

FINER GEOMETRY OF PLANAR SELF-AFFINE SETS

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ABSTRACT. For a planar self-affine set satisfying the strong separation condition it has been recently proved that under mild assumptions the Hausdorff dimension equals the affinity dimension. In this article, we continue this line of research and our objective is to acquire more refined geometric information. In a large class of non-carpet planar self-affine sets, we characterize Ahlfors regularity, determine the Assouad dimension of the set and its projections, and estimate the Hausdorff dimension of slices. We also demonstrate that the Assouad dimension is not necessarily bounded above by the affinity dimension.

1. INTRODUCTION

Let $X \subset \mathbb{R}^2$ be a self-affine set associated to a finite number of invertible affine contractions φ_i on \mathbb{R}^2 . The defining property of X is that it consists of affine copies $\varphi_i(X)$ of itself. The strong separation condition requires the sets $\varphi_i(X)$ to be pairwise disjoint. We write $\varphi_i(x) = A_i x + v_i$ for all $x \in \mathbb{R}^2$, where $A_i \in GL_2(\mathbb{R})$ is the linear part and $v_i \in \mathbb{R}^2$ is the translation vector. To understand geometric properties of X is a surprisingly difficult problem, even if the elements of the matrices are positive and the strong separation condition is assumed, and has attained a lot of interest during recent years; see e.g. Bárány [6], Bárány, Käenmäki, and Koivusalo [9], Hueter and Lalley [37], Käenmäki and Shmerkin [45], Morris and Shmerkin [52], and Rapaport [56] for results in dimension, and Bárány and Käenmäki [8] and Falconer and Kempton [17] for results in projections.

A recent breakthrough is the article of Bárány, Hochman, and Rapaport [7] where the authors proved that under the strong separation condition and mild assumptions on the linear parts, the Hausdorff dimension of X equals the affinity dimension, a natural upper bound stemming from the definition of the self-affine set. The corresponding result for self-similar sets, a sub-class of self-affine sets where the linear parts are assumed to be constant times orthogonal matrices, was proved by Hutchinson [38]. He also showed that the strong separation condition implies the positivity of the Hausdorff measure. A folklore open question, explicitly stated in Falconer [19, §2], asks whether there exist a characterization or even sufficient conditions for the positivity of the Hausdorff measure of self-affine sets.

Question. Is it possible to characterize the positivity of the Hausdorff measure of a self-affine set?

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Bárány, Hochman, and Rapaport [7] also improved what the classical Marstrand's projection theorem gives for planar self-affine sets by showing that the Hausdorff dimension of the projection is either preserved for all directions or is equal to one. A complementary concept to projections is that of slices. The classical Marstrand's slicing theorem shows that almost every slice has Hausdorff dimension at most the surplus dimension of the projection. An immediate question here is whether the slicing theorem on self-affine sets could also be improved for all slices. Perhaps a bit surprisingly, besides some specific cases almost nothing is known about this.

Question. Is the Hausdorff dimension of every slice of a self-affine set at most the surplus dimension of the projection?

In the present article, the main goal is to study these, and related questions on finer geometry, on a large class of self-affine sets.

The detailed main results will be presented in Section 3 but let us now review some of their consequences. Recall that a planar self-affine set X is dominated and irreducible if the linear parts of the defining affine maps of X have positive entries and they do not share a common invariant line. Assuming the strong separation condition, our main results reveal that on such an open and dense collection of self-affine sets it is possible to obtain sharp results analogous to the classical ones known for self-similar sets since 90's. The following theorem applies these sharp results to completely characterize the Ahlfors regularity of X . We denote the s -dimensional Hausdorff measure by \mathcal{H}^s . Ahlfors s -regularity of X means that the $\mathcal{H}^s|_X$ -measure of any ball of radius r centered at X is uniformly comparable to r^s .

Theorem 1.1. *If X is a dominated irreducible planar self-affine set satisfying the strong separation condition, then X is Ahlfors s -regular if and only if $0 \leq s \leq 1$ and $0 < \mathcal{H}^s(X) < \infty$.*

We denote the Hausdorff dimension by $\dim_{\mathbb{H}}$ and the collection of all lines through the origin by \mathbb{RP}^1 . The classical Marstrand's projection theorem [49] for Hausdorff dimension states that, given a Borel set $X \subset \mathbb{R}^2$, we have

$$\dim_{\mathbb{H}}(\text{proj}_{V^\perp}(X)) = \min\{1, \dim_{\mathbb{H}}(X)\}$$

for Lebesgue almost all $V \in \mathbb{RP}^1$. For a general class of self-affine sets, Bárány, Hochman, and Rapaport [7] have proved the above result for all $V \in \mathbb{RP}^1$.

The Assouad dimension $\dim_{\mathbb{A}}$ is the maximal dimension possible to obtain by looking at coverings and it serves as an upper bound for the Hausdorff dimension. Orponen [53] has shown a strong variant of Marstrand's projection theorem for Assouad dimension. It states that, given a set $X \subset \mathbb{R}^2$, we have

$$\dim_{\mathbb{A}}(\text{proj}_{V^\perp}(X)) \geq \min\{1, \dim_{\mathbb{A}}(X)\} \tag{1.1}$$

for all $V \in \mathbb{RP}^1 \setminus E$, where the set $E \subset \mathbb{RP}^1$ satisfies $\dim_{\mathbb{H}}(E) = 0$. It is worth pointing out that in general, besides proving the exceptional set E countable, this result cannot be improved: Fraser and Käenmäki [31] showed that for every upper semi-continuous function $f: \mathbb{RP}^1 \rightarrow [0, 1]$ there exists a compact set $X \subset \mathbb{R}^2$ with $\dim_{\mathbb{A}}(X) = 0$ such that $\dim_{\mathbb{A}}(\text{proj}_V(X)) = f(V)$ for all $V \in \mathbb{RP}^1$. The following theorem can therefore be considered as a manifestation of the rigid structure of self-affine sets. The set $X_F \subset \mathbb{RP}^1$

is the collection of all Furstenberg directions, i.e. the limits of all directions of stronger contraction, and under the assumptions of the theorem, it satisfies $\dim_{\mathbb{H}}(X_F) > 0$.

Theorem 1.2. *If X is a dominated irreducible planar self-affine set satisfying the strong separation condition such that X is not Ahlfors regular, then*

$$\dim_{\mathbb{A}}(\text{proj}_{V^\perp}(X)) = \min\{1, \dim_{\mathbb{A}}(X)\}$$

for all $V \in \mathbb{RP}^1 \setminus E$, where the set $E \subset \mathbb{RP}^1$ satisfies $\dim_{\mathbb{H}}(E) = 0$. Furthermore, if X is Ahlfors regular, then the above holds for all $V \in X_F$.

Marstrand's projection theorem is a strong dimension conservation principle for Hausdorff dimension: if $\dim_{\mathbb{H}}(X) \leq 1$, then the dimension of X is conserved by almost every projection. If $\dim(X) > 1$, then the same cannot be true, as the projections have dimension at most one in every direction. This defect is resolved by the classical Marstrand's slicing theorem [49]. It shows that almost every fiber of a projection do not store more dimension than what is the surplus. The theorem states that, given a Borel set $X \subset \mathbb{R}^2$ and $V \in \mathbb{RP}^1$, we have

$$\dim_{\mathbb{H}}(X \cap (V + x)) \leq \max\{0, \dim_{\mathbb{H}}(X) - 1\}$$

for Lebesgue almost all $x \in V^\perp$. Furstenberg [32] conjectured that for the product of $\times 2$ and $\times 3$ invariant sets all fibers should be small. In our terminology, such sets appear as certain dynamically defined subsets of product-type Bedford-McMullen carpets. Furstenberg's conjecture was resolved by Shmerkin [59] and Wu [60]. It is therefore interesting to ask whether the slices are small also on other self-affine sets. For a Bedford-McMullen carpet X having logarithmically incommensurable contraction ratios, Algom [1] proved that the Hausdorff dimension of any slice not parallel to the principal axes is bounded above by $\max\{0, \dim_{\mathbb{A}}(X) - 1\}$. Besides the following theorem, we are not aware of any results of this type for general classes of non-carpet self-affine sets.

Theorem 1.3. *If X is a dominated irreducible planar self-affine set satisfying the strong separation condition such that $\dim_{\mathbb{H}}(X) \geq 1$, then*

$$\sup_{\substack{x \in X \\ V \in X_F}} \dim_{\mathbb{H}}(X \cap (V + x)) = \dim_{\mathbb{A}}(X) - 1 < 1.$$

The Hausdorff dimension of any self-affine set is bounded above also by the affinity dimension \dim_{aff} , a number which is obtained by looking at the behavior of natural covers in the construction of the set. Prior the following theorem, it has not been known how the Assouad dimension compares to the affinity dimension outside self-affine carpets. It is worth mentioning that on self-similar sets the strong separation condition implies the equality of the Assouad and affinity dimensions.

Theorem 1.4. *If X is a dominated irreducible planar self-affine set satisfying the strong separation condition such that $\dim_{\mathbb{H}}(X) = s < 1$, then either X is Ahlfors regular or $\dim_{\text{aff}}(X) < 1 \leq \dim_{\mathbb{A}}(X)$. In particular, parametrized by the translation vectors and elements of the matrices, there exists a collection with non-empty interior of such planar Ahlfors s -regular self-affine sets and there exists an uncountable collection of such planar self-affine sets X for which $\dim_{\text{aff}}(X) < 1 \leq \dim_{\mathbb{A}}(X)$.*

2. PRELIMINARIES

We introduce rigorous definitions and some preliminaries in this section. The reader familiar with the recent progress in the topic may skip the preliminaries and go directly to Section 3 where we exhibit the main results.

2.1. Ahlfors regularity. We say that a Borel measure μ on \mathbb{R}^2 is *Ahlfors s -regular* if there exists a constant $C \geq 1$ such that

$$C^{-1}r^s \leq \mu(B(x, r)) \leq Cr^s$$

for all $x \in \text{spt}(\mu)$ and $0 < r < \text{diam}(\text{spt}(\mu))$, where $B(x, r)$ is the closed ball centered at x with radius r . A compact set $X \subset \mathbb{R}^2$ is *Ahlfors s -regular* if it supports an Ahlfors s -regular measure. We also say that a measure or a set is *Ahlfors regular* if it is Ahlfors s -regular for some $s \geq 0$. Recall that the *s -dimensional Hausdorff measure* \mathcal{H}^s of a set $X \subset \mathbb{R}^2$ is defined by

$$\mathcal{H}^s(X) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^s(X) = \sup_{\delta > 0} \mathcal{H}_\delta^s(X),$$

where

$$\mathcal{H}_\delta^s(X) = \inf \left\{ \sum_i \text{diam}(U_i)^s : X \subset \bigcup_i U_i \text{ and } \text{diam}(U_i) \leq \delta \right\}$$

is the *s -dimensional Hausdorff δ -content* of A . By [50, Theorem 6.9], an Ahlfors regular set $X \subset \mathbb{R}^2$ has positive and finite s -dimensional Hausdorff measure, $0 < \mathcal{H}^s(X) < \infty$, when $s = \dim_{\text{H}}(X) = \inf\{s \geq 0 : \mathcal{H}^s(X) < \infty\}$ is the Hausdorff dimension of X . In fact, a set $X \subset \mathbb{R}^2$ is Ahlfors s -regular if and only if there exists a constant $C \geq 1$ such that

$$C^{-1}r^s \leq \mathcal{H}^s(X \cap B(x, r)) \leq Cr^s$$

for all $x \in X$ and $0 < r < \text{diam}(X)$.

If a Borel measure μ on \mathbb{R}^2 is Ahlfors s -regular, then we write $\dim(\mu) = s$ and say that s is the *dimension* of μ . More generally, the *upper and lower pointwise dimensions* of μ at $x \in \mathbb{R}^2$ are

$$\begin{aligned} \overline{\dim}_{\text{loc}}(\mu, x) &= \limsup_{r \downarrow 0} \frac{\log B(x, r)}{\log r}, \\ \underline{\dim}_{\text{loc}}(\mu, x) &= \liminf_{r \downarrow 0} \frac{\log B(x, r)}{\log r}, \end{aligned}$$

respectively. For basic properties of pointwise dimensions, we refer to the book of Falconer [20, §10]. If there exists a constant s such that $\overline{\dim}_{\text{loc}}(\mu, x) = \underline{\dim}_{\text{loc}}(\mu, x) = s$ for μ -almost all $x \in \mathbb{R}^2$, then we again write $\dim(\mu) = s$ and say that μ is *exact dimensional*. We remark that in general, most measures do not satisfy this property. But since in the study of self-affine sets, basically all the measures involved are exact dimensional, we use the convention that writing $\dim(\mu)$ implicitly means that the measure μ is known to be exact dimensional; consult [8, 23, 24, 36, 46, 57] for examples. Dimensions of measures introduce us with a way to study the Hausdorff dimension of a given Borel set X : it is well known that

$$\dim_{\text{H}}(X) = \max_{x \sim \mu} \{\text{ess sup } \underline{\dim}_{\text{loc}}(\mu, x) : \mu \text{ is a finite Borel measure on } X\}; \quad (2.1)$$

for example, see [21, §3]. Often with dynamically defined sets, such as in the case of self-affine sets, it is desirable for the approximating measures to adhere to some dynamical properties such as ergodicity and the behavior of entropy and Lyapunov exponents. In this case, the above formula is called the *variational principle*; see [39] for its proof in the self-affine case. Under further assumptions, the approximating measures can be seen to agree also with other properties; see [52, Theorem 1.1] and [11, Proposition 2.4]. These properties were crucial in [7] where the authors were able to determine the Hausdorff dimension of planar self-affine sets in a very general setting; see Theorem 2.16.

2.2. Weak tangents. Let $X \subset \mathbb{R}^2$ be compact. For each $x \in X$ and $r > 0$ we define the *magnification* $M_{x,r}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by setting

$$M_{x,r}(z) = \frac{z - x}{r}$$

for all $z \in \mathbb{R}^2$. We say that a set T intersecting the interior of $B(0,1)$ is a *weak tangent set* of X if there exist sequences $(x_n)_{n \in \mathbb{N}}$ of points in X and $(r_n)_{n \in \mathbb{N}}$ of positive real numbers such that $\lim_{n \rightarrow \infty} r_n = 0$ and $M_{x_n, r_n}(X) \cap B(0,1) \rightarrow T$ in Hausdorff distance. We denote the collection of weak tangent sets of X by $\text{Tan}(X)$.

The *Assouad dimension* of a set $X \subset \mathbb{R}^2$, denoted by $\dim_A(X)$, is the infimum of all $s \geq 0$ satisfying the following: There exists a constant $C \geq 1$ such that for every $x \in X$ and $0 < r < R$ the set $X \cap B(x, R)$ can be covered by at most $C(R/r)^s$ balls of radius r centered at X . It is easy to see that

$$\dim_H(X) \leq \dim_A(X)$$

for all sets $X \subset \mathbb{R}^2$. If $X \subset \mathbb{R}^2$ is compact, then $\dim_A(X) \geq \dim_H(T)$ for all $T \in \text{Tan}(X)$; see [48, Proposition 6.1.5]. The following result of Käenmäki, Ojala, and Rossi [43, Proposition 5.7] shows that there exists a weak tangent set whose Hausdorff dimension attains the maximal possible value. The result introduces a way to calculate the Assouad dimension of a set by considering its weak tangents.

Lemma 2.1. *If $X \subset \mathbb{R}^2$ is compact, then $\dim_A(X) = \max\{\dim_H(T) : T \in \text{Tan}(X)\}$.*

We remark that the first result in this direction is by Furstenberg; see [33, Proposition 5.1]. Together with [44, Proposition 3.13] it shows that the Assouad dimension is realized as a Hausdorff dimension of a weak tangent set or a finite magnification. The above result is needed to guarantee that the Assouad dimension gets realized on a weak tangent. This is particularly important detail in the study of self-affine sets as such sets often undergo a metamorphosis in approaching the weak tangent; see [4, 11, 40, 43].

Analogously, the *lower dimension* of a set $X \subset \mathbb{R}^2$, denoted by $\dim_L(X)$, is the supremum of all $s \geq 0$ satisfying the following: There exists a constant $c > 0$ such that for every $x \in X$ and $0 < r < R < \text{diam}(X)$ covering the set $X \cap B(x, R)$ requires at least $c(r/R)^{-s}$ balls of radius r centered at X . It is easy to see that

$$\dim_L(X) \leq \dim_H(X)$$

for all $X \subset \mathbb{R}^2$. If $X \subset \mathbb{R}^2$ is compact, then $\dim_L(X) \leq \dim_H(T)$ for all $T \in \text{Tan}(X)$; see [30, Proposition 2.3]. The following result of Fraser, Howroyd, Käenmäki, and Yu [43, Theorem 1.1] shows that there exists a weak tangent set whose Hausdorff

dimension attains the minimal possible value. The result introduces a way to calculate the lower dimension of a set by considering its weak tangents.

Lemma 2.2. *If $X \subset \mathbb{R}^2$ is compact, then $\dim_L(X) = \min\{\dim_H(T) : T \in \text{Tan}(X)\}$.*

In the above result, the requirement that the weak tangent set intersects the interior of the unit ball is essential: without this assumption, the weak tangent can be a single point in the boundary. It is straightforward to see that if $X \subset \mathbb{R}^2$ is Ahlfors s -regular, then $\dim_L(X) = \dim_A(X) = s$; see [41, §3]. The converse does not necessarily hold as the following example illustrates.

Example 2.3. We construct a set $X \subset [0, 1]$ with $\dim_L(X) = \dim_A(X)$ and $\mathcal{H}^s(X) = 0$ for $s = \dim_H(X)$. Let $\lambda_n = (3n)^{-1}$ for all $n \in \mathbb{N}$. We start the construction from the unit interval $[0, 1]$. First, we cut out the middle part of length λ_1 . Next, for each of the two small intervals of length $(1 - \lambda_1)/2 = 1/3$, we cut out their middle parts of length $\lambda_2/3$. At step n , we cut out the middle parts of relative length λ_n . We can perform this cutting procedure indefinitely and in the end we obtain a compact set X similarly as with the middle third Cantor set. It is straightforward to see that $\dim_L(X) = 1$ and X has zero Lebesgue measure.

We collect the general implications of Ahlfors regularity in the following lemma:

Lemma 2.4. *If $X \subset \mathbb{R}^2$ is Ahlfors s -regular, then $0 < \mathcal{H}^s(X) < \infty$ where $s = \dim_L(X) = \dim_H(X) = \dim_A(X)$.*

The important observation here is that if the lower and Assouad dimensions of X differ, then X cannot be Ahlfors regular. For other basic properties of the Assouad and lower dimensions, we refer to the book of Fraser [28].

2.3. Irreducibility. If $A \in GL_2(\mathbb{R})$ is an invertible 2×2 -matrix, then we denote the lengths of the major and minor axis of the ellipse $A(B(0, 1))$ by $\alpha_1(A)$ and $\alpha_2(A)$, respectively. Note that $\alpha_1(A) = \|A\|$ and $\alpha_2(A) = \|A^{-1}\|^{-1}$ are the square roots of the non-negative real eigenvalues of the positive semidefinite matrix $A^\top A$. It is also well-known that $|\det(A)| = \alpha_1(A)\alpha_2(A)$ for all $A \in GL_2(\mathbb{R})$. Let \mathbb{RP}^1 be the real projective line, that is, the set of all lines through the origin in \mathbb{R}^2 . Let us denote the Dirac mass at $V \in \mathbb{RP}^1$ by δ_V .

Lemma 2.5. *If $(A_n)_{n \in \mathbb{N}}$ is a sequence of 2×2 matrices such that there exists a non-atomic measure m on \mathbb{RP}^1 for which $\lim_{n \rightarrow \infty} (A_n)_* m = \delta_V$ in the weak* topology, then*

$$\frac{\alpha_1(A_n^\top)}{\alpha_2(A_n^\top)} = \frac{\alpha_1(A_n)}{\alpha_2(A_n)} \rightarrow \infty$$

as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{\|A_n^\top W\|}{\alpha_1(A_n^\top)} = |\cos(\angle(V, W))|$$

for all $W \in \mathbb{RP}^1$, and

$$\angle(A_n^\top W_1, A_n^\top W_2) \rightarrow 0$$

as $n \rightarrow \infty$ for all $W_1, W_2 \in \mathbb{RP}^1 \setminus \{V^\perp\}$.

Proof. Since $\|\det(A_n)^{-1/2}A_n\|^2 = \alpha_1(A_n)/\alpha_2(A_n)$ and

$$\begin{aligned} |\langle A_n^\top W_1, A_n^\top W_2 \rangle| &= \arcsin\left(\frac{|A_n^\top w_1 \wedge A_n^\top w_2|}{|A_n^\top w_1||A_n^\top w_2|}\right) \\ &= \arcsin\left(\frac{|\det(A_n^\top)|}{\|A_n^\top W_1\|\|A_n^\top W_2\|}|\sin(\langle W_1, W_2 \rangle)|\right) \end{aligned}$$

for all $n \in \mathbb{N}$, where $w_1 \in W_1$ and $w_2 \in W_2$ are such that $|w_1| = 1 = |w_2|$, the lemma follows from [15, Proposition II.3.1]. \square

We are primarily interested in semigroups generated by finite collections of matrices. In this context, it is rather standard practise to use separate alphabet to index the elements in the semigroup. Let $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$ be the collection of all infinite words obtained from integers $\{1, \dots, N\}$. If $\mathbf{i} = i_1 i_2 \dots \in \Sigma$, then we define $\mathbf{i}|_n = i_1 \dots i_n$ for all $n \in \mathbb{N}$. If $\mathbf{i} = i_1 \dots i_n$, then we write $\overleftarrow{\mathbf{i}} = i_n \dots i_1$. The empty word $\mathbf{i}|_0$ is denoted by \emptyset . Define $\Sigma_n = \{\mathbf{i}|_n : \mathbf{i} \in \Sigma\}$ for all $n \in \mathbb{N}$ and $\Sigma_* = \bigcup_{n \in \mathbb{N}} \Sigma_n \cup \{\emptyset\}$. Thus Σ_* is the collection of all finite words. The length of $\mathbf{i} \in \Sigma_* \cup \Sigma$ is denoted by $|\mathbf{i}|$. The concatenation of two words $\mathbf{i} \in \Sigma_*$ and $\mathbf{j} \in \Sigma_* \cup \Sigma$ is denoted by $\mathbf{i}\mathbf{j}$ and the longest common prefix of \mathbf{i} and \mathbf{j} by $\mathbf{i} \wedge \mathbf{j}$. Thus $\mathbf{j} = (\mathbf{i} \wedge \mathbf{j})\mathbf{j}'$ for some $\mathbf{j}' \in \Sigma_* \cup \Sigma$. If $\mathbf{A} = (A_1, \dots, A_N) \in GL_2(\mathbb{R})^N$, then we write $A_{\mathbf{i}} = A_{i_1} \dots A_{i_n}$ and

$$\begin{aligned} A_{\overleftarrow{\mathbf{i}}}^\top &= (A_{\overleftarrow{\mathbf{i}}})^\top = A_{i_1}^\top \dots A_{i_n}^\top, \\ A_{\overleftarrow{\mathbf{i}}}^{-1} &= (A_{\overleftarrow{\mathbf{i}}})^{-1} = A_{i_1}^{-1} \dots A_{i_n}^{-1} \end{aligned}$$

for all $\mathbf{i} = i_1 \dots i_n \in \Sigma_n$ and $n \in \mathbb{N}$.

Let σ be the left shift operator defined by $\sigma \mathbf{i} = i_2 i_3 \dots$ for all $\mathbf{i} = i_1 i_2 \dots \in \Sigma$. If $\mathbf{i} \in \Sigma_n$ for some n , then we set $[\mathbf{i}] = \{\mathbf{j} \in \Sigma : \mathbf{j}|_n = \mathbf{i}\}$. The set $[\mathbf{i}]$ is called a *cylinder set*. The *shift space* Σ is compact in the topology generated by the cylinder sets. Moreover, the cylinder sets are open and closed in this topology and they generate the Borel σ -algebra. Let $\mathcal{M}_\sigma(\Sigma)$ be the collection of all σ -invariant Borel probability measures on Σ . We say that a measure μ on Σ is *fully supported* if every cylinder set has positive measure, $\mu([\mathbf{i}]) > 0$ for all $\mathbf{i} \in \Sigma_*$. If (p_1, \dots, p_N) is a probability vector, then the measure $\mu \in \mathcal{M}_\sigma(\Sigma)$ for which

$$\mu([\mathbf{i}]) = p_{i_1} \dots p_{i_n}$$

for all $\mathbf{i} = i_1 \dots i_n \in \Sigma_n$ and $n \in \mathbb{N}$ is called a *Bernoulli measure*. Note that a Bernoulli measure obtained from a probability vector (p_1, \dots, p_N) is fully supported if and only if $p_i > 0$ for all $i \in \{1, \dots, N\}$.

If $\mathbf{A} = (A_1, \dots, A_N) \in GL_2(\mathbb{R})^N$ and $\mu \in \mathcal{M}_\sigma(\Sigma)$ is a fully supported Bernoulli measure obtained from a probability vector (p_1, \dots, p_N) , then the associated *Furstenberg measure* is a Borel probability measure μ_F on \mathbb{RP}^1 satisfying

$$\mu_F = \sum_{i=1}^N p_i (A_i^{-1})_* \mu_F. \quad (2.2)$$

We say that \mathbf{A} is *irreducible* if there does not exist $V \in \mathbb{RP}^1$ such that $A_i V = V$ for all $i \in \{1, \dots, N\}$; otherwise \mathbf{A} is *reducible*. The tuple \mathbf{A} is *strongly irreducible* if there does not exist a finite set $\mathcal{V} \subset \mathbb{RP}^1$ such that $A_i \mathcal{V} = \mathcal{V}$ for all $i \in \{1, \dots, N\}$. A matrix

$A \in GL_2(\mathbb{R})$ is called *proximal* if it has two real eigenvalues with different absolute values. We say that A is *strictly affine* if there is $\mathbf{i} \in \Sigma_*$ such that $A_{\mathbf{i}}$ is proximal. If A is strictly affine, then the set of *Furstenberg directions* is

$$X_F = \{A\mathbb{R}^2 \in \mathbb{RP}^1 : A \in \overline{\{cA_{\mathbf{i}}^{-1} : c \in \mathbb{R} \text{ and } \mathbf{i} \in \Sigma_*\}}\}.$$

The following lemma demonstrates the connection between the Furstenberg measure and directions.

Lemma 2.6. *If $A \in GL_2(\mathbb{R})^N$ is strictly affine and strongly irreducible and $\mu \in \mathcal{M}_\sigma(\Sigma)$ is a fully supported Bernoulli measure, then the associated Furstenberg measure μ_F is unique and non-atomic with $\dim(\mu_F) > 0$, the support of μ_F is X_F , and there exists a measurable function $\vartheta_2: \Sigma \rightarrow X_F$ such that $\mu_F = \int_\Sigma \delta_{\vartheta_2(\mathbf{i})} d\mu(\mathbf{i}) = (\vartheta_2)_*\mu$ and*

$$(A_{\mathbf{i}|n}^{-1})_*\mu_F \rightarrow \delta_{\vartheta_2(\mathbf{i})}$$

in the weak* topology for μ -almost all $\mathbf{i} \in \Sigma$ as $n \rightarrow \infty$. In particular, for every $V \in X_F$ there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of integers and for each $k \in \mathbb{N}$ there is a word $\mathbf{i}_k \in \Sigma_{n_k}$ such that

$$(A_{\mathbf{i}_k}^{-1})_*\mu_F \rightarrow \delta_V.$$

in the weak* topology as $k \rightarrow \infty$.

Proof. By [15, Theorem II.4.1 and Corollary VI.4.2] and [36, Theorem 1.1], the Furstenberg measure μ_F is unique, non-atomic, and satisfies $\dim(\mu_F) > 0$. The fact that the support of μ_F is X_F follows from [11, proof of Lemma 2.3] and the existence of the function $\vartheta_2: \Sigma \rightarrow X_F$ from [15, Proposition II.3.3 and Theorem II.4.1]. To show the last claim, note that, by the definition of the support, there exists a sequence $(\mathbf{i}_k)_{k \in \mathbb{N}}$ of words in Σ such that $\vartheta_2(\mathbf{i}_k) \rightarrow V$. Let d_W be the Wasserstein distance (or any other metric inducing the weak* topology). For each $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ such that

$$d_W((A_{\mathbf{i}_k|n_k}^{-1})_*\mu_F, \delta_{\vartheta_2(\mathbf{i}_k)}) < \frac{1}{k},$$

where $\mathbf{i}_k|n_k \in \Sigma_{n_k}$. Therefore,

$$d_W((A_{\mathbf{i}_k|n_k}^{-1})_*\mu_F, \delta_V) \leq \frac{1}{k} + d_W(\delta_{\vartheta_2(\mathbf{i}_k)}, \delta_V) \rightarrow 0$$

as $k \rightarrow \infty$. □

Let $R: \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ be such that $R(V) = V^\perp$ for all $V \in \mathbb{RP}^1$ and write $\mu_F^\perp = R_*\mu_F$. Observe that if $A \in GL_2(\mathbb{R})^N$ is strictly affine and strongly irreducible, $\mu \in \mathcal{M}_\sigma(\Sigma)$ is a fully supported Bernoulli measure. Then Lemma 2.6 together with the facts that $R^{-1} = R$ and $A^\top V^\perp = (A^{-1}V)^\perp$ for all $A \in GL_2(\mathbb{R})^N$ and $V \in \mathbb{RP}^1$ implies that

$$(A_{\mathbf{i}|n}^\top)_*\mu_F^\perp \rightarrow \delta_{\vartheta_2(\mathbf{i})^\perp} \tag{2.3}$$

in the weak* topology for μ -almost all $\mathbf{i} \in \Sigma$ as $n \rightarrow \infty$, where ϑ_2 is as in Lemma 2.6. We remark that an explicit definition for the function ϑ_2 can be found in [11, §2.3].

Finally, let us analyse the defined reducibility conditions. The following lemma, which classifies the conditions, follows immediately from [11, proof of Lemma 2.2].

Lemma 2.7. *If $A \in GL_2(\mathbb{R})^N$ is strictly affine, then precisely one of the following conditions hold:*

- (1) A is strongly irreducible,
- (2) A is irreducible but not strongly, i.e., the matrices in A are simultaneously diagonal or antidiagonal in some basis so that there is at least one antidiagonal matrix,
- (3) A is reducible, i.e., the matrices in A are simultaneously upper triangular in some basis.

Note that if A consists only of antidiagonal matrices, then their second iterates are diagonal and hence reducible.

2.4. Domination. We say that $A = (A_1, \dots, A_N) \in GL_2(\mathbb{R})^N$ is *dominated* if there exist constants $C > 0$ and $0 < \tau < 1$ such that

$$\alpha_2(A_i) \leq C\tau^{|\mathbf{i}|}\alpha_1(A_i) \quad (2.4)$$

for all $\mathbf{i} \in \Sigma_*$. By [10, Corollary 2.4], a dominated tuple is strictly affine. We call a proper subset $\mathcal{C} \subset \mathbb{RP}^1$ a *multicone* if it is a finite union of closed projective intervals. We say that $\mathcal{C} \subset \mathbb{RP}^1$ is a *strongly invariant multicone* for A if it is a multicone and $A_i\mathcal{C} \subset \mathcal{C}^\circ$ for all $i \in \{1, \dots, N\}$, where \mathcal{C}° is the interior of \mathcal{C} . For example, the first quadrant is strongly invariant for any tuple of positive matrices. By [13, Theorem B], A has a strongly invariant multicone if and only if A is dominated. Furthermore, if $\mathcal{C} \subset \mathbb{RP}^1$ is a strongly invariant multicone for A , then $\overline{\mathbb{RP}^1 \setminus \mathcal{C}}$ and $\{V^\perp : V \in \mathbb{RP}^1 \setminus \mathcal{C}\}$ are strongly invariant multicones for $A^{-1} = (A_1^{-1}, \dots, A_N^{-1})$ and $A^\top = (A_1^\top, \dots, A_N^\top)$, respectively.

Observe that if A is dominated, then the set of Furstenberg directions is the compact set

$$X_F = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \Sigma_n} A_{\mathbf{i}}^{-1} \overline{\mathbb{RP}^1 \setminus \mathcal{C}}. \quad (2.5)$$

If $\Pi: \Sigma \rightarrow \mathbb{RP}^1$ is the *canonical projection* defined by the relation

$$\{\Pi(\mathbf{i})\} = \bigcap_{n=1}^{\infty} A_{\mathbf{i}|_n}^{-1} \overline{\mathbb{RP}^1 \setminus \mathcal{C}}, \quad (2.6)$$

then it is easy to see that $X_F = \bigcup_{\mathbf{i} \in \Sigma} \Pi(\mathbf{i})$. Note that X_F is perfect unless it is a singleton.

Lemma 2.8. *If $A \in GL_2(\mathbb{R})^N$ is dominated, then there exists a constant $D \geq 1$ such that*

$$\|A_{\mathbf{i}}^\top |V^\perp\| \leq \alpha_1(A_{\mathbf{i}}) \leq D \|A_{\mathbf{i}}^{-1} |V^\perp\|$$

for all $\mathbf{i} \in \Sigma_*$ and $V \in X_F$. Furthermore, if $\mathbf{i} \in \Sigma$ and $V = \Pi(\mathbf{i})$, then

$$D^{-1} \|A_{\mathbf{i}|_n}^{-1} |V\| \leq \alpha_2(A_{\mathbf{i}|_n}) \leq \|A_{\mathbf{i}|_n}^{-1} |V\|$$

for all $n \in \mathbb{N}$.

Proof. To prove the first claim, fix $V \in \bigcup_{i=1}^N A_i^{-1} \overline{\mathbb{RP}^1 \setminus \mathcal{C}}$ and let $v \in V^\perp$ be such that $|v| = 1$. Notice that $X_F \subset \bigcup_{i=1}^N A_i^{-1} \overline{\mathbb{RP}^1 \setminus \mathcal{C}}$ and $A_i^\top \{V^\perp : V \in \overline{\mathbb{RP}^1 \setminus \mathcal{C}}\} \subset \{V^\perp : V \in$

$\bigcup_{i=1}^N A_i^{-1} \overline{\mathbb{R}\mathbb{P}^1 \setminus \mathcal{C}}$ for all $\mathbf{i} \in \Sigma_* \setminus \{\emptyset\}$. By [14, Lemma 2.2], there exists a constant $D \geq 1$ such that

$$\|A_{\mathbf{i}}^\top |V^\perp\| = |A_{\mathbf{i}}^\top v| \geq D^{-1} \|A_{\mathbf{i}}\|$$

for all $\mathbf{i} \in \Sigma_*$.

To show the second claim, let $V \in \mathbb{R}\mathbb{P}^1$ be the only element in $\bigcap_{n=1}^\infty A_{\mathbf{i}_n}^{-1} \overline{\mathbb{R}\mathbb{P}^1 \setminus \mathcal{C}}$ and let $v \in V$ be such that $|v| = 1$. Notice that $A_j^{-1} \overline{\mathbb{R}\mathbb{P}^1 \setminus \mathcal{C}} \subset \bigcup_{i=1}^N A_i^{-1} \overline{\mathbb{R}\mathbb{P}^1 \setminus \mathcal{C}}$ for all $j \in \Sigma_* \setminus \{\emptyset\}$. Fixing $n \in \mathbb{N}$ we see that $A_{\mathbf{i}_n}^{-1} v \in \bigcap_{k=1}^\infty A_{\sigma^n \mathbf{i}_k}^{-1} \overline{\mathbb{R}\mathbb{P}^1 \setminus \mathcal{C}} \subset \bigcup_{i=1}^N A_i^{-1} \overline{\mathbb{R}\mathbb{P}^1 \setminus \mathcal{C}}$. Therefore, again by [14, Lemma 2.2], there exists a constant $D \geq 1$ such that

$$1 = |v| = |A_{\mathbf{i}_n}^{-1} A_{\mathbf{i}_n}^{-1} v| \geq D^{-1} \|A_{\mathbf{i}_n}^{-1}\| |A_{\mathbf{i}_n}^{-1} v| = D^{-1} \|A_{\mathbf{i}_n}^{-1}\| \|A_{\mathbf{i}_n}^{-1} v\|$$

which finishes the proof. \square

If $V, W \in \mathbb{R}\mathbb{P}^1$, then the *projection* $\text{proj}_V^W: \mathbb{R}^2 \rightarrow V$ is the linear map such that $\text{proj}_V^W|_V = \text{Id}|_V$ and $\ker(\text{proj}_V^W) = W$. The *orthogonal projection* $\text{proj}_V^{V^\perp}$ onto the subspace V is denoted by proj_V . Note that $(\text{proj}_V A)^\top = A^\top \text{proj}_V$ and hence,

$$\|A^\top |V^\perp\| = \|\text{proj}_{V^\perp} A\| \quad (2.7)$$

for all $A \in GL_2(\mathbb{R})$ and $V \in \mathbb{R}\mathbb{P}^1$. Recall that a 2×2 -matrix A has rank one if and only if there exist $v, w \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $A = vw^\top$ with $\text{im}(A) = \text{span}(v)$ and $\ker(A) = \text{span}(w)^\perp$. It is easy to see that in such a case,

$$A = \begin{cases} \langle v, w \rangle \text{proj}_{\text{im}(A)}^{\ker(A)}, & \text{if } A \text{ is not nilpotent,} \\ |v||w| R \text{proj}_{\ker(A)^\perp}, & \text{if } A \text{ is nilpotent,} \end{cases} \quad (2.8)$$

where $R \in O(2)$ is a rotation by an angle $\pi/2$. In particular, $A(X)$ is bi-Lipschitz equivalent to $\text{proj}_{\ker(A)^\perp}(X)$ for all $X \subset \mathbb{R}^2$. The following lemma guarantees that nilpotent matrices do not appear in the dominated case.

Lemma 2.9. *If $A \in GL_2(\mathbb{R})^N$ is dominated, then the closure of $\{cA_{\mathbf{i}} : c \in \mathbb{R} \text{ and } \mathbf{i} \in \Sigma_*\}$ does not contain non-zero nilpotent elements. In other words, rank one matrices in the above closure are all projections.*

Proof. Let us assume that there exists a non-zero nilpotent matrix P in the closure of $\{cA_{\mathbf{i}} : c \in \mathbb{R} \text{ and } \mathbf{i} \in \Sigma_*\}$. By definitions, $P^2 = 0$ and there exists sequences $(\mathbf{i}_n)_{n \in \mathbb{N}}$ of finite words in Σ_* and $(c_n)_{n \in \mathbb{N}}$ of non-zero real numbers such that $c_n A_{\mathbf{i}_n} \rightarrow P$ as $n \rightarrow \infty$. The domination guarantees that, by possibly going through a sub-sequence, $A_{\mathbf{i}_n} / \|A_{\mathbf{i}_n}\|$ converges to a rank one matrix and so $\lim_{n \rightarrow \infty} c_n / \|A_{\mathbf{i}_n}\| \in \mathbb{R} \setminus \{0\}$. Thus, without loss of generality, we may assume that $c_n = \|A_{\mathbf{i}_n}\|$.

The domination guarantees that, by possibly taking another sub-sequence, also $A_{\mathbf{i}_n}^2 / \|A_{\mathbf{i}_n}^2\|$ converges to a rank one matrix Q . By [10, Corollary 2.4], there exists a constant $C \geq 1$ such that $\|A_{\mathbf{i}_n}^2\| \leq \|A_{\mathbf{i}_n}\|^2 \leq C \|A_{\mathbf{i}_n}^2\|$ for all $n \in \mathbb{N}$. Hence, $A_{\mathbf{i}_n}^2 / \|A_{\mathbf{i}_n}^2\|$ must converge to a constant times P^2 . Since $P^2 = 0$, this contradicts for Q being rank one. \square

A dominated tuple \mathbf{A} is not necessarily irreducible and an irreducible tuple \mathbf{A} is not necessarily dominated. For example, consider a tuple of diagonal matrices and a tuple of diagonal and antidiagonal matrices. The following lemma shows that together the properties imply strong irreducibility.

Lemma 2.10. *If $\mathbf{A} \in GL_2(\mathbb{R})^N$ is dominated and irreducible, then \mathbf{A} is strongly irreducible.*

Proof. Since \mathbf{A} is dominated, it contains only proximal elements and is hence strictly affine; see [10, Corollary 2.4]. Therefore, as \mathbf{A} is irreducible, it suffices to show that the condition (2) in Lemma 2.7 does not hold. Let $\mathcal{C} \subset \mathbb{RP}^1$ be a strongly invariant multicone for \mathbf{A} . If \mathbf{A} contains an antidiagonal matrix A , then A^2 is a constant times the identity matrix and $A^2\mathcal{C} = \mathcal{C}$ which is a contradiction. Hence, \mathbf{A} is strongly irreducible. \square

The following lemma introduces a representation for dominated and reducible tuples for which the collection of Furstenberg directions is non-trivial.

Lemma 2.11. *If $\mathbf{A} = (A_1, \dots, A_N) \in GL_2(\mathbb{R})^N$ is dominated and reducible such that X_F is not a singleton, then, possibly after a change of basis,*

$$A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \quad (2.9)$$

with $0 < |d_i| < |a_i| < 1$ for all $i \in \{1, \dots, N\}$, and the matrices are not simultaneously diagonalizable.

Proof. If X is reducible, then, by Lemma 2.7(3), the matrices A_i are simultaneously upper triangular in some basis and are of the form (2.9). If there exists $i \in \{1, \dots, N\}$ such that $|d_i| \geq |a_i|$, then it is easy to see that for any subspace $W \in \mathbb{RP}^1$, we have $A_i^n W \rightarrow \text{span}(e_1)$ as $n \rightarrow \infty$, where $e_1 = (1, 0)$. Thus, since the matrices are dominated, any strongly invariant multicone \mathcal{C} must contain $\text{span}(e_1)$ as an interior point. But such a cone cannot be strongly invariant for matrices of the form (2.9) unless $|d_i| > |a_i|$ for all $i \in \{1, \dots, N\}$, which contradicts the assumption that X_F is not a singleton. Hence, $|d_i| < |a_i|$ for all $i \in \{1, \dots, N\}$. Finally, as X_F is not a singleton, the matrices cannot be simultaneously diagonalisable. \square

2.5. Equilibrium states. For each $A \in GL_2(\mathbb{R})$ and $s \geq 0$ we define the *singular value function* by setting

$$\varphi^s(A) = \begin{cases} \|A\|^s = \alpha_1(A)^s, & \text{if } 0 \leq s \leq 1, \\ \|A\|^{2-s} |\det(A)|^{s-1} = \alpha_1(A)\alpha_2(A)^{s-1}, & \text{if } 1 < s \leq 2, \\ |\det(A)|^{s/2} = \alpha_1(A)^{s/2}\alpha_2(A)^{s/2}, & \text{if } s > 2. \end{cases}$$

The value $\varphi^s(A)$ represents a measurement of the s -dimensional volume of the ellipse $A(B(0, 1))$. Since the determinant is multiplicative and the operator norm is submultiplicative, we have $\varphi^s(AB) \leq \varphi^s(A)\varphi^s(B)$ for all $A, B \in GL_2(\mathbb{R})$ and $s \geq 0$. For each $\mathbf{A} \in GL_2(\mathbb{R})^N$ and $s \geq 0$ we define the *pressure* by setting

$$P(\mathbf{A}, s) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathbf{i} \in \Sigma_n} \varphi^s(A_{\mathbf{i}}).$$

As the singular value function is sub-multiplicative, the sequence $(\log \sum_{\mathbf{i} \in \Sigma_n} \varphi^s(A_{\mathbf{i}}))_{n \in \mathbb{N}}$ is sub-additive and hence, the limit above exists by Fekete's lemma. It is also easy to see that the pressure $P(\mathbf{A}, s)$ is continuous and strictly decreasing as a function of s with $P(\mathbf{A}, 0) \geq 0$ and $\lim_{s \rightarrow \infty} P(\mathbf{A}, s) = -\infty$. We may thus define the *affinity dimension* by setting $\dim_{\text{aff}}(\mathbf{A})$ to be the unique $s \geq 0$ for which $P(\mathbf{A}, s) = 0$.

Suppose that $\mathbf{A} \in GL_2(\mathbb{R})^N$ is dominated. For each $s \geq 0$ define a function $g_s: \Sigma \rightarrow \mathbb{R}$ by setting

$$g_s(\mathbf{i}) = \begin{cases} \log \|A_{\mathbf{i}|_1}^\top |\Pi(\sigma \mathbf{i})^\perp\|^s, & \text{if } 0 \leq s \leq 1, \\ \log \|A_{\mathbf{i}|_1}^\top |\Pi(\sigma \mathbf{i})^\perp\|^{2-s} |\det(A_{\mathbf{i}|_1})|^{s-1}, & \text{if } 1 < s \leq 2, \\ \log |\det(A_{\mathbf{i}|_1})|^{s/2}, & \text{if } s > 2, \end{cases}$$

for all $\mathbf{i} \in \Sigma$, where Π is as in (2.6). Notice that g_s is Hölder continuous. The *Perron-Frobenius operator* \mathcal{L} for s is the positive linear operator defined by setting

$$\mathcal{L}f(\mathbf{i}) = \sum_{i=1}^N \exp(g_s(i\mathbf{i}))f(i\mathbf{i})$$

for all continuous functions $f: \Sigma \rightarrow \mathbb{R}$. Observe that, by Lemma 2.8, there exists a constant $D \geq 1$ such that

$$\log \|A_{\mathbf{i}|_n}\| - \log D \leq \log \|A_{\mathbf{i}|_n}^\top |\Pi(\sigma^n \mathbf{i})^\perp\| = \sum_{k=0}^{n-1} \log \|A_{\sigma^k \mathbf{i}|_1}^\top |\Pi(\sigma^{k+1} \mathbf{i})^\perp\|$$

and hence, the *Birkhoff sum* of g_s , $\sum_{k=0}^{n-1} g_s(\sigma^k \mathbf{i})$, satisfies

$$\log \varphi^s(A_{\mathbf{i}|_n}) - \log D \leq \sum_{k=0}^{n-1} g_s(\sigma^k \mathbf{i}) \leq \log \varphi^s(A_{\mathbf{i}|_n}) \quad (2.10)$$

for all $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$. The following lemma is a simple consequence of the classical Ruelle's Perron-Frobenius Theorem.

Lemma 2.12. *If $\mathbf{A} \in GL_2(\mathbb{R})^N$ is dominated and \mathcal{L} is the Perron-Frobenius operator for $\dim_{\text{aff}}(\mathbf{A})$, then there exist a unique continuous function $h: \Sigma \rightarrow (0, \infty)$ and a unique Borel probability measure ν on Σ such that*

$$\mathcal{L}h = h, \quad \int_{\Sigma} h(\mathbf{i}) \, d\nu(\mathbf{i}) = 1,$$

and

$$\lim_{n \rightarrow \infty} \sup_{\mathbf{i} \in \Sigma} \left| \mathcal{L}^n f(\mathbf{i}) - h(\mathbf{i}) \int_{\Sigma} f(\mathbf{j}) \, d\nu(\mathbf{j}) \right| = 0$$

for all continuous functions $f: \Sigma \rightarrow \mathbb{R}$. Furthermore, if $\mathbf{A} \in GL_2(\mathbb{R})^N$ is dominated or irreducible, then there exist a unique measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ and a constant $C \geq 1$ such that

$$C^{-1} \varphi^s(A_{\mathbf{i}}) \leq \mu_K([\mathbf{i}]) \leq C \varphi^s(A_{\mathbf{i}})$$

for all $\mathbf{i} \in \Sigma_*$, where $s = \dim_{\text{aff}}(\mathbf{A})$.

Proof. If A is irreducible, then the existence of the claimed measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ follows immediately from [25, Proposition 1.2 and §3]. Thus we assume that A is dominated and write $s = \dim_{\text{aff}}(A)$. By [16, Theorem 1.7], there exist a unique continuous function $h: \Sigma \rightarrow (0, \infty)$ and a unique Borel probability measure ν on Σ such that

$$\mathcal{L}h = \lambda h, \quad \mathcal{L}^*\nu = \lambda\nu, \quad \int_{\Sigma} h(\mathbf{i}) \, d\nu(\mathbf{i}) = 1,$$

and

$$\lim_{n \rightarrow \infty} \sup_{\mathbf{i} \in \Sigma} \left| \lambda^{-n} \mathcal{L}^n f(\mathbf{i}) - h(\mathbf{i}) \int_{\Sigma} f(\mathbf{j}) \, d\nu(\mathbf{j}) \right| = 0$$

for all continuous functions $f: \Sigma \rightarrow \mathbb{R}$, where $\lambda = \mathcal{L}^*\nu(\Sigma) > 0$. By [16, Theorem 1.16], there exists a constant $C \geq 1$ such that the measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ defined by setting

$$\mu_K(\Gamma) = \int_{\Gamma} h(\mathbf{i}) \, d\nu(\mathbf{i})$$

for all Borel sets $\Gamma \subset \Sigma$ is ergodic and satisfies

$$C^{-1} \lambda^{-n} \exp\left(\sum_{k=0}^{n-1} g_s(\sigma^k \mathbf{i})\right) \leq \mu_K([\mathbf{i}|_n]) \leq C \lambda^{-n} \exp\left(\sum_{k=0}^{n-1} g_s(\sigma^k \mathbf{i})\right)$$

for all $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$, where

$$\log \lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathbf{i} \in \Sigma_n} \exp \sup_{\mathbf{j} \in [\mathbf{i}]} \left(\sum_{k=1}^{n-1} g_s(\sigma^k \mathbf{j}) \right).$$

Therefore, by (2.10) and the choice of s , we have $\lambda = \exp(P(A, s)) = 1$ and hence,

$$(CD)^{-1} \varphi^s(A_{\mathbf{i}}) \leq \mu_K([\mathbf{i}|_n]) \leq C \varphi^s(A_{\mathbf{i}})$$

for all $\mathbf{i} \in \Sigma_*$. The uniqueness of μ_K is now evident as two different ergodic measures are mutually singular. \square

The measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ in Lemma 2.12 is called the *equilibrium state* as its canonical projection on the self-affine set X is the natural candidate to attain the maximum in (2.1). Equilibrium states exist also without domination or irreducibility but in this case, they satisfy slightly weaker properties than described in Lemma 2.12 and are not necessarily unique; see [39] and [25].

2.6. Self-affine set. We consider a tuple $\Phi = (A_1 + v_1, \dots, A_N + v_N)$ of contractive invertible affine self-maps on \mathbb{R}^2 , where we have written $A + v$ to denote the affine map $x \mapsto Ax + v$ defined on \mathbb{R}^2 for all 2×2 matrices A and translation vectors $v \in \mathbb{R}^2$. Such a tuple Φ is called an *affine iterated function system*. We also write $\varphi_i = A_i + v_i$ for all $i \in \{1, \dots, N\}$ and $\varphi_{\mathbf{i}} = \varphi_{i_1} \circ \dots \circ \varphi_{i_n}$ for all $\mathbf{i} = i_1 \dots i_n \in \Sigma_n$ and $n \in \mathbb{N}$. Note that the associated tuple of matrices (A_1, \dots, A_N) is an element of $GL_2(\mathbb{R})^N$ and satisfies $\max_{i \in \{1, \dots, N\}} \|A_i\| < 1$. It is a classical result of Hutchinson [38] that for each affine iterated function system there exists a unique non-empty compact set $X \subset \mathbb{R}^2$, called the *self-affine set*, such that

$$X = \bigcup_{i=1}^N \varphi_i(X). \tag{2.11}$$

We use the convention that whenever we speak about a self-affine set X , then it is automatically accompanied with a tuple of affine maps which defines it. This makes it possible to write that e.g. “ X is dominated” which obviously then means that “the associated tuple \mathbf{A} of matrices is dominated”. Similarly, by $\dim_{\text{aff}}(X)$ we mean the affinity dimension $\dim_{\text{aff}}(\mathbf{A})$ defined in §2.5.

We are interested in understanding the geometry of self-affine sets. Relying on (2.11), the self-affine set X can naturally be covered by the sets $\varphi_{\mathbf{i}}(B)$, where B is a ball containing X . Observe that each ellipse $\varphi_{\mathbf{i}}(B)$ can be covered by one ball of radius $\alpha_1(A_{\mathbf{i}}) \text{diam}(B)$ or by $\alpha_1(A_{\mathbf{i}})/\alpha_2(A_{\mathbf{i}})$ many balls of radius $\alpha_2(A_{\mathbf{i}}) \text{diam}(B)$. This motivates us to study the limiting behavior of the sums $\sum_{\mathbf{i} \in \Sigma_n} \varphi^s(A_{\mathbf{i}})$ and indeed, it is straightforward to see that $\dim_{\text{H}}(X) \leq \min\{2, \dim_{\text{aff}}(X)\}$.

Every affine iterated function system is associated with the *canonical projection* $\pi: \Sigma \rightarrow X$ which is defined by $\pi \mathbf{i} = \sum_{n=1}^{\infty} A_{\mathbf{i}|_{n-1}} v_{i_n}$ for all $\mathbf{i} = i_1 i_2 \cdots \in \Sigma$. It is easy to see that π is continuous and the image of Σ is the self-affine set, $\pi(\Sigma) = X$. Separation conditions allow simple interplay between Σ and X . We say that X satisfies the *strong separation condition* if $\varphi_i(X) \cap \varphi_j(X) = \emptyset$ whenever $i \neq j$. In this case, we have

$$\delta = \min_{i \neq j} \text{dist}(\varphi_i(X), \varphi_j(X)) > 0. \quad (2.12)$$

As $\pi([\mathbf{i}]) = \varphi_{\mathbf{i}}(X)$ for all $\mathbf{i} \in \Sigma_*$, the strong separation condition is characterized by the requirement that the canonical projection is one-to-one. We say that X satisfies the *open set condition* if there exists an open set $U \subset \mathbb{R}^2$ such that $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ whenever $i \neq j$ and $\varphi_i(U) \subset U$ for all $i \in \{1, \dots, N\}$. If such a set U also intersects X , then we say that X satisfies the *strong open set condition*. Observe that the strong separation condition implies the strong open set condition.

Let us first survey known results for self-similar sets which are a special case of self-affine sets. If $(\lambda_1 O_1 + v_1, \dots, \lambda_N O_N + v_N)$, where $0 < \lambda_i < 1$ and $O_i \in O(2)$ for all $i \in \{1, \dots, N\}$, is a tuple of contractive similarities on \mathbb{R}^2 , then we call the associated self-affine set X *self-similar*. In this case, the affinity dimension is called *similarity dimension* and we denote it by $\dim_{\text{sim}}(X)$. Notice that $\dim_{\text{sim}}(X)$ is the unique $s \geq 0$ for which $\sum_{i=1}^N \lambda_i^s = 1$. Let us endow the group of all similitudes with the topology of pointwise convergence and define

$$\Sigma(x, r) = \{\mathbf{i} \in \Sigma_* : \text{diam}(\varphi_{\mathbf{i}}(X)) \leq r < \text{diam}(\varphi_{\mathbf{i}^-}(X)) \text{ and } \varphi_{\mathbf{i}}(X) \cap B(x, r) \neq \emptyset\}$$

for all $x \in \mathbb{R}^2$ and $r > 0$.

Theorem 2.13. *If X is a planar self-similar set, then the following conditions are equivalent:*

- (1) X satisfies the open set condition,
- (2) X satisfies the strong open set condition,
- (3) $\sup\{\#\Sigma(x, r) : x \in X \text{ and } r > 0\} < \infty$,
- (4) the identity is not in the closure of $\{\varphi_{\mathbf{i}}^{-1} \circ \varphi_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \Sigma_* \text{ such that } \mathbf{i} \neq \mathbf{j}\}$,
- (5) there is $\eta > 0$ such that $|\varphi_{\mathbf{i}} - \varphi_{\mathbf{j}}| \geq \eta \text{diam}(\varphi_{\mathbf{i}}(X))$ for all $\mathbf{i}, \mathbf{j} \in \Sigma_*$ with $\mathbf{i} \neq \mathbf{j}$,
- (6) $\mathcal{H}^s(X) > 0$ where $s = \dim_{\text{sim}}(X)$,
- (7) X is Ahlfors s -regular where $s = \dim_{\text{sim}}(X)$,

Proof. Notice that (2) \Rightarrow (1) is a triviality and (7) \Rightarrow (6) follows from Lemma 2.4. Hutchinson [38, §5.3] proved the implication (1) \Rightarrow (7), Bandt and Graf [3] showed that (6) \Leftrightarrow (4) \Leftrightarrow (5), and finally, Schief [58, Theorem 2.1] verified the remaining implication (6) \Rightarrow (2). \square

Recall that if X is a self-similar set, then, regardless of separation conditions, [18, Theorem 4] shows that $\mathcal{H}^s(X) < \infty$ where $s = \dim_{\mathbb{H}}(X)$. We say that X satisfies the *weak separation condition* if

$$\sup\{\#\Phi(x, r) : x \in X \text{ and } r > 0\} < \infty,$$

where

$$\Phi(x, r) = \{\varphi_{\mathbf{i}} : \text{diam}(\varphi_{\mathbf{i}}(X)) \leq r < \text{diam}(\varphi_{\mathbf{i}^-}(X)) \text{ and } \varphi_{\mathbf{i}}(X) \cap B(x, r) \neq \emptyset\}$$

for all $x \in \mathbb{R}^2$ and $r > 0$. Note that the open set condition is valid if and only if the weak separation condition holds and $\varphi_{\mathbf{i}} \neq \varphi_{\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \Sigma_*$ with $\mathbf{i} \neq \mathbf{j}$.

Theorem 2.14. *If X is a planar self-similar set then the following conditions are equivalent:*

- (1) X satisfies the weak separation condition,
- (2) the identity is not an accumulation point of $\{\varphi_{\mathbf{i}}^{-1} \circ \varphi_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \Sigma_* \text{ such that } \mathbf{i} \neq \mathbf{j}\}$,
- (3) there is $\eta > 0$ such that $|\varphi_{\mathbf{i}} - \varphi_{\mathbf{j}}| \geq \eta \text{diam}(\varphi_{\mathbf{i}}(X))$ for all $\mathbf{i}, \mathbf{j} \in \Sigma_*$ with $\varphi_{\mathbf{i}} \neq \varphi_{\mathbf{j}}$,

Furthermore, the following conditions follow from the above conditions and, if X is not contained in a line and $\dim_{\mathbb{H}}(X) \leq 1$, or alternatively, if $\dim_{\mathbb{H}}(X) < 1$, then all the conditions are equivalent:

- (4) $\mathcal{H}^s(X) > 0$ where $s = \dim_{\mathbb{H}}(X)$,
- (5) X is Ahlfors regular.
- (6) $\dim_{\mathbb{L}}(X) = \dim_{\mathbb{H}}(X) = \dim_{\mathbb{A}}(X)$.

Proof. It follows from Angelevska, Käenmäki, and Troscheit [2, Theorem 3.2] that (1) \Leftrightarrow (2) \Leftrightarrow (3). Furthermore, by [2, Theorem 3.1], we have (4) \Leftrightarrow (5). Note also that [2, Proposition 3.3] verifies the implication (1) \Rightarrow (4). The implication (5) \Rightarrow (6) follows immediately from Lemma 2.4. Finally, Fraser, Henderson, Olson, and Robinson [29, Theorems 3.1 and 3.2] proved that if X does not satisfy (2), then $\dim_{\mathbb{A}}(X) \geq 1$, and García [34, Theorem 1.4] demonstrated that if, in addition, X is not contained in a line, then $\dim_{\mathbb{A}}(X) > 1$. Therefore, under the mentioned extra assumptions, we have the implication (6) \Rightarrow (2). \square

The assumption $\dim_{\mathbb{H}}(X) \leq 1$ in the above theorem is essential: a slight modification of [22, Proposition 3.3] shows that for each $1 < s \leq 2$ there exists a planar Ahlfors s -regular self-similar set not satisfying the weak separation condition. Furthermore, a line is an Ahlfors 1-regular set and it can be expressed as a self-similar set not satisfying the weak separation condition. This shows that none of the conditions in the second group imply the conditions in the first group without the extra assumption.

Let us next state a dimension results for a special case of self-affine sets, Bedford-McMullen carpets, which are constructed by affine maps sharing a common diagonal matrix as a linear part. Let $q > p \geq 2$ and $N \in \{2, \dots, pq\}$ be integers, and $I \subset \{0, \dots, p-1\} \times \{0, \dots, q-1\}$ a set of N elements. A *Bedford-McMullen carpet* is the self-affine set

$X \subset [0, 1]^2$ associated to a tuple $(\varphi_1, \dots, \varphi_N)$ of affine maps which all have the same linear part $\text{diag}(\frac{1}{p}, \frac{1}{q})$ and the translation part is from the set $\{(\frac{j}{p}, \frac{k}{q}) \in [0, 1]^2 : (j, k) \in I\}$. We assume that each map in the tuple appears there only once. Write $n_j = \#\{k : (j, k) \in I\}$ to denote the number of sets $\varphi_i([0, 1]^2)$ the vertical line $\{(\frac{j}{p}, y) : y \in \mathbb{R}\}$ intersects. We say that the Bedford-McMullen carpet X has *uniform vertical fibers* if there is $n \in \mathbb{N}$ such that $n_j = n$ for all j with $n_j \neq 0$.

Theorem 2.15. *If X is a Bedford-McMullen carpet, then the following conditions are equivalent:*

- (1) X has uniform vertical fibers,
- (2) $\mathcal{H}^s(X) < \infty$ where $s = \dim_{\text{H}}(X)$,
- (3) X is Ahlfors regular,
- (4) $\dim_{\text{L}}(X) = \dim_{\text{H}}(X) = \dim_{\text{A}}(X)$.

Proof. The implication (1) \Rightarrow (3) follows from McMullen [51]. Lemma 2.4 shows (3) \Rightarrow (2) and (3) \Rightarrow (4). Finally, Peres [54, Theorem 1] has shown the implication (2) \Rightarrow (1) and Fraser [27, Corollary 2.14], extending the result of Mackay [47, Theorem 1.1], proved the implication (4) \Rightarrow (1). \square

Let us then turn to the general self-affine case. Recall that the set of all irreducible tuples $A \in GL_2(\mathbb{R})^N$ is open, dense, and full Lebesgue measure in $GL_2(\mathbb{R})^N$. In fact, the set of all reducible tuples $A \in GL_2(\mathbb{R})^N$ is a finite union of $(4N - 1)$ -dimensional algebraic varieties; see [42, Propositions 3.4 and 3.6]. Recall also that the set of all dominated tuples $A \in GL_2(\mathbb{R})^N$ is open in $GL_2(\mathbb{R})^N$.

Theorem 2.13 shows that on self-similar sets the open set condition and the strong open set condition are equivalent. On strictly affine strongly irreducible planar self-affine sets, the open set condition is not a sufficient assumption for any meaningful dimension result; see [52, Example 5.5]. Nevertheless, the strong open set condition still has a role. The following breakthrough result is proven by Bárány, Hochman, and Rapaport [7, Theorems 1.1 and 7.1]:

Theorem 2.16. *If X is a strictly affine strongly irreducible planar self-affine set satisfying the strong open set condition, then*

$$\begin{aligned} \dim_{\text{H}}(X) &= \min\{2, \dim_{\text{aff}}(X)\}, \\ \dim_{\text{H}}(\text{proj}_{V^\perp}(X)) &= \min\{1, \dim_{\text{aff}}(X)\} \end{aligned}$$

for all $V \in \mathbb{RP}^1$.

Recall that if a planar self-affine set X is dominated, then, by [10, Corollary 2.4], it is strictly affine. Therefore, by Lemma 2.10, dominated irreducible planar self-affine sets satisfy the hypothesis of Theorem 2.16. Note that if X is irreducible, then X_F is not a singleton. It turns out that, under domination, the assumption that X_F is not a singleton is enough. Indeed, by recalling Lemma 2.11, we may rely on [7, Proposition 6.6]¹ to arrive at the following theorem:

¹Note that the formulation of [7, Proposition 6.6] has a typo: the proposition should disallow the span of $(1, 0)$, not the span of $(0, 1)$.

Theorem 2.17. *If X is a dominated planar self-affine set satisfying the strong open set condition such that X_F is not a singleton, then*

$$\begin{aligned} \dim_{\mathbb{H}}(X) &= \min\{2, \dim_{\text{aff}}(X)\}, \\ \dim_{\mathbb{H}}(\text{proj}_{V^\perp}(X)) &= \min\{1, \dim_{\text{aff}}(X)\} \end{aligned}$$

for all $V \in \mathbb{RP}^1 \setminus \mathcal{I}$, where $\mathcal{I} = \{W \in \mathbb{RP}^1 : W \text{ is invariant under all the associated matrices}\}$ and contains at most one element.

We remark that Hochman and Rapaport [35] have recently generalized the above results. They showed that the strong open set condition can be replaced by exponential separation, a separation condition which allows overlapping. Our standing assumption is the strong separation condition and therefore Theorems 2.16 and 2.17 suffice for us. We seek more refined information in the setting of Theorems 2.16 and 2.17 analogously to the known results in self-similar sets and Bedford-McMullen carpets. Our first observation in this direction follows immediately from the following lemma.

Lemma 2.18. *If X is a dominated or irreducible planar self-affine set, then $\mathcal{H}^s(X) < \infty$ where $s = \dim_{\text{aff}}(X)$.*

Proof. To prove the first claim, let B be a closed ball containing X and $\mathbf{i} \in \Sigma_*$. To cover the ellipsis $\varphi_{\mathbf{i}}(B)$, we need approximately one ball of radius $\alpha_1(A_{\mathbf{i}})$ or $\alpha_1(A_{\mathbf{i}})/\alpha_2(A_{\mathbf{i}})$ many balls of radius $\alpha_2(A_{\mathbf{i}})$. Thus, by the definitions of the Hausdorff measure and the singular value function, there exists a constant $c > 0$ such that

$$\mathcal{H}^s(X) \leq c \lim_{n \rightarrow \infty} \sum_{\mathbf{i} \in \Sigma_n} \varphi^s(A_{\mathbf{i}}).$$

By Lemma 2.12, there exist a measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ and a constant $C \geq 1$ such that

$$C^{-1} \varphi^s(A_{\mathbf{i}}) \leq \mu_K([\mathbf{i}]) \leq C \varphi^s(A_{\mathbf{i}})$$

for all $\mathbf{i} \in \Sigma_*$. The claim follows. □

3. MAIN RESULTS

Our first result determines the lower dimension of self-affine sets. It will be proved in Section 4. We emphasize that the result does not require X to be dominated.

Theorem 3.1. *If X is a strictly affine strongly irreducible planar self-affine set satisfying the strong separation condition, then $\dim_{\mathbb{L}}(X) = \min\{1, \dim_{\mathbb{H}}(X)\}$.*

We remark that a Bedford-McMullen carpet X with $\dim_{\mathbb{H}}(X) \leq 1$ not having uniform vertical fibers serves as a counter-example for the above result in the reducible case; see Theorem 2.15 and, more precisely, [27, Corollary 2.14]. Recalling Lemma 2.4, it follows from Theorem 3.1 that if $\dim_{\mathbb{H}}(X) > 1$, then X is not Ahlfors regular. As Theorem 2.16 and Lemma 2.18 guarantee finite Hausdorff measure in the dimension, it is natural to ask if, under the assumptions of Theorem 3.1, $s = \dim_{\mathbb{H}}(X) > 1$ implies $\mathcal{H}^s(X) = 0$. Recall that if a Bedford-McMullen carpet is not Ahlfors regular, then, by Theorem 2.15, it has infinite Hausdorff measure in the dimension.

Let us next turn to the Assouad dimension of self-affine sets. The following result is proved in Section 5. It generalizes the result of Bárány, Käenmäki, and Rossi [11, Theorem

3.2] which uses a projection condition, a very restrictive assumption to guarantee that the projection of the self-affine set is a line segment for sufficiently many directions, to overcome several technical difficulties. Recall also the result of Fraser [27, Theorem 2.12] for self-affine carpets.

Theorem 3.2. *If X is a dominated planar self-affine set satisfying the strong separation condition such that $\dim_{\mathbb{H}}(X) \geq 1$ and X_F is not a singleton, then*

$$\dim_{\mathbb{A}}(X) = 1 + \sup_{\substack{x \in X \\ V \in X_F}} \dim_{\mathbb{H}}(X \cap (V + x)) < 2.$$

In the following example, we show that, under the assumptions of Theorem 3.2, it is possible to have $\dim_{\mathbb{L}}(X) < \dim_{\mathbb{H}}(X) = \dim_{\text{aff}}(X) < \dim_{\mathbb{A}}(X)$. This observation answers one of the open questions posed in [28, Question 17.5.2]. The example strongly relies on known results on Bedford-McMullen carpets and it is currently the sole example to demonstrate $\dim_{\text{aff}}(X) < \dim_{\mathbb{A}}(X)$ in the case $\dim_{\mathbb{H}}(X) \geq 1$. If $\dim_{\mathbb{H}}(X) < 1$, then this phenomenon is studied in more detail in Theorem 1.4.

Example 3.3. Let $q > p \geq 2$ and $N \in \{2, \dots, pq\}$ be integers, and $I \subset \{0, \dots, p-1\} \times \{0, \dots, q-1\}$ a set of N elements. Let $A = \text{diag}(\frac{1}{p}, \frac{1}{q})$ and $B_\varepsilon \in GL_2(\mathbb{R})$ be a matrix with positive entries such that $\|B_\varepsilon\| < \varepsilon$ for all $\varepsilon > 0$. It follows that $\varphi^s(B_\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ for all $s \geq 0$. Furthermore, the tuple $(A, B_\varepsilon) \in GL_2(\mathbb{R})^2$ is dominated and irreducible for all $\varepsilon > 0$. Write $\mathbf{A} = (A, \dots, A) \in GL_2(\mathbb{R})^N$ and note that $P(\mathbf{A}, s) = \log(N\varphi^s(A))$. Hence,

$$\dim_{\text{aff}}(\mathbf{A}) = \begin{cases} \frac{\log N}{\log p}, & \text{if } N \in \{2, \dots, p\}, \\ 1 + \frac{\log N/p}{\log q}, & \text{if } N \in \{p+1, \dots, pq\}. \end{cases}$$

Observe also that if $\mathbf{A}_\varepsilon = (A, \dots, A, B_\varepsilon) \in GL_2(\mathbb{R})^{N+1}$, then

$$\begin{aligned} P(\mathbf{A}_\varepsilon, s) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_k \leq n} N^{n-k} \varphi^s(A^{i_1-1} B_\varepsilon A^{i_2-i_1-1} B_\varepsilon \dots B_\varepsilon A^{n-i_k-1}) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_k \leq n} N^{n-k} \varphi^s(A)^{n-k} \varphi^s(B_\varepsilon)^k \\ &= \log(N\varphi^s(A) + \varphi^s(B_\varepsilon)). \end{aligned}$$

Therefore, if $s(\varepsilon)$ is such that $\log(N\varphi^{s(\varepsilon)}(A) + \varphi^{s(\varepsilon)}(B_\varepsilon)) = 0$, we see that $\dim_{\text{aff}}(\mathbf{A}) \leq \dim_{\text{aff}}(\mathbf{A}_\varepsilon) \leq s(\varepsilon)$ for all $\varepsilon > 0$ and $s(\varepsilon) \downarrow \dim_{\text{aff}}(\mathbf{A})$ as $\varepsilon \downarrow 0$.

Recall that the Bedford-McMullen carpet is the self-affine set $X \subset [0, 1]^2$ associated to a tuple $(\varphi_1, \dots, \varphi_N)$ of affine maps which all have the same linear part A and the translation part is from the set $\{(\frac{j}{p}, \frac{k}{q}) \in [0, 1]^2 : (j, k) \in I\}$. We assume that X satisfies the strong separation condition. Write $n_j = \#\{k : (j, k) \in I\}$ to denote the number of sets $\varphi_i([0, 1]^2)$ the vertical line $\{(\frac{j}{p}, y) : y \in \mathbb{R}\}$ intersects. By [47, Theorem 1.1], we have

$$\dim_{\mathbb{A}}(X) = \frac{\log \#\{j \in \{1, \dots, p\} : n_j \neq 0\}}{\log p} + \max_{j \in \{1, \dots, p\}} \frac{\log n_j}{\log q}.$$

For example, if $q = 5$, $p = 4$, and $N = 5$, then, by choosing the translation vectors such that $n_1 = 3$, $n_2 = 0$, and $n_3 = 1 = n_4$, we have $1 < \dim_{\text{aff}}(\mathbf{A}) < \dim_{\mathbb{A}}(X)$. Observe that

several other choices also lead to these strict inequalities. We may now choose $\varepsilon > 0$ such that $s(\varepsilon) < \dim_{\mathbb{A}}(X)$.

Define a contractive affine map $\varphi_{N+1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by setting $\varphi_{N+1}(x) = B_{\varepsilon}x + v$, where $v \in \mathbb{R}^2$ is chosen such that $\varphi_{N+1}([0, 1]^2) \cap \bigcup_{i=1}^N \varphi_i([0, 1]^2) = \emptyset$. Let X' be the dominated irreducible planar self-affine set satisfying the strong separation condition associated to the tuple $(\varphi_1, \dots, \varphi_N, \varphi_{N+1})$. By Theorems 2.16 and 3.1, we have $\dim_{\mathbb{H}}(X') = \dim_{\text{aff}}(\mathbb{A}_{\varepsilon}) \geq \dim_{\text{aff}}(\mathbb{A}) > 1 = \dim_{\mathbb{L}}(X')$. Therefore, as $X \subset X'$,

$$\dim_{\mathbb{L}}(X') < \dim_{\mathbb{H}}(X') = \dim_{\text{aff}}(\mathbb{A}_{\varepsilon}) \leq s(\varepsilon) < \dim_{\mathbb{A}}(X) \leq \dim_{\mathbb{A}}(X')$$

as claimed.

Inspired by Theorem 2.13, we say that a strictly affine planar self-affine set X satisfies a *projective open set condition* if there is $\eta > 0$ such that for every $V \in X_F$ and $\mathbf{i}, \mathbf{j} \in \Sigma_*$ with $\mathbf{i} \neq \mathbf{j}$ there is $x \in X$ such that

$$|\text{proj}_{V^{\perp}}(\varphi_{\mathbf{i}}(x)) - \text{proj}_{V^{\perp}}(\varphi_{\mathbf{j}}(x))| \geq \eta \text{diam}(\text{proj}_{V^{\perp}}(\varphi_{\mathbf{i}}(X))). \quad (3.1)$$

One has to be careful with this definition as, for example, a Bedford-McMullen carpet can be contained in a line parallel to the sole element of X_F in which case the right hand-side in (3.1) is zero. Nevertheless, if X and X_F are not singletons, then there always exists $V \in X_F$ such that $\text{diam}(\text{proj}_{V^{\perp}}(X)) > 0$. In fact, in the proof of Theorem 6.1, we will see that if X is dominated with at least two points and X_F is not a singleton, then the right hand-side in (3.1) is uniformly bounded away from zero. We also define

$$\begin{aligned} \Sigma(V, x, r) = \{ \mathbf{i} \in \Sigma_* : \text{diam}(\text{proj}_{V^{\perp}}(\varphi_{\mathbf{i}}(X))) < r \leq \text{diam}(\text{proj}_{V^{\perp}}(\varphi_{\mathbf{i}^-}(X))) \\ \text{and } \text{proj}_{V^{\perp}}(\varphi_{\mathbf{i}}(X)) \cap B(\text{proj}_{V^{\perp}}(x), r) \neq \emptyset \} \end{aligned} \quad (3.2)$$

for all $V \in X_F$, $x \in X$, and $r > 0$. The projective open set condition is characterized in the following theorem. The result is analogous to Theorems 2.13 and 2.14 in the self-similar case. Its proof consists of Sections 6 and 7.

Theorem 3.4. *If X is a dominated planar self-affine set satisfying the strong separation condition such that X_F is not a singleton and $\dim_{\mathbb{H}}(X) \leq 1$, then the followings conditions are equivalent:*

- (1) X satisfies projective open set condition.
- (2) $\sup\{\#\Sigma(V, x, r) : V \in X_F, x \in X, \text{ and } r > 0\} < \infty$,
- (3) $\mathcal{H}^s(X) > 0$ where $s = \dim_{\mathbb{H}}(X)$,
- (4) $\inf_{V \in X_F} \mathcal{H}_{\infty}^s(\text{proj}_{V^{\perp}}(X)) > 0$ where $s = \dim_{\mathbb{H}}(X)$,
- (5) X is Ahlfors regular,
- (6) $\text{proj}_{V^{\perp}}(X)$ is Ahlfors regular for all $V \in X_F$.

Furthermore, if $\dim_{\mathbb{H}}(X) < 1$, then the following condition can be added to the list:

- (7) $\dim_{\mathbb{L}}(X) = \dim_{\mathbb{H}}(X) = \dim_{\mathbb{A}}(X)$.

If the self-affine set X has positive Hausdorff measure, then Theorem 3.4 guarantees that there are no exact overlaps in the projections onto orthogonal complements of the Furstenberg directions. In other words, if under the assumptions of Theorem 3.4 it holds that $\mathcal{H}^s(X) > 0$ for $s = \dim_{\mathbb{H}}(X)$, then $\text{proj}_{V^{\perp}} \varphi_{\mathbf{i}} \neq \text{proj}_{V^{\perp}} \varphi_{\mathbf{j}}$ for all $V \in X_F$ and $\mathbf{i}, \mathbf{j} \in \Sigma_*$ with $\mathbf{i} \neq \mathbf{j}$.

Theorem 3.5. *If X is a dominated planar self-affine set satisfying the strong separation condition, but not the projective open set condition, such that X_F is not a singleton, then $\dim_{\mathbb{A}}(X) \geq 1$.*

It would be interesting to know if the above theorem can be improved to show $\dim_{\mathbb{A}}(X) > 1$. The theorem would then be analogous to the result of García [34, Theorem 1.4] in the self-similar case; recall also Theorem 2.14.

Our final result shows that, in a topological sense typical self-affine sets satisfy the assumptions of Theorem 3.5. The proof of Theorem 3.6 will be postponed until Section 8. Let $\mathbf{A} = (A_1, \dots, A_N) \in GL_2(\mathbb{R})^N$ and consider the affine iterated function systems $\Phi_{\mathbf{v}} = (A_1 + v_1, \dots, A_N + v_N)$ parametrized by the translation vector $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^2)^N$. Let $\pi_{\mathbf{v}}: \Sigma \rightarrow X_{\mathbf{v}}$ be the associated canonical projection onto the self-affine set $X_{\mathbf{v}}$. If \mathbf{A} is strictly affine, fix $\delta > 0$ and define

$$\begin{aligned} \mathcal{N}(\mathbf{A}) = \{ \mathbf{v} \in (\mathbb{R}^2)^N : X_{\mathbf{v}} \text{ satisfies the strong separation condition} \\ \text{with uniform } \delta > 0 \text{ in (2.12) and there are} \\ V \in X_F \text{ and } \mathbf{i}, \mathbf{j} \in \Sigma \text{ with } \mathbf{i}|_1 \neq \mathbf{j}|_1 \text{ such} \\ \text{that } \text{proj}_{V^\perp}(\pi_{\mathbf{v}}(\mathbf{i})) = \text{proj}_{V^\perp}(\pi_{\mathbf{v}}(\mathbf{j})) \}. \end{aligned} \quad (3.3)$$

Since $\delta > 0$ above is uniform, it is easy to see that $\mathcal{N}(\mathbf{A})$ is complete. Recall that a *residual* set is an intersection of countably many sets with dense interiors.

Theorem 3.6. *If $\mathbf{A} \in GL_2(\mathbb{R})^N$ is strictly affine such that $\max_{i \in \{1, \dots, N\}} \|A_i\| < \frac{1}{2}$, then there exists a residual set $\mathcal{R}(\mathbf{A}) \subset \mathcal{N}(\mathbf{A})$ such that for each $\mathbf{v} \in \mathcal{R}(\mathbf{A})$ the planar self-affine set $X_{\mathbf{v}}$ does not satisfy the projective open set condition.*

Let us next prove Theorems 1.1–1.4 by relying on the above stated results:

Proof of Theorem 1.1. Since X is irreducible and dominated, it is also strongly irreducible; see Lemma 2.10. Therefore, it follows from Theorem 3.1 and Lemma 2.4 that if $s = \dim_{\mathbb{H}}(X) > 1$, then X is not Ahlfors s -regular. Recalling Lemma 2.4, we have thus proven that if X is Ahlfors s -regular, then $0 \leq s \leq 1$ and $0 < \mathcal{H}^s(X) < \infty$. Conversely, as X is strongly irreducible and dominated, Theorem 3.4 shows that $\mathcal{H}^s(X) > 0$ where $s = \dim_{\mathbb{H}}(X) \leq 1$ implies that X is Ahlfors s -regular. \square

Proof of Theorem 1.2. By (1.1), we have $\dim_{\mathbb{A}}(\text{proj}_{V^\perp}(X)) \geq \min\{1, \dim_{\mathbb{A}}(X)\}$ for all $V \in \mathbb{RP}^1 \setminus E$, where the set $E \subset \mathbb{RP}^1$ satisfies $\dim_{\mathbb{H}}(E) = 0$. If $\dim_{\mathbb{A}}(X) \geq 1$, then, as $\dim_{\mathbb{A}}(\text{proj}_{V^\perp}(X)) \leq 1$ for all $V \in \mathbb{RP}^1$, the claim is a direct consequence of this result. We may thus assume that $\dim_{\mathbb{H}}(X) \leq 1$. If X is not Ahlfors regular, then Theorem 3.4 implies that X does not satisfy the projective open set condition. Therefore, by Theorem 3.5, we have $\dim_{\mathbb{A}}(X) \geq 1$ and we have shown the first claim. Furthermore, if X is Ahlfors regular, then, by Theorem 3.4, $\text{proj}_{V^\perp}(X)$ is Ahlfors regular for all $V \in X_F$. Therefore, Lemma 2.4 and the fact that the Hausdorff dimension cannot increase under Lipschitz maps, gives $\dim_{\mathbb{A}}(\text{proj}_{V^\perp}(X)) = \dim_{\mathbb{H}}(\text{proj}_{V^\perp}(X)) \leq \min\{1, \dim_{\mathbb{H}}(X)\} \leq \min\{1, \dim_{\mathbb{A}}(X)\}$ for all $V \in X_F$ and finishes the proof. \square

Proof of Theorem 1.3. This is a direct consequence of Theorem 3.2. \square

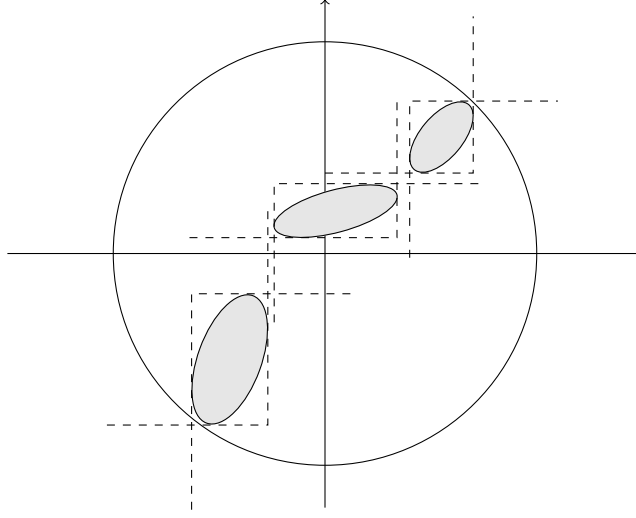


FIGURE 1. Construction of dominated Ahlfors regular self-affine sets.

Proof of Theorem 1.4. Let $A \in GL_2(\mathbb{R})^N$ be dominated such that $\max_{i \in \{1, \dots, N\}} \|A_i\| < \frac{1}{2}$, X_F is not a singleton, and $\dim_{\text{aff}}(A) < 1$. Let $v \in (\mathbb{R}^2)^N$ be a translation vector such that X_v satisfies the strong separation condition. If X_v satisfies the projective open set condition, then, by Theorem 3.4, X_v is Ahlfors regular. On the other hand, if X_v does not satisfy the projective open set condition, then, by Theorem 3.5, $\dim_{\text{aff}}(A) < 1 \leq \dim_A(X_v)$. By Theorem 3.6, there is a residual set $\mathcal{R}(A) \subset \mathcal{N}(A)$ such that for every $v \in \mathcal{R}(A)$ the associated planar self-affine set X_v satisfies the strong separation condition, but not the projective open set condition. Therefore, $\dim_{\text{aff}}(A) < 1 \leq \dim_A(X_v)$ for all $v \in \mathcal{R}(A)$. As $\mathcal{R}(A)$ is residual, we see that $\mathcal{R}(A) = \bigcap_{q \in \mathbb{N}} \mathcal{R}_q(A)$, where $\mathcal{R}_q(A)^\circ$ is dense in $\mathcal{N}(A)$. If $\mathcal{R}(A)$ is countable, say $\mathcal{R}(A) = \{v_1, v_2, \dots\}$, then we have $\bigcap_{q, k \in \mathbb{N}} \mathcal{R}_q(A)^\circ \cap (\mathcal{N}(A) \setminus \{v_k\}) = \emptyset$ contradicting the Baire category theorem. Therefore, $\mathcal{R}(A)$ is uncountable.

It remains to show that such self-affine sets with projective open set condition have non-empty interior in the set of parameters formed by the translation vectors and elements of the matrices. Fix a dominated tuple $A = (A_1, \dots, A_N) \in GL_2(\mathbb{R})^N$ and let $\mathcal{C} \subset \mathbb{RP}^1$ be a strongly invariant multicone for A . Let us choose the tuple $v = (v_1, \dots, v_N)$ of translation vectors such that $v_i \in B(0, 1)^\circ$ and $\text{span}(v_i - v_j) \in \mathcal{C}^\circ$ whenever $i \neq j$. Thus, by choosing sufficiently small positive constants c_1, \dots, c_N and defining $\varphi_i: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by setting $\varphi_i(x) = c_i A_i x + v_i$, we have $\varphi_i(B(0, 1)) \subset B(0, 1)^\circ$ for all $i \in \{1, \dots, N\}$ and $\varphi_i(x) - \varphi_j(y) \in \mathcal{C}^\circ$ for all $x, y \in B(0, 1)$ whenever $i \neq j$. We consider the self-affine set associated to $(\varphi_1, \dots, \varphi_N)$. It is evident that the mentioned properties are open in the set of parameters. Moreover, since $X_F \subset \mathbb{RP}^1 \setminus \mathcal{C}$ by compactness, there exists a constant $\eta > 0$ such that

$$|\text{proj}_{V^\perp}(\varphi_i(x)) - \text{proj}_{V^\perp}(\varphi_j(y))| \geq \eta$$

for all $V \in X_F$, $i, j \in \{1, \dots, N\}$ with $i \neq j$, and $x, y \in X$. The projective open set condition (3.1) follows. For illustration in the case where all the matrices have positive entries, see Figure 1. \square

Let us finish this section by formulating precise open problems raising from our work. Recall that in the introduction, we presented two slightly vague questions for motivation. The first question asked whether it is possible to characterize when the Hausdorff measure of a self-affine set is positive. If X is a dominated irreducible planar self-affine set satisfying the strong separation condition such that $\dim_{\mathbb{H}}(X) \leq 1$, then Theorem 3.4 offers several characterizations. On the other hand, if $s = \dim_{\mathbb{H}}(X) > 1$, then Lemma 2.10, Theorem 3.1, and Lemma 2.4 imply that X is not Ahlfors regular. As Theorem 2.16 and Lemma 2.18 show that $\mathcal{H}^s(X) < \infty$, the following question is natural:

Question 3.7. Let X be a strictly affine strongly irreducible self-affine set satisfying the strong separation condition such that $s = \dim_{\mathbb{H}}(X) > 1$. Is it possible to have $\mathcal{H}^s(X) > 0$?

We remark that Przytycki and Urbański [55, Section 6 and Remark 13] have shown that the reducible planar self-affine set X associated to $((x, y) \mapsto (x/2, \lambda y + 1), (x, y) \mapsto ((x + 1)/2, \lambda y - 1))$ where $\lambda > 1/2$ has $\mathcal{H}^s(X) > 0$ for $s = \dim_{\mathbb{H}}(X) > 1$ if the canonical projection of the equidistributed Bernoulli measure onto the orthogonal complement of the sole element in X_F is absolutely continuous and has L^∞ -density.

The second question posed in the introduction asked whether all the slices of self-affine sets are small. If X is a dominated irreducible planar self-affine set satisfying the strong separation condition such that $\dim_{\mathbb{H}}(X) \geq 1$, then Theorem 3.2 shows that all the slices parallel to Furstenberg directions are small in terms of the Assouad dimension. It also shows that the largest such slice has Hausdorff dimension close to the Assouad dimension minus one. Therefore, one cannot hope to obtain the same upper bound as in the Marstrand's slicing theorem for all slices. Indeed, Example 3.3 serves as a counter-example for this. But since the example strongly relies on the rigid structure of Bedford-McMullen carpets, a class of self-affine sets which can be considered exceptional in several ways, the following question is natural:

Question 3.8. For a “typical” strictly affine strongly irreducible self-affine set X satisfying the strong separation condition, is it true that

$$\sup_{\substack{x \in X \\ V \in \mathbb{R}\mathbb{P}^1}} \dim_{\mathbb{H}}(X \cap (V + x)) \leq \max\{0, \dim_{\mathbb{H}}(X) - 1\}?$$

In particular, is it true that $\dim_{\Lambda}(X) = \dim_{\mathbb{H}}(X)$ for a “typical” self-affine set X with $\dim_{\mathbb{H}}(X) \geq 1$?

We speculate that to address the above question, one is obliged to further develop the theory of scenery flows introduced by Furstenberg [33] for self-affine sets and possibly strengthen it by the new methods introduced by Hochman and Rapaport [35]. We expect that our present work and the machinery we have developed serves as an intrinsic groundwork in this.

García [34, Theorem 1.4] proved that if a planar self-similar set X not satisfying the weak separation condition is not contained in a line, then $\dim_{\Lambda}(X) > 1$. Theorem 3.5 shows that if X is a dominated planar self-affine set satisfying the strong separation condition, but not the projective open set condition, such that X_F is not a singleton, then $\dim_{\Lambda}(X) \geq 1$. Note that if X_F is not a singleton, then X is not contained in a line. Therefore, the following question is natural:

Question 3.9. Let X be a dominated planar self-affine set satisfying the strong separation condition, but not the projective open set condition, such that X_F is not a singleton and $\dim_{\mathbb{H}}(X) \leq 1$. Is $\dim_{\mathbb{A}}(X) > 1$?

If for “typical” self-affine sets it is possible to obtain a positive answer in Question 3.9, then [11, Proposition 2.4 and Theorem 5.2] imply a negative answer to Question 3.8.

4. LOWER DIMENSION OF SELF-AFFINE SETS

In this section, we prove Theorem 3.1. The proof relies on analysis on weak tangent sets and it is worth emphasizing that it does not require domination. The upper bound is done in Proposition 4.4 below. The main idea is to find more and more narrow parts of the self-affine set which eventually result in a weak tangent set contained in a line. Proposition 4.6 below gives the lower bound. Since, by Theorem 2.16, the dimension of a self-affine set is preserved under projections, the task there is to compare weak tangent sets to projections. Let us denote the convex hull of a set $A \subset \mathbb{R}^2$ by $\text{conv}(A)$ and the boundary of A by ∂A . We say that a set $A \subset \mathbb{R}^2$ has *positive length* if $\mathcal{H}^1(A) > 0$.

Lemma 4.1. *If X is a planar self-affine set satisfying the strong separation condition, then $\partial \text{conv}(X)$ contains at most countably many line segments with positive length.*

Proof. Write $S(V, t) = (V + t) \cap \partial \text{conv}(X)$ for all $t \in V^\perp$ and $V \in \mathbb{RP}^1$. Let

$$\mathcal{I} = \{(V, t) \in \mathbb{RP}^1 \times \mathbb{R}^2 : \mathcal{H}^1(S(V, t)) > 0 \text{ and } t \in V^\perp\} \quad (4.1)$$

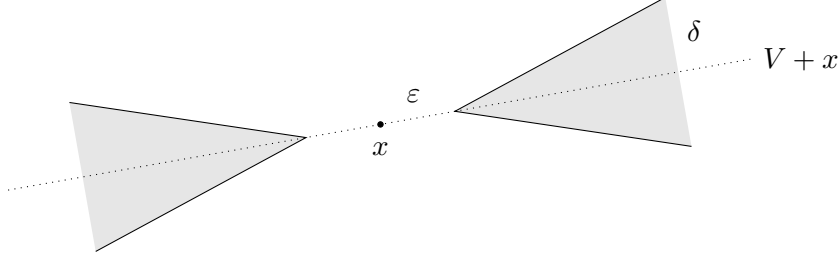
and observe that if $(V, t) \in \mathcal{I}$, then, by the convexity of $\text{conv}(X)$, $S(V, t)$ is a proper line segment in $\partial \text{conv}(X)$. Let us show that \mathcal{I} is a countable set. If $W \in \mathbb{RP}^1$ is the x -axis, then, again by the convexity of $\text{conv}(X)$, it is easy to see that there are at most two points $t_1, t_2 \in W$ such that $(W^\perp, t_1) \in \mathcal{I}$ and $(W^\perp, t_2) \in \mathcal{I}$. It is thus enough to show that $\mathcal{I}' = \{(V, t) \in \mathcal{I} : V \neq W^\perp\}$ is a countable set.

Observe that $\text{proj}_W(S(V, t))$ has positive length for all $(V, t) \in \mathcal{I}'$. Thus, for every $(V, t) \in \mathcal{I}'$ there exists $q \in \mathbb{Q}$ such that $q \in \text{proj}_W(S(V, t))$. Moreover, using the convexity of $\text{conv}(X)$ and the definition of \mathcal{I}' once more, for every $q \in \mathbb{Q}$ there are at most two $(V_1, t_1), (V_2, t_2) \in \mathcal{I}'$ such that $q \in \text{proj}_W(S(V_j, t_j))$ for both $j \in \{1, 2\}$. Hence, $\#\mathcal{I}'$ is indeed at most countable. \square

Lemma 4.2. *If X is a planar self-affine set satisfying the strong separation condition, then for every except possibly countably many $V \in \mathbb{RP}^1$ there exists $y \in X$ such that $(V + y) \cap X = \{y\}$ and $X \setminus \{y\}$ is contained in one of the open half-planes defined by $V + y$.*

Proof. Let $\mathcal{S} = \{V \in \mathbb{RP}^1 : (V + t) \cap \partial \text{conv}(X) \text{ has positive length for some } t \in V^\perp\}$ and notice that \mathcal{S} is at most countable by Lemma 4.1. Fix $V \in \mathbb{RP}^1 \setminus \mathcal{S}$ and let $v \in V^\perp$ be such that $|v| = 1$. Since the set $\text{proj}_{V^\perp}(X)$ is compact, there exist unique $t_1, t_2 \in \mathbb{R}$ such that $\text{proj}_{V^\perp}(X) \subset \{tv : t_1 \leq t \leq t_2\}$ and $t_1v, t_2v \in \text{proj}_{V^\perp}(X)$. In particular, $\{tv : t_1 \leq t \leq t_2\} = \text{proj}_{V^\perp}(\text{conv}(X))$. To finish the proof, it suffices to show that $\text{proj}_{V^\perp}^{-1}(t_1v) \cap X$ is a singleton.

Suppose to the contrary that there are $x_1, x_2 \in \text{proj}_{V^\perp}^{-1}(t_1v) \cap X$ such that $x_1 \neq x_2$. But then $x_1, x_2 \in \partial \text{conv}(X)$ and in particular, the line segment connecting x_1, x_2 must be also

FIGURE 2. Illustration for the set $H(x, V, \delta, \varepsilon)$.

contained in $\partial \text{conv}(X)$. Since this line segment is parallel to V , this is a contradiction as $V \notin \mathcal{S}$. \square

For $x \in \mathbb{R}^2$, $V \in \mathbb{RP}^1$, $v \in V$ such that $|v| = 1$, and $0 \leq \delta \leq 1$, we set

$$C(x, v, \delta) = \{y \in \mathbb{R}^2 : (y - x) \cdot v < \sqrt{1 - \delta^2}|y - x|\},$$

$$H(x, V, \delta, \varepsilon) = C(x + \varepsilon v, v, \delta) \cup C(x - \varepsilon v, -v, \delta).$$

For illustration, see Figure 2. Observe that $\bigcap_{\delta > 0} H(x, V, \delta, \varepsilon) = (V + x) \setminus B(x, \varepsilon)$.

Lemma 4.3. *If X is a planar self-affine set satisfying the strong separation condition and $\varepsilon > 0$, then for every except possibly countably many $V \in \mathbb{RP}^1$ there exist $y \in X$ and $\delta > 0$ such that $X \subset \mathbb{R}^2 \setminus H(y, V, \delta, \varepsilon)$.*

Proof. Let \mathcal{S} be the at most countable subset of \mathbb{RP}^1 for which the conclusion of Lemma 4.2 fails. Let us argue by contradiction that there exist $\varepsilon > 0$ and $V \in \mathbb{RP}^1 \setminus \mathcal{S}$ such that for every $y \in X$ and $\delta > 0$ it holds that $X \cap H(y, V, \delta, \varepsilon) \neq \emptyset$. Let $y \in X$ be the point given by Lemma 4.2 such that $(V + y) \cap X = \{y\}$. For each $n \in \mathbb{N}$ let us now choose a point $x_n \in X \cap H(y, V, \frac{1}{n}, \varepsilon)$. By the compactness of X , going into a subsequence if required, the sequence $(x_n)_{n \in \mathbb{N}}$ converges to a point $x \in X$. As the sequence of sets $(H(y, V, \frac{1}{n}, \varepsilon))_{n \in \mathbb{N}}$ is decreasing, we must have $x \in \bigcap_{n \in \mathbb{N}} H(y, V, \frac{1}{n}, \varepsilon) = (V + y) \setminus B(y, \varepsilon)$. Therefore, $y \neq x \in (V + y) \cap X$ which is a contradiction. \square

We remark that in general it is not possible to choose $\varepsilon = 0$ above as a planar self-affine set can be contained for example in a parabola; see [5, 26]. The following proposition gives the upper bound in Theorem 3.1.

Proposition 4.4. *If X is a strictly affine strongly irreducible planar self-affine set satisfying the strong separation condition, then $\dim_L(X) \leq \min\{1, \dim_H(X)\}$.*

Proof. Since $\dim_L(X) \leq \dim_H(X)$, it is enough to show that $\dim_L(X) \leq 1$. Therefore, by Lemma 2.2, it suffices to show that there exists a weak tangent set T such that $\dim_H(T) \leq 1$. Let $\mu \in \mathcal{M}_\sigma(\Sigma)$ be a fully supported Bernoulli measure obtained from a probability vector (p_1, \dots, p_N) and μ_F be the associated Furstenberg measure; see (2.2).

Let $\mathcal{S} \subset \mathbb{RP}^1$ be the at most countable exceptional set of Lemma 4.3. By Lemma 2.6 and (2.3), for μ -almost every $i \in \Sigma$ there exists $V \in X_F \setminus \mathcal{S}$ such that

$$(A_{i|n}^\top)_* \mu_F^\perp \rightarrow \delta_{V^\perp} \quad (4.2)$$

in the weak* topology as $n \rightarrow \infty$. Fix $\mathbf{i} \in \Sigma$ and $V \in \mathbb{RP}^1 \setminus \mathcal{S}$ such that the above convergence holds and let $\varepsilon > 0$. By Lemma 4.3, there exist $y \in X$ and $\delta > 0$ such that

$$X \subset \mathbb{R}^2 \setminus H(y, V, \delta, \varepsilon).$$

Relying on the strong separation condition, let $d > 0$ be as in (2.12) and write

$$T_n = M_{\varphi_{\mathbf{i}|n}(y), \|A_{\mathbf{i}|n}^{-1}V\|d}(X) \cap B(0, d) \subset \frac{1}{\|A_{\mathbf{i}|n}^{-1}V\|d} A_{\mathbf{i}|n}^{-1}(\mathbb{R}^2 \setminus H(0, V, \delta, \varepsilon)) \quad (4.3)$$

for all $n \in \mathbb{N}$. Let $W_1, W_2 \in \mathbb{RP}^1 \setminus \{V\}$ be parallel to the lines appearing in $\partial H(0, V, \delta, \varepsilon)$. By going into a sub-sequence, if necessary, we see that there is $T \in \text{Tan}(X)$ such that $T_n \rightarrow T$ in Hausdorff distance. By (4.2) and Lemma 2.5, there is $W \in \mathbb{RP}^1$ such that, by possibly going into a sub-sequence once more, $A_{\mathbf{i}|n}^{-1}W_1 \rightarrow W$ and $A_{\mathbf{i}|n}^{-1}W_2 \rightarrow W$ as $n \rightarrow \infty$. Therefore, by (4.3), T is contained in an $\frac{2\varepsilon}{d}$ -neighbourhood of W . Letting $\varepsilon \downarrow 0$, it follows that T is contained in a line and $\dim_{\mathbb{H}}(T) \leq 1$ as required. \square

By a rank of an affine map, we mean the rank of its linear part. We use the same convention also for the kernel and image. Let us state a useful lemma which is not stated explicitly but is contained in the proof of [11, Theorem 5.2].

Lemma 4.5. *If X is a planar self-affine set satisfying the strong separation condition, then for every $T \in \text{Tan}(X)$ there exist affine maps $G_1, G_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ having rank at least one such that $G_1(X) \subset T$ and $G_2(T) \subset X$.*

Proof. Let $(\mathbf{i}_k)_{k \in \mathbb{N}}$ be a sequence of infinite words in Σ and $(r_k)_{k \in \mathbb{N}}$ be a sequence of positive real numbers converging to zero such that $M_{\pi \mathbf{i}_k, r_k}(X) \cap B(0, 1) \rightarrow T$ in Hausdorff distance. For each $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that $\alpha_1(A_{\mathbf{i}_k|n_k}) \leq r_k < \alpha_1(A_{\mathbf{i}_k|n_k-1})$. The affine map G_1 is now obtained as an accumulation point of the sequence $M_{\pi \mathbf{i}_k, r_k} \circ \varphi_{\mathbf{i}_k|n_k}$. Similarly, for each $k \in \mathbb{N}$, choose $m_k \in \mathbb{N}$ such that $\alpha_2(A_{\mathbf{i}_k|m_k}) \leq r_k < \alpha_2(A_{\mathbf{i}_k|m_k-1})$. The affine map G_2 is now obtained as an accumulation point of the sequence $\varphi_{\mathbf{i}_k|m_k}^{-1} \circ M_{\pi \mathbf{i}_k, r_k}^{-1}$. For more details, consult the proof of [11, Theorem 5.2]. \square

Finally, the following proposition gives the lower bound in Theorem 3.1.

Proposition 4.6. *If X is a strongly irreducible planar self-affine set satisfying the strong separation condition, then $\dim_{\mathbb{L}}(X) \geq \min\{1, \dim_{\mathbb{H}}(X)\}$.*

Proof. By Theorem 2.16 and Lemma 2.2, it is enough to show that for every $T \in \text{Tan}(X)$ there exists $V \in \mathbb{RP}^1$ such that

$$\dim_{\mathbb{H}}(T) \geq \dim_{\mathbb{H}}(\text{proj}_{V^\perp}(X)).$$

To that end, fix $T \in \text{Tan}(X)$. By Lemma 4.5, there exists an affine map $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\text{rank}(G) \geq 1$ such that $G(X) \subset T$. If $\text{rank}(G) = 2$, then $\dim_{\mathbb{H}}(T) \geq \dim_{\mathbb{H}}(X) \geq \dim_{\mathbb{H}}(\text{proj}_{V^\perp}(X))$ for any $V \in \mathbb{RP}^1$. If $\text{rank}(G) = 1$, then the linear part of G is a projection as described in (2.8). In particular, $G(X)$ and $\text{proj}_{\ker(G)^\perp}(X)$ are bi-Lipschitz equivalent. Thus, if $V = \ker(G)$, then $\dim_{\mathbb{H}}(T) \geq \dim_{\mathbb{H}}(G(X)) = \dim_{\mathbb{H}}(\text{proj}_{V^\perp}(X))$ also in this case. \square

5. ASSOUD DIMENSION OF SELF-AFFINE SETS HAVING LARGE HAUSDORFF DIMENSION

In this section, we prove Theorem 3.2. We begin by an auxiliary lemma which shows that all the slices of X have dimension strictly smaller than one.

Lemma 5.1. *If X is a planar self-affine set satisfying the strong separation condition, then*

$$\sup_{\substack{x \in X \\ V \in \mathbb{RP}^1}} \dim_{\mathbb{H}}(X \cap (V + x)) < 1.$$

Proof. Relying on the strong separation condition, let $\delta > 0$ be as in (2.12) and U be the open $\frac{\delta}{3}$ -neighbourhood of X . Note that, as X is compact, U has finitely many connected components. Define

$$\Sigma_n(V, x) = \{\mathbf{i} \in \Sigma_n : \varphi_{\mathbf{i}}(X) \cap (V + x) \neq \emptyset\}$$

for all $x \in X$, $V \in \mathbb{RP}^1$, and $n \in \mathbb{N}$. Note that $\varphi_{\mathbf{i}}(U) \cap (V + x)$ consists of line segments where the number of line segments is bounded above by the number of the connected components of U for all $\mathbf{i} \in \Sigma_n(V, x)$. Fix $x \in X$ and $V \in \mathbb{RP}^1$. It is easy to see that, by the strong separation condition,

$$\mathcal{H}^1(\varphi_{\mathbf{i}}(U) \cap (V + x)) \geq \frac{2\delta \|A_{\mathbf{i}}|A_{\mathbf{i}}^{-1}V\|}{3} = \frac{2\delta}{3\|A_{\mathbf{i}}^{-1}V\|} \quad (5.1)$$

and

$$\mathcal{H}^1(\varphi_{\mathbf{i}}(U) \cap (V + x)) \leq \text{diam}(\varphi_{\mathbf{i}}(U) \cap (V + x)) \leq \frac{3 \text{diam}(X) + 2\delta}{3\|A_{\mathbf{i}}^{-1}V\|} \quad (5.2)$$

for all $\mathbf{i} \in \Sigma_n(V, x)$. Note that (5.1) and (5.2) together give

$$\mathcal{H}^1(\varphi_{\mathbf{i}}(U) \cap (V + x)) \geq \frac{2\delta}{3 \text{diam}(X) + 2\delta} \text{diam}(\varphi_{\mathbf{i}}(U) \cap (V + x)) \quad (5.3)$$

for all $\mathbf{i} \in \Sigma_n(V, x)$.

Write $M_{\mathbf{i}} = \#\{j \in \{1, \dots, N\} : \mathbf{i}j \in \Sigma_{n+1}(V, x)\} \geq 1$ for all $\mathbf{i} \in \Sigma_n(V, x)$ and notice that, by the strong separation condition and (5.2),

$$\begin{aligned} \sum_{\mathbf{i}j \in \Sigma_{n+1}(V, x)} \mathcal{H}^1(\varphi_{\mathbf{i}j}(U) \cap (V + x)) &\leq \mathcal{H}^1(\varphi_{\mathbf{i}}(U) \cap (V + x)) - \frac{(M_{\mathbf{i}} - 1)\delta}{3\|A_{\mathbf{i}}^{-1}V\|} \\ &\leq \left(1 - \frac{(M_{\mathbf{i}} - 1)\delta}{3 \text{diam}(X) + 2\delta}\right) \mathcal{H}^1(\varphi_{\mathbf{i}}(U) \cap (V + x)) \end{aligned}$$

for all $\mathbf{i} \in \Sigma_n(V, x)$. Observe that

$$0 < (M_{\mathbf{i}} - 1)\delta \leq \text{diam}(X) + 2\delta < 3 \text{diam}(X) + 2\delta.$$

Therefore, by Hölder's inequality, we have

$$\begin{aligned}
& \sum_{ij \in \Sigma_{n+1}(V,x)} \mathcal{H}^1(\varphi_{ij}(U) \cap (V+x))^s \\
& \leq M_i^{1-s} \left(\sum_{ij \in \Sigma_{n+1}(V,x)} \mathcal{H}^1(\varphi_{ij}(U) \cap (V+x)) \right)^s \\
& \leq M_i^{1-s} \left(1 - \frac{(M_i - 1)\delta}{3 \operatorname{diam}(X) + 2\delta} \right)^s \mathcal{H}^1(\varphi_i(U) \cap (V+x))^s
\end{aligned} \tag{5.4}$$

for all $i \in \Sigma_n(V,x)$ and $0 \leq s \leq 1$. For each $M \in \{1, \dots, N\}$ define a function $f_M: [0, 1] \rightarrow \mathbb{R}$ by setting

$$f_M(s) = M^{1-s} \left(1 - \frac{(M-1)\delta}{3 \operatorname{diam}(X) + 2\delta} \right)^s.$$

If $M = 1$, then $f_M \equiv 1$. If $M \geq 2$, then it is easy to see that f_M is continuous and strictly decreasing with $f_M(0) = M \geq 2$ and $f_M(1) < 1$. Hence for every $M \in \{2, \dots, N\}$ there exists a unique $0 < s_M < 1$ such that $f_M(s_M) = 1$.

Choosing now $\max_{M \in \{2, \dots, N\}} s_M \leq s < 1$, the estimate (5.4) gives

$$\sum_{ij \in \Sigma_{n+1}(V,x)} \mathcal{H}^1(\varphi_{ij}(U) \cap (V+x))^s \leq \mathcal{H}^1(\varphi_i(U) \cap (V+x))^s.$$

Thus, by induction,

$$\begin{aligned}
& \sum_{j \in \Sigma_{n+1}(V,x)} \mathcal{H}^1(\varphi_j(U) \cap (V+x))^s = \sum_{i \in \Sigma_n(V,x)} \sum_{ij \in \Sigma_{n+1}(V,x)} \mathcal{H}^1(\varphi_{ij}(U) \cap (V+x))^s \\
& \leq \sum_{i \in \Sigma_n(V,x)} \mathcal{H}^1(\varphi_i(U) \cap (V+x))^s \leq \dots \leq \sum_{i \in \Sigma_1(V,x)} \mathcal{H}^1(\varphi_i(U) \cap (V+x))^s.
\end{aligned} \tag{5.5}$$

So, by considering the natural cover $\{\varphi_i(U) \cap (V+x)\}_{i \in \Sigma_n(V,x)}$ of $X \cap (V+x)$, the estimates (5.3) and (5.5) yield

$$\begin{aligned}
\mathcal{H}^s(X \cap (V+x)) & \leq \lim_{n \rightarrow \infty} \sum_{i \in \Sigma_n(V,x)} \operatorname{diam}(\varphi_i(U) \cap (V+x))^s \\
& \leq \left(1 + \frac{3 \operatorname{diam}(X)}{2\delta} \right)^s \lim_{n \rightarrow \infty} \sum_{i \in \Sigma_n(V,x)} \mathcal{H}^1(\varphi_i(U) \cap (V+x))^s \\
& \leq \left(1 + \frac{3 \operatorname{diam}(X)}{2\delta} \right)^s \sum_{i \in \Sigma_1(V,x)} \mathcal{H}^1(\varphi_i(U) \cap (V+x))^s < \infty.
\end{aligned}$$

This implies $\dim_{\mathbb{H}}(X \cap (V+x)) \leq s < 1$. As the upper bound s for the dimension does not depend on $x \in X$ or $V \in \mathbb{R}\mathbb{P}^1$, we have finished the proof. \square

The following lemma examines the slices of weak tangent sets. This allows us to determine the dimension of the set.

Lemma 5.2. *If X is a dominated planar self-affine set satisfying the strong separation condition such that $\dim_{\mathbb{H}}(X) \geq 1$ and X_F is not a singleton, then for every $x \in X$ and $V \in X_F$ there exist $T \in \text{Tan}(X)$ and $W, L \in \mathbb{RP}^1$ with $W \neq L$ such that*

$$\begin{aligned} \dim_{\mathbb{H}}(T \cap L) &\geq \dim_{\mathbb{H}}(X \cap (V + x)), \\ \dim_{\mathbb{H}}(T \cap (W + y)) &= 1 \end{aligned}$$

for all $y \in T$.

Proof. Let us first define the weak tangent set $T \in \text{Tan}(X)$. Let $x \in X$ and $V \in X_F$. By (2.6), there exists $\mathbf{i} \in \Sigma$ such that $\Pi(\mathbf{i}) = V$. Write

$$T_k = M_{\varphi_{\mathbf{i}|k}^{\leftarrow}(x), \|A_{\mathbf{i}|k}^{\leftarrow}|V\|} \circ \varphi_{\mathbf{i}|k}^{\leftarrow}(X) \cap B(0, 1)$$

for all $k \in \mathbb{N}$. By going into a sub-sequence n_k , if necessary, we see that there is $T \in \text{Tan}(X)$ such that $T_{n_k} \rightarrow T$ in Hausdorff distance.

Let us then show that there exists $L \in \mathbb{RP}^1$ such that $X \cap (V + x) \subset T \cap L$ and for every $y \in T$ there exists $W \in \mathbb{RP}^1$ such that $W \neq L$ and $\dim_{\mathbb{H}}(T \cap (W + y)) = 1$. Fix $y \in T$ and note that, possibly passing through a sub-sequence, there exists a sequence $(\mathbf{j}_k)_{k \in \mathbb{N}}$ of words in Σ such that $\mathbf{j}_k \in [\mathbf{i}|_{n_k}]$ for all $k \in \mathbb{N}$ and $M_{\varphi_{\mathbf{i}|n_k}^{\leftarrow}(x), \|A_{\mathbf{i}|n_k}^{\leftarrow}|V\|}(\pi(\mathbf{j}_k)) \rightarrow y$. Let $m_k \geq n_k$ be the unique integer such that

$$\alpha_1(A_{\mathbf{j}_k|m_k}) \leq \|A_{\mathbf{i}|n_k}^{\leftarrow}|V\| \leq \alpha_1(A_{\mathbf{j}_k|m_{k-1}}).$$

Then, possibly passing again through a sub-sequence, there exist $L \in \mathbb{RP}^1$ and an affine map P such that

$$\begin{aligned} M_{\varphi_{\mathbf{i}|n_k}^{\leftarrow}(x), \|A_{\mathbf{i}|n_k}^{\leftarrow}|V\|} \circ \varphi_{\mathbf{i}|n_k}^{\leftarrow}(V + x) &= A_{\mathbf{i}|n_k}^{\leftarrow}V \rightarrow L, \\ M_{\varphi_{\mathbf{i}|n_k}^{\leftarrow}(x), \|A_{\mathbf{i}|n_k}^{\leftarrow}|V\|} \circ \varphi_{\mathbf{j}_k|m_k} &\rightarrow P. \end{aligned} \quad (5.6)$$

Since $V \in X_F$, we have $L \in X_F$. Also, by compactness, we see that $X \cap (V + x) \subset T \cap L$ and $y \in P(X) \subset T$. Let us show that $\text{rank}(P) = 1$. Observe that

$$\frac{\alpha_2(A_{\mathbf{j}_k|m_k})}{\alpha_1(A_{\mathbf{j}_k|m_k})} \leq \frac{\|A_{\mathbf{i}|n_k}^{\leftarrow}|V\| \alpha_1(A_{\sigma^{n_k} \mathbf{j}_k|m_k})}{\alpha_1(A_{\mathbf{j}_k|m_k})} \leq \left(\min_{i \in \{1, \dots, N\}} \alpha_2(A_i) \right)^{-1} \alpha_1(A_{\sigma^{n_k} \mathbf{j}_k|m_k}). \quad (5.7)$$

If the sequence $(|\sigma^{n_k} \mathbf{j}_k|m_k|)_{k \in \mathbb{N}}$ of natural numbers was bounded by some $K \in \mathbb{N}$, then

$$\begin{aligned} \alpha_1(A_{\mathbf{i}|n_k}^{\leftarrow}) \min_i \alpha_2(A_i)^K &\leq \alpha_1(A_{\mathbf{i}|n_k}^{\leftarrow}) \alpha_2(A_{\sigma^{n_k} \mathbf{j}_k|m_k}) \leq \alpha_1(A_{\mathbf{i}|n_k}^{\leftarrow}) \alpha_2(A_{\sigma^{n_k} \mathbf{j}_k|m_k}) \\ &= \alpha_1(A_{\mathbf{j}_k|m_k}) \leq \|A_{\mathbf{i}|n_k}^{\leftarrow}|V\| \leq D \alpha_2(A_{\mathbf{i}|n_k}^{\leftarrow}), \end{aligned}$$

where the last inequality follows from Lemma 2.8. As this contradicts with (2.4), the sequence $(|\sigma^{n_k} \mathbf{j}_k|m_k|)_{k \in \mathbb{N}}$ must be unbounded. Therefore, it follows from (5.7) that

$$\frac{\alpha_2(A_{\mathbf{j}_k|m_k})}{\alpha_1(A_{\mathbf{j}_k|m_k})} \rightarrow 0$$

as $k \rightarrow \infty$. Hence $\det(\|A_{\mathbf{i}|n_k}^{\leftarrow}|V\|^{-1} A_{\mathbf{j}_k|m_k}) \rightarrow 0$ as $k \rightarrow \infty$ and $\text{rank}(P) = 1$. By Lemma 2.9 and (2.8), we see that the linear part of P is a constant times $\text{proj}_{\text{im}(P)}^{\ker(P)}$. Let us choose $W = \text{im}(P)$ and show that $W \neq L$ and $\dim_{\mathbb{H}}(T \cap (W + y)) = 1$. Relying on

domination, let $\mathcal{C} \subset \mathbb{RP}^1$ be a strongly invariant multicone. Fix $Q \in \mathcal{C}$ and notice that, as $A_{j_k|_{m_k}} \mathcal{C} \subset \mathcal{C}^o$ for all $k \in \mathbb{N}$, by going into a sub-sequence, if necessary, $A_{j_k|_{m_k}} Q \rightarrow \text{im}(P) \in \mathcal{C}^o$. Since, by (2.5), $\mathcal{C} \cap X_F = \emptyset$, we see that $W \neq L$. Furthermore, since $P(X)$ and $\text{proj}_{\ker(P)^\perp}(X)$ are bi-Lipschitz equivalent by (2.8), the assumption $\dim_{\text{H}}(X) \geq 1$, Lemma 2.10, and Theorem 2.17 give $\dim_{\text{H}}(T \cap (W + y)) \geq \dim_{\text{H}}(P(X) \cap (W + y)) = \dim_{\text{H}}(\text{proj}_{\ker(P)^\perp}(X)) = 1$.

To finish the proof, let us show that $W \in \mathbb{RP}^1$ does not depend on the choice of $y \in T$. Suppose to the contrary that there exist $y_1, y_2 \in T$ such that the associated lines $W_1, W_2 \in \mathbb{RP}^1$ satisfy $W_1 \neq W_2$. Let P_1 and P_2 be the affine maps associated to y_1 and y_2 defined in (5.6), respectively. By Lemma 4.5, there exists an affine map $G: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\text{rank}(G) \geq 1$ and $G(T) \subset X$. If $\text{rank}(G) = 2$, then

$$\begin{aligned} \dim_{\text{H}}(X \cap (GW_i + Gy_i)) &\geq \dim_{\text{H}}(GP_i(X) \cap (GW_i + Gy_i)) \\ &= \dim_{\text{H}}(G \text{proj}_{W_i}(X)) = 1 \end{aligned} \quad (5.8)$$

for both $i \in \{1, 2\}$. If $\text{rank}(G) = 1$, then G is bi-Lipschitz equivalent with $\text{proj}_{\ker(G)^\perp}$. If $i \in \{1, 2\}$ is such that $W_i \neq \ker(G)$, then (5.8) holds for this i . In any case, there exists a slice of X with dimension one, which is impossible by Lemma 5.1. \square

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. It suffices to prove the lower bound for $\dim_{\text{A}}(X)$ as the upper bound follows from [11, Proposition 2.4 and Theorem 5.2]. The fact that the upper bound is strictly smaller than 2 follows immediately from Lemma 5.1. Fix $\varepsilon > 0$ and choose $x \in X$ and $V \in X_F$ be such that

$$\dim_{\text{H}}(X \cap (V + x)) \geq \sup_{\substack{z \in X \\ W \in X_F}} \dim_{\text{H}}(X \cap (W + z)) - \varepsilon. \quad (5.9)$$

By Lemma 5.2, there exist $T \in \text{Tan}(X)$ and $W, L \in \mathbb{RP}^1$ with $W \neq L$ such that

$$\begin{aligned} \dim_{\text{H}}(T \cap L) &\geq \dim_{\text{H}}(X \cap (V + x)), \\ \dim_{\text{H}}(T \cap (W + y)) &= 1 \end{aligned}$$

for all $y \in T$. Notice that if $\dim_{\text{H}}(T \cap L) = 0$, then trivially $\dim_{\text{H}}(T) \geq 1 = 1 + \dim_{\text{H}}(X \cap (V + x))$. Let us therefore assume that $0 < s < \dim_{\text{H}}(T \cap L)$. Relying on Frostman's lemma, see e.g. [50, Theorem 8.8], let μ be a Borel probability measure on $T \cap L$ such that for some constant $C \geq 1$ it holds that $\mu(B(y, r)) \leq Cr^s$ for all $y \in T \cap L$ and $r > 0$. Now Marstrand's slicing theorem [12, Theorem 3.3.1] implies that

$$1 = \dim_{\text{H}}(T \cap (W + y)) \leq \dim_{\text{H}}(T) - s$$

for μ -almost all $y \in T \cap L$. Therefore, by letting $s \uparrow \dim_{\text{H}}(T \cap L)$, we get

$$\dim_{\text{H}}(T) \geq 1 + \dim_{\text{H}}(T \cap L) \geq 1 + \dim_{\text{H}}(X \cap (V + x)). \quad (5.10)$$

By Lemma 2.1, (5.10), and (5.9), we thus have

$$\dim_{\text{A}}(X) \geq \dim_{\text{H}}(T) \geq 1 + \sup_{\substack{z \in X \\ W \in X_F}} \dim_{\text{H}}(X \cap (W + z)) - \varepsilon$$

and the lower bound for $\dim_{\text{A}}(X)$ follows by letting $\varepsilon \downarrow 0$. \square

6. ASSOUD DIMENSION OF SELF-AFFINE SETS HAVING SMALL HAUSDORFF DIMENSION

In this section, we show that the conditions (1), (2), (5), and (7) in Theorem 3.4 are equivalent and prove Theorem 3.5. The implication (1) \Rightarrow (2) is shown in Theorem 6.1 below and, by recalling Theorem 2.17, Theorem 6.2 verifies the implication (2) \Rightarrow (5). The implication (5) \Rightarrow (7) follows from Lemma 2.4. Finally, Theorem 6.5 below proves Theorem 3.5 and also the implication (7) \Rightarrow (1).

Theorem 6.1. *If X is a dominated planar self-affine set satisfying the projective open set condition, then there exists a constant $C \geq 1$ such that*

$$\sup\{\#\Sigma(V, x, r) : x \in X \text{ and } r > 0\} \leq C \frac{\exp(\text{diam}(\text{proj}_{V^\perp}(X))^{-2})}{\text{diam}(\text{proj}_{V^\perp}(X))} \quad (6.1)$$

for all $V \in X_F$ with $\text{diam}(\text{proj}_{V^\perp}(X)) > 0$, where $\Sigma(V, x, r)$ is as in (3.2). In particular, if X_F is not a singleton, then $\sup\{\#\Sigma(V, x, r) : V \in X_F, x \in X, \text{ and } r > 0\} < \infty$.

Proof. Let us first assume that $V \in X_F$ is such that $\text{diam}(\text{proj}_{V^\perp}(X)) > 0$. Recall that, by (2.7), $\|\text{proj}_{V^\perp} A_i\| = \|A_i^\top |V^\perp\|$ for all $i \in \Sigma_*$. By the projective open set condition (3.1), there is $\eta > 0$ such that for every $V \in X_F, x \in X, r > 0$, and $j, k \in \Sigma(V, x, r)$ with $j \neq k$ there is $z \in X$ such that

$$|\text{proj}_{V^\perp}(\varphi_j(z) - \varphi_k(z))| \geq \eta \text{diam}(\text{proj}_{V^\perp}(X)) \|A_j^\top |V^\perp\|. \quad (6.2)$$

As A is dominated, Lemma 2.8 shows that there exists a constant $D \geq 1$ such that

$$\|A_i^\top |V^\perp\| \leq \alpha_1(A_i) \leq D \|A_i^\top |V^\perp\| \quad (6.3)$$

for all $i \in \Sigma_*$ and $V \in X_F$.

Fix $x \in X$ and $r > 0$. Let $j, k \in \Sigma(V, x, r)$ with $j \neq k$ and $z \in X$ be such that (6.2) holds, and choose $y \in B(z, \gamma \text{diam}(\text{proj}_{V^\perp}(X)))$, where $\gamma = \kappa\eta/6D$ and $\kappa = \min_{i \in \{1, \dots, N\}} \alpha_2(A_i)$. Observe that, by (6.3), $\|A_j^\top |V^\perp\| \geq D^{-1} \alpha_1(A_j) \geq \kappa D^{-1} \alpha_1(A_{j^-}) \geq \kappa D^{-1} \|A_{j^-}^\top |V^\perp\|$ and hence, by (6.2), the triangle inequality, and (3.2),

$$\begin{aligned} |\text{proj}_{V^\perp}(\varphi_j(y) - \varphi_k(y))| &\geq |\text{proj}_{V^\perp}(\varphi_j(z) - \varphi_k(z))| - |\text{proj}_{V^\perp}(A_j - A_k)(z - y)| \\ &\geq \eta \text{diam}(\text{proj}_{V^\perp}(X)) \|A_j^\top |V^\perp\| - (\|A_j^\top |V^\perp\| + \|A_k^\top |V^\perp\|) |z - y| \\ &\geq (\kappa D^{-1} \eta - 2\gamma) r = 4\gamma r. \end{aligned} \quad (6.4)$$

As X is compact, there are finitely many points $z_1, \dots, z_k \in X$ such that

$$X \subset \bigcup_{i=1}^k B(z_i, \gamma \text{diam}(\text{proj}_{V^\perp}(X))).$$

By a simple volume argument, the points can be chosen such that

$$k \leq C \left(\frac{\text{diam}(X)}{\gamma \text{diam}(\text{proj}_{V^\perp}(X))} \right)^2,$$

where $C \geq 1$ does not depend on V . Therefore, for every $j, k \in \Sigma(V, x, r)$ with $j \neq k$ and the associated $z \in X$ satisfying (6.2) there is $i \in \{1, \dots, k\}$ such that $z_i \in$

$B(x, \gamma \operatorname{diam}(\operatorname{proj}_{V^\perp}(X)))$. Writing $\xi_{\mathbf{1}} = (\operatorname{proj}_{V^\perp}(\varphi_{\mathbf{1}}(z_1)), \dots, \operatorname{proj}_{V^\perp}(\varphi_{\mathbf{1}}(z_k))) \in (\mathbb{R}^2)^k$ for all $\mathbf{1} \in \Sigma_*$, we see that (6.4) gives $|\xi_j - \xi_k| \geq 4\gamma r$ and hence,

$$B(\xi_j, \gamma r) \cap B(\xi_k, \gamma r) = \emptyset \quad (6.5)$$

for all $j, k \in \Sigma(V, x, r)$ with $j \neq k$. On the other hand, if $j \in \Sigma(V, x, r)$, then

$$|\operatorname{proj}_{V^\perp}(x) - \operatorname{proj}_{V^\perp}(\varphi_j(z))| \leq 2r$$

for all $z \in X$, and so $|\operatorname{proj}_{V^\perp}(x) - \xi_j| \leq 2\sqrt{k}r$. It follows that

$$\bigcup_{j \in \Sigma(V, x, r)} B(\xi_j, \gamma r) \subset B((\operatorname{proj}_{V^\perp}(x), \dots, \operatorname{proj}_{V^\perp}(x)), 2\sqrt{k}r). \quad (6.6)$$

Therefore, by (6.5) and (6.6), we see that

$$\#\Sigma(V, x, r) \leq \left(\frac{2\sqrt{k}}{\gamma}\right)^{2k} = \left(\frac{12\sqrt{k}D}{\kappa\eta}\right)^{2k}.$$

As the upper bound does not depend on $x \in X$ nor $r > 0$, we have shown (6.1).

To show the second claim, let us assume that X_F is not a singleton. By (6.1), it suffices to show that

$$\inf_{V \in X_F} \operatorname{diam}(\operatorname{proj}_{V^\perp}(X)) > 0.$$

If this was not the case, then, by the continuity of $V \mapsto \operatorname{diam}(\operatorname{proj}_{V^\perp}(X))$ and the compactness of X_F , there is $V \in X_F$ such that $\operatorname{diam}(\operatorname{proj}_{V^\perp}(X)) = 0$. It follows that there exist $x, y \in X$ such that $X \subset V + x$ and $y \neq x$. Furthermore, since X_F is not a singleton, there exists $i \in \{1, \dots, N\}$ such that $A_i^{-1}V \neq V$. Note that $\varphi_i(x) \neq \varphi_i(y)$ and since $\varphi_i(X) \subset X \subset V + x$, we have $\varphi_i(x) - \varphi_i(y) \in V$. But then $x - y = A_i^{-1}(\varphi_i(x) - \varphi_i(y)) \notin V$, which is a contradiction. \square

Theorem 6.2. *If X is a dominated planar self-affine set such that $s = \dim_{\text{aff}}(X) \leq 1$ and $\sup\{\#\Sigma(V, x, r) : x \in X \text{ and } r > 0\} < \infty$ for some $V \in X_F$, then X is Ahlfors s -regular.*

Proof. By Lemma 2.12, there exist a measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ and a constant $C \geq 1$ such that

$$C^{-1}\alpha_1(A_{\mathbf{i}})^s \leq \mu_K([\mathbf{i}]) \leq C\alpha_1(A_{\mathbf{i}})^s \quad (6.7)$$

for all $\mathbf{i} \in \Sigma_*$. Let $V \in X_F$ be such that $\sup\{\#\Sigma(V, x, r) : x \in X \text{ and } r > 0\} < \infty$. By Lemma 2.8, there exists a constant $D \geq 1$ such that

$$\|A_{\mathbf{i}}^\top |V^\perp\| \leq \alpha_1(A_{\mathbf{i}}) \leq D\|A_{\mathbf{i}}^\top |V^\perp\| \quad (6.8)$$

for all $\mathbf{i} \in \Sigma_*$.

Fix $x \in X$ and $0 < r < \operatorname{diam}(X)$. Let $\mathbf{i} \in \Sigma$ be such that $\pi \mathbf{i} = x$ and choose $n \in \mathbb{N}$ such that $\varphi_{\mathbf{i}|_n}(X) \subset B(x, r)$ but $\varphi_{\mathbf{i}|_n}(X) \setminus B(x, r) \neq \emptyset$. Note that

$$\kappa r \leq \kappa \operatorname{diam}(\varphi_{\mathbf{i}|_{n-1}}(X)) = \kappa \alpha_1(A_{\mathbf{i}|_{n-1}}) \operatorname{diam}(X) \leq \operatorname{diam}(\varphi_{\mathbf{i}|_n}(X)) \leq 2r.$$

Therefore, by (6.7),

$$\begin{aligned} \pi_* \mu_K(B(x, r)) &\geq \pi_* \mu_K(\varphi_{\mathbf{i}|_n}(X)) \geq \mu_K([\mathbf{i}|_n]) \\ &\geq C^{-1} \alpha_1(A_{\mathbf{i}|_n})^s \geq C^{-1} \min_{i \in \{1, \dots, N\}} \alpha_2(A_i)^s \operatorname{diam}(X)^{-s} r^s \end{aligned} \quad (6.9)$$

and, by (6.8),

$$\begin{aligned} \pi_*\mu_K(B(x, r)) &\leq \pi_*\mu_K(\text{proj}_{V^\perp}^{-1}(\text{proj}_{V^\perp}(B(x, r)))) \leq \sum_{j \in \Sigma(V, x, r)} \mu_K([j]) \\ &\leq C \sum_{j \in \Sigma(V, x, r)} \alpha_1(A_j)^s \leq CD^s \sum_{j \in \Sigma(V, x, r)} \|\text{proj}_{V^\perp} A_j\|^s \\ &\leq CD^s \sup\{\#\Sigma(V, x, r) : x \in X \text{ and } r > 0\} \lambda^{-s} r^s. \end{aligned}$$

The measure $\pi_*\mu_K$ is thus Ahlfors s -regular. As (6.7) guarantees that the support of $\pi_*\mu_K$ is X , Lemma 2.4 finishes the proof. \square

Before proving Theorem 6.5, let us exhibit two auxiliary lemmas. The first one shows the existence of distinct maps whose projections are arbitrarily close to each other in the relative scale. Relying on the strong separation condition, let $\delta > 0$ be as in (2.12). If the self-affine set X is not contained in a line, let $z_1, z_2, z_3 \in X$ be in a general position such that $X \cap \text{conv}(\{z_1, z_2, z_3\})^o \neq \emptyset$ and $|z_i - z_j| \leq \frac{\delta}{3}$ whenever $i \neq j$. Write

$$Z = \text{conv}(\{z_1, z_2, z_3\}) \quad (6.10)$$

and bear in mind that if X and X_F are not singletons, then X is not contained in a line.

Lemma 6.3. *If X is a dominated planar self-affine set satisfying the strong separation condition, but not projective open set condition, such that X_F is not a singleton, then there are $\eta_0 > 0$ and $C \geq 1$ such that for every $0 < \eta < \eta_0$ there exist $V \in X_F$ and $\mathbf{i}, \mathbf{j} \in \Sigma_*$ with $\mathbf{i}|_1 \neq \mathbf{j}|_1$ such that*

$$C^{-1}\eta\|A_{\mathbf{i}}^\top|V^\perp\| \leq |\text{proj}_{V^\perp}(\varphi_{\mathbf{i}}(x) - \varphi_{\mathbf{j}}(x))| \leq C\eta\|A_{\mathbf{i}}^\top|V^\perp\|$$

for all $x \in Z$, where Z is as in (6.10).

Proof. Define $\eta_0 = \frac{1}{6} \text{diam}(X)^{-1}\delta$, where δ is as in (2.12), and choose $0 < \eta < \eta_0$. Since the projective open set condition (3.1) is not satisfied, there exist $L \in X_F$ and $\mathbf{h}, \mathbf{k} \in \Sigma_*$ with $\mathbf{h} \neq \mathbf{k}$ such that

$$\begin{aligned} |\text{proj}_{A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp}(\varphi_{\mathbf{i}}(x) - \varphi_{\mathbf{j}}(x))| \|A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp\| &= |\text{proj}_{L^\perp}(\varphi_{\mathbf{h}}(x) - \varphi_{\mathbf{k}}(x))| \\ &\leq \eta \text{diam}(\text{proj}_{L^\perp}(X)) \|A_{\mathbf{h}}^\top|L^\perp\|, \end{aligned}$$

where $\mathbf{i} = \sigma^{|\mathbf{h} \wedge \mathbf{k}|}(\mathbf{h})$ and $\mathbf{j} = \sigma^{|\mathbf{h} \wedge \mathbf{k}|}(\mathbf{k})$, and

$$\begin{aligned} |\text{proj}_{A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp}(\varphi_{\mathbf{i}}(x) - \varphi_{\mathbf{j}}(x))| &\leq \eta \text{diam}(\text{proj}_{L^\perp}(X)) \|A_{\mathbf{i}}^\top|A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp\| \\ &\leq \eta \text{diam}(X) \alpha_1(A_{\mathbf{i}}) \end{aligned}$$

for all $x \in X$. Since Z is the convex hull of $z_1, z_2, z_3 \in X$ we see that for any $x \in Z$ there is a probability vector (p_1, p_2, p_3) such that $x = \sum_i p_i z_i$. Hence,

$$\begin{aligned} |\text{proj}_{A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp}(\varphi_{\mathbf{i}}(x) - \varphi_{\mathbf{j}}(x))| &= \left| \sum_i p_i \text{proj}_{A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp}(\varphi_{\mathbf{i}}(z_i) - \varphi_{\mathbf{j}}(z_i)) \right| \\ &\leq \eta \text{diam}(X) \alpha_1(A_{\mathbf{i}}) \end{aligned} \quad (6.11)$$

for all $x \in Z$. Moreover, since $\text{diam}(Z) \leq \frac{\delta}{3}$ we have

$$|\varphi_{\mathbf{i}}(x) - \varphi_{\mathbf{j}}(x)| \geq \frac{\delta}{3}$$

for every $x \in Z$. Since

$$\begin{aligned} |\text{proj}_{A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp}(\varphi_i(x) - \varphi_j(x))| &= |\varphi_i(x) - \varphi_j(x)| |\cos(\angle(A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp, \varphi_i(x) - \varphi_j(x)))| \\ &\geq \frac{\delta}{3} |\cos(\angle(A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp, \varphi_i(x) - \varphi_j(x)))|, \end{aligned}$$

the estimate (6.11) implies

$$|\cos(\angle(A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp, \varphi_i(x) - \varphi_j(x)))| \leq \eta 3\delta^{-1} \text{diam}(X) \alpha_1(A_i) \quad (6.12)$$

for all $x \in Z$.

Write $W = A_{\mathbf{h} \wedge \mathbf{k}}^{-1} L \in X_F$ and notice that $W^\perp = A_{\mathbf{h} \wedge \mathbf{k}}^\top L^\perp$. Recall that since \mathbf{A} is dominated, Lemma 2.8 implies that there exists a constant $D \geq 1$ such that $\alpha_1(A_i) \leq D \|A_i^\top |V^\perp|\|$ for all $V \in X_F$. Recalling that X_F is compact and perfect, let $P, Q \in X_F$ be such that $|\sin(\angle(P, Q))| = \text{diam}(X_F) > 0$ and write

$$K = \frac{24D^3}{|\sin(\angle(P, Q))| \min_{i \in \{1, \dots, N\}} \alpha_2(A_i)}.$$

Then, by (6.11), for any $V \in X_F$ with $|\sin(\angle(W, V))| \leq K\eta\delta^{-1} \text{diam}(X) \alpha_1(A_i)$ we have

$$\begin{aligned} |\text{proj}_{V^\perp}(\varphi_i(x) - \varphi_j(x))| &= |\varphi_i(x) - \varphi_j(x)| |\cos(\angle(V^\perp, \varphi_i(x) - \varphi_j(x)))| \\ &= |\varphi_i(x) - \varphi_j(x)| |\cos(\angle(W^\perp, \varphi_i(x) - \varphi_j(x)) \cos(\angle(W, V)) \\ &\quad \pm \sin(\angle(W^\perp, \varphi_i(x) - \varphi_j(x)) \sin(\angle(W, V))| \\ &\leq |\text{proj}_{W^\perp}(\varphi_i(x) - \varphi_j(x))| + |\varphi_i(x) - \varphi_j(x)| |\sin(\angle(W, V))| \\ &\leq \eta \text{diam}(X) \alpha_1(A_i) + K\eta\delta^{-1} \text{diam}(X) (\text{diam}(X) + \delta/3) \alpha_1(A_i) \\ &\leq (1 + K\delta^{-1} (\text{diam}(X) + \delta/3)) \text{diam}(X) D\eta \|A_i^\top |V^\perp|\| \end{aligned} \quad (6.13)$$

for all $x \in Z$ giving the claimed upper bound for any such V .

Let us then show the lower bound. Since

$$|\sin(\angle(A_1^{-1}P, A_1^{-1}Q))| = \frac{|A_1^{-1}v \wedge A_1^{-1}w|}{|A_1^{-1}v| |A_1^{-1}w|} = \frac{|\det(A_1^{-1})|}{\|A_1^{-1}|P|\|A_1^{-1}|Q|\|} |\sin(\angle(P, Q))|,$$

where $v \in P$ and $w \in Q$ such that $|v| = 1 = |w|$, we see that

$$\begin{aligned} \frac{\alpha_2(A_1)}{\alpha_1(A_1)} |\sin(\angle(P, Q))| &\leq |\sin(\angle(A_1^{-1}P, A_1^{-1}Q))| \\ &\leq D^2 \frac{|\det(A_1^{-1})|}{\alpha_1(A_1^{-1})^2} = D^2 \frac{\alpha_2(A_1)}{\alpha_1(A_1)} \end{aligned} \quad (6.14)$$

for all $\mathbf{1} \in \Sigma_*$. Let $\mathbf{1} \in [\mathbf{h} \wedge \mathbf{k}] \subset \Sigma$ be such that $W \in A_{\mathbf{1}|_n}^{-1} X_F$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ such that

$$D^2 \frac{\alpha_2(A_{\mathbf{1}|_n})}{\alpha_1(A_{\mathbf{1}|_n})} \leq K\eta\delta^{-1} \text{diam}(X) \alpha_1(A_i) < D^2 \frac{\alpha_2(A_{\mathbf{1}|_{n-1}})}{\alpha_1(A_{\mathbf{1}|_{n-1}})} \quad (6.15)$$

and observe that, by the fact that $|\sin(\angle(A_{1|n}^{-1}P, A_{1|n}^{-1}Q))| = \text{diam}(A_{1|n}^{-1}X_F)$, the estimates (6.14) and (6.15) show that there exists $V \in \{A_{1|n}^{-1}P, A_{1|n}^{-1}Q\} \subset A_{1|n}^{-1}X_F$ such that

$$\frac{1}{2} \frac{\alpha_2(A_{1|n})}{\alpha_1(A_{1|n})} |\sin(\angle(P, Q))| \leq |\sin(\angle(W, V))| \leq K\eta\delta^{-1} \text{diam}(X)\alpha_1(A_i). \quad (6.16)$$

It follows that the upper bound found in (6.13) is valid for this $V \in X_F$. As $0 < \eta < \eta_0$, we have $\sqrt{1 - (\eta 3\delta^{-1} \text{diam}(X)\alpha_1(A_i))^2} > \frac{1}{2}$. Therefore, by (6.12), (6.16), (6.11), and (6.15),

$$\begin{aligned} |\text{proj}_{V^\perp}(\varphi_i(x) - \varphi_j(x))| &= |\varphi_i(x) - \varphi_j(x)| \cos(\angle(W^\perp, \varphi_i(x) - \varphi_j(x))) \cos(\angle(W, V)) \\ &\quad \pm \sin(\angle(W^\perp, \varphi_i(x) - \varphi_j(x))) \sin(\angle(W, V)) \\ &\geq |\varphi_i(x) - \varphi_j(x)| \sin(\angle(W^\perp, \varphi_i(x) - \varphi_j(x))) |\sin(\angle(W, V))| \\ &\quad - |\text{proj}_{W^\perp}(\varphi_i(x) - \varphi_j(x))| \cos(\angle(W, V)) \\ &\geq \frac{\delta}{3} \sqrt{1 - (\eta 3\delta^{-1} \text{diam}(X)\alpha_1(A_i))^2} \frac{1}{2} \frac{\alpha_2(A_{1|n})}{\alpha_1(A_{1|n})} |\sin(\angle(P, Q))| \\ &\quad - \eta \text{diam}(X)\alpha_1(A_i) \\ &\geq \left(\frac{K \min_{i \in \{1, \dots, N\}} \alpha_2(A_i) |\sin(\angle(P, Q))|}{12D^2} - D \right) \text{diam}(X)\eta \|A_i^\top\| |V^\perp|. \end{aligned}$$

The choice of K guarantees that the coefficient obtained in the end is positive and therefore, the claimed lower bound holds for this $V \in X_F$. \square

If X is dominated and $\mathcal{C} \subset \mathbb{RP}^1$ is a strongly invariant multicone, then, by (2.5), we have $\mathcal{C} \cap X_F = \emptyset$. Fix

$$L \in \bigcap_{n=1}^{\infty} \bigcup_{i \in \Sigma_n} A_i \mathcal{C} \subset \mathcal{C} \quad (6.17)$$

and notice that L is uniformly transverse to every $V \in X_F$.

Lemma 6.4. *If X is dominated planar self-affine set satisfying the strong separation condition, but not projective open set condition, such that X_F is not a singleton, then there exist $x_0 \in X$ and $C \geq 1$ so that for each $k \in \mathbb{N}$ there are $V_k \in X_F$, a natural number $n_k \geq k$, and finite words $\mathbf{k}_0, \dots, \mathbf{k}_{n_k} \in \Sigma_*$ such that*

$$C^{-1} \frac{\|A_{\mathbf{k}_0}^\top\| |V_k^\perp|}{n_k} \leq |\text{proj}_{V_k^\perp}(\varphi_{\mathbf{k}_{i-1}}(x_0) - \varphi_{\mathbf{k}_i}(x_0))| \leq C \frac{\|A_{\mathbf{k}_0}^\top\| |V_k^\perp|}{n_k}$$

for all $i \in \{1, \dots, n_k\}$. Furthermore, the set $\{\text{proj}_{V_k^\perp}(\varphi_{\mathbf{k}_{i-1}}(x_0) - \varphi_{\mathbf{k}_i}(x_0)) : i \in \{1, \dots, n_k\}\}$ is completely contained in one of the two halfplanes determined by L^\perp , where $L \in \mathbb{RP}^1$ is as in (6.17).

Proof. Let Z be as in (6.10). Recalling that $X \cap Z^\circ \neq \emptyset$, we choose $x_0 \in X \cap Z^\circ$ and let $\mathbf{f} \in \Sigma$ be such that $\pi(\mathbf{f}) = x_0$. By Lemma 6.3, there exist $n_0 \in \mathbb{N}$ and $C \geq 1$ such that for every $n \geq n_0$ there exist $V_n \in X_F$ and $\mathbf{i}_n, \mathbf{j}_n \in \Sigma_*$ with $\mathbf{i}_n|_1 \neq \mathbf{j}_n|_1$ such that

$$C^{-1} \frac{\|A_{\mathbf{i}_n}^\top\| |V_n^\perp|}{n} \leq |\text{proj}_{V_n^\perp}(\varphi_{\mathbf{i}_n}(x) - \varphi_{\mathbf{j}_n}(x))| \leq C \frac{\|A_{\mathbf{i}_n}^\top\| |V_n^\perp|}{n} \quad (6.18)$$

for all $x \in Z$. Notice also that for every $n \in \mathbb{N}$, by the continuity of the projection $V \mapsto \text{proj}_{V^\perp}(\varphi_{i_n}(x) - \varphi_{j_n}(x))$ and the compactness of Z , there exists $r_n > 0$ such that

$$\frac{1}{2} \leq \frac{|\text{proj}_{V_n^\perp}(\varphi_{i_n}(x) - \varphi_{j_n}(x))|}{|\text{proj}_{W^\perp}(\varphi_{i_n}(x) - \varphi_{j_n}(x))|} \leq 2 \quad (6.19)$$

for all $x \in Z$ and $W \in B(V_n, r_n)$.

Recalling (2.6), let $\mathbf{d}_n \in \Sigma_*$ be such that

$$A_{\mathbf{d}_n}^{-1}(X_F) \subset B(V_n, r_n).$$

Since Z is connected, $\text{proj}_{V_n^\perp}(\varphi_{i_n}(x) - \varphi_{j_n}(x))$ is contained in the same halfplane determined by L^\perp for all $x \in Z$. Hence, there exists a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers such that the points $\text{proj}_{V_{n_k}^\perp}(\varphi_{i_{n_k}}(x) - \varphi_{j_{n_k}}(x))$ are contained in the same halfplane determined by L^\perp for all $x \in Z$ and $k \in \mathbb{N}$. Fix $K \in \mathbb{N}$ and write $M = n_K$. Let $k_1, \dots, k_M \in \mathbb{N}$ and $l_2, \dots, l_M \in \mathbb{N}$ be arbitrary. We will specify these numbers later. Let us define words $\mathbf{k}_0, \dots, \mathbf{k}_M \in \Sigma_*$ by setting

$$\begin{aligned} \mathbf{k}_0 &= \mathbf{d}_{n_{k_M}} \mathbf{i}_{n_{k_M}} \mathbf{f}|_{l_M} \cdots \mathbf{d}_{n_{k_2}} \mathbf{i}_{n_{k_2}} \mathbf{f}|_{l_2} \mathbf{d}_{n_{k_1}} \mathbf{i}_{n_{k_1}}, \\ &\vdots \\ \mathbf{k}_i &= \mathbf{d}_{n_{k_M}} \mathbf{i}_{n_{k_M}} \mathbf{f}|_{l_M} \cdots \mathbf{d}_{n_{k_{i+1}}} \mathbf{i}_{n_{k_{i+1}}} \mathbf{f}|_{l_{i+1}} \mathbf{d}_{n_{k_i}} \mathbf{j}_{n_{k_i}} \mathbf{f}|_{l_i} \cdots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}, \\ &\vdots \\ \mathbf{k}_M &= \mathbf{d}_{n_{k_M}} \mathbf{j}_{n_{k_M}} \mathbf{f}|_{l_M} \cdots \mathbf{d}_{n_{k_2}} \mathbf{j}_{n_{k_2}} \mathbf{f}|_{l_2} \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}. \end{aligned}$$

To simplify notation, write

$$\begin{aligned} \mathbf{l}_i &= \mathbf{d}_{n_{k_K}} \mathbf{i}_{n_{k_M}} \mathbf{f}|_{l_M} \cdots \mathbf{d}_{n_{k_{i+1}}} \mathbf{i}_{n_{k_{i+1}}} \mathbf{f}|_{l_{i+1}} \mathbf{d}_{n_{k_i}}, \\ \mathbf{h}_i &= \mathbf{i}_{n_{k_i}} \mathbf{f}|_{l_i} \cdots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}, \\ \mathbf{g}_i &= \mathbf{j}_{n_{k_i}} \mathbf{f}|_{l_i} \cdots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}, \end{aligned}$$

and observe that then

$$|\text{proj}_{V_{n_K}^\perp}(\varphi_{\mathbf{k}_{i-1}}(x_0) - \varphi_{\mathbf{k}_i}(x_0))| = |\text{proj}_{A_{\mathbf{l}_i}^\top V_{n_K}^\perp}(\varphi_{\mathbf{h}_i}(x_0) - \varphi_{\mathbf{g}_i}(x_0))| \|A_{\mathbf{l}_i}^\top|V_{n_K}^\perp\| \quad (6.20)$$

for all $i \in \{1, \dots, M\}$. Without loss of generality, we may assume that $A_{\mathbf{l}_i}^\top|V_{n_K}^\perp$ is orientation preserving. Indeed, adding an extra symbol in the front of $\mathbf{d}_{n_{k_i}}$ leaves the properties of $\mathbf{d}_{n_{k_i}}$ defined above unchanged but may change the orientation of $A_{\mathbf{l}_i}^\top|V_{n_K}^\perp$ if necessary. Thus, $\text{proj}_{V_{n_K}^\perp}(\varphi_{\mathbf{k}_{i-1}}(x_0) - \varphi_{\mathbf{k}_i}(x_0))$ is in the same halfplane determined by L^\perp as $\text{proj}_{A_{\mathbf{l}_i}^\top V_{n_K}^\perp}(\varphi_{\mathbf{h}_i}(x_0) - \varphi_{\mathbf{g}_i}(x_0))$. Furthermore, since $A_{\mathbf{l}_i}^\top = A_{\mathbf{d}_{n_{k_i}}}^\top A_{\mathbf{d}_{n_{k_M}} \mathbf{i}_{n_{k_M}} \mathbf{f} \cdots \mathbf{d}_{n_{k_{i+1}}} \mathbf{i}_{n_{k_{i+1}}} \mathbf{f}}$, we have $A_{\mathbf{l}_i}^\top V_{n_K}^\perp \in B(V_{n_{k_i}}^\perp, r_{n_{k_i}})$. Therefore, if $\varphi_{\mathbf{f}|_{l_i} \cdots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}}(x_0) \in Z$, then, by (6.20),

(6.19), and (6.18),

$$\begin{aligned} (2C)^{-1} \frac{\|A_{1_i}^\top |V_{n_{k_i}}^\perp\| \|A_{1_i}^\top |V_{n_K}^\perp\|}{n_{k_i}} &\leq |\text{proj}_{V_{n_K}^\perp}(\varphi_{\mathbf{k}_{i-1}}(x_0) - \varphi_{\mathbf{k}_i}(x_0))| \\ &\leq 2C \frac{\|A_{1_i}^\top |V_{n_{k_i}}^\perp\| \|A_{1_i}^\top |V_{n_K}^\perp\|}{n_{k_i}} \end{aligned}$$

for all $i \in \{1, \dots, M\}$. Hence, to finish the proof, it suffices to recall Lemma 2.8 and choose the numbers $k_1, \dots, k_M \in \mathbb{N}$ such that

$$\begin{aligned} c_1 \frac{\alpha_1(A_{\mathbf{f}|l_i \mathbf{d}_{n_{k_{i-1}}} \mathbf{j}_{n_{k_{i-1}}} \mathbf{f}|l_{i-1} \dots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}})}{n_K} &\leq \frac{1}{n_{k_i}} \\ &\leq c_2 \frac{\alpha_1(A_{\mathbf{f}|l_i \mathbf{d}_{n_{k_{i-1}}} \mathbf{j}_{n_{k_{i-1}}} \mathbf{f}|l_{i-1} \dots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}})}{n_K} \end{aligned} \quad (6.21)$$

for some constants $c_1, c_2 > 0$ and the numbers $l_2, \dots, l_M \in \mathbb{N}$ such that the points $\varphi_{\mathbf{f}|l_i \dots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}}(x_0)$ are in Z .

Let us give the precise definition for the numbers k_1, k_2, \dots, k_M and l_2, \dots, l_M . Let $k_1 = K$ and choose l_2 to be the smallest integer such that $\varphi_{\mathbf{f}|l_2}(X) \subset Z$ and

$$\left[\frac{\alpha_1(A_{\mathbf{f}|l_2+1 \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}})}{n_K}, \frac{\alpha_1(A_{\mathbf{f}|l_2 \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}})}{n_K} \right) \cap \{n_{k_1+1}^{-1}, n_{k_1+2}^{-1}, \dots\} \neq \emptyset,$$

and choose k_2 such that n_{k_2} is the largest element of the above set. We continue inductively. If k_i and l_i has been defined for $i \in \{2, \dots, M-1\}$, then let l_{i+1} be the smallest integer such that $\varphi_{\mathbf{f}|l_{i+1}}(X) \subset Z$ and the intersection

$$\left[\frac{\alpha_1(A_{\mathbf{f}|l_{i+1}+1 \mathbf{d}_{n_{k_i}} \mathbf{j}_{n_{k_i}} \mathbf{f}|l_i \dots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}})}{n_K}, \frac{\alpha_1(A_{\mathbf{f}|l_{i+1} \mathbf{d}_{n_{k_i}} \mathbf{j}_{n_{k_i}} \mathbf{f}|l_i \dots \mathbf{d}_{n_{k_1}} \mathbf{j}_{n_{k_1}}})}{n_K} \right) \cap \{n_{k_i+1}^{-1}, n_{k_i+2}^{-1}, \dots\}$$

is non-empty, and choose k_{i+1} such that $n_{k_{i+1}}$ is the largest element of the above set. As these choices clearly satisfy (6.21), we have finished the proof. \square

Theorem 6.5. *If X is a dominated planar self-affine set satisfying the strong separation condition, but not the projective open set condition, such that X_F is not a singleton, then $\dim_A(X) \geq 1$.*

Proof. Let $x_0 \in X$ and $C \geq 1$ be as in Lemma 6.4. Fix $k \in \mathbb{N}$ and let $V_k \in X_F$, $n_k \geq k$, and $\mathbf{k}_0, \dots, \mathbf{k}_{n_k} \in \Sigma_*$ be as in Lemma 6.4. Write $B_k = \{\varphi_{\mathbf{k}_0}(x_0), \dots, \varphi_{\mathbf{k}_{n_k}}(x_0)\}$ and let L be as in (6.17). Let $a_i^L \in L$ and $a_i^V \in V_k$ be such that $a_i^L + a_i^V = \varphi_{\mathbf{k}_i}(x_0) - \varphi_{\mathbf{k}_{i-1}}(x_0)$ for all $i \in \{1, \dots, n_k\}$. Recalling (2.6), let $\mathbf{1}_k \in \Sigma$ be such that $V_k = \Pi(\mathbf{1}_k)$. Let $D \geq 1$ be as in Lemma 2.8. Relying on the definition of domination, choose m_k to be the smallest integer for which

$$D\alpha_2(A_{\mathbf{1}_k|_{m_k}}) \text{diam}(X) \leq \frac{1}{n_k^2} \alpha_1(A_{\mathbf{1}_k|_{m_k}}) \|A_{\mathbf{k}_0}^\top |V_k^\perp\|. \quad (6.22)$$

By Lemma 2.8,

$$|A_{\mathbb{1}_k|m_k}^{\leftarrow} a_i^V| = \|A_{\mathbb{1}_k|m_k}^{\leftarrow} |V_k|\| |a_i^V| \leq D\alpha_2(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \text{diam}(X). \quad (6.23)$$

Therefore, by (6.23) and (6.22), we have

$$\left| \sum_{j=1}^i A_{\mathbb{1}_k|m_k}^{\leftarrow} a_j^V \right| \leq \frac{1}{n_k} \alpha_1(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \|A_{\mathbb{k}_0}^\top |V_k^\perp|\| \quad (6.24)$$

for all $i \in \{1, \dots, n_k\}$. Clearly, each $A_{\mathbb{1}_k|m_k}^{\leftarrow} a_i^L$ is contained in the subspace $A_{\mathbb{1}_k|m_k}^{\leftarrow} L$ and hence, by (6.24),

$$\begin{aligned} & \text{dist}(\varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_i}(x_0)) - \varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_0}(x_0)), A_{\mathbb{1}_k|m_k}^{\leftarrow} L) \\ & \leq \left| \varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_i}(x_0)) - \varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_0}(x_0)) - \sum_{j=1}^i A_{\mathbb{1}_k|m_k}^{\leftarrow} a_j^L \right| \\ & = \left| \sum_{j=1}^i A_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_j}(x_0) - \varphi_{\mathbb{k}_{j-1}}(x_0)) - \sum_{j=1}^i A_{\mathbb{1}_k|m_k}^{\leftarrow} a_j^L \right| \\ & = \left| \sum_{j=1}^i A_{\mathbb{1}_k|m_k}^{\leftarrow} a_j^V \right| \leq \frac{1}{n_k} \alpha_1(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \|A_{\mathbb{k}_0}^\top |V_k^\perp|\|. \end{aligned} \quad (6.25)$$

Thus, $\varphi_{\mathbb{1}_k|m}^{\leftarrow}(B_k)$ is in the $\frac{1}{n_k} \alpha_1(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \|A_{\mathbb{k}_0}^\top |V_k^\perp|\|$ -neighbourhood of the line $A_{\mathbb{1}_k|m_k}^{\leftarrow} L + \varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_0}(x_0))$.

By Lemma 6.4,

$$C^{-1} \frac{\|A_{\mathbb{k}_0}^\top |V_k^\perp|\|}{n_k} \leq |\text{proj}_{V_k^\perp}(\varphi_{\mathbb{k}_{i-1}}(x_0) - \varphi_{\mathbb{k}_i}(x_0))| \leq C \frac{\|A_{\mathbb{k}_0}^\top |V_k^\perp|\|}{n_k}$$

for all $i \in \{1, \dots, n_k\}$. Since L is uniformly transverse to every $V \in X_F$, we may, by possibly adjusting the constant C , replace $|\text{proj}_{V_k^\perp}(\varphi_{\mathbb{k}_{i-1}}(x_0) - \varphi_{\mathbb{k}_i}(x_0))|$ in the above estimate by $|a_i^L|$. Therefore, by Lemma 2.8,

$$(CD)^{-1} \frac{\alpha_1(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \|A_{\mathbb{k}_0}^\top |V_k^\perp|\|}{n_k} \leq |A_{\mathbb{1}_k|m_k}^{\leftarrow} a_i^L| \leq C \frac{\alpha_1(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \|A_{\mathbb{k}_0}^\top |V_k^\perp|\|}{n_k} \quad (6.26)$$

for all $i \in \{1, \dots, n_k\}$. Let

$$\begin{aligned} T_k &= M_{\varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_0}(x_0)), \alpha_1(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \|A_{\mathbb{k}_0}^\top |V_k^\perp|\|}(X) \cap B(0, 1) \\ &\supset M_{\varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(\varphi_{\mathbb{k}_0}(x_0)), \alpha_1(A_{\mathbb{1}_k|m_k}^{\leftarrow}) \|A_{\mathbb{k}_0}^\top |V_k^\perp|\|}(\varphi_{\mathbb{1}_k|m_k}^{\leftarrow}(B_k)) \cap B(0, 1). \end{aligned}$$

Since the vectors a_i^L are contained in the same halfplane determined by L^\perp , there exists, by possibly going through a subsequence, a weak tangent set T such that $T_k \rightarrow T$ as $k \rightarrow \infty$ in Hausdorff distance such that, by (6.25) and (6.26), T contains a line segment of length at least $(CD)^{-1}$. The claim follows now from Lemma 2.1. \square

7. HAUSDORFF MEASURE OF SELF-AFFINE SETS HAVING SMALL HAUSDORFF DIMENSION

In this section, we conclude the proof of Theorem 3.4. In Section 6, we proved that the conditions (1), (5), and (7) are equivalent. Recall that (5) \Rightarrow (3) by Lemma 2.4. Recalling Theorem 2.17, Theorem 7.2 below verifies the implication (3) \Rightarrow (4) and Theorem 7.3 shows the implication (4) \Rightarrow (6) \Rightarrow (5). Before proving the aforementioned theorems, we verify a technical lemma. Let X be a dominated planar self-affine set, $s = \dim_{\text{aff}}(X) \leq 1$, and define a function $H: \Sigma \rightarrow [0, \infty)$ by setting

$$H(\mathbf{i}) = \mathcal{H}_{\infty}^s(\text{proj}_{\Pi(\mathbf{i})^{\perp}}(X))$$

for all $\mathbf{i} \in \Sigma$. The function H measures the s -dimensional Hausdorff content of the projection in different directions. The fact that H takes finite values follows from Lemma 2.18. Recall that the Perron-Frobenius operator \mathcal{L} for s is the positive linear operator defined by setting

$$\mathcal{L}f(\mathbf{i}) = \sum_{i=1}^N \|A_i^{\top} | \Pi(\mathbf{i})^{\perp} \|^s f(i\mathbf{i})$$

for all continuous functions $f: \Sigma \rightarrow \mathbb{R}$. Let the continuous function $h: \Sigma \rightarrow (0, \infty)$ and the Borel probability measure ν on Σ be as in Lemma 2.12. Although omitted in notation, bear in mind that H , \mathcal{L} , h , and ν depend on s .

Lemma 7.1. *If X is a dominated planar self-affine set with $s = \dim_{\text{aff}}(X) \leq 1$, then*

$$H(\mathbf{i}) = h(\mathbf{i}) \int_{\Sigma} H(\mathbf{j}) \, d\nu(\mathbf{j})$$

for all $\mathbf{i} \in \Sigma$. In particular, either $\max_{\mathbf{i} \in \Sigma} H(\mathbf{i}) = 0$ or $\inf_{\mathbf{i} \in \Sigma} H(\mathbf{i}) > 0$.

Proof. Let us first show that H is upper semi-continuous. Fix $\mathbf{j} \in \Sigma$ and choose $\varepsilon > 0$. By the definition of the Hausdorff content, there exists a countable open cover $\{U_i\}_i$ of $\text{proj}_{\Pi(\mathbf{j})^{\perp}}(X)$ such that

$$\sum_i \text{diam}(U_i)^s \leq H(\mathbf{j}) + \varepsilon.$$

Since $\text{proj}_{\Pi(\mathbf{j})^{\perp}}(X)$ is compact, we may assume that the cover $\{U_i\}_i$ is finite. Write $\varrho = \text{dist}(\text{proj}_{\Pi(\mathbf{j})^{\perp}}(X), \mathbb{R} \setminus \bigcup_i U_i) > 0$ and $r = \arcsin(\varrho / \text{diam}(X))$. Hence, for every $\mathbf{i} \in \Sigma$ with $\angle(\Pi(\mathbf{i}), \Pi(\mathbf{j})) < r$, we have $|\text{proj}_{\Pi(\mathbf{i})^{\perp}}(x) - \text{proj}_{\Pi(\mathbf{j})^{\perp}}(x)| < \varrho$ for all $x \in X$. This means that $\{U_i\}_i$ covers each such $\text{proj}_{\Pi(\mathbf{i})^{\perp}}(X)$ and consequently,

$$\sup_{\angle(\Pi(\mathbf{j}), \Pi(\mathbf{i})) < r} H(\mathbf{i}) \leq \sum_i \text{diam}(U_i)^s \leq H(\mathbf{j}) + \varepsilon.$$

The claim follows now by letting $\varrho \downarrow 0$, in which case also $r \downarrow 0$, and then $\varepsilon \downarrow 0$.

As H is upper semi-continuous, a standard measure-theoretical argument applying Urysohn's lemma shows that for each $n \in \mathbb{N}$ there exists a continuous function $f_n: \Sigma \rightarrow \mathbb{R}$ such that $f_n \geq H$ and

$$\int_{\Sigma} f_n(\mathbf{i}) \, d\nu(\mathbf{i}) \leq \int_{\Sigma} H(\mathbf{i}) \, d\nu(\mathbf{i}) + \frac{1}{n}. \quad (7.1)$$

Since $\text{proj}_V(Ax + y) = \pm \|A^\top |V|\| \text{proj}_{A^\top V}(x) + \text{proj}_V(y)$ for all $V \in \mathbb{RP}^1$, $A \in GL_2(\mathbb{R})$, and $x, y \in \mathbb{R}^2$, where \pm indicates that the equality holds with one of the signs, we have

$$H(\mathbf{i}) = \mathcal{H}_\infty^s \left(\bigcup_{i=1}^N \text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_i(X)) \right) \leq \sum_{i=1}^N \|A_i^\top | \Pi(\mathbf{i})^\perp \|^s H(i\mathbf{i}) = \mathcal{L}H(\mathbf{i}) \quad (7.2)$$

for all $\mathbf{i} \in \Sigma$. Applying (7.2), Lemma 2.12, and (7.1), we see that

$$H(\mathbf{i}) \leq \mathcal{L}^k H(\mathbf{i}) \leq \mathcal{L}^k f_n(\mathbf{i}) \rightarrow h(\mathbf{i}) \int_\Sigma f_n(\mathbf{j}) \, d\nu(\mathbf{j}) \leq h(\mathbf{i}) \left(\int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) + \frac{1}{n} \right)$$

uniformly as $k \rightarrow \infty$ for all $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$. By letting $n \rightarrow \infty$, it follows that

$$H(\mathbf{i}) \leq h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) \quad (7.3)$$

for all $\mathbf{i} \in \Sigma$. To show the equality, write

$$\Gamma_n = \left\{ \mathbf{i} \in \Sigma : H(\mathbf{i}) \leq h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) - \frac{1}{n} \right\}$$

for all $n \in \mathbb{N}$ and observe that, by the definition of Γ_n , (7.3), and Lemma 2.12,

$$\begin{aligned} \int_\Sigma H(\mathbf{i}) \, d\nu(\mathbf{i}) &= \int_{\Gamma_n} H(\mathbf{i}) \, d\nu(\mathbf{i}) + \int_{\Sigma \setminus \Gamma_n} H(\mathbf{i}) \, d\nu(\mathbf{i}) \\ &\leq \int_{\Gamma_n} h(\mathbf{i}) \, d\nu(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) - \frac{\nu(\Gamma_n)}{n} + \int_{\Sigma \setminus \Gamma_n} h(\mathbf{i}) \, d\nu(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) \\ &= \int_\Sigma h(\mathbf{i}) \, d\nu(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) - \frac{\nu(\Gamma_n)}{n} = \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) - \frac{\nu(\Gamma_n)}{n}. \end{aligned}$$

It follows that $\nu(\Gamma_n) = 0$ for all $n \in \mathbb{N}$ and consequently,

$$\nu \left(\left\{ \mathbf{i} \in \Sigma : H(\mathbf{i}) < h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) \right\} \right) = \nu \left(\bigcup_{n \in \mathbb{N}} \Gamma_n \right) \leq \sum_{n \in \mathbb{N}} \nu(\Gamma_n) = 0.$$

Therefore, by (7.3),

$$H(\mathbf{i}) = h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) \quad (7.4)$$

for ν -almost all $\mathbf{i} \in \Sigma$. To see that this holds for all words, fix $\mathbf{i} \in \Sigma$. Since ν is fully supported, there exists a sequence $(\mathbf{j}_n)_{n \in \mathbb{N}}$ of words in Σ converging to \mathbf{i} such that (7.4) holds for each \mathbf{j}_n . Thus, by the continuity of h , (7.4), the upper semi-continuity of H , and (7.3),

$$\begin{aligned} h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) &= \lim_{n \rightarrow \infty} h(\mathbf{j}_n) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) \\ &= \lim_{n \rightarrow \infty} H(\mathbf{j}_n) \leq H(\mathbf{i}) \leq h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) \, d\nu(\mathbf{j}) \end{aligned}$$

which finishes the proof of the first claim. Since $h(\mathbf{i}) > 0$ for all $\mathbf{i} \in \Sigma$, the dichotomy in the last claim follows immediately from the first claim. \square

Theorem 7.2. *If X is a dominated planar self-affine set with $s = \dim_{\text{aff}}(X) \leq 1$, then there exists a constant $c > 0$ such that*

$$\mathcal{H}^s(X) \leq c \max_{\mathbf{i} \in \Sigma} H(\mathbf{i}).$$

In particular, if $\mathcal{H}^s(X) > 0$, then $\inf_{\mathbf{i} \in \Sigma} H(\mathbf{i}) > 0$.

Proof. In the proof of Lemma 7.1, we showed H upper semi-continuous. This ensures the existence of the maximum $\max_{\mathbf{i} \in \Sigma} H(\mathbf{i})$. Furthermore, if the claimed inequality holds and $\mathcal{H}^s(X) > 0$, then there is $\mathbf{i} \in \Sigma$ such that $H(\mathbf{i}) > 0$. By Lemma 7.1, this implies $\inf_{\mathbf{i} \in \Sigma} H(\mathbf{i}) > 0$.

Let us then prove the claimed inequality. Fix $\varepsilon > 0$ and notice that for every $V \in X_F$ there exists an open cover \mathcal{U}_V of $\text{proj}_{V^\perp}(X)$ such that

$$\sum_{U \in \mathcal{U}_V} \text{diam}(U)^s \leq \mathcal{H}_\infty^s(\text{proj}_{V^\perp}(X)) + \varepsilon. \quad (7.5)$$

Since $\text{proj}_{V^\perp}(X)$ is compact, we may assume that \mathcal{U}_V is finite and $\max_{U \in \mathcal{U}_V} \text{diam}(U) \leq \text{diam}(X)$. Write $\varrho_V = \min_{U \in \mathcal{U}_V} \text{diam}(U) > 0$. Note that there is a constant $c > 0$ such that $\|\text{proj}_V - \text{proj}_W\| \leq c \angle(V, W)$ for all $V, W \in \mathbb{R}\mathbb{P}^1$. Since X_F is compact, there exist $m \in \mathbb{N}$ and $V_1, \dots, V_m \in X_F$ such that

$$X_F \subset \bigcup_{i=1}^m B(V_i, \varrho_{V_i}/c).$$

Thus, for every $V \in X_F$ there exists $i \in \{1, \dots, m\}$ such that $V \in B(V_i, \varrho_{V_i}/c)$ and, in particular,

$$\text{proj}_{V^\perp}(X) \subset \bigcup_{U \in \mathcal{U}_{V_i}} U.$$

Let $P(V)$ to denote this V_i and write $\varrho = \min_{i \in \{1, \dots, m\}} \varrho_{V_i}$.

Define

$$\begin{aligned} \Sigma(r) = \{ \mathbf{i} \in \Sigma_* : \alpha_2(A_{\mathbf{i}}) \text{diam}(X) < \varrho \alpha_1(A_{\mathbf{i}}) \text{ and } \alpha_1(A_{\mathbf{i}}) \text{diam}(X) < r \\ \text{but } \alpha_2(A_{\mathbf{i}-}) \text{diam}(X) \geq \varrho \alpha_1(A_{\mathbf{i}-}) \text{ or } \alpha_1(A_{\mathbf{i}-}) \text{diam}(X) \geq r \} \end{aligned}$$

for all $r > 0$. Note that $\{[\mathbf{i}] : \mathbf{i} \in \Sigma(r)\}$ is a partition of Σ for all $r > 0$. Fix $r > 0$ and for each $\mathbf{i} \in \Sigma(r)$ choose $\mathbf{i}' \in \Sigma$ such that $\mathbf{i}' = \overleftarrow{\mathbf{i}} \mathbf{j}$ for some $\mathbf{j} \in \Sigma$. Hence,

$$\bigcup_{\mathbf{i} \in \Sigma(r)} \bigcup_{U \in \mathcal{U}_{P(\Pi(\mathbf{i}'))}} \varphi_{\mathbf{i}}(X \cap (\text{proj}_{\Pi(\mathbf{i}')^\perp})^{-1}(U))$$

is clearly a cover of X . Observe that

$$\begin{aligned} & \text{diam}(\varphi_{\mathbf{i}}(X \cap (\text{proj}_{\Pi(\mathbf{i}')^\perp})^{-1}(U))) \\ & \leq \|A_{\mathbf{i}}\| \Pi(\mathbf{i}') \text{diam}(X) + \|A_{\mathbf{i}}\| \Pi(\mathbf{i}')^\perp \text{diam}(U) \\ & \leq \alpha_1(A_{\mathbf{i}})(c\varrho + \text{diam}(U)) \leq 2c\alpha_1(A_{\mathbf{i}}) \text{diam}(U) \leq 2cr \end{aligned} \quad (7.6)$$

for all $\mathbf{i} \in \Sigma(r)$. Recall also that, by Lemma 2.12, there exist a measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ and a constant $C \geq 1$ such that

$$C^{-1} \alpha_1(A_{\mathbf{i}})^s \leq \mu_K([\mathbf{i}]) \leq C \alpha_1(A_{\mathbf{i}})^s \quad (7.7)$$

for all $\mathbf{i} \in \Sigma_*$. Therefore, by (7.6), (7.5), and (7.7),

$$\begin{aligned}
 \mathcal{H}_{2cr}^s(X) &\leq \sum_{\mathbf{i} \in \Sigma(r)} \sum_{U \in \mathcal{U}_{P(\Pi(\mathbf{i}'))}} \text{diam}(\varphi_{\mathbf{i}}(X \cap (\text{proj}_{\Pi(\mathbf{i}')^\perp}^{-1}(U)))^s \\
 &\leq (2c)^s \sum_{\mathbf{i} \in \Sigma(r)} \sum_{U \in \mathcal{U}_{P(\Pi(\mathbf{i}')})} \alpha_1(A_{\mathbf{i}})^s \text{diam}(U)^s \\
 &\leq (2c)^s \sum_{\mathbf{i} \in \Sigma(r)} \alpha_1(A_{\mathbf{i}})^s (\mathcal{H}_\infty^s(\text{proj}_{P(\Pi(\mathbf{i}')^\perp}(X)) + \varepsilon) \\
 &\leq C(2c)^s (\max_{\mathbf{k} \in \Sigma} H(\mathbf{k}) + \varepsilon) \sum_{\mathbf{i} \in \Sigma(r)} \mu_K([\mathbf{i}]) \\
 &= C(2c)^s (\max_{\mathbf{k} \in \Sigma} H(\mathbf{k}) + \varepsilon).
 \end{aligned}$$

The claim follows by letting $r \downarrow 0$ and then $\varepsilon \downarrow 0$. \square

Theorem 7.3. *If X is a dominated planar self-affine set with $s = \dim_{\text{aff}}(X) \leq 1$ and $\inf_{\mathbf{i} \in \Sigma} H(\mathbf{i}) > 0$, then X and $\text{proj}_{V^\perp}(X)$ are Ahlfors s -regular for all $V \in X_F$. In fact, X is Ahlfors s -regular if and only if $\text{proj}_{V^\perp}(X)$ is Ahlfors s -regular for all $V \in X_F$.*

Proof. Fix $\delta > 0$ and choose $n \in \mathbb{N}$ such that $\text{diam}(\varphi_j(X)) < \delta$ for all $j \in \Sigma_n$. By Lemmas 7.1 and 2.12, we have

$$\begin{aligned}
 \mathcal{H}_\delta^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X)) &= \mathcal{H}_\delta^s\left(\bigcup_{j \in \Sigma_n} \text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_j(X))\right) \\
 &\leq \sum_{j \in \Sigma_n} \mathcal{H}_\infty^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_j(X))) \\
 &= \sum_{j \in \Sigma_n} \|A_j^\top | \Pi(\mathbf{i})^\perp \|^s H(\overleftarrow{j} \mathbf{i}) \\
 &= \mathcal{L}^n H(\mathbf{i}) = \mathcal{L}^n h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) d\nu(\mathbf{j}) \\
 &= h(\mathbf{i}) \int_\Sigma H(\mathbf{j}) d\nu(\mathbf{j}) = H(\mathbf{i}).
 \end{aligned} \tag{7.8}$$

By letting $\delta \downarrow 0$, we see that

$$\mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X)) = H(\mathbf{i}) \tag{7.9}$$

for all $\mathbf{i} \in \Sigma$. In particular, it follows from (7.9) that

$$\begin{aligned}
 &\mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X) \cap B(\text{proj}_{\Pi(\mathbf{i})^\perp}(x), r)) \\
 &= \mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X)) - \mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X) \setminus B(\text{proj}_{\Pi(\mathbf{i})^\perp}(x), r)) \\
 &\leq H(\mathbf{i}) - \mathcal{H}_\infty^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X) \setminus B(\text{proj}_{\Pi(\mathbf{i})^\perp}(x), r)) \\
 &\leq \mathcal{H}_\infty^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X) \cap B(\text{proj}_{\Pi(\mathbf{i})^\perp}(x), r)) \leq (2r)^s
 \end{aligned} \tag{7.10}$$

for all $x \in X$, $r > 0$, and $\mathbf{i} \in \Sigma$. Furthermore, fix $\mathbf{h}, \mathbf{k} \in \Sigma_*$ such that $[\mathbf{h}] \cap [\mathbf{k}] = \emptyset$ and write $n = \max\{|\mathbf{h}|, |\mathbf{k}|\}$. Let $\Sigma'_n = \{\mathbf{h}, \mathbf{k}\} \cup \{j \in \Sigma_n : [j] \cap [\mathbf{h}] = \emptyset \text{ and } [j] \cap [\mathbf{k}] = \emptyset\}$.

Relying on (7.9) and (7.8), we get

$$\begin{aligned}
\mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X)) &= \mathcal{H}^s\left(\bigcup_{j \in \Sigma'_n} \text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_j(X))\right) \\
&\leq \sum_{j \in \Sigma'_n} \mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_j(X))) - \mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{h}}(X)) \cap \text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{k}}(X))) \\
&\leq \mathcal{L}^n H(\mathbf{i}) - \mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{h}}(X)) \cap \text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{k}}(X))) \\
&= \mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(X)) - \mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{h}}(X)) \cap \text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{k}}(X))).
\end{aligned}$$

It follows that

$$\mathcal{H}^s(\text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{h}}(X)) \cap \text{proj}_{\Pi(\mathbf{i})^\perp}(\varphi_{\mathbf{k}}(X))) = 0 \quad (7.11)$$

for all $\mathbf{h}, \mathbf{k} \in \Sigma_*$ with $[\mathbf{h}] \cap [\mathbf{k}] = \emptyset$ and $\mathbf{i} \in \Sigma$.

To prove the claim, it suffices to show that $\pi_*\mu_K$ and $\mathcal{H}^s|_{\text{proj}_{V^\perp}(X)}$ are simultaneously Ahlfors s -regular for all $V \in X_F$. Recall that, by Lemma 2.12, there exist a measure $\mu_K \in \mathcal{M}_\sigma(\Sigma)$ and a constant $C \geq 1$ such that

$$C^{-1}\alpha_1(A_{\mathbf{i}})^s \leq \mu_K([\mathbf{i}]) \leq C\alpha_1(A_{\mathbf{i}})^s \quad (7.12)$$

for all $\mathbf{i} \in \Sigma_*$. Fix $V \in X_F$ and notice first that, by (6.9) and (7.10), there exists a constant $c > 0$ such that

$$\pi_*\mu_K(B(x, r)) \geq cr^s \quad \text{and} \quad \mathcal{H}^s|_{\text{proj}_{V^\perp}(X)}(B(x, r)) \leq 2^s r^s \quad (7.13)$$

for all $x \in X$ and $0 < r < \text{diam}(X)$. To show the other inequalities, let $\Sigma(V, x, r)$ be as in (3.2) for all $x \in X$ and $r > 0$. Observe that, by (7.13), (7.12), Lemma 2.8, (7.9), and (7.10), we have

$$\begin{aligned}
c \inf_{\mathbf{k} \in \Sigma} H(\mathbf{k})r^s &\leq \inf_{\mathbf{k} \in \Sigma} H(\mathbf{k})\pi_*\mu_K(B(x, r)) \\
&\leq \inf_{\mathbf{k} \in \Sigma} H(\mathbf{k}) \sum_{j \in \Sigma(V, x, r)} \mu_K([j]) \leq C \inf_{\mathbf{k} \in \Sigma} H(\mathbf{k}) \sum_{j \in \Sigma(V, x, r)} \|A_j\|^s \\
&\leq CD^s \inf_{\mathbf{k} \in \Sigma} H(\mathbf{k}) \sum_{j \in \Sigma(V, x, r)} \|A_j^\top|V^\perp\|^s \\
&\leq CD^s \sum_{j \in \Sigma(V, x, r)} \|A_j^\top|V^\perp\|^s \mathcal{H}^s(\text{proj}_{A_j^\top V^\perp}(X)) \\
&\leq CD^s \sum_{j \in \Sigma(V, x, r)} \mathcal{H}^s(\text{proj}_{A_j^\top V^\perp}(\varphi_j(X))) \\
&= CD^s \mathcal{H}^s\left(\bigcup_{j \in \Sigma(V, x, r)} \text{proj}_{A_j^\top V^\perp}(\varphi_j(X))\right) \\
&\leq CD^s \mathcal{H}^s(\text{proj}_{V^\perp}(X) \cap B(\text{proj}_{V^\perp}(x), 2r)) \leq C(4D)^s r^s
\end{aligned}$$

for all $x \in X$ and $0 < r < \text{diam}(X)$. Since $\inf_{\mathbf{k} \in \Sigma} H(\mathbf{k}) > 0$, we have finished the proof. \square

8. ASSOUD AND AFFINITY DIMENSIONS OF SELF-AFFINE SETS

In this section, we prove Theorem 3.6. We begin by a technical lemma whose proof is a standard transversality argument. The lemma investigates how, for a given affine iterated function system $\Phi_{\mathbf{v}}$, the projection of $\pi_{\mathbf{v}+\theta \otimes w}$ behaves locally in a given direction $w \in S^1$ for all $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$. Recall that $\theta \otimes w = (\theta_1 w, \dots, \theta_N w) \in (\mathbb{R}^2)^N$ is the Kronecker product.

Lemma 8.1. *If $\mathbf{A} = (A_1, \dots, A_N) \in GL_2(\mathbb{R})^N$ is strictly affine such that $\max_{\{1, \dots, N\}} \|A_i\| < \frac{1}{2}$, $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^2)^N$, and $w \in S^1$, then there exists $\delta > 0$ such that*

$$\frac{d}{d\tau_{\mathbf{i}|_1}} \left| \text{proj}_{\text{span}(w)^\perp}(\pi_{\mathbf{v}+\tau \otimes w}(\mathbf{i})) - \text{proj}_{\text{span}(w)^\perp}(\pi_{\mathbf{v}+\tau \otimes w}(\mathbf{j})) \right|_{\tau=\theta} > \delta$$

for all $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$ and $\mathbf{i}, \mathbf{j} \in \Sigma$ with $\mathbf{i}|_1 \neq \mathbf{j}|_1$.

Proof. Write $W = \text{span}(w) \in \mathbb{R}\mathbb{P}^1$ and $\mathbf{i} = i_1 i_2 \dots$ and $\mathbf{j} = j_1 j_2 \dots$. It is easy to see that

$$\begin{aligned} \text{proj}_{W^\perp}(\pi_{\mathbf{v}+\theta \otimes w}(\mathbf{i})) &= \text{proj}_{W^\perp} \left(\sum_{n=1}^{\infty} A_{\mathbf{i}|_{n-1}}(v_{i_n} + \theta_{i_n} w) \right) \\ &= \theta_{i_1} + \text{proj}_{W^\perp}(v_{i_1}) + \sum_{n=1}^{\infty} \|A_{\mathbf{i}|_{n-1}}^\top\| |W^\perp| \text{proj}_{A_{\mathbf{i}|_{n-1}}^\top W^\perp}(v_{i_n} + \theta_{i_n} w) \end{aligned}$$

Hence,

$$\frac{d}{d\theta_j} \text{proj}_{W^\perp}(\pi_{\mathbf{v}+\theta \otimes w}(\mathbf{i})) = \delta_{i_1, j} + \sum_{n=1}^{\infty} \|A_{\mathbf{i}|_{n-1}}^\top\| |W^\perp| \text{proj}_{A_{\mathbf{i}|_{n-1}}^\top W^\perp}(w) \delta_{i_n, j}$$

for all $j \in \{1, \dots, N\}$, where $\delta_{i, j} = 1$, if $i = j$, and $\delta_{i, j} = 0$, if $i \neq j$. Since $\max_{\{1, \dots, N\}} \|A_i\| < \frac{1}{2}$, it follows that

$$\begin{aligned} &\frac{d}{d\tau_{\mathbf{i}|_1}} \left| \text{proj}_{W^\perp}(\pi_{\mathbf{v}+\tau \otimes w}(\mathbf{i})) - \text{proj}_{W^\perp}(\pi_{\mathbf{v}+\tau \otimes w}(\mathbf{j})) \right|_{\tau=\theta} \\ &= \left| \sum_{n=1}^{\infty} \|A_{\mathbf{i}|_{n-1}}^\top\| |W^\perp| \text{proj}_{A_{\mathbf{i}|_{n-1}}^\top W^\perp}(w) \delta_{i_n, j} \right. \\ &\quad \left. - \sum_{n=1}^{\infty} \|A_{\mathbf{j}|_{n-1}}^\top\| |W^\perp| \text{proj}_{A_{\mathbf{j}|_{n-1}}^\top W^\perp}(w) \delta_{i_n, j} \right| \\ &\geq 1 - \sum_{n=1}^{\infty} \|A_{\mathbf{i}|_{n-1}}^\top\| - \sum_{n=1}^{\infty} \|A_{\mathbf{j}|_{n-1}}^\top\| \geq \frac{1 - 2 \max_i \|A_i\|}{1 - \max_i \|A_i\|} > 0 \end{aligned}$$

as claimed. \square

We are now ready to prove Theorem 3.6.

Proof of Theorem 3.6. To simplify notation, we assume that $X \subset B(0, 1)$. Let $\mathcal{N}(\mathbf{A})$ be as in (3.3). For each $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^2)^N$ we write $\varphi_i^\mathbf{v} = A_i + v_i$ and $\varphi_{\mathbf{i}}^\mathbf{v} = \varphi_{i_1}^\mathbf{v} \circ \dots \circ \varphi_{i_n}^\mathbf{v}$

for all $\mathbf{i} = i_1 \cdots i_n \in \Sigma_n$ and $n \in \mathbb{N}$. It is easy to see that, for every $q \in \mathbb{N}$, $W \in X_F$, and $\mathbf{i}, \mathbf{j} \in \Sigma_*$ with $\mathbf{i}|_1 \neq \mathbf{j}|_1$, the set

$$\{\mathbf{v} \in \mathcal{N}(\mathbf{A}) : \max_{x \in B(0,1)} |\text{proj}_{W^\perp}(\varphi_{\mathbf{i}}^\mathbf{v}(x) - \varphi_{\mathbf{j}}^\mathbf{v}(x))| < 5q^{-1} \|A_{\mathbf{i}}^\top |W^\perp\|\}$$

is an open subset of $\mathcal{N}(\mathbf{A})$. Therefore, the set

$$\mathcal{R}_q(\mathbf{A}) = \bigcup_{W \in X_F} \bigcup_{\substack{\mathbf{i}, \mathbf{j} \in \Sigma_* \\ \mathbf{i}|_1 \neq \mathbf{j}|_1}} \{\mathbf{v} \in \mathcal{N}(\mathbf{A}) : \max_{x \in B(0,1)} |\text{proj}_{W^\perp}(\varphi_{\mathbf{i}}^\mathbf{v}(x) - \varphi_{\mathbf{j}}^\mathbf{v}(x))| < 5q^{-1} \|A_{\mathbf{i}}^\top |W^\perp\|\}$$

is open and, recalling the definition of the projective open set condition (3.1), it is enough to show that $\mathcal{R}_q(\mathbf{A})$ is dense in $\mathcal{N}(\mathbf{A})$ for all $q \in \mathbb{N}$ since then the set $\mathcal{R}(\mathbf{A}) = \bigcap_{q \in \mathbb{N}} \mathcal{R}_q(\mathbf{A})$ is residual.

Fix $\mathbf{v} \in \mathcal{N}(\mathbf{A})$ and $\varepsilon > 0$. By the definition of $\mathcal{N}(\mathbf{A})$, there are $W \in X_F$ and $\mathbf{h}, \mathbf{1} \in \Sigma$ with $\mathbf{h}|_1 \neq \mathbf{1}|_1$ such that $\text{proj}_{W^\perp}(\pi_{\mathbf{v}}(\mathbf{h})) = \text{proj}_{W^\perp}(\pi_{\mathbf{v}}(\mathbf{1}))$. We may thus choose $n \in \mathbb{N}$ large enough so that $d_H(\text{proj}_{W^\perp}(\varphi_{\mathbf{h}|_n}^\mathbf{v}(X)), \text{proj}_{W^\perp}(\varphi_{\mathbf{1}|_n}^\mathbf{v}(X))) < \varepsilon$, where d_H denotes the Hausdorff distance. Note that there is a constant $c > 0$ such that $\min\{|x - y|, |x + y|\} \leq c \triangleleft(\text{span}(x), \text{span}(y))$ for all $x, y \in S^1$. Following (6.14) and recalling the definition of domination, we see that there are constants $D, C \geq 1$ and $0 < \tau < 1$ such that

$$\begin{aligned} \max_{x \in B(0,1)} |\text{proj}_{A_{\mathbf{k}}^\top W^\perp}(x) - \text{proj}_{A_{\mathbf{k}}^\top V^\perp}(x)| &\leq \|\text{proj}_{A_{\mathbf{k}}^\top W^\perp} - \text{proj}_{A_{\mathbf{k}}^\top V^\perp}\| \\ &\leq c \triangleleft(A_{\mathbf{k}}^\top W^\perp, A_{\mathbf{k}}^\top V^\perp) \leq cD \frac{\alpha_2(A_{\mathbf{k}})}{\alpha_1(A_{\mathbf{k}})} \leq cDC\tau^{|\mathbf{k}|}. \end{aligned} \quad (8.1)$$

for all $\mathbf{k} \in \Sigma_*$ and $V \in X_F$. Recalling (2.6), let $\mathbf{f} \in \Sigma$ be such that $W = \Pi(\mathbf{f})$. Fix $q \in \mathbb{N}$, choose $m \in \mathbb{N}$ large enough so that $\max\{1 + cDC\tau^m, (1 + cD^2C\tau^m)/(1 - cD^2C\tau^m)\} < 1 + q^{-1}$, and write $\mathbf{k} = \mathbf{f}|_m$. Observe also that

$$\| \|A_{\mathbf{g}}|W\| - \|A_{\mathbf{g}}|V\| \| \leq c \|A_{\mathbf{g}}\| \triangleleft(V, W)$$

and hence, by Lemma 2.8,

$$\left| \frac{\|A_{\mathbf{g}}|W\|}{\|A_{\mathbf{g}}|V\|} - 1 \right| \leq \frac{c \|A_{\mathbf{g}}\| \triangleleft(V, W)}{\|A_{\mathbf{g}}|V\|} \leq cD \triangleleft(V, W). \quad (8.2)$$

for all $\mathbf{g} \in \Sigma_*$ and $V \in X_F$. Writing $\mathbf{i} = \mathbf{h}|_n \mathbf{k}|_n$ and $\mathbf{j} = \mathbf{1}|_n \mathbf{k}|_n$, let $y_{\mathbf{i}}$ and $y_{\mathbf{j}}$ be the fixed points of $\varphi_{\mathbf{i}}^\mathbf{v}$ and $\varphi_{\mathbf{j}}^\mathbf{v}$, respectively, and $v_{\mathbf{i}} = (\text{Id} - A_{\mathbf{i}})y_{\mathbf{i}}$ and $v_{\mathbf{j}} = (\text{Id} - A_{\mathbf{j}})y_{\mathbf{j}}$ be the translation vectors of $\varphi_{\mathbf{i}}^\mathbf{v}$ and $\varphi_{\mathbf{j}}^\mathbf{v}$, respectively. Simple algebraic manipulations show that

$$\begin{aligned} \text{proj}_{W^\perp}(\varphi_{\mathbf{i}}^\mathbf{w}(x) - \varphi_{\mathbf{j}}^\mathbf{w}(x)) &= \|A_{\mathbf{i}}^\top |W^\perp\| (\text{proj}_{A_{\mathbf{i}}^\top W^\perp}(x) - \text{proj}_{A_{\mathbf{j}}^\top W^\perp}(x)) \\ &\quad + \|A_{\mathbf{i}}^\top |W^\perp\| \left(1 - \frac{\|A_{\mathbf{j}}^\top |W^\perp\|}{\|A_{\mathbf{i}}^\top |W^\perp\|} \right) \text{proj}_{A_{\mathbf{j}}^\top W^\perp}(x) \\ &\quad + \text{proj}_{W^\perp}(w_{\mathbf{i}}) - \text{proj}_{W^\perp}(w_{\mathbf{j}}) \end{aligned} \quad (8.3)$$

for all $\mathbf{w} = (w_1, \dots, w_N) \in (\mathbb{R}^2)^N$ and $x \in B(0,1)$, where $w_{\mathbf{i}} = \sum_{k=1}^{|\mathbf{i}|} A_{\mathbf{i}|_{k-1}} w_{i_k}$ and $w_{\mathbf{j}} = \sum_{k=1}^{|\mathbf{j}|} A_{\mathbf{j}|_{k-1}} w_{j_k}$ are the translation vectors of $\varphi_{\mathbf{i}}^\mathbf{w}$ and $\varphi_{\mathbf{j}}^\mathbf{w}$, respectively. Furthermore,

by using (8.2), (8.1), and recalling the choice of m ,

$$\begin{aligned} \frac{\|A_j^\top |W^\perp\|}{\|A_i^\top |W^\perp\|} &= \frac{\|A_k^\top A_{h|n}^\top |A_k^\top A_{1|n}^\top W^\perp\| \|A_k^\top A_{1|n}^\top |W^\perp\|}{\|A_k^\top A_{1|n}^\top |A_k^\top A_{h|n}^\top W^\perp\| \|A_k^\top A_{h|n}^\top |W^\perp\|} \\ &\leq \frac{1 + cD \angle(W^\perp, A_k^\top A_{1|n}^\top W^\perp)}{1 - cD \angle(W^\perp, A_k^\top A_{h|n}^\top W^\perp)} \leq \frac{1 + cD^2 C \tau^m}{1 - cD^2 C \tau^m} < 1 + q^{-1}. \end{aligned} \quad (8.4)$$

Let $\delta > 0$ be as in Lemma 8.1. By Lemma 8.1, there exists $\mathbf{w} = (w_1, \dots, w_N) \in (\mathbb{R}^2)^N$ in the $\varepsilon\delta^{-1}$ -neighborhood of $\mathbf{v} \in \mathcal{N}(\mathbf{A})$ such that

$$\text{proj}_{W^\perp}(z_i) = \text{proj}_{W^\perp}(z_j),$$

where z_i and z_j are the fixed points of $\varphi_i^{\mathbf{w}}$ and $\varphi_j^{\mathbf{w}}$, respectively, and $w_i = (\text{Id} - A_i)z_i$ and $w_j = (\text{Id} - A_j)z_j$ are the translation vectors of $\varphi_i^{\mathbf{w}}$ and $\varphi_j^{\mathbf{w}}$, respectively. Hence, by (8.1) and (8.4),

$$\begin{aligned} &|\text{proj}_{W^\perp}(w_i) - \text{proj}_{W^\perp}(w_j)| \\ &= \|A_i^\top |W^\perp\| \text{proj}_{A_i^\top W^\perp}(z_i) - \|A_j^\top |W^\perp\| \text{proj}_{A_j^\top W^\perp}(z_j) \\ &\leq (\|A_i^\top |W^\perp\| + \|A_j^\top |W^\perp\|) cDC\tau^m \\ &\quad + \|A_i^\top |W^\perp\| \text{proj}_{W^\perp}(z_i) - \|A_j^\top |W^\perp\| \text{proj}_{W^\perp}(z_j) \quad (8.5) \\ &\leq \|A_i^\top |W^\perp\| \left(\left(1 + \frac{\|A_j^\top |W^\perp\|}{\|A_i^\top |W^\perp\|}\right) cDC\tau^m + \left|1 - \frac{\|A_j^\top |W^\perp\|}{\|A_i^\top |W^\perp\|}\right| \right) \\ &\leq 3q^{-1} \|A_i^\top |W^\perp\|. \end{aligned}$$

Thus, by (8.1), (8.3), and (8.5), we see that

$$\max_{x \in B(0,1)} |\text{proj}_{W^\perp}(\varphi_i^{\mathbf{w}}(x) - \varphi_j^{\mathbf{w}}(x))| < 5q^{-1} \|A_i^\top |W^\perp\|$$

and consequently, $\mathbf{w} \in \mathcal{R}_q(\mathbf{A})$. \square

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