ON ITERATED FUNCTION SYSTEMS WITH PLACE-DEPENDENT PROBABILITIES

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Abstract. In this paper we study a family of invariant measures of parameterized iterated function systems where the corresponding probabilities are place-dependent. We prove that the Hausdorff dimension of the measure is equal to Entropy/Lyapunov exponent whenever it is less than 1 and the measure is absolute continuous w.r.t. the Lebesgue measure if Entropy/Lyapunov exponent is greater than 1 for Lebesgue almost every parameters.

1. Introduction and Statements

Let X be a compact interval on the real line and let $\{\psi_i\}_{i=1}^k$ be a family of contractive maps mapping X into itself. We call the set $\{\psi_i\}_{i=1}^k$ an *iterated function system* (IFS) on X. It is well known that there exists a unique non-empty compact set $\Lambda \subseteq X$ such that it is invariant w.r.t the IFS, that is $\Lambda = \cup_{i=1}^k \psi_i(\Lambda)$. We call the set Λ the *attractor* of the IFS. Moreover, for any probability weights $\{p_i\}_{i=1}^k$ such that $0 < p_i < 1$ and $\sum_{i=1}^k p_i = 1$ there exists a unique probability measure μ that satisfies $\text{supp}\mu = \Lambda$ and

$$\mu = \sum_{i=1}^k p_i \mu \circ \psi_i^{-1}. \tag{1.1}$$

The measure μ is called the *invariant measure* of the IFS (see e.g. Hutchinson [5], Falconer [2]).

In this paper we focus on an extended class of invariant measures. We consider the probability measures on the set Λ that satisfy the equation

$$\int f(x)d\mu(x) = \sum_{i=1}^k \int p_i(x) f(\psi_i(x)) d\mu(x) \text{ for every } f \in C(X), \tag{1.2}$$

where $p_i : X \mapsto (0,1)$ are Hölder continuous for every $i = 1 \dots k$ and $\sum_{i=1}^k p_i(x) \equiv 1$. Fan and Lau proved that there exists a unique probability measure which satisfies the equation (1.2) and $\text{supp}\mu = \Lambda$, see [3]. Let us call the measure μ *place-dependent invariant measure*.

Place-dependent invariant measures were studied in several papers, see e.g. [4, 6, 7, 8, 19]. Our goal is to determine the *Hausdorff dimension* of such measures and to give a sufficient condition for the absolute continuity w.r.t the Lebesgue measure. For the basic properties and definition of the Hausdorff dimension see for example [2].

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For further analysis let us introduce two notations. Let us define the *entropy* h_μ of the measure μ as

$$h_\mu := -\int \sum_{i=1}^{k} p_i(x) \log p_i(x) d\mu(x). \tag{1.3}$$

Moreover, denote χ_μ the *Lyapunov exponent* of the IFS w.r.t μ. That is,

$$\chi_\mu := -\int \sum_{i=1}^{k} p_i(x) \log \left|\psi_i'(x)\right| d\mu(x). \tag{1.4}$$

Fan and Lau showed that if the IFS $\{\psi_i\}_{i=1}^k$ satisfies the so-called *open set condition* (i.e. there exists an open set U such that $\psi_i(U) \cap \psi_j(U) = \emptyset$ for every $i \neq j$) then

$$\dim_H \mu = \frac{h_\mu}{\chi_\mu},$$

where $\dim_H \mu$ denotes the Hausdorff dimension of the measure μ, see [3, Corollary 3.5]. Jaroszewska and Rams proved that without any separation condition the entropy divided by the Lyapunov exponent is always an upper bound for the Hausdorff dimension, see [7, Theorem 1]. Furthermore, Lau, Ngai and Wang constructed absolutely continuous place-dependent invariant measures defined by non-linear IFSs with overlaps, see [9, Section 7]. However, sufficient condition for absolute continuity is not known in general.

In the case of ordinary (not place-dependent) invariant measures (see (1.1)) to prove absolute continuity in general, essentially the only approach is the so-called *transversality method* which was first introduced by Pollicott and Simon [15]. Simon, Solomyak and Urbański considered parameterized families of iterated function systems and proved that the Hausdorff dimension of the invariant measure is the minimum of the entropy divided by the Lyapunov exponent and 1, further the measure is absolute continuous if the entropy/Lyapunov exponent is strictly greater than 1 for Lebesgue almost every parameters whenever the IFS satisfies the *transversality condition*, see [18]. Our main theorem establishes this result for place-dependent invariant measures.

Let $U \subset \mathbb{R}^d$ be an open, bounded set. Let us consider a family of IFSs $\Psi_{\boldsymbol{\lambda}} = \{\psi_i^{\boldsymbol{\lambda}}\}_{i=1}^k$, $\boldsymbol{\lambda} \in \overline{U}$. Denote $\mathcal{S} = \{1, \dots, k\}$ the set of symbols and $\Sigma = \mathcal{S}^{\mathbb{N}}$ the *symbolic space*. Let us define the *natural projection* as

$$\pi_{\boldsymbol{\lambda}}(\mathbf{i}) := \lim_{n\to\infty} \psi_{i_0}^{\boldsymbol{\lambda}} \circ \dots \circ \psi_{i_n}^{\boldsymbol{\lambda}}(0), \text{ for } \mathbf{i} = (i_0 i_1 \cdots) \in \Sigma. \tag{1.5}$$

Let us suppose that $\Psi_{\boldsymbol{\lambda}}$ and the weights $\{p_i(x)\}_{i\in\mathcal{S}}$ satisfy the following conditions.

Principal Assumptions:

(A1) Continuity: the maps $\boldsymbol{\lambda} \mapsto \psi_i^{\boldsymbol{\lambda}}$ are continuous from $\overline{U}$ to $C^{1+\theta}(X)$ for every $i \in \mathcal{S}$.

(A2) Hyperbolicity: there exist $0 < \gamma < \kappa < 1$ such that $\gamma < |\left(\psi_i^{\boldsymbol{\lambda}}\right)'(x)| < \kappa$ for every $\boldsymbol{\lambda} \in \overline{U}$, $x \in X$ and $i \in \mathcal{S}$.

(A3) Transversality: there exists a constant C_1 such that for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_0 \neq j_0$

$$\mathcal{L}_d \{\boldsymbol{\lambda} \in U : |\pi_{\boldsymbol{\lambda}}(\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{j})| < r\} < C_1 r \text{ for all } r > 0.$$

(A4) PROBABILITIES: the maps $p_i : X \mapsto (0,1)$ are Hölder continuous and bounded away from zero for every $i \in \mathcal{S}$. Moreover, $\sum_{i=1}^{k} p_i(x) \equiv 1$.

Theorem 1.1. *Suppose that the family $\{\Psi_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in\overline{U}}$ of iterated function systems satisfies the assumptions (A1),(A2) and (A3). Moreover, let $\{p_i(x)\}_{i\in\mathcal{S}}$ be place-dependent probability weights satisfying (A4) and let $\mu_{\boldsymbol{\lambda}}$ be the place-dependent invariant measure according to the weights $\{p_i(x)\}_{i\in\mathcal{S}}$. Then*

(1) *for Lebesgue almost every $\boldsymbol{\lambda} \in U$*

$$\dim_H \mu_{\boldsymbol{\lambda}} = \min\left\{\frac{h_{\mu_{\boldsymbol{\lambda}}}}{\chi_{\mu_{\boldsymbol{\lambda}}}}, 1\right\},$$

(2) $\mu_{\boldsymbol{\lambda}} \ll \mathcal{L}_1$ *for Lebesgue almost every* $\boldsymbol{\lambda} \in \left\{\boldsymbol{\lambda} \in U : \frac{h_{\mu_{\boldsymbol{\lambda}}}}{\chi_{\mu_{\boldsymbol{\lambda}}}} > 1\right\}$.

As an easy consequence we have the following corollary.

Corollary 1.2. *The measure $\mu_{\boldsymbol{\lambda}}$ is equivalent with $\mathcal{L}_1|_{\Lambda_{\boldsymbol{\lambda}}}$ for Lebesgue almost every $\boldsymbol{\lambda} \in \left\{\boldsymbol{\lambda} \in U : \frac{h_{\mu_{\boldsymbol{\lambda}}}}{\chi_{\mu_{\boldsymbol{\lambda}}}} > 1\right\}$.*

Proof. The statement follows immediately from Theorem 1.1(2) and [4, Theorem 1.1]. □

Remark 1.3. *The statement of Theorem 1.1 is also valid when the probabilities depends on the parameters continuously. Precisely, when the function $\boldsymbol{\lambda} \mapsto p_i^{\boldsymbol{\lambda}}$ is continuous from $\overline{U}$ to $C^{\theta}(X)$ for every $i \in \mathcal{S}$.*

Remark 1.4. *In the assumption (A4) the Hölder continuity implies the uniqueness of the measure. Jaroszewska showed that if the probability weights are only continuous then the place-dependent invariant measure is not necessarily unique, see* [6, Theorem 4].

The organization of the paper is as follows: Section 2 is devoted for preliminaries and we prove our main theorem in Section 3. The proof follows the idea and the method of [18]. Simon, Solomyak and Urbański [18] investigated the dimension theory of the push-down measures of fixed left-shift invariant measures on the symbolic space. Since our measure is place-dependent, the induced measure on the symbolic space will depend on the natural projection and therefore on the parameters. So it is not possible to apply the known methods directly. To avoid the difficulty that the induced measure on the symbolic space depends also on the parameters we apply the method introduced by Persson [13]. In Section 4 we apply our results for place-dependent Bernoulli convolutions.

2. Preliminaries

First of all, we introduce some standard notations. Denote σ the left-shift operator on Σ, that is

$$\sigma(i_0 i_1 \dots) = (i_1 i_2 \dots).$$

It is easy to see from the definition of the natural projection (1.5) that

$$\pi_{\boldsymbol{\lambda}}(\mathbf{i}) = \psi_{i_0}^{\boldsymbol{\lambda}}(\pi_{\boldsymbol{\lambda}}(\sigma\mathbf{i})). \tag{2.1}$$

Let us define the set $\mathcal{S}^* = \cup_{n=0}^{\infty}\mathcal{S}^n$ the set of finite length words of symbols and for a $\mathbf{i} = (i_0 i_1 \dots) \in \Sigma$ let $\mathbf{i}|_n = (i_0 i_1 \dots i_{n-1})$. Moreover, for a finite length word $(i_0 \dots i_n)$ let $[i_0 \dots i_n]$ be the corresponding cylinder set,

$$[i_0 \dots i_n] := \{\mathbf{j} = (j_0 j_1 \dots) \in \Sigma : j_0 = i_0, \dots, j_n = i_n\}.$$

Furthermore, let $\mathbf{i} \wedge \mathbf{j} = \min\{n : i_n \neq j_n\}$.

Let us consider a set of place-dependent probability weights $\{p_i(x)\}_{i\in\mathcal{S}}$ and a family of parameterized IFS $\Psi_{\boldsymbol{\lambda}} = \{\psi_i^{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda}\in\mathcal{S}}$. Let us suppose that satisfy (A1), (A2) and (A4). Then without loss of generality, we can assume that there exist constants $C, q > 0$ and $0 < \theta < 1$ such that

$$|p_i(x) - p_i(y)| \leq C|x-y|^{\theta} \text{ and } p_i(x) > q > 0 \text{ for every } i \in \mathcal{S} \text{ and } x, y \in X, \quad (2.2)$$

and

$$\left|(\psi_i^{\boldsymbol{\lambda}})'(x) - (\psi_i^{\boldsymbol{\lambda}})'(y)\right| \leq C|x-y|^{\theta} \text{ for every } x, y \in X,\ i \in \mathcal{S} \text{ and } \boldsymbol{\lambda} \in \overline{U}. \quad (2.3)$$

Define the corresponding Ruelle operator $T_{\boldsymbol{\lambda}} : C(X) \mapsto C(X)$, where $C(X)$ is the set of continuous functions and

$$(T_{\boldsymbol{\lambda}} f)(x) = \sum_{i=1}^{k} p_i(x) f(\psi_i^{\boldsymbol{\lambda}}(x)), \quad (2.4)$$

and let $T_{\boldsymbol{\lambda}}^* : M(X) \mapsto M(X)$ be the adjoint operator, where $M(X)$ denotes the set of Borel probability measures on X.

Proposition 2.1 (Fan, Lau). *Suppose that $\{p_i(x)\}_{i\in\mathcal{S}}$ satisfies (2.2) and let $\{\psi_i^{\boldsymbol{\lambda}}\}_{i=1}^{k}$ be an IFS on X satisfying (A2). Then for every $\boldsymbol{\lambda} \in \overline{U}$ there exists a unique probability measure $\mu_{\boldsymbol{\lambda}}$ such that*

$$T_{\boldsymbol{\lambda}}^* \mu_{\boldsymbol{\lambda}} = \mu_{\boldsymbol{\lambda}}.$$

Moreover, for every $f \in C(X)$, $T_{\boldsymbol{\lambda}}^n f$ converges uniformly to $\int f(x) d\mu_{\boldsymbol{\lambda}}(x)$.

Proof. The proposition follows from [3, Theorem 1.1]. □

The uniqueness of the measure $\mu_{\boldsymbol{\lambda}}$ implies that it is necessarily in *pure type*, that is, the measure $\mu_{\boldsymbol{\lambda}}$ is either singular or absolutely continuous w.r.t the Lebesgue measure.

Now we give an important characterization of the measure $\mu_{\boldsymbol{\lambda}}$. We will show that $\mu_{\boldsymbol{\lambda}}$ is a push-down measure of a Gibbs measure on the symbolic space. First, we need the following lemma.

Lemma 2.2. *There exist a unique, ergodic, left-shift invariant probability measure $\nu_{\boldsymbol{\lambda}}$ on Σ and a constant $c > 1$ independent of $\boldsymbol{\lambda}$ such that for every $\mathbf{i} \in \Sigma$ and $n \geq 1$*

$$c^{-1} \leq \frac{\nu_{\boldsymbol{\lambda}}([\mathbf{i}|_n])}{\prod_{m=0}^{n-1} p_{i_m}(\pi_{\boldsymbol{\lambda}}(\sigma^{m+1}\mathbf{i}))} \leq c. \quad (2.5)$$

Moreover, the entropy $h_{\nu_{\boldsymbol{\lambda}}}$ of $\nu_{\boldsymbol{\lambda}}$ satisfies

$$h_{\nu_{\boldsymbol{\lambda}}} = -\int_{\Sigma} \sum_{i=1}^{k} p_i(\pi_{\boldsymbol{\lambda}}(\mathbf{i})) \log p_i(\pi_{\boldsymbol{\lambda}}(\mathbf{i})) d\nu_{\boldsymbol{\lambda}}(\mathbf{i}). \quad (2.6)$$

Proof. Let $\varphi_{\boldsymbol{\lambda}}(\mathbf{i}) := \log p_{i_0}(\pi_{\boldsymbol{\lambda}}(\sigma\mathbf{i}))$. It is easy to see that

$$|\varphi_{\boldsymbol{\lambda}}(\mathbf{i}) - \varphi_{\boldsymbol{\lambda}}(\mathbf{j})| \leq b\alpha^{\mathbf{i}\wedge\mathbf{j}} \text{ for every } \mathbf{i}, \mathbf{j} \in \Sigma, \tag{2.7}$$

with the choose $b = \max\left\{\frac{2}{q}, \frac{C}{q\kappa^\theta}\right\}$ and $\alpha = \kappa^\theta$, where C, q, θ are from (2.2) and κ is from (A2). Then it follows from [1, Theorem 1.4] that there exists a unique σ invariant prob. measure $\nu_{\boldsymbol{\lambda}}$ (which is called the the Gibbs measure of the potential $\varphi_{\boldsymbol{\lambda}}$) for one can find constant $c_1(\boldsymbol{\lambda}), c_2(\boldsymbol{\lambda}) > 1$ and P such that

$$c_1(\boldsymbol{\lambda})^{-1} \leq \frac{\nu_{\boldsymbol{\lambda}}([\mathbf{i}|_n])}{e^{-nP+\sum_{i=0}^{n-1}\varphi_{\boldsymbol{\lambda}}(\sigma^i\mathbf{i})}} \leq c_2(\boldsymbol{\lambda}).$$

The ergodicity of $\nu_{\boldsymbol{\lambda}}$ follows from [1, Proposition 1.14]. To prove (2.5) and (2.6), first let us define the operator $\mathcal{T}_{\boldsymbol{\lambda}} : C(\Sigma) \mapsto C(\Sigma)$

$$(\mathcal{T}_{\boldsymbol{\lambda}} f)(\mathbf{i}) := \sum_{i=1}^{k} e^{\varphi_{\boldsymbol{\lambda}}(i\mathbf{i})} f(i\mathbf{i}) = \sum_{i=1}^{k} p_i(\pi_{\boldsymbol{\lambda}}(\mathbf{i})) f(i\mathbf{i}). \tag{2.8}$$

Then it is easy to see that the constant function $h(\mathbf{i}) \equiv 1$ is an eigenfunction corresponding to the maximal eigenvalue 1. This implies that $P = 0$, see [1, p. 26]. Then the equation (2.6) is an easy consequence of [1, Theorem 1.22]. Finally, to get the inequality (2.5) one can check that

$$c_1(\boldsymbol{\lambda}), c_2(\boldsymbol{\lambda}) \leq c := e^{3b/(1-\alpha)} = e^{3\max\left\{\frac{2}{q}, \frac{C}{q\kappa^\theta}\right\}/(1-\kappa^\theta)},$$

see the proof of [1, Theorem 1.16]. □

Lemma 2.3. *The measure $\mu_{\boldsymbol{\lambda}}$ is the push-down measure of $\nu_{\boldsymbol{\lambda}}$. That is,*

$$\mu_{\boldsymbol{\lambda}} = (\pi_{\boldsymbol{\lambda}})_* \nu_{\boldsymbol{\lambda}} = \nu_{\boldsymbol{\lambda}} \circ \pi_{\boldsymbol{\lambda}}^{-1}.$$

Proof. Let $f : X \mapsto \mathbb{R}$ continuous. Then

$$\int_X f(x) d(\pi_{\boldsymbol{\lambda}})_* \nu_{\boldsymbol{\lambda}}(x) = \int_\Sigma f(\pi_{\boldsymbol{\lambda}}(\mathbf{i})) d\nu_{\boldsymbol{\lambda}}(\mathbf{i}).$$

Applying the operator $\mathcal{T}_{\boldsymbol{\lambda}}$ defined in the proof of Lemma 2.2, we have

$$\int_\Sigma f(\pi_{\boldsymbol{\lambda}}(\mathbf{i})) d\nu_{\boldsymbol{\lambda}}(\mathbf{i}) = \int_\Sigma \sum_{i=1}^{k} p_i(\pi_{\boldsymbol{\lambda}}(\mathbf{i})) f(\pi_{\boldsymbol{\lambda}}(i\mathbf{i})) d\nu_{\boldsymbol{\lambda}}(\mathbf{i}) =$$

$$\int_\Sigma \sum_{i=1}^{k} p_i(\pi_{\boldsymbol{\lambda}}(\mathbf{i})) f(\psi_i^{\boldsymbol{\lambda}}(\pi_{\boldsymbol{\lambda}}(\mathbf{i}))) d\nu_{\boldsymbol{\lambda}}(\mathbf{i}) = \int_X \sum_{i=1}^{k} p_i(x) f(\psi_i^{\boldsymbol{\lambda}}(x)) d(\pi_{\boldsymbol{\lambda}})_* \nu_{\boldsymbol{\lambda}}(x) =$$

$$\int_X (T_{\boldsymbol{\lambda}} f)(x) d(\pi_{\boldsymbol{\lambda}})_* \nu_{\boldsymbol{\lambda}}(x).$$

In the second equality we used the identity (2.1). This implies that $(\pi_{\boldsymbol{\lambda}})_* \nu_{\boldsymbol{\lambda}} = T_{\boldsymbol{\lambda}}^* (\pi_{\boldsymbol{\lambda}})_* \nu_{\boldsymbol{\lambda}}$. The statement of the lemma follows from Proposition 2.1. □

For the simplicity let us introduce the following notation, for an $\mathbf{i} \in \Sigma$ let

$$\psi_{\mathbf{i}|_n}^{\boldsymbol{\lambda}}(x) := \psi_{i_0}^{\boldsymbol{\lambda}} \circ \cdots \circ \psi_{i_{n-1}}^{\boldsymbol{\lambda}}(x).$$

Lemma 2.4. *There exists a constant $C_2 > 1$ such that for every $n \geq 1$, $\mathbf{i} \in \Sigma$, $\boldsymbol{\lambda} \in \overline{U}$ and $x, y \in X$ we have*

$$C_2^{-1} \leq \frac{\left(\psi^{\boldsymbol{\lambda}}_{\mathbf{i}|_n}\right)'(x)}{\left(\psi^{\boldsymbol{\lambda}}_{\mathbf{i}|_n}\right)'(y)} \leq C_2, \tag{2.9}$$

Proof. For the proof we refer to [17, Lemma 5.8]. □

Finally, the we prove the continuity of the entropy and Lyapunov exponent.

Proposition 2.5. *The maps $\boldsymbol{\lambda} \mapsto h_{\mu_{\boldsymbol{\lambda}}}$ and $\boldsymbol{\lambda} \mapsto \chi_{\mu_{\boldsymbol{\lambda}}}$ are continuous on $\overline{U}$.*

Before we prove the proposition we state an auxiliary lemma about the uniform convergence of the Perron-Frobenius operator on the symbolic space. Let

$$\beta := \max\left\{\frac{2}{-\kappa^{\theta} q \log q}, \frac{C}{-q\kappa^{2\theta}\log q}, \frac{2}{-\kappa^{\theta}\gamma\log\gamma}, \frac{C}{\kappa^{2\theta}\gamma\log\gamma}\right\} \text{ and } \alpha := \kappa^{\theta},$$

where the constant $C, q, \kappa, \gamma, \theta$ are the constants in (A2),(2.2) and (2.3). Furthermore, let $B_m := e^{2\beta\sum_{n=m+1}^{\infty}\alpha^n}$ and

$$\Gamma := \{f \in C(\Sigma) : f \geq 0, f(\mathbf{i}) \leq B_m f(\mathbf{j}) \text{ when } \mathbf{i} \wedge \mathbf{j} = m\}.$$

For brevity, let us introduce the notation $\nu_{\boldsymbol{\lambda}}(f) = \int f(\mathbf{i})d\nu_{\boldsymbol{\lambda}}(\mathbf{i})$.

Lemma 2.6. *There exist a $C' > 0$ and $0 < \eta < 1$ universal constants such that for every $f \in \Gamma$*

$$|\mathcal{T}^n_{\boldsymbol{\lambda}} f(\mathbf{i}) - \nu_{\boldsymbol{\lambda}}(f)| < C'\nu_{\boldsymbol{\lambda}}(f)(1-\eta)^n \textit{ for every } n \geq 1,\ \mathbf{i} \in \Sigma \textit{ and } \boldsymbol{\lambda} \in \overline{U},$$

where $\mathcal{T}_{\boldsymbol{\lambda}}$ is the operator defined in (2.8).

To prove the lemma we will apply the so-called *cone method* which was first introduced in [1].

Proof. If $f \in \Gamma$ and $\mathbf{i} \wedge \mathbf{j} = m$ then

$$e^{\varphi_{\boldsymbol{\lambda}}(i\mathbf{i})} f(i\mathbf{i}) \leq e^{\varphi_{\boldsymbol{\lambda}}(i\mathbf{j})} e^{b\alpha^{m+1}} B_{m+1} f(i\mathbf{j}) \leq B_m e^{\varphi_{\boldsymbol{\lambda}}(i\mathbf{j})} f(i\mathbf{j}),$$

where b is the constant defined in (2.7). Therefore $\mathcal{T}_{\boldsymbol{\lambda}}\Gamma \subseteq \Gamma$.

Let

$$\eta := \frac{(2\beta - b)(1-\alpha)}{2\beta B_0^2} \text{ and } C' := B_0 + 1.$$

Now, fix an $f \in \Gamma$ and suppose that $\nu_{\boldsymbol{\lambda}}(f) = 1$. Define f_1 such that it satisfies $\nu_{\boldsymbol{\lambda}}(f_1) = 1$ and

$$\mathcal{T}_{\boldsymbol{\lambda}} f = \eta + (1-\eta) f_1. \tag{2.10}$$

We claim that $f_1 \in \Gamma$. It is enough to see that $(\mathcal{T}_{\boldsymbol{\lambda}} f - \eta) / (1-\eta) \in \Gamma$. Since $\mathcal{T}_{\boldsymbol{\lambda}} f(\mathbf{i}) \geq B_0^{-1} \geq \eta$ therefore $f_1 \geq 0$. Moreover, to see that $f_1(\mathbf{i}) \leq B_m f_1(\mathbf{j})$ whenever $\mathbf{i} \wedge \mathbf{j} = m$ it is enough to prove that

$$\eta \leq \frac{B_m \mathcal{T}_{\boldsymbol{\lambda}} f(\mathbf{j}) - \mathcal{T}_{\boldsymbol{\lambda}} f(\mathbf{i})}{B_m - 1}.$$

It follows easily from the definitions that

$$\frac{B_m \mathcal{T}_{\boldsymbol{\lambda}} f(\mathbf{j}) - \mathcal{T}_{\boldsymbol{\lambda}} f(\mathbf{i})}{B_m - 1} \geq \frac{B_m - B_{m+1}e^{b\alpha^{m+1}}}{B_m - 1}\mathcal{T}_{\boldsymbol{\lambda}} f(\mathbf{j}) \geq \frac{B_m - B_{m+1}e^{b\alpha^{m+1}}}{B_m - 1} B_0^{-1}.$$

By Lagrange theorem,

$$\text{for every } x, y \in [0, \log B_0], x > y \text{ we have } x - y < e^x - e^y < B_0(x-y).$$

Applying Lagrange theorem we get

$$\frac{B_m - B_{m+1}e^{b\alpha^{m+1}}}{B_m - 1}B_0^{-1} > \frac{\log B_m - \log B_{m+1}e^{b\alpha^{m+1}}}{\log B_m}B_0^{-2} = \frac{(2\beta-b)(1-\alpha)}{2\beta B_0^2},$$

which is exactly the definition of η. Iterating (2.10) we obtain that for every $n \geq 1$ there exists an $f_n \in \Gamma$ such that $\nu_{\boldsymbol{\lambda}}(f_n) = 1$ and

$$\mathcal{T}_{\boldsymbol{\lambda}}^n f = 1 - (1-\eta)^n + (1-\eta)^n f_n.$$

Hence we get that for every $f \in \Gamma$ with $\nu_{\boldsymbol{\lambda}}(f) = 1$

$$|\mathcal{T}_{\boldsymbol{\lambda}}^n f(\mathbf{i}) - 1| \leq (1-\eta)^n(1+B_0).$$

This implies the statement of the lemma. □

Proof of Proposition 2.5. We will only prove that the entropy is continuous, the proof for the Lyapunov exponent is similar. First, we prove that the function $-\varphi_{\boldsymbol{\lambda}} = -\log p_{i_0}(\pi_{\boldsymbol{\lambda}}(\sigma\mathbf{i}))$ is in Γ. From equation (2.7) it follows that for $\mathbf{i} \wedge \mathbf{j} = m$

$$\frac{-\varphi_{\boldsymbol{\lambda}}(\mathbf{i})}{-\varphi_{\boldsymbol{\lambda}}(\mathbf{j})} \leq \frac{-\varphi_{\boldsymbol{\lambda}}(\mathbf{j}) + b\alpha^m}{-\varphi_{\boldsymbol{\lambda}}(\mathbf{j})} \leq 1 + \frac{b}{-\log q\alpha}\alpha^{m+1} \leq e^{\frac{b}{-\log q\alpha}\alpha^{m+1}} \leq B_m.$$

Therefore, by applying Lemma 2.6 we get for every $\boldsymbol{\lambda} \in \overline{U}$

$$|\mathcal{T}_{\boldsymbol{\lambda}}^n(-\varphi_{\boldsymbol{\lambda}}) - \nu_{\boldsymbol{\lambda}}(-\varphi_{\boldsymbol{\lambda}})| \leq C' \log k(1-\eta)^n \text{ for every } n \geq 1.$$

Fix $\varepsilon > 0$. Since the function $\boldsymbol{\lambda} \mapsto \mathcal{T}_{\boldsymbol{\lambda}}^n(-\varphi_{\boldsymbol{\lambda}})$ is continuous for every $n \in \mathbb{N}$ one can choose $N \geq 1$ and $\delta > 0$ such that $C' \log k(1-\eta)^N < \varepsilon/3$ and $|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2| < \delta \Rightarrow \left|\mathcal{T}_{\boldsymbol{\lambda}_1}^N(-\varphi_{\boldsymbol{\lambda}_1}) - \mathcal{T}_{\boldsymbol{\lambda}_2}^N(-\varphi_{\boldsymbol{\lambda}_2})\right| < \varepsilon/3$. Therefore,

$$|\nu_{\boldsymbol{\lambda}_1}(-\varphi_{\boldsymbol{\lambda}_1}) - \nu_{\boldsymbol{\lambda}_2}(-\varphi_{\boldsymbol{\lambda}_2})| <$$
$$\left|\nu_{\boldsymbol{\lambda}_1}(-\varphi_{\boldsymbol{\lambda}_1}) - \mathcal{T}_{\boldsymbol{\lambda}_1}^N(-\varphi_{\boldsymbol{\lambda}_1})\right| + \left|\mathcal{T}_{\boldsymbol{\lambda}_1}^N(-\varphi_{\boldsymbol{\lambda}_1}) - \mathcal{T}_{\boldsymbol{\lambda}_2}^N(-\varphi_{\boldsymbol{\lambda}_2})\right| + \left|\mathcal{T}_{\boldsymbol{\lambda}_2}^N(-\varphi_{\boldsymbol{\lambda}_2}) - \nu_{\boldsymbol{\lambda}_2}(-\varphi_{\boldsymbol{\lambda}_2})\right| < \varepsilon.$$

The statement follows from Lemma 2.3 and the fact that $h_{\nu_{\boldsymbol{\lambda}}} = \nu_{\boldsymbol{\lambda}}(-\varphi_{\boldsymbol{\lambda}})$. □

3. Proof of Theorem 1.1

During the proof of Theorem 1.1 we follow the method of Simon, Solomyak and Urbański [18] with a relevant modification based on the idea of Persson [13]. The method introduced by Pollicott and Simon [15] and later extended by Peres and Solomyak [12] and Simon, Solomyak and Urbański [18] is applied for the push-down measures of left-shift invariant ergodic measures on the symbolic space. More precisely, for a fixed ergodic σ-invariant measure on the symbolic space, the dimension of its push-down measure is the minimum of the entropy/Lyapunov exponent and 1, whenever the transversality condition holds. That is, the measure on the symbolic space is independent of the parameters. In our case, the Gibbs measure on the symbolic space depends on the parameters of the IFS as well. To avoid the difficulties of this fact we need the following lemma according to [13, Lemma 3].

Lemma 3.1. *Let $\mathbf{i}, \mathbf{j} \in \Sigma$ be such that $i_0 \neq j_0$. Then for every $r > 0$ there exists a function $G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda})$ such that*

$$\mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\mathbf{i}) - \pi_{\boldsymbol{\lambda}}(\mathbf{j})| < r\}} \leq G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda}) \tag{3.1}$$

and

$$\int G_r(\mathbf{i}, \mathbf{j}, \boldsymbol{\lambda}) d\mathcal{L}_d(\boldsymbol{\lambda}) \leq 2C_1 r, \tag{3.2}$$

where $\mathbb{1}$ *denotes the indicator function and* C_1 *is the constant from (A3). Moreover, the function* $G_r(\mathbf{i},\mathbf{j},\boldsymbol{\lambda})$ *is constant on cylinders* $[\mathbf{i}|_{N+1}]\times[\mathbf{j}|_{N+1}]$*, where* $N=\left\lceil\frac{\log(r/(2\mathrm{diam}X)}{\log\kappa}\right\rceil$.

Proof. Let us suppose that $\mathbf{i}\in[i_0\dots i_N]$ and $\mathbf{j}\in[j_0\dots j_N]$. Let $\mathbf{i}^*:=(i_0\dots i_N\mathbf{1})$ and $\mathbf{j}^*:=(j_0\dots j_N\mathbf{1})$, where $\mathbf{1}$ denotes the word $(11\dots)$. Define $G_r(\mathbf{i},\mathbf{j},\boldsymbol{\lambda})$ as

$$G_r(\mathbf{i},\mathbf{j},\boldsymbol{\lambda}):=\mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\mathbf{i}^*)-\pi_{\boldsymbol{\lambda}}(\mathbf{j}^*)|<2r\}}.$$

Then by (2.1) and Lagrange mean value theorem

$$||\pi_{\boldsymbol{\lambda}}(\mathbf{i})-\pi_{\boldsymbol{\lambda}}(\mathbf{j})|-|\pi_{\boldsymbol{\lambda}}(\mathbf{i}^*)-\pi_{\boldsymbol{\lambda}}(\mathbf{j}^*)||\leq|(\pi_{\boldsymbol{\lambda}}(\mathbf{i})-\pi_{\boldsymbol{\lambda}}(\mathbf{i}^*))-(\pi_{\boldsymbol{\lambda}}(\mathbf{j})-\pi_{\boldsymbol{\lambda}}(\mathbf{j}^*))|\leq$$
$$\left|\left(\psi^{\boldsymbol{\lambda}}_{\mathbf{i}|_{N+1}}\right)'(\xi_1)\right|\left|\pi_{\boldsymbol{\lambda}}(\sigma^{N+1}\mathbf{i})-\pi_{\boldsymbol{\lambda}}(\mathbf{1})\right|+\left|\left(\psi^{\boldsymbol{\lambda}}_{\mathbf{j}|_{N+1}}\right)'(\xi_2)\right|\left|\pi_{\boldsymbol{\lambda}}(\sigma^{N+1}\mathbf{j})-\pi_{\boldsymbol{\lambda}}(\mathbf{1})\right|\leq$$
$$2\mathrm{diam}X\kappa^{N+1}\leq r.$$

This implies the inequality (3.1). The other inequality follows from (A3). □

Now we recall some classical result. By Shannon-McMillan-Breiman Theorem,

$$-\frac{1}{n}\log\nu_{\boldsymbol{\lambda}}([\mathbf{i}|_n])\to h_{\nu_{\boldsymbol{\lambda}}}\text{ for }\nu_{\boldsymbol{\lambda}}\text{ a. e. }\mathbf{i}\in\Sigma,$$

and by Birkhoff's Ergodic Theorem

$$-\frac{1}{n}\log\left|(\psi^{\boldsymbol{\lambda}}_{\mathbf{i}|_n})'(\pi_{\boldsymbol{\lambda}}(\sigma^n\mathbf{i}))\right|\to\chi_{\nu_{\boldsymbol{\lambda}}}\text{ for }\nu_{\boldsymbol{\lambda}}\text{ a. e. }\mathbf{i}\in\Sigma.$$

Lemma 3.2. *For every* $\varepsilon>0$ *and* $\varepsilon_1>0$*,* $\varepsilon_2>0$ *there exist a set* $J\subseteq U$ *and a constant* $\widetilde{C}>1$ *such that* $\mathcal{L}_d(U\backslash J)<\varepsilon_2$ *and for every* $\boldsymbol{\lambda}\in U$ *there exists a set* $\Omega_{\boldsymbol{\lambda}}\subseteq\Sigma$ *such that* $\nu_{\boldsymbol{\lambda}}(\Omega_{\boldsymbol{\lambda}})>1-\varepsilon_1$*,*

$$\widetilde{C}^{-1}e^{-n(h_{\mu_{\boldsymbol{\lambda}}}+\varepsilon)}\leq\nu_{\boldsymbol{\lambda}}([\mathbf{i}|_n]\leq\widetilde{C}e^{-n(h_{\mu_{\boldsymbol{\lambda}}}-\varepsilon)}\textit{ and}\tag{3.3}$$

$$\widetilde{C}^{-1}e^{-n(\chi_{\mu_{\boldsymbol{\lambda}}}+\varepsilon)}\leq\left|\left(\psi^{\boldsymbol{\lambda}}_{\mathbf{i}|_n}\right)'(\pi_{\boldsymbol{\lambda}}(\sigma^{n+1}\mathbf{i}))\right|\leq\widetilde{C}e^{-n(\chi_{\mu_{\boldsymbol{\lambda}}}-\varepsilon)}\textit{ for every }n\geq1,$$
$$\boldsymbol{\lambda}\in J\textit{ and }\mathbf{i}\in\Omega_{\boldsymbol{\lambda}}.\tag{3.4}$$

Proof. By Egorov's Theorem, for every $\varepsilon>0$, $\varepsilon_1>0$ and $\boldsymbol{\lambda}\in U$ there exists a set $\Omega_{\boldsymbol{\lambda}}\subseteq\Sigma$ such that $\nu_{\boldsymbol{\lambda}}(\Omega_{\boldsymbol{\lambda}})>1-\varepsilon_1$,

$$\widetilde{C}_{\boldsymbol{\lambda}}^{-1}e^{-n(h_{\mu_{\boldsymbol{\lambda}}}+\varepsilon)}\leq\nu_{\boldsymbol{\lambda}}([\mathbf{i}|_n]\leq\widetilde{C}_{\boldsymbol{\lambda}}e^{-n(h_{\mu_{\boldsymbol{\lambda}}}-\varepsilon)}\text{ and}$$

$$\widetilde{C}_{\boldsymbol{\lambda}}^{-1}e^{-n(\chi_{\mu_{\boldsymbol{\lambda}}}+\varepsilon)}\leq\left|\left(\psi^{\boldsymbol{\lambda}}_{\mathbf{i}|_n}\right)'(\pi_{\boldsymbol{\lambda}}(\sigma^{n+1}\mathbf{i}))\right|\leq\widetilde{C}_{\boldsymbol{\lambda}}e^{-n(\chi_{\mu_{\boldsymbol{\lambda}}}-\varepsilon)}\text{ for every }n\geq1\text{ and}$$
$$\mathbf{i}\in\Omega_{\boldsymbol{\lambda}}.$$

An application of Lusin's Theorem shows that for every $\varepsilon_2>0$ there exist a set $J\subseteq U$ and a constant $\widetilde{C}$ such that $\mathcal{L}_d(U\backslash J)<\varepsilon_2$ and $\widetilde{C}_{\boldsymbol{\lambda}}\leq\widetilde{C}$ for every $\boldsymbol{\lambda}\in J$. □

Denote $\widetilde{\nu}_{\boldsymbol{\lambda}}$ the restriction of the measure $\nu_{\boldsymbol{\lambda}}$ to $\Omega_{\boldsymbol{\lambda}}$ and $\widetilde{\mu}_{\boldsymbol{\lambda}}$ its push-down measure. That is,

$$\widetilde{\nu}_{\boldsymbol{\lambda}}:=\nu_{\boldsymbol{\lambda}}|_{\Omega_{\boldsymbol{\lambda}}}\text{ and }\widetilde{\mu}_{\boldsymbol{\lambda}}:=\widetilde{\nu}_{\boldsymbol{\lambda}}\circ\pi_{\boldsymbol{\lambda}}^{-1},\tag{3.5}$$

where $\Omega_{\boldsymbol\lambda}$ is defined in Lemma 3.2. For a finite length word $(l_0 \dots l_{n-1}) \in \mathcal{S}^*$ let us define

$$A^{\boldsymbol\lambda}_{(l_0 \dots l_{n-1})} = \left\{(\mathbf{i},\mathbf{j}) \in \Omega^2_{\boldsymbol\lambda} : \mathbf{i}\wedge\mathbf{j} = n \text{ and } i_m = j_m = l_m \text{ for } 0 \le m \le n-1\right\}.$$

We note that for an empty word $A^{\boldsymbol\lambda}_{\emptyset} = \left\{(\mathbf{i},\mathbf{j}) \in \Omega^2_{\boldsymbol\lambda} : i_0 \neq j_0\right\}$.

Frostman's Theorem, see [2, Theorem 4.13], implies

$$\dim_H \mu_{\boldsymbol\lambda} \ge \sup\left\{s > 0 : \iint_{\mathbb{R}^2} |x-y|^{-s}\, d\mu_{\boldsymbol\lambda}(x) d\mu_{\boldsymbol\lambda}(y) < \infty\right\}. \tag{3.6}$$

Lemma 3.3. *For every* $0 < s < 1$ *and* $\mathbf{i},\mathbf{j} \in \Sigma$ *such that* $i_0 \neq j_0$ *we have*

$$|\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})|^{-s} \le \sum_{n=0}^{\infty} \frac{2^{s(n+1)}}{\operatorname{diam} X^s} G_{\frac{\operatorname{diam} X}{2^n}}(\mathbf{i},\mathbf{j},\boldsymbol\lambda).$$

Proof. If $|\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})| = 0$ then the right hand side of the inequality is divergent. Otherwise, if $|\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})| > 0$ then

$$|\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})|^{-s} \le \sum_{n=0}^{\infty} \frac{2^{s(n+1)}}{\operatorname{diam} X^s} \mathbb{1}_{\left\{\frac{2^{sn}}{\operatorname{diam} X^s} < |\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})|^{-s} \le \frac{2^{s(n+1)}}{\operatorname{diam} X^s}\right\}} =$$

$$\sum_{n=0}^{\infty} \frac{2^{s(n+1)}}{\operatorname{diam} X^s} \mathbb{1}_{\left\{\frac{\operatorname{diam} X}{2^{n+1}} \le |\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})| < \frac{\operatorname{diam} X}{2^n}\right\}} \le \sum_{n=0}^{\infty} \frac{2^{s(n+1)}}{\operatorname{diam} X^s} \mathbb{1}_{\left\{|\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})| < \frac{\operatorname{diam} X}{2^n}\right\}}.$$

The statement follows from (3.1). □

Proposition 3.4. *For every* $\boldsymbol\lambda_0 \in U$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\dim_H \mu_{\boldsymbol\lambda} \ge \min\left\{\frac{h_{\mu_{\boldsymbol\lambda_0}} - 2\varepsilon}{\chi_{\mu_{\boldsymbol\lambda_0}} + 2\varepsilon}, 1-\varepsilon\right\} \text{ for Lebesgue-a.e. } \boldsymbol\lambda \in B_\delta(\boldsymbol\lambda_0), \tag{3.7}$$

where $B_\delta(\boldsymbol\lambda_0)$ *denotes the ball with center at* $\boldsymbol\lambda_0$ *and radius* δ.

Proof. Fix an $\varepsilon > 0$ and a $\boldsymbol\lambda_0 \in U$. By Proposition 2.5 let $\delta > 0$ be such that if $|\boldsymbol\lambda_0 - \boldsymbol\lambda| < \delta$

$$\left|h_{\mu_{\boldsymbol\lambda_0}} - h_{\mu_{\boldsymbol\lambda}}\right| < \varepsilon \text{ and } \left|\chi_{\mu_{\boldsymbol\lambda_0}} - \chi_{\mu_{\boldsymbol\lambda}}\right| < \varepsilon. \tag{3.8}$$

Let $s < \min\left\{1-\varepsilon, \frac{h_{\mu_{\boldsymbol\lambda}} - 2\varepsilon}{\chi_{\mu_{\boldsymbol\lambda}} + 2\varepsilon}\right\}$ and let $\varepsilon_2 > 0$ and J as in Lemma 3.2. For the simplicity, $J_\delta(\boldsymbol\lambda_0) := B_\delta(\boldsymbol\lambda_0) \cap J$. Since $\dim_H \mu_{\boldsymbol\lambda} \ge \dim_H \widetilde{\mu}_{\boldsymbol\lambda}$ and $\widetilde{\mu}_{\boldsymbol\lambda}$ is the push-down measure of $\widetilde{\nu}_{\boldsymbol\lambda}$, to prove (3.7) it is enough to show according to (3.6) that

$$\mathcal{I} := \int_{J_\delta(\boldsymbol\lambda_0)} \iint_{\Omega^2_{\boldsymbol\lambda}} |\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})|^{-s}\, d\widetilde{\nu}_{\boldsymbol\lambda}(\mathbf{i}) d\widetilde{\nu}_{\boldsymbol\lambda}(\mathbf{j}) d\mathcal{L}_d(\boldsymbol\lambda) < \infty.$$

For the simplicity denote $\widetilde{\nu}_{\boldsymbol\lambda} \times \widetilde{\nu}_{\boldsymbol\lambda}$ by $\nu^2_{\boldsymbol\lambda}$ then

$$\mathcal{I} = \sum_{n=0}^{\infty} \sum_{\bar\imath \in \mathcal{S}^n} \int_{J_\delta(\boldsymbol\lambda_0)} \iint_{A^{\boldsymbol\lambda}_{\bar\imath}} |\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})|^{-s}\, d\nu^2_{\boldsymbol\lambda}(\mathbf{i},\mathbf{j}) d\mathcal{L}_d(\boldsymbol\lambda)$$

By (2.9) we have

$$\mathcal{I} \le$$

$$\sum_{n=0}^{\infty} \sum_{\bar\imath \in \mathcal{S}^n} \int_{J_\delta(\boldsymbol\lambda_0)} \iint_{A^{\boldsymbol\lambda}_{\bar\imath}} C_2^s \left|\left(\psi^{\boldsymbol\lambda}_{\mathbf{i}|_n}\right)'(\pi_{\boldsymbol\lambda}(\sigma^n \mathbf{i}))\right|^{-s} |\pi_{\boldsymbol\lambda}(\sigma^n \mathbf{i}) - \pi_{\boldsymbol\lambda}(\sigma^n \mathbf{j})|^{-s}\, d\nu^2_{\boldsymbol\lambda}(\mathbf{i},\mathbf{j}) d\mathcal{L}_d(\boldsymbol\lambda).$$

By Lemma 3.2(3.4) and (3.8)

$$\mathcal{I} \leq$$
$$C_2^s\widetilde{C}\sum_{n=0}^{\infty} e^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)s}\sum_{\bar{\imath}\in\mathcal{S}^n}\int_{J_\delta(\boldsymbol{\lambda}_0)}\iint_{A_{\bar{\imath}}^{\boldsymbol{\lambda}}}|\pi_{\boldsymbol{\lambda}}(\sigma^n\mathbf{i})-\pi_{\boldsymbol{\lambda}}(\sigma^n\mathbf{j})|^{-s}\,d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i},\mathbf{j})d\mathcal{L}_d(\boldsymbol{\lambda}).$$

By applying Lemma 3.3 and Fatou's Lemma we get

$$\mathcal{I} \leq$$
$$C_2^s\widetilde{C}\sum_{n=0}^{\infty} e^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)s}\sum_{\bar{\imath}\in\mathcal{S}^n}\sum_{m=0}^{\infty}\frac{2^{s(m+1)}}{\mathrm{diam}X}\int_{J_\delta(\boldsymbol{\lambda}_0)}\iint_{A_{\bar{\imath}}^{\boldsymbol{\lambda}}}G_{\frac{\mathrm{diam}X}{2^m}}(\sigma^n\mathbf{i},\sigma^n\mathbf{j},\boldsymbol{\lambda})d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i},\mathbf{j})d\mathcal{L}_d(\boldsymbol{\lambda}).$$

Lemma 3.1 implies that

$$\iint_{A_{\bar{\imath}}^{\boldsymbol{\lambda}}}G_{\frac{\mathrm{diam}X}{2^m}}(\sigma^n\mathbf{i},\sigma^n\mathbf{j},\boldsymbol{\lambda})d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i},\mathbf{j})=\sum_{\bar{\jmath}_1,\bar{\jmath}_2\in\mathcal{S}^{N(m)}}\sum_{\substack{l,p\in\mathcal{S}\\ l\neq p}}G_{\frac{\mathrm{diam}X}{2^m}}(l\bar{\jmath}_1\mathbf{1},p\bar{\jmath}_2\mathbf{1},\boldsymbol{\lambda})\nu_{\boldsymbol{\lambda}}^2([\bar{\imath}l\bar{\jmath}_1]\times[\bar{\imath}p\bar{\jmath}_2]),$$

where $N(m)=\left\lceil (m+1)\frac{\log 2}{-\log\kappa}\right\rceil$ according to Lemma 3.1. Therefore,

$$\sum_{\bar{\imath}\in\mathcal{S}^n}\sum_{\bar{\jmath}_1,\bar{\jmath}_2\in\mathcal{S}^{N(m)}}\sum_{\substack{l,p\in\mathcal{S}\\ l\neq p}}\int_{J_\delta(\boldsymbol{\lambda}_0)}G_{\frac{\mathrm{diam}X}{2^m}}(l\bar{\jmath}_1\mathbf{1},p\bar{\jmath}_2\mathbf{1},\boldsymbol{\lambda})\nu_{\boldsymbol{\lambda}}^2([\bar{\imath}l\bar{\jmath}_1]\times[\bar{\imath}p\bar{\jmath}_2])d\mathcal{L}_d(\boldsymbol{\lambda})\leq$$
$$\sum_{\bar{\imath}\in\mathcal{S}^n}\sum_{\bar{\jmath}_1,\bar{\jmath}_2\in\mathcal{S}^{N(m)}}\sum_{\substack{l,p\in\mathcal{S}\\ l\neq p}}\max_{\hbar_1,\hbar_2\in\mathcal{S}^{N(m)}}\int_{J_\delta(\boldsymbol{\lambda}_0)}G_{\frac{\mathrm{diam}X}{2^m}}(l\hbar_1\mathbf{1},p\hbar_2\mathbf{1},\boldsymbol{\lambda})\nu_{\boldsymbol{\lambda}}^2([\bar{\imath}l\bar{\jmath}_1]\times[\bar{\imath}p\bar{\jmath}_2])d\mathcal{L}_d(\boldsymbol{\lambda})=$$
$$\max_{\hbar_1,\hbar_2\in\mathcal{S}^{N(m)}}\int_{J_\delta(\boldsymbol{\lambda}_0)}G_{\frac{\mathrm{diam}X}{2^m}}(l\hbar_1\mathbf{1},p\hbar_2\mathbf{1},\boldsymbol{\lambda})\sum_{\bar{\imath}\in\mathcal{S}^n}\nu_{\boldsymbol{\lambda}}^2(A_{\bar{\imath}})d\mathcal{L}_d(\boldsymbol{\lambda})\leq$$
$$\sup_{\boldsymbol{\lambda}\in J_\delta(\boldsymbol{\lambda}_0)}\sum_{\bar{\imath}\in\mathcal{S}^n}\nu_{\boldsymbol{\lambda}}^2(A_{\bar{\imath}})\max_{\hbar_1,\hbar_2\in\mathcal{S}^{N(m)}}\int_{J_\delta(\boldsymbol{\lambda}_0)}G_{\frac{\mathrm{diam}X}{2^m}}(l\hbar_1\mathbf{1},p\hbar_2\mathbf{1},\boldsymbol{\lambda})d\mathcal{L}_d(\boldsymbol{\lambda})\leq$$
$$\frac{2C_1\mathrm{diam}X}{2^m}\sup_{\boldsymbol{\lambda}\in J_\delta(\boldsymbol{\lambda}_0)}\sum_{\bar{\imath}\in\mathcal{S}^n}\nu_{\boldsymbol{\lambda}}^2(A_{\bar{\imath}}),$$

where we used in the last inequality (3.2). Hence,

$$\mathcal{I}\leq C_2^s\widetilde{C}\sum_{n=0}^{\infty}e^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)s}\sum_{m=0}^{\infty}\frac{2^{s(m+1)}}{\mathrm{diam}X^s}\frac{2C_1\mathrm{diam}X}{2^m}\sup_{\boldsymbol{\lambda}\in J_\delta(\boldsymbol{\lambda}_0)}\sum_{\bar{\imath}\in\mathcal{S}^n}\nu_{\boldsymbol{\lambda}}^2(A_{\bar{\imath}}).$$

Applying Lemma 3.2(3.3) and (3.8) we get

$$\nu_{\boldsymbol{\lambda}}^2(A_{\bar{\imath}})\leq\widetilde{C}e^{-n(h_{\mu_{\boldsymbol{\lambda}_0}}-2\varepsilon)}\nu_{\boldsymbol{\lambda}}([\bar{\imath}])$$

and this implies that

$$\mathcal{I}\leq C_2^s\widetilde{C}^2C_1\mathrm{diam}X^{1-s}2^{s+1}\sum_{n=0}^{\infty}e^{n\left((\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)s-(h_{\mu_{\boldsymbol{\lambda}_0}}-2\varepsilon)\right)}\sum_{m=0}^{\infty}2^{(s-1)m}.$$

Since $s<\min\left\{1-\varepsilon,\frac{h_{\mu_{\boldsymbol{\lambda}_0}}-2\varepsilon}{\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon}\right\}$ the right-hand side of the inequality is finite.

Moreover, $\varepsilon_2 > 0$ was arbitrary and therefore $\dim_H \mu_{\boldsymbol\lambda} \geq \min\left\{1-\varepsilon, \frac{h_{\mu_{\boldsymbol\lambda_0}}-2\varepsilon}{\chi_{\mu_{\boldsymbol\lambda_0}}+2\varepsilon}\right\}$ for Lebesgue almost every $\boldsymbol\lambda \in B_\delta(\boldsymbol\lambda_0)$ which was to be proven. □

Proof of Theorem 1.1(1). Since $\dim_H \mu_{\boldsymbol\lambda} \leq \frac{h_{\mu_{\boldsymbol\lambda}}}{\chi_{\mu_{\boldsymbol\lambda}}}$ for all $\boldsymbol\lambda \in \overline{U}$ by [7, Theorem 1], we only need to establish the estimate from below.

Let us argue by contradiction. Suppose that there exist an $\varepsilon > 0$ and a positive measure set $\widehat{U}$ of parameters such that

$$\dim_H \mu_{\boldsymbol\lambda} < \min\left\{1, \frac{h_{\mu_{\boldsymbol\lambda}}}{\chi_{\mu_{\boldsymbol\lambda}}}\right\} - \varepsilon \text{ for almost every } \boldsymbol\lambda \in \widehat{U}.$$

Let $\boldsymbol\lambda_0$ be a density point of $\widehat{U}$. Then there exists a $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$

$$\mathcal{L}_d\left(\boldsymbol\lambda \in B_\delta(\boldsymbol\lambda_0) : \dim_H \mu_{\boldsymbol\lambda} < \min\left\{1, \frac{h_{\mu_{\boldsymbol\lambda}}}{\chi_{\mu_{\boldsymbol\lambda}}}\right\} - \varepsilon\right) > 0.$$

Using the continuity of the entropy and the Lyapunov exponent (Proposition 2.5) we get for sufficiently small $\delta > 0$

$$\mathcal{L}_d\left(\boldsymbol\lambda \in B_\delta(\boldsymbol\lambda_0) : \dim_H \mu_{\boldsymbol\lambda} < \min\left\{1, \frac{h_{\mu_{\boldsymbol\lambda_0}}}{\chi_{\mu_{\boldsymbol\lambda_0}}}\right\} - \frac{\varepsilon}{2}\right) > 0.$$

This contradicts to Proposition 3.4. □

Proof of Theorem 1.1(2). Let $U' = U \cap \left\{\boldsymbol\lambda \in U : \frac{h_{\mu_{\boldsymbol\lambda}}}{\chi_{\mu_{\boldsymbol\lambda}}} > 1\right\}$, which is by Proposition 2.5 open. Fix an arbitrary $\boldsymbol\lambda_0 \in U'$.

Let $\varepsilon > 0$ be such that $\frac{h_{\mu_{\boldsymbol\lambda_0}}-2\varepsilon}{\chi_{\mu_{\boldsymbol\lambda_0}}+2\varepsilon} > 1$. By Proposition 2.5 let $\delta > 0$ be such that if $|\boldsymbol\lambda_0 - \boldsymbol\lambda| < \delta$

$$\left|h_{\mu_{\boldsymbol\lambda_0}} - h_{\mu_{\boldsymbol\lambda}}\right| < \varepsilon \text{ and } \left|\chi_{\mu_{\boldsymbol\lambda_0}} - \chi_{\mu_{\boldsymbol\lambda}}\right| < \varepsilon. \tag{3.9}$$

Let $\varepsilon_1, \varepsilon_2 > 0$ and let J and $\Omega_{\boldsymbol\lambda}$ as in Lemma 3.2 and $J_\delta(\boldsymbol\lambda_0) := B_\delta(\boldsymbol\lambda_0) \cap J$. We are going to prove that $\widetilde{\mu}_{\boldsymbol\lambda}$ is absolutely continuous for Lebesgue almost every $\boldsymbol\lambda \in J_\delta(\boldsymbol\lambda_0)$ with density in L^2. Letting $\varepsilon_2 \to 0$ we get that $\widetilde{\mu}_{\boldsymbol\lambda}$ is absolutely continuous for Lebesgue-a. e. $\boldsymbol\lambda \in B_\delta(\boldsymbol\lambda_0)$ with density in L^2, then letting $\varepsilon_1 \to 0$ along a countable set we get that $\mu_{\boldsymbol\lambda}$ is abs. cont. for almost every $B_\delta(\boldsymbol\lambda_0)$, but the L^2 property may disappear.

Let

$$\underline{D}(\widetilde{\mu}_{\boldsymbol\lambda}, x) := \liminf_{r\to 0} \frac{\widetilde{\mu}_{\boldsymbol\lambda}((x-r, x+r))}{2r}.$$

the lower density of the measure $\widetilde{\mu}_{\boldsymbol\lambda}$ at the point x. By [10, Theorem 2.12], if $\underline{D}(\widetilde{\mu}_{\boldsymbol\lambda}, x) < \infty$ for $\widetilde{\mu}_{\boldsymbol\lambda}$-a.e. x then the measure is absolutely continuous. To prove that it is sufficient to show

$$\mathcal{J} := \int_{J_\delta(\boldsymbol\lambda_0)} \int_{\mathbb{R}} \underline{D}(\widetilde{\mu}_{\boldsymbol\lambda}, x) d\widetilde{\mu}_{\boldsymbol\lambda}(x) d\mathcal{L}_d(\boldsymbol\lambda) < \infty.$$

Applying (3.5) and Fubini's Lemma we get

$$\mathcal{J} \leq \liminf_{r\to 0} \frac{1}{2r} \int_{J_\delta(\boldsymbol\lambda_0)} \iint_{\Omega^2_{\boldsymbol\lambda}} \mathbb{1}_{\{|\pi_{\boldsymbol\lambda}(\mathbf{i}) - \pi_{\boldsymbol\lambda}(\mathbf{j})| < r\}} d\nu^2_{\boldsymbol\lambda}(\mathbf{i}, \mathbf{j}) d\mathcal{L}_d(\boldsymbol\lambda),$$

where $\nu_{\boldsymbol{\lambda}}^2 = \widetilde{\nu}_{\boldsymbol{\lambda}} \times \widetilde{\nu}_{\boldsymbol{\lambda}}$. Then

$$\mathcal{J} \leq \liminf_{r\to 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{\overline{\imath}\in\mathcal{S}^n} \int_{J_\delta(\boldsymbol{\lambda}_0)} \iint_{A_{\overline{\imath}}} \mathbb{1}_{\{|\pi_{\boldsymbol{\lambda}}(\mathbf{i})-\pi_{\boldsymbol{\lambda}}(\mathbf{j})|<r\}} d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i},\mathbf{j}) d\mathcal{L}_d(\boldsymbol{\lambda}),$$

and by (2.9), (3.4) and (3.9)

$$\mathcal{J} \leq \liminf_{r\to 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{\overline{\imath}\in\mathcal{S}^n} \int_{J_\delta(\boldsymbol{\lambda}_0)} \iint_{A_{\overline{\imath}}} \mathbb{1}_{\left\{|\pi_{\boldsymbol{\lambda}}(\sigma^n\mathbf{i})-\pi_{\boldsymbol{\lambda}}(\sigma^n\mathbf{j})|<C_2\widetilde{C}re^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)}\right\}} d\nu_{\boldsymbol{\lambda}}^2(\mathbf{i},\mathbf{j}) d\mathcal{L}_d(\boldsymbol{\lambda}).$$

Applying Lemma 3.1 we get

$$\mathcal{J} \leq$$

$$\liminf_{r\to 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{\overline{\imath}\in\mathcal{S}^n} \sum_{\substack{p,l\in\mathcal{S}\\ p\neq l}} \sum_{\overline{\jmath}_1,\overline{\jmath}_2\in\mathcal{S}^{N(n,r)}} \int_{J_\delta(\boldsymbol{\lambda}_0)} G_{C_2\widetilde{C}re^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)}}(l\overline{\jmath}_1\mathbf{1}, p\overline{\jmath}_2\mathbf{1}, \boldsymbol{\lambda}) \nu_{\boldsymbol{\lambda}}^2([\overline{\imath}l\overline{\jmath}_1]\times[\overline{\imath}p\overline{\jmath}_2]) d\mathcal{L}_d(\boldsymbol{\lambda}),$$

where $N(n,r) = \left\lceil \frac{\log C_2\widetilde{C}re^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)}/2\mathrm{diam}X}{\log\kappa} \right\rceil$ according to Lemma 3.1. Similarly, as in the proof of Proposition 3.4 one can get

$$\sum_{\overline{\imath}\in\mathcal{S}^n} \sum_{\substack{p,l\in\mathcal{S}\\ p\neq l}} \sum_{\overline{\jmath}_1,\overline{\jmath}_2\in\mathcal{S}^{N(n,r)}} \int_{J_\delta(\boldsymbol{\lambda}_0)} G_{C_2\widetilde{C}re^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)}}(l\overline{\jmath}_1\mathbf{1}, p\overline{\jmath}_2\mathbf{1}, \boldsymbol{\lambda}) \nu_{\boldsymbol{\lambda}}^2([\overline{\imath}l\overline{\jmath}_1]\times[\overline{\imath}p\overline{\jmath}_2]) d\mathcal{L}_d(\boldsymbol{\lambda}) \leq$$

$$2C_1C_2\widetilde{C}re^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}+2\varepsilon)} \sup_{\boldsymbol{\lambda}\in J_\delta(\boldsymbol{\lambda}_0)} \sum_{\overline{\imath}\in\mathcal{S}^n} \nu_{\boldsymbol{\lambda}}^2(A_{\overline{\imath}})$$

and

$$\sup_{\boldsymbol{\lambda}\in J_\delta(\boldsymbol{\lambda}_0)} \sum_{\overline{\imath}\in\mathcal{S}^n} \nu_{\boldsymbol{\lambda}}^2(A_{\overline{\imath}}) \leq \widetilde{C}e^{-n(h_{\mu_{\boldsymbol{\lambda}_0}}-2\varepsilon)}.$$

Therefore,

$$\mathcal{J} \leq 2C_1C_2\widetilde{C}^2 \sum_{n=0}^{\infty} e^{n(\chi_{\mu_{\boldsymbol{\lambda}_0}}-h_{\mu_{\boldsymbol{\lambda}_0}}+4\varepsilon)}.$$

Since $\chi_{\mu_{\boldsymbol{\lambda}_0}} - h_{\mu_{\boldsymbol{\lambda}_0}} + 4\varepsilon < 0$ the right-hand side is finite. This completes the proof. □

4. An example: place-dependent Bernoulli convolutions

In this section we show an application for our main theorem. Let $0 < \rho < \frac{1}{2}$ and $0.5 < \lambda < 1$ and let us consider the following function $f : [-1,1]\times[0,1] \mapsto [-1,1]\times[0,1]$

$$f(x,y) = \begin{cases} (\lambda x - (1-\lambda), \frac{2y}{1+2\rho x}) & \text{if } 0 \leq y < \frac{1}{2} + \rho x \\ (\lambda x + (1-\lambda), \frac{2y-2\rho x-1}{1-2\rho x}) & \text{if } \frac{1}{2} + \rho x \leq y \leq 1. \end{cases}$$

For the action of f on the rectangle $[-1,1]\times[0,1]$ see Figure 1.

It follows by [14] that there exists an f-invariant measure μ_{SBR} which is called as the SBR-measure such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \overline{\mathcal{L}}_2 \circ f^{-k} \to \mu_{\mathrm{SBR}} \text{ weakly},$$

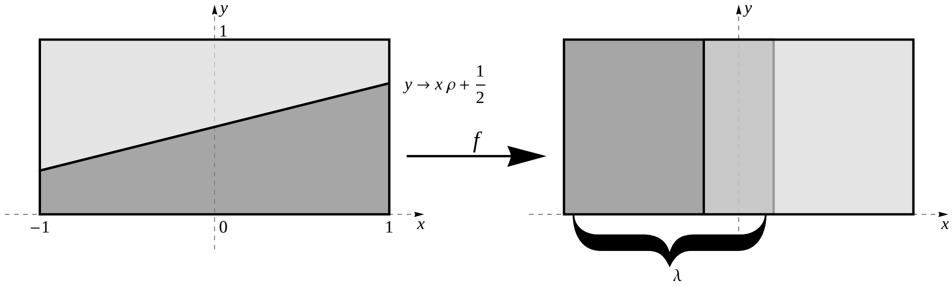

FIGURE 1. The map f acting on the rectangle $[-1,1] \times [0,1]$.

where $\overline{\mathcal{L}}_2$ is the normalized Lebesgue measure to the rectangle. Our aim to find a set of parameters such that $\mu_{\mathrm{SBR}} \ll \mathcal{L}_2$. It is easy to see that

$$d\mu_{\mathrm{SBR}}(x,y) = d\mu_{\lambda,\rho}(x) d\mathcal{L}_1(y),$$

where $\mu_{\lambda,\rho}$ is the place-dependent invariant measure of the IFS

$$\Psi_\lambda = \left\{ \psi_0^\lambda(x) = \lambda x - (1-\lambda), \psi_1^\lambda(x) = \lambda x + (1-\lambda) \right\}$$

with probabilities $\left\{ p_0(x) = \frac{1}{2} + \rho x, p_1(x) = \frac{1}{2} - \rho x \right\}$. Then the property $\mu_{\mathrm{SBR}} \ll \mathcal{L}_2$ is equivalent to $\mu_{\lambda,\rho} \ll \mathcal{L}_1$. It was proven by Peres and Solomyak [11] that if $\rho = 0$ then $\mu_{\lambda,0}$ is absolutely continuous w.r.t Lebesgue measure for Leb-a.e. $\lambda \in (0.5, 1)$.

According to (1.3) and (1.4) we have

$$\chi_{\mu_{\lambda,\rho}} = -\log \lambda \text{ and}$$
$$h_{\mu_{\lambda,\rho}} = -\int_{\mathbb{R}} \left(\tfrac{1}{2} + \rho x\right) \log\left(\tfrac{1}{2} + \rho x\right) + \left(\tfrac{1}{2} - \rho x\right) \log\left(\tfrac{1}{2} - \rho x\right) d\mu_{\lambda,\rho}(x).$$

To simplify the entropy let us observe that $|2\rho x| < 1$, therefore

$$h_{\mu_{\lambda,\rho}} = \log 2 + \frac{1}{2} \int_{\mathbb{R}} (1 + 2\rho x) \sum_{n=1}^{\infty} \frac{(-2\rho x)^n}{n} + (1 - 2\rho x) \sum_{n=1}^{\infty} \frac{(2\rho x)^n}{n} d\mu_{\lambda,\rho}(x) =$$
$$\log 2 - \sum_{n=1}^{\infty} \frac{(2\rho)^{2n}}{2n(2n-1)} \int_{\mathbb{R}} x^{2n} d\mu_{\lambda,\rho}(x). \quad (4.1)$$

For brevity, let $F_n = \int_{\mathbb{R}} x^{2n} d\mu_{\lambda,\rho}(x)$. Using (2.4) one can write an inductive formula for the series $\{F_n\}_{n=0}^{\infty}$. Precisely,

$$F_n = (1-\lambda)^{2n} + \sum_{m=1}^{n} 2m(1-\lambda)^{2n-2m} \lambda^{2m-1} \binom{2n}{2m} \left(\frac{\lambda}{2m} - \frac{2\rho(1-\lambda)}{2n-2m+1} \right) F_m,$$

and therefore

$$F_n = \frac{(1-\lambda)^{2n}}{1 + \lambda^{2n-1}(4n\rho(1-\lambda) - \lambda)} +$$
$$\sum_{m=1}^{n-1} \frac{2m(1-\lambda)^{2n-2m} \lambda^{2m-1}}{1 + \lambda^{2n-1}(4n\rho(1-\lambda) - \lambda)} \binom{2n}{2m} \left(\frac{\lambda}{2m} - \frac{2\rho(1-\lambda)}{2n-2m+1} \right) F_m.$$

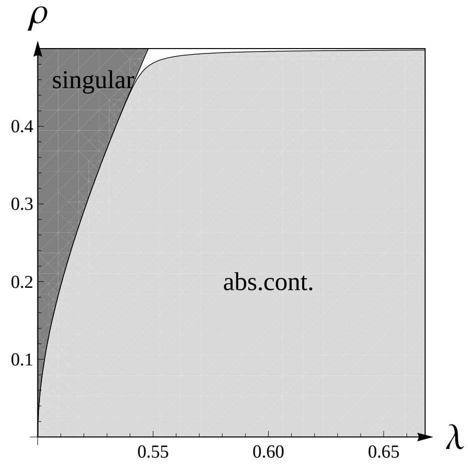

FIGURE 2. The singularity and absolute continuity region of the measure $\mu_{\lambda,\rho}$.

For example,

$$F_1 = \frac{(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)},$$

$$F_2 = \frac{(1-\lambda)^4}{1+\lambda^3(8\rho(1-\lambda)-\lambda)} + \frac{2(1-\lambda)^4\lambda(3\lambda-4\rho(1-\lambda))}{(1+\lambda^3(8\rho(1-\lambda)-\lambda))(1+\lambda(4\rho(1-\lambda)-\lambda))}, \text{ etc.}$$

It follows from (4.1) that for every $N \geq 1$

$$\log 2 - \sum_{n=1}^{N} \frac{(2\rho)^{2n}}{2n(2n-1)} F_n - \frac{(2\rho)^{N+1}}{(2N+2)(2N+1)(1-(2\rho)^2)} \leq h_{\mu_{\lambda,\rho}} \leq \log 2 - \sum_{n=1}^{N} \frac{(2\rho)^{2n}}{2n(2n-1)} F_n. \tag{4.2}$$

Theorem 4.1. *For every* $0 \leq \rho < 0.5$ *and Lebesgue almost every* $\lambda \in (0.5, 0.6684755)$

$$\frac{\log 2 - \frac{2\rho^2(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)} - \frac{\rho^2}{3(1-4\rho^2)}}{-\log\lambda} \leq \dim_H \mu_{\lambda,\rho} \leq \frac{\log 2 - \frac{2\rho^2(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)}}{-\log\lambda}.$$

Moreover, $\mu_{\lambda,\rho}$ *is absolutely continuous for Lebesgue almost every*

$$\lambda \in \left\{\lambda \in (0.5, 0.6684755) : \log 2 - \frac{2\rho^2(1-\lambda)^2}{1+\lambda(4\rho(1-\lambda)-\lambda)} - \frac{\rho^2}{3(1-4\rho^2)} > -\log\lambda\right\}.$$

We note that if $0 < \lambda < 0.5$ then the attractor of the IFS is a Cantor set with dimension strictly less than one. Moreover, it satisfies the open set condition trivially, therefore without loss of generality we may assume that $\lambda > 0.5$.

Proof. The system Ψ_λ and the probabilities $\left\{\frac{1}{2}+\rho x, \frac{1}{2}-\rho x\right\}$ satisfy the conditions (A1),(A2) and (A4) trivially.

It follows from [12, Section 5] and [16, Theorem 2.6] that Ψ_λ satisfies the condition (A3) on the interval $(0.5, 0.6684755)$. The theorem now is an easy consequence of Theorem 1.1 and the bound (4.2) with $N = 1$. □

For more precise characterization of the measure $\mu_{\lambda,\rho}$, see Figure 2.

References

[1] R. Bowen: *Equilibrium states and the ergodic theory of Anosov Diffeomorphisms*, Springer-Verlag, 1975.

[2] K. Falconer: *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley and Sons, 1990.

[3] A. H. Fan and K.-S. Lau: Iterated Function System and Ruelle Operator, *J. Math. Anal. & Appl.* **231** (1999), 319-344.

[4] T.-Y. Hu, K.-S. Lau and X.-Y. Wang: On the absolute continuity of a class of invariant measures, *Proc. Amer. Math. Soc.* **130** No. 3 (2002), 759-767.

[5] J. E. Hutchinson: Fractals and Self-Similarity, *Indiana Univ. Math. Journal* **30** No. 5, (1981), 713-747.

[6] J. Jaroszewska: Iterated Function Systems with Continuous Place Dependent Probabilities, *Univ. Iagel. Acta Math.* **40** (2002), 137-146.

[7] J. Jaroszewska and M. Rams: On the Hausdorff Dimension of Invariant Measures of Weakly Contracting on Average Measurable IFS, *J. Stat. Phys.* **132** (2008), 907-919.

[8] A. A. Kwiecińska and W. Słomczyński: Random dynamical systems arising from iterated function systems with place-dependent probabilities, *Stat. & Prob. Letters* **50** (2000), 401-407.

[9] K.-S. Lau, S.-M. Ngai and X.-Y. Wang: Separation conditions for conformal iterated function systems, *Monatsh. Math.* **156** No. 4, (2009), 325-355.

[10] P. Mattila: *Geometry of sets and measures in Euclidean spaces*, C.U.P., Cambridge, 1995.

[11] Y. Peres and B. Solomyak: Absolute continuity of Bernoulli convolutions, a simple proof, *Math. Research Letters* **3** No. 2, (1996), 231-239.

[12] Y. Peres and B. Solomyak, Self-similar measures and intersections of Cantor sets, *Trans. Amer. Math. Soc.* **350** No. 10 (1998), 4065–4087.

[13] T. Persson: Absolutely continuous invariant measures for some piecewise hyperbolic affine maps, *Erg. Th. & Dyn. Sys.* **28** No. 1 (2008), 211-228.

[14] Ya. B. Pesin: Dynamical systems with generalized hyperbolic attractors: hiperbolic, ergodic and topological properties, *Erg. Th. & Dyn. Sys.* **12**, (1992), 123-151.

[15] M. Pollicott and K. Simon: The Hausdorff dimension of λ-expansions with deleted digits, *Trans. Amer. Math. Soc.* **347** (1995), 967-983.

[16] P. Shmerkin and B. Solomyak: Zeros of $\{-1, 0, 1\}$ power series and connectedness loci for self-affine sets, *Experiment. Math.* **15** No. 4 (2006), 499-511.

[17] K. Simon, B. Solomyak and M. Urbański: Hausdorff dimension of limit sets for parabolic IFS with overlaps, *Pacific J. Math.* **201** No. 2 (2001), 441-478.

[18] K. Simon, B. Solomyak, and M. Urbański: Invariant measures for parabolic IFS with overlaps and random continued fractions, *Trans. Amer. Math. Soc.* **353** (2001), 5145-5164.

[19] I. Werner: Contractive Markov systems, *J. Lond. Math. Soc.* **71** No. 2 (2005), 236-258.

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