4.4.6. *For every $\alpha(\theta) < \delta \leq b_{\max}$,*

$$\dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} \geq \delta\} = \inf_{t>0} \left\{-\delta t + \frac{P(t)}{\log 2}\right\}.$$

Proof. Let us observe by Lemma 4.4.4 that

$$\inf_t \left\{ -\delta t + \frac{P(t)}{\log 2} \right\} = \inf_{t>0} \left\{ -\delta t + \frac{P(t)}{\log 2} \right\}.$$

First, we will prove the upper bound with the method of [Wi, Lemma 3.18]. Let us define

$$\mathbf{A}_n(\varepsilon) = \left\{ (\xi_1, \ldots, \xi_k) : k \geq n,\ \delta - \varepsilon \leq \frac{\log \underline{e} A_{\xi_1} \cdots A_{\xi_k} \underline{e}}{k \log 2} \right\}.$$

It is easy to see that the set

$$\bigcup_{j=1}^{p+q} \bigcup_{(\xi_1,\ldots,\xi_k)\in \mathbf{A}_n(\varepsilon)} I_j^{\xi_1,\ldots,\xi_k}$$

covers the set $\mathcal{G}_\delta := \{a \in \Lambda_\theta : \delta \leq \underline{\dim}_B E_{\theta,a}\}$. Let $\mathbf{B}_n(\varepsilon)$ be the set of disjoint cylinders of $\mathbf{A}_n(\varepsilon)$ such that

$$\bigcup_{j=1}^{p+q} \bigcup_{(\xi_1,\ldots,\xi_k)\in \mathbf{B}_n(\varepsilon)} I_j^{\xi_1,\ldots,\xi_k} = \bigcup_{j=1}^{p+q} \bigcup_{(\xi_1,\ldots,\xi_k)\in \mathbf{A}_n(\varepsilon)} I_j^{\xi_1,\ldots,\xi_k}.$$

Then for any $t > 0$ and $\varepsilon' > 0$ we have

$$\mathcal{H}_{2^{-n}}^{-\delta t + \frac{P(t)}{\log 2} + \varepsilon' t}(\mathcal{G}_\delta) \leq \sum_{j=1}^{p+q} \sum_{(\xi_1,\ldots,\xi_k)\in \mathbf{B}_n(\varepsilon)} \left| I_j^{\xi_1,\ldots,\xi_k} \right|^{-\delta t + \frac{P(t)}{\log 2} + \varepsilon' t} \leq$$

$$(p+q) 2^{(\varepsilon - \varepsilon')nt} \sum_{(\xi_1,\ldots,\xi_k)\in \mathbf{B}_n(\varepsilon)} \left(\underline{e} A_{\xi_1} \cdots A_{\xi_k} \underline{e} \right)^t e^{-kP(t)}.$$

By Lemma 4.4.5

$$\mathcal{H}_{2^{-n}}^{-\delta t + \frac{P(t)}{\log 2} + \varepsilon'}(\mathcal{G}_\delta) \leq C(p+q) 2^{(\varepsilon - \varepsilon')nt} \sum_{(\xi_1,\ldots,\xi_k)\in \mathbf{B}_n(\varepsilon)} \mu_t((\xi_1, \ldots, \xi_k)) \leq C(p+q) 2^{(\varepsilon - \varepsilon')nt}.$$

This implies that

$$\dim_H \{a \in \Lambda_\theta : \delta \leq \dim_H E_{\theta,a}\} \leq \dim_H \{a \in \Lambda_\theta : \delta \leq \underline{\dim}_B E_{\theta,a}\} \leq -\delta t + \frac{P(t)}{\log 2} + \varepsilon' t$$

for any $t > 0$ and $\varepsilon' > \varepsilon > 0$. This proves the upper bound.

Now, we prove the lower bound. By Lemma 4.4.2, for every $\alpha(\theta) < \delta < b_{\max}$ there exists a $t > 0$ such that $P'(t) = \delta \log 2$. By Lemma 4.4.5, let μ_t be the

Gibbs measure. The measure μ_t is shift invariant and ergodic. Moreover, by the Gibbs property, μ_t satisfies the assumption of Proposition 4.3.8 and we have

$$\dim_H E_{\theta,a} = \dim_B E_{\theta,a} \text{ for } \mu_t\text{-almost all } (\xi_1, \xi_2, \dots),$$

where $a = 1 - \frac{k-1}{q} - \frac{1}{q}\sum_{i=1}^{\infty} \frac{\xi_i}{2^i}$ for some $1 \le k \le p+q$. Then by (4.4.2) and (4.4.3) we have

$$\dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} \ge \delta\} \ge \dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} = \dim_B E_{\theta,a} = \delta\} \ge \\ \dim_H \mu_t = -t\delta + \frac{P(t)}{\log 2} \ge \inf_{t>0} \left\{-t\delta + \frac{P(t)}{\log 2}\right\}.$$

If $\delta = b_{\max}$ then

$$\dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} \ge b_{\max}\} \le \lim_{\delta \to b_{\max}+} \dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} \ge \delta\} = \\ \lim_{\delta \to b_{\max}+} \inf_{t>0} \left\{-t\delta + \frac{P(t)}{\log 2}\right\} = \inf_{t>0} \left\{-tb_{\max} + \frac{P(t)}{\log 2}\right\} = 0.$$

In the last two equations we used the continuity property [Fe2, Theorem 1.1] and the definition of $b_{\max}$. □

Proof of Theorem 4.1.9(1). The proof is the combination of Lemma 4.4.3 and Lemma 4.4.6. □

Proof of Theorem 4.1.9(2). By the observation

$$\dim_H \{a \in \Lambda_\theta : \underline{\dim}_B E_{\theta,a} \ge \delta\} \ge \dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} = \delta\} \ge \\ \dim_H \{a \in \Lambda_\theta : \dim_B E_{\theta,a} = \dim_H E_{\theta,a} = \delta\}$$

one can finish the proof as Lemma 4.4.6. □

Chapter 5

The absolute continuity of the invariant measure of Random Iterated Function Systems

5.1 Definitions and Statements

In this last chapter, we study the invariant measure of random iterated function systems. Let $\{f_1, \dots, f_l\}$ be an iterated function system (IFS) on the real line, where the maps are applied according to the probabilities $(p_1, \dots, p_l)$, with the choice of the map random and independent at each step.

Suppose that for each $i \in \{1, \dots, l\}$, f_i maps $[-1, 1)$ into itself, such that $f_i([-1, 1))$ is bounded away from -1 and 1, $f_i \in C^{1+\alpha}([-1, 1))$ and

$$0 < \lambda_{i,\min} \leq |f_i'(x)| \leq \lambda_{i,\max} < 1 \tag{5.1.1}$$

for every $x \in [-1, 1)$. Moreover let us assume that for every i the fixed point of f_i is $a_i \in (-1, 1)$, and

$$i \neq j \;\Rightarrow\; a_i \neq a_j. \tag{5.1.2}$$

Denote the invariant measure of the IFS $\{f_1, \dots, f_l\}$ with respect to the probability vector $(p_1, \dots, p_l)$ by ν. That is

$$\nu = \sum_{i=1}^{l} p_i \nu \circ f_i^{-1}. \tag{5.1.3}$$

Let $\mu = (p_1, \dots, p_l)^{\mathbb{N}}$ be a Bernoulli measure on the space $\Sigma = \{1, \dots, l\}^{\mathbb{N}}$ and let Y_ε be uniformly distributed on $[1 - \varepsilon, 1 + \varepsilon]$. Denote the probability measure of Y_ε by η_ε. Let

$$f_{i,Y_\varepsilon}(x) = Y_\varepsilon f_i(x) + a_i(1 - Y_\varepsilon) \tag{5.1.4}$$

for every $i \in \{1, \ldots, l\}$. Then $f_{i,Y_\varepsilon}(x)$ is in $[-1, 1)$ for all values of $x \in [-1, 1)$ and Y_ε, provided ε is sufficiently small. The iterated maps are applied randomly according to the stationary measure μ, with the sequence of independent and identically distributed errors $y_1, y_2, \ldots$, distributed as Y_ε, independent of the choice of the function. The Lyapunov exponent of the IFS is defined by

$$\chi(\mu, \eta_\varepsilon) = \mathbb{E}(\log(Y_\varepsilon f'))$$

and it is easy to see that

$$\chi(\mu, \eta_\varepsilon) < \sum_{i=1}^{l} p_i \log((1 + \varepsilon)\lambda_{i,\max}) < 0,$$

for sufficiently small $\varepsilon > 0$. Let Z_ε be the following random variable

$$Z_\varepsilon := \lim_{n\to\infty} f_{i_1,y_{1,\varepsilon}} \circ f_{i_2,y_{2,\varepsilon}} \circ \cdots \circ f_{i_n,y_{n,\varepsilon}}(0), \tag{5.1.5}$$

where the numbers i_k are i.i.d., with the distribution μ on $\{1, \ldots, l\}$, and $y_{k,\varepsilon}$ are pairwise independent with distribution of Y_ε and also independent of the choice of i_k. Let ν_ε be the distribution of Z_ε.

One can easily prove the following theorem.

Theorem 5.1.1. *The measure ν_ε converges weakly to the measure ν as $\varepsilon \to 0$, see (5.1.3).*

Theorem 5.1.2. *Let ν_ε be the distribution of the limit (5.1.5). We assume that (5.1.1) and (5.1.2) hold, and*

$$\sum_{i=1}^{l} p_i^2 \frac{\lambda_{i,\max}}{\lambda_{i,\min}^2} < 1. \tag{5.1.6}$$

Then for every sufficiently small $\varepsilon > 0$, we have that ν_ε is absolutely continuous with respect to the Lebesgue measure, with density in L^2, and there exists a constant C such that the density of ν_ε satisfies

$$\|\nu_\varepsilon\|_2 \le \frac{C}{\sqrt{\varepsilon}}.$$

Remark 5.1.1. Let

$$C'_\varepsilon = \sqrt{\frac{32}{\left(1 - \sum_{i=1}^{l} p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\right) C''_\varepsilon}}$$

and

$$C''_\varepsilon = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1 - \varepsilon^2} \right\}.$$

The proof of Theorem 5.1.2 will show that we have $\|\nu_\varepsilon\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$. Hence we can choose any $C > \lim_{\varepsilon \to 0} C'_\varepsilon$.

Remark 5.1.2. Actually the proof of Theorem 5.1.1 shows that Z_ε conditioned on the perturbations $y_{1,\varepsilon}, y_{2,\varepsilon}, \ldots$ has density in L^2 for almost all $y_{1,\varepsilon}, y_{2,\varepsilon}, \ldots$.

We can state an easy corollary of the theorem.

Corollary 5.1.3. *Let $\{\lambda_i Y_\varepsilon x + a_i(1 - \lambda_i Y_\varepsilon)\}_{i=1}^{l}$ be a random iterated function system. We assume that (5.1.2) holds, and*

$$\sum_{i=1}^{l} \frac{p_i^2}{\lambda_i} < 1. \tag{5.1.7}$$

Then for every sufficiently small $\varepsilon > 0$, we have that ν_ε is absolutely continuous with respect to the Lebesgue measure with density in L^2, and there exists a constant C such that

$$\|\nu_\varepsilon\|_2 \leq \frac{C}{\sqrt{\varepsilon}}.$$

We study another case of random perturbation, namely let $\widetilde{\lambda}_{i,\varepsilon}$ be uniformly distributed on $[\lambda_i - \varepsilon, \lambda_i + \varepsilon]$. Let $\left\{\widetilde{\lambda}_{i,\varepsilon} x + a_i(1 - \widetilde{\lambda}_{i,\varepsilon})\right\}_{i=1}^{l}$ be our random iterated function system, where $a_i \neq a_j$ for every $i \neq j$. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_l)$, and $X_{\underline{\lambda},\varepsilon}$ be the following random variable

$$X_{\underline{\lambda},\varepsilon} = \sum_{k=1}^{\infty} (a_{i_k}(1 - \widetilde{\lambda}_{i_k,\varepsilon})) \prod_{j=1}^{k-1} \widetilde{\lambda}_{i_j,\varepsilon} \tag{5.1.8}$$

where the numbers i_k are i.i.d., with the distribution μ on $\{1, \ldots, l\}$, and $\widetilde{\lambda}_{i_k,\varepsilon}$ are pairwise independent. Let $\nu_{\underline{\lambda},\varepsilon}$ denote the distribution of the random variable $X_{\underline{\lambda},\varepsilon}$. Moreover let $\nu_{\underline{\lambda}}$ be the invariant measure of the iterated function system $\{\lambda_i x + a_i(1 - \lambda_i)\}_{i=1}^{l}$ according to μ.

Theorem 5.1.4. *The measure $\nu_{\underline{\lambda},\varepsilon}$ converges weakly to the measure $\nu_{\underline{\lambda}}$ as $\varepsilon \to 0$.*

To have a similar statement as in Theorem 5.1.2 we need a technical assumption, namely

$$\min_{i\neq j}\left|\frac{a_j\lambda_i - a_i\lambda_j}{\lambda_i-\lambda_j}\right| > 1. \tag{5.1.9}$$

Theorem 5.1.5. *Let us suppose that (5.1.9) and (5.1.2) hold, and moreover that*

$$\sum_{i=1}^{l}\frac{p_i^2}{\lambda_i} < 1. \tag{5.1.10}$$

Then for every sufficiently small $\varepsilon > 0$, the measure $\nu_{\underline{\lambda},\varepsilon}$ is absolutely continuous with respect to the Lebesgue measure, with density in L^2, and there exists a constant C such that

$$\|\nu_{\underline{\lambda},\varepsilon}\|_2 \leq \frac{C}{\sqrt{\varepsilon}}.$$

Remark 5.1.3. Let

$$C'_\varepsilon = \sqrt{\frac{32}{\left(1-\sum_{i=1}^{l} p_i^2\frac{\lambda_i+\varepsilon}{(\lambda_i-\varepsilon)^2}\right)C''_\varepsilon}}$$

and

$$C''_\varepsilon = \sigma\min_{i\neq j}\frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i-\lambda_j|}{\lambda_i\lambda_j},$$

where $0 < \sigma < 1$. As in Remark 5.1.1, the proof of Theorem 5.1.5 will show that we have $\|\nu_{\underline{\lambda},\varepsilon}\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$ for small ε.

The main difference between Theorem 5.1.5 and Corollary 5.1.3 is the random perturbation. Namely, in Theorem 5.1.5 we choose the contraction ratio uniformly in the ε neighborhood of λ_i, but in Corollary 5.1.3 we choose the contraction ratio uniformly in the $\lambda_i\varepsilon$ neighborhood of λ_i.

The chapter is based on [BP] which is a joint work with Tomas Persson.

5.2 Proof of Theorem 5.1.2

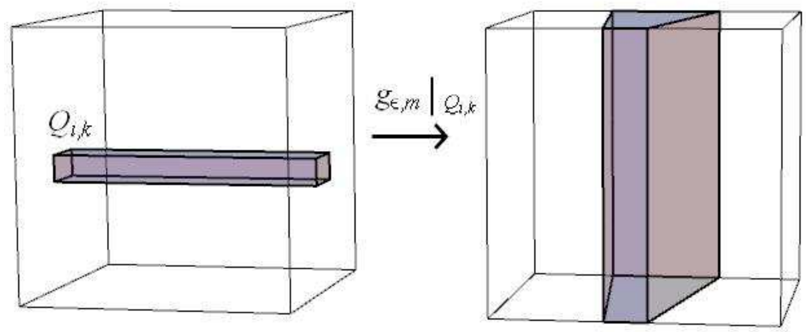

Figure 5.1: Picture showing the action of $g_{\varepsilon,m}$ restricted to $Q_{i,k}$.

Let $Q = [-1,1)^3$ and $m \in \mathbb{N}$. We partition the cube Q into the rectangles $\{Q_{1,k}, \ldots, Q_{l,k}\}_{k=0}^{2^m-1}$, where

$$Q_{i,k} = \Bigg\{ (x,y,z) \in Q : -1 + 2\sum_{j=1}^{i-1} p_j \leq y < -1 + 2\sum_{j=1}^{i} p_i,$$

$$-1 + k2^{-m+1} \leq z < -1 + (k+1)2^{-m+1} \Bigg\},$$

where we use the convention that an empty sum is 0. Hence we slice Q in 2^m slices along the z-axis and l slices along the y-axis. We thereby get $2^m l$ pieces which we call $Q_{i,k}$, according to the definition above.

Let

$$Q_i = \bigcup_{k=0}^{2^m-1} Q_{i,k}.$$

On each of the slices $Q_{i,k}$, we define the map $g_{\varepsilon,m}$ to map $Q_{i,k}$ into Q such that $Q_{i,k}$ is expanded as much as possible in the second and third coordinate. In the first coordinate it is mapped according to a perturbation of f_i, and hence contracted. Which perturbation is chosen depends on the third coordinate. There is a picture of this in Figure 5.1.

More precisely, for $(x,y,z) \in Q_{i,k}$, we define $g_{\varepsilon,m} \colon Q \to Q$ by

$$g_{\varepsilon,m} \colon (x,y,z) \mapsto \left(d(z) f_i(x) + a_i(1 - d(z)),\ \frac{1}{p_i} y + b(y),\ 2^m z + c(z) \right),$$

where

$$
\begin{aligned}
d(z) &= 1 + 2^m \varepsilon (z - (-1 + (k + \frac{1}{2}) 2^{-m+1}), && \text{for } (x,y,z) \in Q_{i,k}, \\
b(y) &= 1 - \frac{1}{p_i} \left(-1 + 2 \sum_{j=1}^{i} p_j \right), && \text{for } (x,y,z) \in Q_{i,k}, \\
c(z) &= 2^m - 2k - 1, && \text{for } (x,y,z) \in Q_{i,k}.
\end{aligned}
$$

Hence $g_{\varepsilon,m}$ maps each of the pieces $Q_{i,j}$ so that it is contracted in the x-direction and fully expanded in the y- and z-directions.

Let $\mathcal{L}_3$ be the normalised Lebesgue measure on Q. The measures

$$
\gamma_{\varepsilon,m,n} = \frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}_3 \circ g_{\varepsilon,m}^{-k}
$$

converge weakly to an SRB-measure $\gamma_{\varepsilon,m}$ as $n \to \infty$, see [Pes2] and [ST]. The measure $\gamma_{\varepsilon,m}$ is ergodic by the Hopf argument, since $g_{\varepsilon,m}$ is hyperbolic and the stable and unstable manifolds are parallel to the coordinate axes and have maximal extension in the box Q. Moreover, let $\nu_{\varepsilon,m}$ be the projection of $\gamma_{\varepsilon,m}$ onto the first coordinate. More precisely, if $E \subset [-1,1)$ is a measurable set, then we define $\nu_{\varepsilon,m}(E) = \gamma_{\varepsilon,m}(E \times [-1,1) \times [-1,1))$.

The measure $\nu_{\varepsilon,m}$ is the distribution of the limit

$$
\lim_{n \to \infty} f_{i_1, y_{1,\varepsilon}} \circ f_{i_2, y_{2,\varepsilon}} \circ \cdots \circ f_{i_n, y_{n,\varepsilon}}(0),
$$

where $y_{i,\varepsilon}$ are uniformly distributed on $[1-\varepsilon, 1+\varepsilon]$, but not independent. However, one can easily prove the following lemma.

Lemma 5.2.1. *The measure $\nu_{\varepsilon,m}$ converges weakly to ν_ε as $m \to \infty$.*

Let

$$
A_i = \{(i,0), (i,1), \ldots, (i, 2^m - 1)\}
$$

and

$$
A = \bigcup_{i=1}^{l} A_i.
$$

If $a = (i,k) \in A$ we will use the notation $\hat{Q}_a$ to denote $Q_{i,k}$. With this notation we have

$$
Q = \bigcup_{a \in A} \hat{Q}_a \quad \text{and} \quad Q_i = \bigcup_{a \in A_i} \hat{Q}_a, \quad i = 0, 1, \ldots, l.
$$

Let $\Theta_0 = A^{\mathbb{N}\cup\{0\}}$. If $p \in Q$ then there is a unique sequence $\rho_0(p) = \{\rho_0(p)_k\}_{k=0}^{\infty} \in \Theta_0$ such that

$$g_{\varepsilon,m}^k(p) \in Q_{\rho_0(p)_k}, \quad k = 0, 1, \ldots$$

The map $\rho_0 \colon Q \to \Theta_0$ is not injective. We have $\rho_0(x,y,z) = \rho_0(x',y',z')$ if $y = y'$ and $z = z'$, but $\rho_0(x,y,z) \neq \rho_0(x',y',z')$ otherwise. Hence we can (and will) use the notation $\rho_0(y,z)$ instead of $\rho_0(x,y,z)$.

We will denote elements in Θ_0 by $\boldsymbol{a}$, $\boldsymbol{b}$ and so on. We let σ denote the left shift on Θ_0, defined in the usual way.

We can transfer the measures $\gamma_{\varepsilon,m}$ to a measure γ_{Θ_0} by $\gamma_{\Theta_0} = \gamma_{\varepsilon,m} \circ \rho_0^{-1}$.

We let Θ denote the natural extension of Θ_0. That is, Θ is the set of all two sided infinite sequences such that any one sided infinite subsequence of a sequence in Θ is a sequence in Θ_0. The measures γ_{Θ_0} defines an ergodic measure γ_Θ on Θ in a natural way. If $\xi \colon \Theta \to \Theta_0$ is defined by $\xi(\{i_k\}_{k\in\mathbb{Z}}) = \{i_k\}_{k\in\mathbb{N}\cup\{0\}}$, then we define $\gamma_\Theta(\xi^{-1}E) = \gamma_{\Theta_0}(E)$. We can define a map $\rho^{-1} \colon \Theta \to Q$ such that $\rho^{-1}(\sigma(\boldsymbol{a})) = g_{\varepsilon,m}(\rho^{-1}(\boldsymbol{a}))$ holds for any sequence $\boldsymbol{a} \in \Theta$.

We note that the L^2 norm of the density $\nu_{\varepsilon,m}$ is not larger than twice that of the density of $\gamma_{\varepsilon,m}$. If $h_{\nu_{\varepsilon,m}}(x)$ and $h_{\gamma_{\varepsilon,m}}(x,y,z)$ denote the density of $\nu_{\varepsilon,m}$ and $\gamma_{\varepsilon,m}$ respectively, then by Cauchy–Schwarz's inequality

$$\begin{aligned}
\|\nu_{\varepsilon,m}\|_2^2 \leq \int_{-1}^{1} h_{\nu_{\varepsilon,m}}(x)^2 \, dx &= 32 \int_{-1}^{1} \left(\int_{-1}^{1} \int_{-1}^{1} h_{\gamma_{\varepsilon,m}}(x,y,z) \, \frac{dy}{2} \frac{dz}{2} \right)^2 \frac{dx}{2} \\
&\leq 32 \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} h_{\gamma_{\varepsilon,m}}(x,y,z)^2 \, \frac{dy}{2} \frac{dz}{2} \frac{dx}{2} = 4\|\gamma_{\varepsilon,m}\|_2^2.
\end{aligned}$$

This proves that if $\gamma_{\varepsilon,m}$ has L^2 density, then so has $\nu_{\varepsilon,m}$, and

$$\|\nu_{\varepsilon,m}\|_2 \leq 2\|\gamma_{\varepsilon,m}\|_2. \tag{5.2.1}$$

If p is a point in Q, then we let T_pQ denote the tangent space at p. For each p in Q we define the following cone in the tangent space T_pQ:

$$C_p = \left\{ (u,v,w) \in T_pQ : \left|\frac{u}{w}\right|, \left|\frac{v}{w}\right| < \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right\},$$

where $\lambda_{\max,\max} = \max_i \lambda_{i,\max} = \max_i \sup_{x\in[-1,1)} |f_i'(x)|$. The following lemma states that the set of cones C_p, defines a family of unstable cones, and that images of certain curves intersect transversally. There is an illustration of the transversality in Figure 5.2.

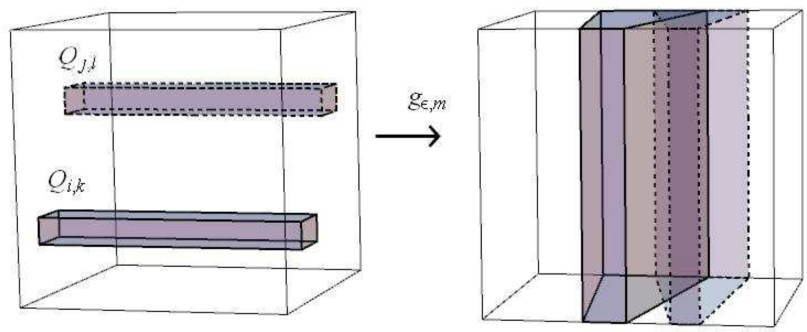

Figure 5.2: Every two different $Q_{i,k}$ and $Q_{j,l}$ on the same height ($i = j$) share the same image, but in the case when $i \neq j$ their images have transversal intersection if they intersect.

Lemma 5.2.2. *The cones C_p make up a family of unstable cones, that is $d_p g_{\varepsilon,m}(C_p) \subset C_{g_{\varepsilon,m}(p)}$.*

Moreover, for sufficiently large m and every $0 < \varepsilon < \min_{i \neq j} \frac{|a_i - a_j|}{2+|a_i+a_j|}$, if $\zeta_1 \subset Q_{\xi_1}$ and $\zeta_2 \subset Q_{\xi_2}$ are two curve segments with tangents in C_p such that $\xi_1 \in A_i$ and $\xi_2 \in A_j$, $i \neq j$, then if $g_{\varepsilon,m}(\zeta_1)$ and $g_{\varepsilon,m}(\zeta_2)$ intersect, and if $(u_1, v_1, 1)$ and $(u_2, v_2, 1)$ are tangents to $g_{\varepsilon,m}(\zeta_1)$ and $g_{\varepsilon,m}(\zeta_2)$ respectively, it holds $|u_1 - u_2| > C_{\varepsilon,m}\varepsilon$, where

$$C_{\varepsilon,m} = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1 - \varepsilon^2} - \frac{4(1+\varepsilon)\lambda_{\max,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)} \right\}.$$

Proof of Lemma 5.2.2. The Jacobian of $g_{\varepsilon,m}$ is

$$d_p g_{\varepsilon,m} = \begin{pmatrix} d(z)f_i'(x) & 0 & 2^m\varepsilon(f_i(x) - a_i) \\ 0 & \frac{1}{p_i} & 0 \\ 0 & 0 & 2^m \end{pmatrix}$$

where $p = (x, y, z) \in Q_{i,k}$. If $(u, v, w) \in C_p$, then

$$d_p g_{\varepsilon,m}(u, v, w) = \begin{pmatrix} d(z)f_i'(x)u + 2^m\varepsilon(f_i(x) - a_i)w \\ \frac{1}{p_i}v \\ 2^m w \end{pmatrix}$$

We just need to check that this vector is in C_p, provided that m is large. This is easily checked, using that $|d(z)| \leq 1 + \varepsilon$, $|f_i'(x)| \leq \lambda_{i,\max}$

and $|f_i(x) - a_i| \leq 2$. Indeed,

$$\frac{|d(z)f_i'(x)u + 2^m\varepsilon(f_i(x) - a_i)w|}{|2^m w|} \leq \frac{(1+\varepsilon)\lambda_{i,\max}}{2^m}\frac{|u|}{|w|} + 2\varepsilon$$
$$\leq \frac{(1+\varepsilon)\lambda_{i,\max}}{2^m}\frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}} + 2\varepsilon \leq \frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}}$$

and
$$\frac{|\frac{1}{p_i}v|}{|2^m w|} \leq \frac{1}{p_i 2^m}\frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}} \leq \frac{2^{m+1}\varepsilon}{2^m - (1+\varepsilon)\lambda_{\max,\max}}$$
proves that $d_p g_{\varepsilon,m}(C_p) \subset C_{g_{\varepsilon,m}(p)}$ if m is sufficiently large, so that $2^m - (1+\varepsilon)\lambda_{\max,\max} > 0$ and $p_i 2^m > 1$.

To prove the other statement of the Lemma, assume that $p = (x_p, y_p, z_p) \in Q_i$ and $q = (x_q, y_q, z_q) \in Q_j$, $i \neq j$, are such that $g_{\varepsilon,m}(p) = g_{\varepsilon,m}(q) = (x, y, z)$. Then, if $p \in Q_i$

$$d_p g_{\varepsilon,m}\colon (u, v, 1) \mapsto 2^m\left(\frac{d(z_p)f_i'(x_p)}{2^m}u + (f_i(x_p) - a_i)\varepsilon,\ \frac{v}{p_i},\ 1\right)$$

Then

$$f_i(x_p) = \frac{x - a_i(1 - d(z_p))}{d(z_p)} \quad \text{and} \quad f_j(x_q) = \frac{x - a_j(1 - d(z_q))}{d(z_q)}.$$

Without loss of generality, let us assume that $a_i > a_j$. For simplicity we study the case $x \geq a_i > a_j$. The proofs of the cases $a_i \geq x \geq a_j$ and $a_i > a_j \geq x$ are similar. Then

$$d_p g_{\varepsilon,m}(C_p) \subset \left\{ w(u, v, 1) : \frac{x - a_i}{1+\varepsilon}\varepsilon - \Delta_i\varepsilon \leq u \leq \frac{x - a_i}{1-\varepsilon}\varepsilon + \Delta_i\varepsilon \right\},$$

where $\Delta_i = \frac{2(1+\varepsilon)\lambda_{i,\max}}{2^m - \lambda_{\max,\max}(1+\varepsilon)}$. Therefore

$$|u_2 - u_1| \geq \frac{x - a_j}{1+\varepsilon}\varepsilon - \frac{x - a_i}{1-\varepsilon}\varepsilon - (\Delta_i + \Delta_j)\varepsilon$$
$$\geq \left(\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} - 2\max_i \Delta_i\right)\varepsilon$$

for every $x \geq a_i > a_j$. Let $\Delta_{\max} = \max_i \Delta_i$. Since $0 < \varepsilon < \min_{i \neq j} \frac{|a_i - a_j|}{2 + |a_i + a_j|}$, we have
$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} > 0.$$

Therefore
$$\frac{a_i - a_j + \varepsilon(a_i + a_j - 2)}{1 - \varepsilon^2} - 2\Delta_{\max} > 0,$$
for sufficiently large m. By similar methods, we have for $a_i \geq x \geq a_j$
$$|u_2 - u_1| \geq \left(\frac{a_i - a_j}{1 + \varepsilon} - 2\Delta_{\max} \right) \varepsilon,$$
and for $a_i > a_j \geq x$
$$|u_2 - u_1| \geq \left(\frac{a_i - a_j - \varepsilon(a_i + a_j + 2)}{1 - \varepsilon^2} - 2\Delta_{\max} \right) \varepsilon.$$
Therefore we can choose
$$C_{\varepsilon,m} = \min_{i \neq j} \left\{ \frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1 - \varepsilon^2} - 2\Delta_{\max} \right\}. \qquad \square$$

The rest of the section will follow the method of Tsujii's article [T].

Proof of Theorem 5.1.2. For any $r > 0$ we define the bilinear form $(\cdot, \cdot)_r$ of signed measures on $\mathbb{R}$ by
$$(\rho_1, \rho_2)_r = \int_{\mathbb{R}} \rho_1(B_r(x))\rho_2(B_r(x))\, dx$$
where $B_r(x) = [x - r, x + r]$. It is easy to see that if
$$\liminf_{r \to 0} \frac{1}{r^2}(\rho, \rho)_r < \infty$$
then the measure ρ has density in L^2, see [T]. Moreover
$$\|\rho\|_2^2 \leq \liminf_{r \to 0} \frac{1}{r^2}(\rho, \rho)_r.$$

Let γ_z denote the conditional measure of $\gamma_{\varepsilon,m}$ on the set $R_z = \{(u, v, w) \in Q : v = y, w = z\}$. Since the one-dimensional Lebesgue measure is invariant under the action of $g_{\varepsilon,m}$ projected to the second coordinate, we conclude that γ_z is independent of y almost everywhere. Therefore, it follows that
$$\|\gamma_{\varepsilon,m}\|_2^2 = \int_{-1}^{1} \|\gamma_z\|_2^2\, dz. \tag{5.2.2}$$

Let

$$J(r) := \frac{1}{r^2}\int_{-1}^{1}(\gamma_z,\gamma_z)_r\,dz.$$

By the invariance of $\gamma_{\varepsilon,m}$ it follows that

$$\gamma_z = 2^{-m}\sum_{i=1}^{l} p_i \sum_{a\in A_i}\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m}, \tag{5.2.3}$$

where $g^{-a}_{\varepsilon,m}$ denotes the inverse branch of $g_{\varepsilon,m}$ such that the image of $g^{-a}_{\varepsilon,m}$ is in $\hat{Q}_a$. Recall that $a\in A_i$ means that $a=(i,k)$ for some k, so that $\hat{Q}_a = Q_{i,k}$ for some k. We denote the measure $\gamma_{g^{-a}_{\varepsilon,m}(z)}\circ g^{-a}_{\varepsilon,m}$ by $\sigma_{a,z}$. Then by (5.2.3) and the definition of $J(r)$

$$J(r) = \frac{1}{2^{2m}r^2}\sum_{i=1}^{l}\sum_{j=1}^{l}p_ip_j\sum_{a\in A_i}\sum_{b\in A_j}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz. \tag{5.2.4}$$

For fixed $a,b\in A_i$ it holds,

$$\begin{aligned}
&(\sigma_{a,z},\sigma_{b,z})_r \le (\sigma_{a,z},\sigma_{a,z})_r^{\frac12}(\sigma_{b,z},\sigma_{b,z})_r^{\frac12}\\
&\quad\le (1+\varepsilon)\lambda_{i,\max}\left(\gamma_{g^{-a}_{\varepsilon,m}(z)},\gamma_{g^{-a}_{\varepsilon,m}(z)}\right)^{\frac12}_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\times\left(\gamma_{g^{-b}_{\varepsilon,m}(z)},\gamma_{g^{-b}_{\varepsilon,m}(z)}\right)^{\frac12}_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\\
&\quad\le (1+\varepsilon)\lambda_{i,\max}\frac{\left(\gamma_{g^{-a}_{\varepsilon,m}(z)},\gamma_{g^{-a}_{\varepsilon,m}(z)}\right)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}+\left(\gamma_{g^{-b}_{\varepsilon,m}(z)},\gamma_{g^{-b}_{\varepsilon,m}(z)}\right)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}}{2}.
\end{aligned} \tag{5.2.5}$$

Moreover, if $a\in A_i$ and $b\in A_j$, $i\neq j$, then

$$\begin{aligned}
(\sigma_{a,z},&\sigma_{b,z})_r\\
&= \int \sigma_{a,z}(B_r(x))\sigma_{b,z}(B_r(x))\,dx\\
&= \iiint \mathbb{I}_{\{\,s:|s-x|<r\,\}}(s)\mathbb{I}_{\{\,t:|t-x|<r\,\}}(t)\,d\sigma_{a,z}(s)d\sigma_{b,z}(t)dx\\
&\le \iint 2r\mathbb{I}_{\{\,(s,t):|s-t|<2r\,\}}(s,t)\,d\sigma_{a,z}(s)d\sigma_{b,z}(t)\\
&= \iint \mathbb{I}_{\{\,(\boldsymbol{c},\boldsymbol{d}):|\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z))-\rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))|<2r\,\}}(\boldsymbol{c},\boldsymbol{d})\\
&\qquad\qquad\qquad\qquad\qquad\qquad d\gamma_\Theta(\boldsymbol{c})d\gamma_\Theta(\boldsymbol{d}).
\end{aligned}$$

Let us comment on the notation $\rho_0(z)$. Actually $\rho_0(z)$ is not defined, but rather $\rho_0(x,y,z)$. Recall that $\rho_0(x,y,z)$ is independent of x and that we therefore have introduced the notation $\rho_0(y,z)$. Moreover, as noticed above, the measures γ_z, and therefore also $\sigma_{a,z}$, are independent of y. Hence we can choose arbitrary x, y and let $\rho_0(z)$ denote $\rho_0(x,y,z) = \rho_0(y,z)$. Since all the estimates below will be independent of this choice of y, we will use the notation $\rho_0(z)$ instead of $\rho_0(x,y,z)$.

By a change of order of integration we get that

$$\begin{aligned}\int_{-1}^{1} &(\sigma_{a,z}, \sigma_{b,z})_r \, dz \\ &\leq 2r \iint \mathcal{L}_1(\{\, z : |\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)) \\ &\qquad - \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))| < 2r \,\}) \, d\gamma_\Theta(\boldsymbol{c}) d\gamma_\Theta(\boldsymbol{d}). \end{aligned} \tag{5.2.6}$$

We will now use Lemma 5.2.2 on (5.2.6). Note that

$$\begin{aligned} z &\mapsto \rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)) \\ z &\mapsto \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z)) \end{aligned}$$

defines two curves with tangents in the cones C_p. Lemma 5.2.2 states that these curves have a transversal intersection, if they intersect, so that

$$\mathcal{L}_1(\big\{\, z : |\rho^{-1}(\cdots c_{-2}c_{-1}a\rho_0(z)) - \rho^{-1}(\cdots d_{-2}d_{-1}b\rho_0(z))| < 2r \,\big\}) \leq \frac{4r}{C_{\varepsilon,m}}.$$

Hence

$$\int_{-1}^{1} (\sigma_{a,z}, \sigma_{b,z})_r \, dz \leq \frac{8r^2}{C_{\varepsilon,m}\varepsilon}. \tag{5.2.7}$$

By using (5.2.4) we have

$$\begin{aligned} J(r) = \frac{1}{2^{2m}r^2} \sum_{i=1}^{l} p_i^2 \sum_{a,b \in A_i} \int_{-1}^{1} (\sigma_{a,z}, \sigma_{b,z})_r \, dz \\ + \frac{1}{2^{2m}r^2} \sum_{i \neq j} p_i p_j \sum_{a \in A_i} \sum_{b \in A_j} \int_{-1}^{1} (\sigma_{a,z}, \sigma_{b,z})_r \, dz. \end{aligned} \tag{5.2.8}$$

We first give an upper bound for the first part of the sum in (5.2.8), using

(5.2.5) and an integral transformation. By (2.2.7) we have

$$\begin{aligned}
&\sum_{a,b\in A_i}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz\\
&\qquad\le(1+\varepsilon)\lambda_{i,\max}2^m\sum_{a\in A_i}\int_{-1}^{1}(\gamma_{g^{-a}_{\varepsilon,m}(z)},\gamma_{g^{-a}_{\varepsilon,m}(z)})_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz\\
&\qquad=(1+\varepsilon)\lambda_{i,\max}2^m\sum_{k=0}^{2^m-1}2^m\int_{-1+k2^{-m+1}}^{-1+(k+1)2^{-m+1}}(\gamma_z,\gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{2^{2m}r^2}\sum_{i=1}^{l}p_i^2\sum_{a,b\in A_i}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz\\
&\le\frac{1}{2^{2m}r^2}\sum_{i=1}^{l}p_i^2(1+\varepsilon)\lambda_{i,\max}2^m\sum_{k=0}^{2^m-1}2^m\int\limits_{-1+k2^{-m+1}}^{-1+(k+1)2^{-m+1}}(\gamma_z,\gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz\\
&\le\sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\frac{1}{\left(\frac{r}{(1-\varepsilon)\lambda_{i,\min}}\right)^2}\int_{-1}^{1}(\gamma_z,\gamma_z)_{\frac{r}{(1-\varepsilon)\lambda_{i,\min}}}\,dz\\
&\le\max_i J\left(\frac{r}{\lambda_{i,\min}(1-\varepsilon)}\right)\sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}. \qquad (5.2.9)
\end{aligned}$$

For the second part of the sum in (5.2.8), we use (5.2.7), to prove that it is bounded by

$$\begin{aligned}
&\frac{1}{2^{2m}r^2}\sum_{i\ne j}p_ip_j\sum_{a\in A_i}\sum_{b\in A_j}\int_{-1}^{1}(\sigma_{a,z},\sigma_{b,z})_r\,dz\\
&\qquad\qquad\le\frac{1}{2^{2m}r^2}\sum_{i\ne j}p_ip_j\sum_{a\in A_i}\sum_{b\in A_j}\frac{8r^2}{C_{\varepsilon,m}\varepsilon}\le\frac{8}{C_{\varepsilon,m}\varepsilon}. \qquad (5.2.10)
\end{aligned}$$

By combining (5.2.9) and (5.2.10) we have

$$J(r)\le\frac{8}{C_{\varepsilon,m}\varepsilon}+\beta\max_i J\left(\frac{r}{\lambda_{i,\min}(1-\varepsilon)}\right) \qquad (5.2.11)$$

where $\beta=\sum_{i=1}^{l}p_i^2\frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}$ is less than 1 by (5.1.6).

We define a strictly monotone decreasing sequence r_k. Let $r_0 < 1/2$ be fixed and define r_k recursively. Assume that r_{k-1} has been defined. Then we define $r_k = (1-\varepsilon)\lambda_{i_k,\min} r_{k-1}$, where i_k is chosen such that

$$\max_i J\left(\frac{r_k}{(1-\varepsilon)\lambda_{i,\min}}\right) = J\left(\frac{r_k}{(1-\varepsilon)\lambda_{i_k,\min}}\right) = J(r_{k-1}).$$

Hence we have $r_k = r_0(1-\varepsilon)^k \prod_{n=1}^k (\lambda_{i_n,\min})$.

We note that r_k is a well defined sequence. By induction and (5.2.11), we have

$$J(r_k) \leq \frac{8}{C_{\varepsilon,m}\varepsilon}\frac{1-\beta^k}{1-\beta} + \beta^k J(r_0) \tag{5.2.12}$$

for every $k \geq 1$. Hence by (5.2.1), (5.2.2) and (5.2.12) we get

$$\begin{aligned}\|\nu_{\varepsilon,m}\|_2^2 \leq 4 \liminf_{r\to 0} J(r) \leq 4 \liminf_{k\to\infty} J(r_k) \\ \leq \frac{32}{C_{\varepsilon,m}\varepsilon} \frac{1}{1-\sum_{i=1}^l p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}}.\end{aligned} \tag{5.2.13}$$

We now use that L^2 is a Hilbert space, and that in a Hilbert space, a closed ball is compact in the weak topology. (See for instance [Yos, Theorem V.2.1].) Hence, if $h_{\nu_{\varepsilon,m}}$ is the density of $\nu_{\varepsilon,m}$, then $h_{\nu_{\varepsilon,m}}$ is in L_2, and from above we know that there is a constant C'_ε such that $\|h_{\nu_{\varepsilon,m}}\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$.

By the compactness statement above, there is an h with $\|h\|_2 \leq C'_\varepsilon/\sqrt{\varepsilon}$, such that some subsequence of $h_{\nu_{\varepsilon,m}}$ converges weakly to h. Moreover h defines a probability measure since $1 = \int 1 \cdot h_{\nu_{\varepsilon,m}}\, d\mathcal{L}_3 \to \int 1 \cdot h\, d\mathcal{L}_3$.

Since $\nu_{\varepsilon,m}$ converges weakly to ν_ε we get that ν_ε has density in L^2 and that

$$\|\nu_\varepsilon\|_2 \leq \frac{1}{\sqrt{\varepsilon}} C'_\varepsilon, \tag{5.2.14}$$

where

$$C'_\varepsilon = \sqrt{\frac{32}{\left(1-\sum_{i=1}^l p_i^2 \frac{(1+\varepsilon)\lambda_{i,\max}}{((1-\varepsilon)\lambda_{i,\min})^2}\right) C''_\varepsilon}}$$

and

$$C''_\varepsilon = \lim_{m\to\infty} C_{\varepsilon,m} = \min_{i\neq j}\left\{\frac{|a_i - a_j| + \varepsilon(-|a_i + a_j| - 2)}{1-\varepsilon^2}\right\}. \qquad \square$$

5.3 Proof of Theorem 5.1.5

We do not give the whole proof of Theorem 5.1.5, because it is similar to the proof of Theorem 5.1.2. We prove only the modification of Lemma 5.2.2, which is important as it proves transversality.

First we define a new dynamical system $\widetilde{g}_{\varepsilon,m}\colon Q \to Q$, similar to the dynamical system $g_{\varepsilon,m}\colon Q \to Q$. Let $Q_{i,k}$ and $A_{i,k}$ be as in Section 5.2. Let $\widetilde{g}_{\varepsilon,m}\colon Q \to Q$ be defined by

$$\widetilde{g}_{\varepsilon,m}\colon (x,y,z) \mapsto \left(\widetilde{d}(z)x + a_i(1-\widetilde{d}(z)),\ \frac{1}{p_i}y + b(y),\ 2^m z + c(z)\right),$$

for $(x,y,z) \in Q_i$, where

$$\begin{aligned}
\widetilde{d}(z) &= \lambda_i + 2^m\varepsilon(z - (-1 + (k+\tfrac{1}{2})2^{-m+1})), && \text{for } (x,y,z) \in Q_{i,k},\\
b(y) &= 1 - \frac{1}{p_i}\left(-1 + 2\sum_{j=1}^{i} p_j\right), && \text{for } (x,y,z) \in Q_{i,k},\\
c(z) &= 2^m - 2k - 1, && \text{for } (x,y,z) \in Q_{i,k}.
\end{aligned}$$

Hence the only difference between $\widetilde{g}_{\varepsilon,m}$ and $g_{\varepsilon,m}$ is in the first coordinate, where the perturbations of f_i are made. Figure 5.1 also serves in visualizing the action of $\widetilde{g}_{\varepsilon,m}$.

We define the cones

$$C_p = \left\{ (u,v,w) \in T_pQ : \left|\frac{u}{w}\right|, \left|\frac{v}{w}\right| < \frac{2^{m+1}\varepsilon}{2^m - \lambda_{\max} - \varepsilon} \right\},$$

where $p \in Q$ and $\lambda_{\max} = \max_i \lambda_i$. Similar to Lemma 5.2.2, we show that these cones define a family of unstable cones, and that a certain transversality property holds.

Lemma 5.3.1. *Let us suppose that (5.1.9) holds. The cones C_p defines a family of unstable cones, that is $d_p\widetilde{g}_{\varepsilon,m}(C_p) \subset C_{\widetilde{g}_{\varepsilon,m}(p)}$.*

Moreover, for sufficiently large m and every sufficiently small $0 < \varepsilon$, if $\zeta_1 \subset Q_{\xi_1}$ and $\zeta_2 \subset Q_{\xi_2}$ are two line segments with tangents in C_p such that $\xi_1 \in A_i$ and $\xi_2 \in A_j$, $i \neq j$, then if $\widetilde{g}_{\varepsilon,m}(\zeta_1)$ and $\widetilde{g}_{\varepsilon,m}(\zeta_2)$ intersects, and if $(u_1,v_1,1)$ and $(u_2,v_2,1)$ are tangents to $\widetilde{g}_{\varepsilon,m}(\zeta_1)$ and $\widetilde{g}_{\varepsilon,m}(\zeta_2)$ respectively, there exists a constant $C_{\varepsilon,m}$, depending on ε and m, but bounded away from 0 and infinity, such that $|u_1 - u_2| > C_{\varepsilon,m}\varepsilon$.

Proof of Lemma 5.3.1. The Jacobian of $\widetilde{g}_{\varepsilon,m}$

$$d_p\widetilde{g}_{\varepsilon,m} = \begin{pmatrix} \widetilde{d}(z) & 0 & 2^m\varepsilon(x-a_i) \\ 0 & \frac{1}{p_i} & 0 \\ 0 & 0 & 2^m \end{pmatrix},$$

where $p=(x,y,z)\in Q_{i,k}$. If $(u,v,w)\in C_p$, then

$$d_p\widetilde{g}_{\varepsilon,m}(u,v,w)=\begin{pmatrix}\widetilde{d}(z)u+2^m\varepsilon(x-a_i)w\\ \frac{1}{p_i}v\\ 2^m w\end{pmatrix}.$$

The estimate

$$\begin{aligned}\frac{|\widetilde{d}(z)u+2^m\varepsilon(x-a_i)w|}{|2^m w|}&\le\frac{\widetilde{d}(z)|u|}{2^m|w|}+2\varepsilon\\&\le\frac{\lambda_i+\varepsilon}{2^m}\frac{2^{m+1}\varepsilon}{2^m-\lambda_{\max}-\varepsilon}+2\varepsilon\le\frac{2^{m+1}\varepsilon}{2^m-\lambda_{\max}-\varepsilon}\end{aligned}$$

shows that $d_p\widetilde{g}_{\varepsilon,m}(C_p)\subset C_{\widetilde{g}_{\varepsilon,m}(p)}$. Now we prove the other statement of the Lemma. Assume that $p=(x_p,y_p,z_p)\in Q_i$ and $q=(x_q,y_q,z_q)\in Q_j$, $i\neq j$, are such that $\widetilde{g}_{\varepsilon,m}(p)=\widetilde{g}_{\varepsilon,m}(q)=(x,y,z)$. Then

$$p\in Q_i\quad\Rightarrow\quad d_p\widetilde{g}_{\varepsilon,m}:(u,v,1)\mapsto 2^m\left(\frac{\widetilde{d}(z_p)}{2^m}u+(x_p-a_i)\varepsilon,\ \frac{v}{p_i},\ 1\right),$$

and

$$x_p=\frac{x-a_i(1-\widetilde{d}(z_p))}{\widetilde{d}(z_p)},\quad x_q=\frac{x-a_j(1-\widetilde{d}(z_q))}{\widetilde{d}(z_q)}.$$

Let $\widetilde{\Delta}_i=\frac{2(\lambda_i+\varepsilon)}{2^m-\lambda_{\max}-\varepsilon}$. Then

$$d_p\widetilde{g}_{\varepsilon,m}(C_p)\ \subset\ \left\{w(u,v,1)\ :\ \frac{x-a_i}{\widetilde{d}(z_p)}\varepsilon-\widetilde{\Delta}_i\varepsilon\ \le\ u\ \le\ \frac{x-a_i}{\widetilde{d}(z_p)}\varepsilon+\widetilde{\Delta}_i\varepsilon\right\}.$$

Therefore

$$|u_2-u_1|\ge\left(\left|\frac{x-a_i}{\widetilde{d}(z_p)}-\frac{x-a_j}{\widetilde{d}(z_q)}\right|-(\widetilde{\Delta}_i+\widetilde{\Delta}_j)\right)\varepsilon.$$

The term

$$\left|\frac{x-a_i}{\widetilde{d}(z_p)}-\frac{x-a_j}{\widetilde{d}(z_q)}\right|$$

can be estimated by

$$\left|\frac{x-a_i}{\widetilde{d}(z_p)}-\frac{x-a_j}{\widetilde{d}(z_q)}\right|\ge\left|\frac{|\widetilde{d}(z_p)-\widetilde{d}(z_q)||x|-|a_j\widetilde{d}(z_p)-a_i\widetilde{d}(z_q)|}{\widetilde{d}(z_p)\widetilde{d}(z_q)}\right|.$$

Hence, this term is positive provided that

$$|a_j\widetilde{d}(z_p) - a_i\widetilde{d}(z_q)| > |\widetilde{d}(z_p) - \widetilde{d}(z_q)|.$$

Since $\lambda_i - \varepsilon \leq \widetilde{d}(z_p) \leq \lambda_i + \varepsilon$ and $\lambda_j - \varepsilon \leq \widetilde{d}(z_q) \leq \lambda_j + \varepsilon$, this is implied by (2.1.1) if ε is sufficiently small.

If we let

$$C_{\varepsilon,m} = \frac{1}{2}\min_{i\neq j}\frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i\lambda_j},$$

then

$$|u_2 - u_1| \geq C_{\varepsilon,m}\varepsilon,$$

provided that ε is small and m large.

In fact we can let

$$C_{\varepsilon,m} = \sigma\min_{i\neq j}\frac{|a_i\lambda_j - a_j\lambda_i| - |\lambda_i - \lambda_j|}{\lambda_i\lambda_j},$$

for $0 < \sigma < 1$. $\square$

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