SCALING LIMITS OF SELF-CONFORMAL MEASURES

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ABSTRACT. We prove that every self-conformal measure on $\mathbb{R}$ and every self-similar measure on $\mathbb{R}^d$ is uniformly scaling and generate an ergodic fractal distribution, generalizing previous results by eliminating the need for separation conditions. Additionally, we establish applications to the prevalence of normal numbers in self-conformal sets, resonance between self-conformal measures on the line, and projections of self-affine measures on carpets.

RÉSUMÉ. Nous prouvons que toute mesure auto-conforme sur $\mathbb{R}$ et toute mesure auto-similaire sur $\mathbb{R}^d$ sont uniformément échelonnées et engendre une distribution fractale ergodique, généralisant les résultats antérieurs en éliminant le besoin de conditions de séparation. De plus, nous établissons des applications à la prévalence des nombres normaux dans les ensembles auto-conformes, à la résonance entre les mesures auto-conformes sur la droite, et aux projections des mesures auto-affines sur les tapis.

1. INTRODUCTION

Analyzing an object through its small-scale structure via tangents is a classical approach in analysis, with significant applications in geometric measure theory. For a Radon measure μ on $\mathbb{R}^d$, a point x in its support $\operatorname{spt}\mu$, and a number $t \geqslant 0$, the *magnification* of μ at x by 2^t is the probability measure $\mu_{x,t}$ on the unit ball $B(0,1)$, defined by

$$\mu_{x,t}(A) = \frac{\mu(2^{-t}A + x)}{\mu(B(x, 2^{-t}))}$$

for every Borel set $A \subseteq B(0,1)$. Here and throughout the paper, $B(x,r)$ denotes the closed ball of radius r centered at x. The family $(\mu_{x,t})_{t\geqslant 0}$ is called the *scenery* of μ at x. The statistical properties of the scenery reveal the local geometry of μ at x. To capture these statistics, we define the *scenery flow* of μ at x as the one-parameter family $(\langle\mu\rangle_{x,T})_{T>0}$, where

$$\langle\mu\rangle_{x,T} = \frac{1}{T}\int_0^T \delta_{\mu_{x,t}}\,\mathrm{d}t \tag{1.1}$$

and δ_ν denotes the Dirac measure at ν. The accumulation points of (1.1) in the weak*-topology are called *tangent distributions* of μ at x. If there exists a measure P such

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that, for μ-almost every x, P is the unique tangent distribution at x, we say that μ is *uniformly scaling* and *generates* P.

The concept of uniform scaling was introduced by Gavish [14], who proved that self-similar measures on $\mathbb{R}^d$ satisfying the open set condition are uniformly scaling. A Borel probability measure μ on $\mathbb{R}^d$ is *self-similar* if there exists a *similarity iterated function system* $\Phi = (f_i)_{i\in\Lambda}$, where $\Lambda = \{1,\dots,N\}$, consisting of contracting, affine, and angle-preserving maps $f_i\colon \mathbb{R}^d \to \mathbb{R}^d$, and a probability vector $(p_i)_{i\in\Lambda}$ such that

$$\mu = \sum_{i=1}^{N} p_i (f_i)_*\mu. \tag{1.2}$$

Gavish's proof utilized Furstenberg's CP-chains, while Hochman [15] provided a more direct ergodic-theoretic proof in his comprehensive study of tangent distributions. Subsequently, Hochman and Shmerkin [17], employing techniques inspired by Gavish [14] and extended by Hochman [15], showed that self-conformal measures on $\mathbb{R}$ satisfying the open set condition are also uniformly scaling. A Borel probability measure μ is *self-conformal* if there exists a bounded, open, convex set $U \subset \mathbb{R}^d$ such that μ satisfies (1.2) with maps $f_i\colon U \to U$ that are contractive, injective, and differentiable, with Hölder continuous derivatives $x \mapsto Df_i|_x$, where the differentials $Df_i|_x$ are non-zero and the normalized differentials $\|Df_i|_x\|^{-1}Df_i|_x$ are orthogonal for all $x \in U$. The system $\Phi = (f_i)_{i\in\Lambda}$ is called a *conformal iterated function system*. In [25], the third author demonstrated that for a self-similar measure to be uniformly scaling, the weak separation condition is sufficient.

Self-affine measures are Borel probability measures that satisfy (1.2) with affine maps f_i. Ferguson, Fraser, and Sahlsten [12] established that self-affine measures on Bedford-McMullen carpets in $\mathbb{R}^2$ are uniformly scaling. This result largely represents the current understanding of the scenery flow for self-affine measures. While Kempton [20] showed that self-affine measures with an irreducibility condition are not necessarily uniformly scaling, the question for general reducible self-affine measures remains open.

In this work, we fully resolve the scaling properties of self-conformal measures in $\mathbb{R}^d$, eliminating the need for separation conditions assumed in [14, 17, 25], and establish their uniform scaling property on $\mathbb{R}$. Consequently, we derive applications to the prevalence of normal numbers in self-conformal sets and the dimensions of convolutions of self-conformal measures on the line, both active areas of research. Additionally, we prove that self-affine measures on carpets satisfying the strong separation condition are uniformly scaling and show that the dimension of their non-principal orthogonal projections is preserved.

To present the main results, we introduce the following notation. Let Λ be a finite set with at least two elements, and let $\Phi = (f_i)_{i\in\Lambda}$ be a conformal iterated function system on $\mathbb{R}^d$ with attractor K. A measure $\bar{\mu}$ on $\Lambda^{\mathbb{N}}$ is called *quasi-Bernoulli* if there exists a constant $C \geqslant 1$ such that

$$C^{-1}\bar{\mu}([\mathtt{i}])\bar{\mu}([\mathtt{j}]) \leqslant \bar{\mu}([\mathtt{ij}]) \leqslant C\bar{\mu}([\mathtt{i}])\bar{\mu}([\mathtt{j}]) \tag{1.3}$$

for all finite words $\mathtt{i},\mathtt{j} \in \Lambda^*$ formed from the elements of Λ, where $[\mathtt{i}]$ denotes the collection of infinite words with prefix $\mathtt{i}$. We assume, without loss of generality, that $\bar{\mu}$ is fully supported on $\Lambda^{\mathbb{N}}$, adjusting Φ by removing maps if necessary. If $C = 1$, then $\bar{\mu}$ is a *Bernoulli measure* with marginals $(\bar{\mu}([i]))_{i\in\Lambda}$. Let $\Pi\colon \Lambda^{\mathbb{N}} \to K$ be the canonical

projection; see §2.1 for the definition. If $\bar{\mu}$ is quasi-Bernoulli, the pushforward measure $\mu = \Pi_* \bar{\mu}$ on K is also called quasi-Bernoulli. Specifically, any measure satisfying (1.2) is a Bernoulli measure, as it is the canonical projection of the Bernoulli measure on $\Lambda^{\mathbb{N}}$ with marginals $(p_i)_{i \in \Lambda}$.

Our first main result establishes that every self-conformal measure on $\mathbb{R}$ and every self-similar measure on $\mathbb{R}^d$ is uniformly scaling, independent of separation conditions. For the definition of an ergodic fractal distribution, see §2.2.

Theorem 1.1. *Let Φ be a conformal iterated function system on $\mathbb{R}$ or a similarity iterated function system on $\mathbb{R}^d$ with attractor K, and let μ be a quasi-Bernoulli measure on K. Then μ is uniformly scaling and generates an ergodic fractal distribution.*

This follows from Theorem 1.2, which establishes the result under a more general assumption on the generated orthogonal group. For each $\mathtt{j} \in \Lambda^*$, we define

$$\mathcal{O}_\Phi(\mathtt{j}) = \overline{\{\|Df_{\mathtt{ji}}|_x\|^{-1} Df_{\mathtt{ji}}|_x : \mathtt{i} \in \Lambda^* \text{ and } x \in K\}},$$

where the closure is taken in the operator norm. Similarly, we define $\mathcal{O}_\Phi$ as above, but without reference to $\mathtt{j}$.

Theorem 1.2. *Let Φ be a conformal iterated function system on $\mathbb{R}^d$ with attractor K, and let μ be a quasi-Bernoulli measure on K. If $\mathrm{Id} \in \mathcal{O}_\Phi(\mathtt{j})$ for all $\mathtt{j} \in \Lambda^*$, then μ is uniformly scaling and generates an ergodic fractal distribution.*

The proof of Theorem 1.1 follows from a straightforward compactness argument in the orthogonal group. The identity map Id belongs to $\mathcal{O}_\Phi(\mathtt{j})$ for all $\mathtt{j} \in \Lambda^*$, for instance, when $d = 1$ or when Φ is a similarity iterated function system.

In higher dimensions, self-conformal measures are generally not uniformly scaling. Fraser and Pollicott [13] provide a counterexample: the upper half of the unit circle's boundary in $\mathbb{R}^2$ is a self-conformal set, and the normalized one-dimensional Lebesgue measure μ on this semicircle is a self-conformal measure. At any point x on the semicircle, all tangent measures of μ at x are supported on lines parallel to the tangent line at x. Consequently, the collections of tangent measures at distinct points x and y are disjoint, implying that μ is not uniformly scaling.

In the general case, our main result is the following theorem.

Theorem 1.3. *Let Φ be a conformal iterated function system on $\mathbb{R}^d$ with attractor K, and let μ be a quasi-Bernoulli measure on K. There exists an ergodic fractal distribution P such that, for μ-almost every x, every tangent distribution Q at x satisfies: if $Q = \int Q_\omega \, \mathrm{d}Q(\omega)$ is the ergodic decomposition of Q, then for Q-almost every ω, there exists an orthogonal matrix $U_\omega \in \mathcal{O}_\Phi$ such that $Q_\omega = (U_\omega)_{**} P$.*

This follows from Theorem 1.4, which establishes the existence of a conformal map h such that the pushforward measure $h_*\mu$ is uniformly scaling along a subsequence. The class $\mathcal{C}^{1+\alpha}_{\mathrm{conf}}(U)$ of conformal maps in $\mathbb{R}^d$ is defined in §2.

Theorem 1.4. *Let Φ be a conformal iterated function system on $[-1,1]^d$ with attractor K, and let μ be a quasi-Bernoulli measure on K. There exists an ergodic fractal distribution P, a sequence $(n_k)_{k \in \mathbb{N}}$, and a function $h \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$ such that*

$$\lim_{k \to \infty} \langle h_*\mu \rangle_{x, n_k} = \lim_{k \to \infty} \frac{1}{n_k} \int_0^{n_k} \delta_{(h_*\mu)_{x,t}} \, \mathrm{d}t = P$$

in the weak-topology for $h_*\mu$-almost all x.*

The proofs of Theorems 1.1–1.4 are presented in §3. These proofs are indirect, relying on a result of Hochman [15, Theorem 1.7] stating that, for any Radon measure μ, typical tangent measures—weak*-accumulation points of the scenery $(\mu_{x,t})_{t\geqslant 0}$—are uniformly scaling and generate fractal distributions. For a self-conformal measure μ, these tangent measures have a specific structure: Lemma 3.6 shows they take the form

$$\int \frac{(h_*\mu)|_{B(0,1)}}{(h_*\mu)(B(0,1))}\,\mathrm{d}\nu(h),$$

where ν is a measure on the space of conformal maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. For a quasi-Bernoulli measure μ, its tangent measures are equivalent to measures of this form. Additionally, Hochman [15] proves that many tangent measures have Hausdorff dimension at most that of μ. The proof is completed by Proposition 3.5, a key technical result, which establishes that the sceneries of a measure of the form above, with dimension close to that of μ, are asymptotic to the sceneries of μ at many points, enabling the transfer of regularity from the local statistics of typical tangent measures to those of μ.

We provide a detailed sketch of the proof of Theorem 1.1 for the case where μ is a homogeneous self-similar measure on $\mathbb{R}$. Let $\mu = \sum_{i\in\Lambda} p_i (f_i)_*\mu$ be a homogeneous self-similar measure on $\mathbb{R}$, where homogeneity means that the maps f_i have a common scaling factor. Our goal is to show that there exists an ergodic fractal distribution P such that, for μ-almost every x,

$$\langle\mu\rangle_{x,T} = \frac{1}{T}\int_0^T \delta_{\mu_{x,t}}\,\mathrm{d}t \to P.$$

To this end, it suffices to prove that for $\mu\times\mu$-almost every (x,y), if $\langle\mu\rangle_{x,T_{1,i}} \to P_x$ and $\langle\mu\rangle_{x,T_{1,i}} \to P_y$, where P_x and P_y are fractal distributions, along some sequences $(T_{1,i})_{i\in\mathbb{N}}$ and $(T_{2,i})_{i\in\mathbb{N}}$ diverging to infinity, then $P_x = P_y$. By the self-similarity of μ, Lemma 3.6 shows that for P_x-almost every measure ν, there exists a Borel probability measure ν' on $\mathbb{R}$ and a rescaling $S_*\mu$ of μ such that

$$\nu = \frac{(\nu' * S_*\mu)|_{B(0,1)}}{(\nu' * S_*\mu)(B(0,1))},$$

where $*$ denotes the classical convolution of measures. The same holds for P_y.

For simplicity, assume P_x and P_y are ergodic fractal distributions. By the properties of P_x, for P_x-almost every ν, the scenery flow of ν at ν-almost every z converges to P_x. Proposition 3.5, the key novel result of this work, establishes that for P_x-almost every ν, for all sufficiently large T, and for ν'-almost every z_1 and $S_*\mu$-almost every z_2, the distributions

$$\langle\nu\rangle_{z,T} = \frac{1}{T}\int_0^T \delta_{\nu_{z,t}}\,\mathrm{d}t \quad\text{and}\quad \langle S_*\mu\rangle_{z_2,T} = \frac{1}{T}\int_0^T \delta_{(S_*\mu)_{z_2,t}}\,\mathrm{d}t,$$

where $z = z_1 + z_2$, are arbitrarily close. This relies on ν and μ having the same Hausdorff dimension. Since $\langle\nu\rangle_{z,T}$ approximates P_x, it follows that for μ-almost every z_2, $\langle\mu\rangle_{z_2,T}$ is close to P_x. Similarly, for P_y, we obtain that for μ-almost every z_2, $\langle\mu\rangle_{z_2,T}$ is close to P_y. Thus, for μ-almost every z_2, the distribution $\langle\mu\rangle_{z_2,T}$ is close to both P_x and P_y, implying $P_x = P_y$.

We now discuss applications of Theorem 1.2. For a contracting continuously differentiable map $f\colon \mathbb{R} \to \mathbb{R}$, we define its *asymptotic contraction ratio* as $\lambda(f) = |Df|_p|$, where p is the fixed point of f. For $\beta > 1$, the $\times\beta$ map, defined by $x \mapsto \beta x \mod 1$, has a unique absolutely continuous invariant probability measure, denoted by μ_β, as shown in [22]. When $\beta \in \mathbb{N}$, this measure is the Lebesgue measure on $[0, 1)$. A real number x is *β-normal* if the sequence $(\beta^n x \mod 1)_{n\in\mathbb{N}}$ equidistributes with respect to μ_β. Following Hochman and Shmerkin [17], an iterated function system $\Phi = (f_i)_{i\in\Lambda}$ is *totally non-linear* if it is not conjugate to a linear iterated function system via a diffeomorphism; that is, there exists no diffeomorphism g such that $g\Phi = (g \circ f_i \circ g^{-1})_{i\in\Lambda}$ consists solely of linear maps.

By combining Theorem 1.2 with the methods of Hochman and Shmerkin [17], we obtain a result on the prevalence of normal numbers in self-conformal sets, with the proof provided in §4. A conformal iterated function system $\Phi = (f_i)_{i\in\Lambda}$ on $\mathbb{R}$ is *arithmetically independent* of a real number $\beta > 1$ if zero is an accumulation point of the set

$$\{\log \lambda(f_{\mathtt{i}}) + n \log \beta : \mathtt{i} \in \Lambda^* \text{ and } n \in \mathbb{N}\},$$

where log denotes the base-2 logarithm throughout this paper. For instance, Φ is arithmetically independent of β if $\frac{\log \lambda(f_i)}{\log \beta} \notin \mathbb{Q}$ for some $i \in \Lambda$. A *Pisot number* is a real algebraic integer greater than 1 whose Galois conjugates all have absolute value less than 1. In particular, integers greater than 1 are Pisot numbers.

Theorem 1.5. *Let Φ be a conformal iterated function system on $\mathbb{R}$ with attractor K, and let μ be a non-atomic quasi-Bernoulli measure on K. For a Pisot number $\beta > 1$, if Φ is arithmetically independent of β, then for any diffeomorphism g, $g_*\mu$-almost every x is β-normal. Furthermore, if Φ is totally non-linear and consists of real analytic maps, this holds for every Pisot number $\beta > 1$.*

Theorem 1.5 extends the result of Algom, Baker, and Shmerkin [1] to the self-conformal setting. When μ is a self-conformal measure, β is an integer, and Φ exhibits sufficient non-linearity, the conclusion of Theorem 1.5 can also be derived by combining Fourier decay results for self-conformal measures from [2–4] with the Davenport-Erdős-LeVeque criterion for pointwise normality. However, these Fourier decay results rely on μ being the projection of a Bernoulli measure on $\Lambda^{\mathbb{N}}$ and do not appear applicable to β-normal numbers for non-integer β.

By combining Theorem 1.2 with a version of the local entropy averages from Hochman and Shmerkin [16], as extended in [15], we obtain a result on the resonance between quasi-Bernoulli measures on self-conformal sets. Two exact-dimensional probability measures μ and ν on $\mathbb{R}$ are said to *resonate* if

$$\dim(\mu * \nu) < \min\{1, \dim \mu + \dim \nu\};$$

otherwise, they are said to *dissonate*. Two conformal iterated function systems $\Psi = (f_i)_{i\in\Gamma}$ and $\Phi = (g_j)_{j\in\Lambda}$ on $\mathbb{R}$ are *independent* if zero is an accumulation point of the set

$$\{\log \lambda(f_{\mathtt{i}}) - \log \lambda(g_{\mathtt{j}}) : \mathtt{i} \in \Gamma^* \text{ and } \mathtt{j} \in \Lambda^*\}.$$

For example, Ψ and Φ are independent if $\frac{\log \lambda(f_i)}{\log \lambda(g_j)} \notin \mathbb{Q}$ for some $(i, j) \in \Gamma \times \Lambda$.

Theorem 1.6. *Let Ψ and Φ be conformal iterated function systems on $\mathbb{R}$ with attractors K_Ψ and K_Φ, respectively. If Ψ and Φ are independent, then for any quasi-Bernoulli measures μ on K_Ψ and ν on K_Φ,*

$$\dim(\mu * \nu) = \min\{1, \dim \mu + \dim \nu\},$$

meaning that μ and ν dissonate.

The proof of Theorem 1.6 is deferred to §5. The study of resonance between dynamically defined sets and measures has a rich history, originating with Furstenberg's conjecture in the 1960s on sums of $\times 2$- and $\times 3$-invariant sets. Broadly, resonance is expected only when the sets or their defining dynamics are arithmetically similar. Theorem 1.6 aligns with analogous results for various dynamically defined sets and measures, as explored in [10, 16, 21, 24]. Recently, Bruce and Jin [9] established a version of Theorem 1.6 for a broad class of dynamically defined measures on homogeneous self-similar sets, including canonical projections of ergodic measures. However, Theorem 1.6 does not readily extend to ergodic measures, and the results of [9] do not appear to apply to measures on self-conformal sets. Previously, Theorem 1.6 was known for self-conformal measures only under the strong separation condition, as shown in [15, Theorem 1.39]. Whether an analogue of Theorem 1.6 holds in higher dimensions remains an open question, though recent work by the third author [26] confirms this for a class of planar self-affine measures.

Consider a *diagonal iterated function system* $\Phi = (f_i)_{i\in\Lambda}$ on $\mathbb{R}^2$, where each map $f_i \colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $f_i(x) = A_i x + a_i$, is affine with

$$A_i = \begin{pmatrix} \rho_i & 0 \\ 0 & \lambda_i \end{pmatrix}.$$

We call Φ *proximal* if $\rho_i \neq \lambda_i$ for at least one $i \in \Lambda$. The following projection theorem extends the result of Ferguson, Fraser, and Sahlsten [12] to measures without grid structure, i.e., without assumptions on the translation vectors a_i. We say Φ satisfies the *rectangular strong separation condition* if $f_i([-1,1]^2) \cap f_j([-1,1]^2) = \emptyset$ for all $i \neq j$. Additionally, Φ has *independent eigenvalues* if zero is an accumulation point of the set

$$\{\log \lambda_{\mathtt{i}} - \log \rho_{\mathtt{j}} : \mathtt{i}, \mathtt{j} \in \Lambda^*\}.$$

For example, Φ has independent eigenvalues if $\frac{\log \lambda_i}{\log \rho_i} \notin \mathbb{Q}$ for some $i \in \Lambda$.

Theorem 1.7. *Let Φ be a proximal, diagonal iterated function system on $\mathbb{R}^2$ with attractor K, satisfying the rectangular strong separation condition. If Φ has independent eigenvalues, then for any self-affine measure μ on K,*

$$\dim \pi_* \mu = \min\{1, \dim \mu\}$$

for all non-principal orthogonal projections π.

The paper is organized as follows. In §2, we introduce notation and preliminaries. The proofs of Theorems 1.1–1.4 are presented in §3. Theorems 1.5 and 1.6, concerning normal numbers and resonance, are proven in §4 and §5, respectively. In §6, we establish that self-affine measures on attractors of diagonal iterated function systems are uniformly scaling, concluding the proof of Theorem 1.7.

2. Preliminaries

Let $U \subset \mathbb{R}^d$ be a bounded, open, convex set, and let $\alpha > 0$. Since U is bounded, the space $\mathcal{C}^1(U)$ of continuously differentiable maps $f\colon U \to \mathbb{R}^d$, equipped with the $\mathcal{C}^1$-norm

$$\|f\|_{\mathcal{C}^1} = \sup_{x\in U} |f(x)| + \sup_{x\in U} \|Df|_x\|$$

is a Banach space. Define the space

$$\mathcal{C}^{1+\alpha}(U) = \{f \in \mathcal{C}^1(U) : [Df]_\alpha < \infty\},$$

where

$$[Df]_\alpha = \sup_{\substack{x,y\in U\\ x\neq y}} \frac{\|Df|_x - Df|_y\|}{|x-y|^\alpha}$$

is the *α-Hölder seminorm*, to be the vector space of all continuously differentiable maps $f\colon U \to \mathbb{R}^d$ for which $x \mapsto Df|_x$ is α-Hölder continuous on U. The α-Hölder seminorm vanishes for affine functions, so it is not a norm. Define the space

$$\mathcal{C}^{1+\alpha}_{\mathrm{conf}}(U) = \Bigg\{ f \in \mathcal{C}^{1+\alpha}(U) : f \text{ is injective, } f(U) \subseteq U, \\ \text{and } \frac{Df|_x}{\|Df|_x\|} \in O(d) \text{ for all } x \in U \Bigg\},$$

where $O(d)$ denotes the orthogonal group in $\mathbb{R}^d$. Elements of $\mathcal{C}^{1+\alpha}_{\mathrm{conf}}(U)$ are called *conformal maps*. For $d = 1$, these are simply $\mathcal{C}^{1+\alpha}$ functions. For $d = 2$, conformal maps are complex analytic, and for $d \geqslant 3$, by Liouville's theorem (see, e.g., [27, Theorem 4.1]), they are Möbius transformations. A conformal map f is *contractive* if $\sup_{x\in U} \|Df|_x\| < 1$. For $M > 0$, define the space

$$\mathcal{C}^{1+\alpha}_M(U) = \{f \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}(U) : [Df]_\alpha \leqslant M \text{ and } M^{-1} \leqslant \|Df|_x\| \leqslant M \text{ for all } x \in U\}.$$

By uniform continuity, maps in $\mathcal{C}^{1+\alpha}_M(U)$ extend to the closure of U. Therefore, by the Arzelá-Ascoli theorem, $\mathcal{C}^{1+\alpha}_M(U)$ is a compact metric space under the $\mathcal{C}^1$-norm. We endow $\mathcal{C}^{1+\alpha}_{\mathrm{conf}}(U) = \bigcup_{M=1}^\infty \mathcal{C}^{1+\alpha}_M(U)$ with the Borel σ-algebra $\mathcal{B}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}(U))$, generated by the topology induced by the union of the topologies on $\mathcal{C}^{1+\alpha}_M(U)$ under the $\mathcal{C}^1$-norm.

Let X be a metric space, and denote by $\mathcal{P}(X)$ the set of all Borel probability measures on X. Unless otherwise stated, all measures in this paper are Borel probability measures. The *weak*-topology* on $\mathcal{P}(\mathbb{R}^d)$ is the coarsest topology such that the evaluation functionals

$$\mu \mapsto \int_{\mathbb{R}^d} f(x)\,\mathrm{d}\mu(x)$$

are continuous for every bounded continuous function $f\colon \mathbb{R}^d \to \mathbb{R}$. The *Borel σ-algebra* $\mathcal{B}(\mathcal{P}(\mathbb{R}^d))$ on $\mathcal{P}(\mathbb{R}^d)$ is the σ-algebra generated by the open sets in the weak*-topology. For $\mu, \nu \in \mathcal{P}(X)$, the *Lévy-Prokhorov metric* is defined as

$$d_{\mathrm{LP}}(\mu,\nu) = \inf\{\varepsilon > 0 : \mu(A) \leqslant \nu(A^\varepsilon) + \varepsilon \text{ for all Borel sets } A\},$$

where A^ε is the ε-neighborhood of A in X. The definition of d_{LP} is symmetric, as for any Borel set A, we have $(X \setminus A^\varepsilon)^\varepsilon \subseteq X \setminus A$ and hence,

$$\nu(A) = 1 - \nu(X \setminus A) \leqslant 1 - \nu((X \setminus A^\varepsilon)^\varepsilon) \leqslant 1 - (\mu(X \setminus A^\varepsilon) - \varepsilon) = \mu(A^\varepsilon) + \varepsilon$$

ensuring d_{LP} is a metric. If X is separable, convergence in Lévy-Prokhorov metric is equivalent to the weak*-convergence and also the space $(\mathcal{P}(X), d_{\mathrm{LP}})$ is separable; for example, see [8, Theorem 6.8]. For a measure μ and a measurable function f, the pushforward measure is defined as $f_*\mu = \mu \circ f^{-1}$. Let $\mathcal{D}_n(\mathbb{R}^d)$ denote the partition of $\mathbb{R}^d$ into dyadic cubes of side length 2^{-n}, homothetic to $[0,1)^d$, with vertex at the origin. For $x \in \mathbb{R}^d$, let $\mathcal{D}_n(x) \in \mathcal{D}_n(\mathbb{R}^d)$ be the cube containing x. For a Radon measure μ on $\mathbb{R}^d$ and $x \in \operatorname{spt}\mu$, the *dyadic magnification* of μ at x is the Borel probability measure

$$\mu^{\mathcal{D}_n(x)} = \frac{1}{\mu(\mathcal{D}_n(x))}(f_{\mathcal{D}_n(x)})_*(\mu|_{\mathcal{D}_n(x)}), \tag{2.1}$$

where $\mu|_{\mathcal{D}_n(x)}$ is the restriction of μ to $\mathcal{D}_n(x)$, and $f_{\mathcal{D}_n(x)}$ is the unique homothety mapping $\mathcal{D}_n(x)$ to $[0,1)^d$. If μ is a Radon measure on $\mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ is a Borel set with $0 < \mu(A) < \infty$, then the measure μ conditioned to A is the Borel probability measure $\mu_A = \mu(A)^{-1}\mu|_A$. We will use the convention that if $\mu(\mathcal{D}_n(x)) = 0$ then $\mu^{\mathcal{D}_n(x)}$ is the zero measure. For Radon measures μ and ν and $C \geqslant 1$, we write $\mu \sim_C \nu$ to indicate that μ and ν are mutually absolutely continuous and $C^{-1} \leqslant \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \leqslant C$, where $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ is the Radon-Nikodym derivative.

2.1. **Symbolic space.** Let Λ be a finite set with $\#\Lambda \geqslant 2$. Define $\Lambda^* = \bigcup_{n\in\mathbb{N}} \Lambda^n$ as the set of finite words, and for $\mathtt{i} \in \Lambda^*$, let $|\mathtt{i}|$ denote the unique integer such that $\mathtt{i} \in \Lambda^{|\mathtt{i}|}$. For a finite word $\mathtt{i} = i_1 \cdots i_n \in \Lambda^n$, let $\mathtt{i}^- = i_1 \cdots i_{n-1} \in \Lambda^{n-1}$. For an infinite word $\mathtt{j} \in \Lambda^{\mathbb{N}}$ and $k \in \mathbb{N}$, let $\mathtt{j}|_k \in \Lambda^k$ denote the projection onto the first k coordinates. The *left shift* $\sigma\colon \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ is defined by $\sigma\mathtt{j} = j_2 j_3 \cdots$ for $\mathtt{j} = j_1 j_2 \cdots$. Equip $\Lambda^{\mathbb{N}}$ with the topology generated by the *cylinder sets* defined as

$$[\mathtt{i}] = \{\mathtt{j} \in \Lambda^{\mathbb{N}} : \ \mathtt{j}|_{|\mathtt{i}|} = \mathtt{i}\}$$

for all $\mathtt{i} \in \Lambda^*$. For an iterated function system $\Phi = (f_i)_{i\in\Lambda}$ with attractor K, the *canonical projection* $\Pi\colon \Lambda^{\mathbb{N}} \to K$ is defined as

$$\Pi(\mathtt{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0)$$

for all $\mathtt{i} = i_1 i_2 i_3 \cdots \in \Lambda^{\mathbb{N}}$. Note that Π is a continuous surjection. For $\mathtt{i} = i_1 \cdots i_n \in \Lambda^*$, we write $f_{\mathtt{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$, and if $p_i \in \mathbb{R}$ for all $i \in \Lambda$, we define $p_{\mathtt{i}} = p_{i_1} \cdots p_{i_n}$.

2.2. **Tangent distributions.** We recall the theory of tangent distributions from [15], with extensions from [19]. Let $(S_t)_{t\geqslant 0}$ be the semi-flow on $\{\mu \in \mathcal{P}(\mathbb{R}^d) : 0 \in \operatorname{spt}\mu\}$ defined by

$$S_t\mu = \mu_{0,t}.$$

For $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $x \in \operatorname{spt}\mu$, the scenery of μ at x is the orbit of $(T_x)_*\mu$ under $(S_t)_{t\geqslant 0}$, given by $\mu_{x,t} = S_t(T_x)_*\mu$ for all $t \geqslant 0$, where $T_x\colon \mathbb{R}^d \to \mathbb{R}^d$ is the translation $y \mapsto y - x$. For a measurable map $f\colon B(0,1) \to B(0,1)$ and a Borel probability measure $P \in \mathcal{P}(\mathcal{P}(B(0,1)))$, the pushforward distribution $f_{**}P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ is given by

$$(f_{**}P)(\mathcal{A}) = P(\{\mu \in \mathcal{P}(\mathbb{R}^d) : f_*\mu \in \mathcal{A}\})$$

for all Borel sets $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}^d)$. We also denote by $(S_t)_{t\geqslant 0}$ the semi-flow on $\mathcal{P}(\{\mu \in \mathcal{P}(\mathbb{R}^d) : 0 \in \operatorname{spt}\mu\})$ defined by $S_tP = (S_t)_{**}P$ for $P \in \mathcal{P}(\{\mu \in \mathcal{P}(\mathbb{R}^d) : 0 \in \operatorname{spt}\mu\})$.

Let $P \in \mathcal{P}(\mathcal{P}(B(0,1)))$ be a Borel probability measure. We say P is a *fractal distribution* if it is invariant under the semi-flow $(S_t)_{t\geqslant 0}$ and, for any $\mathcal{A} \subseteq \mathcal{P}(B(0,1))$ with $P(\mathcal{A}) = 1$, the scenery $\mu_{x,t} \in \mathcal{A}$ for μ-almost every $x \in B(0,1)$ and every $t \geqslant 0$ such that $B(x, 2^{-t}) \subseteq B(0,1)$, for P-almost every $\mu \in \mathcal{A}$. Although the semi-flow $(S_t)_{t\geqslant 0}$ is not continuous and the fractal distribution property is not closed, the family of fractal distributions is closed in the weak*-topology; see [19, Theorem 1.1].

Theorem 2.1. *The family of fractal distributions is closed in the weak*-topology.*

We say P is an *ergodic fractal distribution* if it is a fractal distribution and ergodic under the semi-flow $(S_t)_{t\geqslant 0}$. Let $\mathcal{FD}$ denote the family of all fractal distributions. The set of ergodic fractal distributions is dense in $\mathcal{FD}$ with respect to the weak*-topology; see [19, Theorem 1.2].

Recall that tangent distributions of μ at x are the accumulation points of the scenery flow $(\langle\mu\rangle_{x,T})_{T>0}$ in the weak*-topology. We say μ is *uniformly scaling* and generates $P \in \mathcal{P}(\mathcal{P}(B(0,1)))$ if P is the unique tangent distribution of μ for μ-almost every $x \in \operatorname{spt}\mu$. By [15, Theorems 1.3 and 1.6], typical measures drawn from fractal distributions are much more structured than arbitrary measures. The ergodic decomposition of the dynamical system $(\mathcal{P}(B(0,1)), P, (S_t)_{t\geqslant 0})$ yields a measurable map $\omega \mapsto P_\omega$ from $\mathcal{P}(B(0,1))$ to $\mathcal{P}(\mathcal{P}(B(0,1)))$, defined with respect to the σ-algebra $\mathcal{I}$ of $(S_t)_{t\geqslant 0}$-invariant sets, such that $P = \int P_\omega \,\mathrm{d}P(\omega)$ and, for P-almost every ω, P_ω is an invariant, ergodic measure under $(S_t)_{t\geqslant 0}$, supported on the atom of $\mathcal{I}$ containing ω.

Theorem 2.2. *Let P be a fractal distribution. Then P-almost every ν is uniformly scaling and generates the ergodic component of P into which it belongs. In particular, almost all ergodic components of a fractal distribution are fractal distributions.*

In fact, every fractal distribution is generated by some uniformly scaling measure; see [19, Theorem 1.3]. Conversely, by [15, Theorem 1.7], the tangent distributions of any Radon measure exhibit the structure of a fractal distribution.

In [15], Hochman introduces two types of fractal distributions: restricted and extended. Our definition of a fractal distribution corresponds to Hochman's restricted fractal distribution. While most results in [15], including Theorem 2.2, are stated for extended fractal distributions, they also hold for restricted fractal distributions, as verified by [15, Lemma 3.1].

Theorem 2.3. *Let μ be a Radon measure. Then for μ-almost every x, every tangent distribution of μ at x is a fractal distribution.*

In particular, tangent distributions are invariant under the semi-flow $(S_t)_{t\geqslant 0}$. The following theorem is due to Hochman [15, Proposition 1.9].

Proposition 2.4. *Let μ be a Radon measure on $\mathbb{R}^d$, $x \in \mathbb{R}^d$, and suppose that $\lim_{i\to\infty}\langle\mu\rangle_{x,T_i} = P$ in the weak*-topology for some fractal distribution P. Then, for any diffeomorphism $f\colon \mathbb{R}^d \to \mathbb{R}^d$ and all sufficiently large $t_0 \geqslant 0$,*

$$\lim_{i\to\infty} \langle f_*\mu\rangle_{f(x),T_i} = S_{t_0}(Df|_x)_{**}P$$

in the weak*-topology.

We also recall an application of the Besicovitch density theorem, as detailed in the proof of [15, Proposition 3.8].

Proposition 2.5. *Let μ and ν be non-trivial Radon measures on $\mathbb{R}^d$ such that $\mu \ll \nu$. Then, for μ-almost every $x \in \mathbb{R}^d$,*

$$\lim_{t\to\infty} d_{\mathrm{LP}}(\mu_{x,t}, \nu_{x,t}) = 0.$$

2.3. **Pointwise dimensions.** For a Radon measure μ on $\mathbb{R}^d$, the *upper and lower pointwise dimensions* at $x \in \mathbb{R}^d$ are defined as

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$
$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

respectively. These pointwise dimensions naturally induce various dimensions for a measure, such as the *lower Hausdorff dimension*, given by $\underline{\dim}_{\mathrm{H}}\,\mu = \operatorname{ess\,inf}_{x\sim\mu} \underline{\dim}_{\mathrm{loc}}(\mu, x)$. If there exists a constant $s \in \mathbb{R}$ such that $\overline{\dim}_{\mathrm{loc}}(\mu, x) = \underline{\dim}_{\mathrm{loc}}(\mu, x) = s$ for μ-almost every $x \in \mathbb{R}^d$, we say μ is *exact-dimensional* and write $\dim \mu = s$. For instance, quasi-Bernoulli measures on self-conformal sets are exact-dimensional; see [11, Theorem 2.8]. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be a probability measure and $\mathcal{A}$ be a finite partition of $\mathbb{R}^d$ into Borel sets. The *Shannon entropy* of μ with respect to $\mathcal{A}$ is

$$H(\mu, \mathcal{A}) = -\sum_{A\in\mathcal{A}} \mu(A) \log \mu(A),$$

where log denotes the base-2 logarithm throughout this paper, and we adopt the convention that $0 \log 0 = 0$. If μ is exact-dimensional, then, by Young's theorem (see, e.g., [7, Theorem 2.6.4 and Proposition 2.6.11]), we have

$$\lim_{n\to\infty} \frac{1}{n} H(\mu, \mathcal{D}_n) = \dim \mu. \tag{2.2}$$

By [15, Lemma 1.18], for a fractal distribution P, P-almost every measure ν is exact-dimensional. The *dimension* of a fractal distribution P is defined as

$$\dim P = \int \dim \nu \, \mathrm{d}P(\nu),$$

which, by [15, Lemma 1.18], equals $\int \log \nu(B(0,r))/\log r \, \mathrm{d}P(\nu)$ for all $0 < r < 1$. Although measure dimensions are typically discontinuous, the map $P \mapsto \dim P$ on $\mathcal{FD}$ is continuous; see [18, Lemma 3.20].

Lemma 2.6. *The function $P \mapsto \dim P$ defined on fractal distributions is continuous with respect to the weak*-topology.*

Moreover, Hochman [15, Proposition 1.19] (see also the remark following [18, Theorem 3.21]) demonstrated that tangent distributions fully capture the dimensional information of measures.

Proposition 2.7. *Let μ be a Radon measure on $\mathbb{R}^d$. Then, for μ-almost every $x \in \mathbb{R}^d$,*

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \sup\{\dim P : P \in \mathcal{FD} \text{ is a tangent distribution of } \mu \text{ at } x\},$$
$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \inf\{\dim P : P \in \mathcal{FD} \text{ is a tangent distribution of } \mu \text{ at } x\},$$

In particular, if μ is uniformly scaling and generates $P \in \mathcal{FD}$, then μ is exact-dimensional with $\dim \mu = \dim P$.

We also recall the following result from [15, Lemma 6.7], relating the lower pointwise dimension to the entropy of dyadic partitions. The dyadic magnification $\mu^{\mathcal{D}_k(x)} \in \mathcal{P}(\mathbb{R}^d)$ of a Radon measure μ is defined in (2.1).

Lemma 2.8. *Let μ be a Radon measure on $\mathbb{R}^d$. Then, for μ-almost every $x \in \mathbb{R}^d$,*

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} H(\mu^{\mathcal{D}_k(x)}, \mathcal{D}_1).$$

Additionally, we recall a result stating that fractal distributions with positive dimension assign zero mass to measures charging boundaries of balls; see [18, Theorem 3.22].

Theorem 2.9. *Let P be a fractal distribution with $\dim P > 0$. Then, for P-almost every ν, we have $\nu(\partial B) = 0$ for all balls $B \subset \mathbb{R}^d$.*

3. Self-conformal measures are uniformly scaling

In this section, we prove Theorems 1.2–1.4. For a finite index set $\mathcal{I}$, let $(p_i)_{i\in\mathcal{I}}$ and $(q_i)_{i\in\mathcal{I}}$ be probability vectors such that $q_i = 0$ implies $p_i = 0$ for all $i \in \mathcal{I}$. The *Kullback-Leibler divergence* between $(p_i)_{i\in\mathcal{I}}$ and $(q_i)_{i\in\mathcal{I}}$ is defined as

$$D_{\mathrm{KL}}((p_i)_i \,\|\, (q_i)_i) = -\sum_{i\in\mathcal{I}} p_i \log \frac{q_i}{p_i}.$$

Lemma 3.1 (Gibbs' inequality)**.** *Let $\mathcal{I}$ be a finite index set, and let $(p_i)_{i\in\mathcal{I}}$ and $(q_i)_{i\in\mathcal{I}}$ probability vectors such that $q_i = 0$ implies $p_i = 0$ for all $i \in \mathcal{I}$. Then*

1. *$D_{\mathrm{KL}}((p_i)_{i\in\mathcal{I}} \,\|\, (q_i)_{i\in\mathcal{I}}) \geqslant 0$,*
2. *$D_{\mathrm{KL}}((p_i)_{i\in\mathcal{I}} \,\|\, (q_i)_{i\in\mathcal{I}}) < \varepsilon$ implies $\sum_{i\in\mathcal{I}} |p_i - q_i| < \sqrt{2\varepsilon \#\mathcal{I}}$ for all $\varepsilon > 0$.*

Proof. It is well known that $-\log(1-x) \geqslant x + \frac{x^2}{2\max\{1,1-x\}}$ for all $x < 1$. Hence,

$$\begin{aligned} D_{\mathrm{KL}}((p_i)_{i\in\mathcal{I}} \,\|\, (q_i)_{i\in\mathcal{I}}) &\geqslant \sum_{i\in\mathcal{I}} p_i \left(\frac{p_i - q_i}{p_i} + \frac{\left(\frac{p_i-q_i}{p_i}\right)^2}{2\max\left\{1, 1 - \frac{p_i-q_i}{p_i}\right\}} \right) \\ &= \sum_{i\in\mathcal{I}} \frac{(p_i - q_i)^2}{2\max\{p_i, q_i\}} \geqslant \frac{1}{2} \sum_{i\in\mathcal{I}} (p_i - q_i)^2, \end{aligned}$$

from which the first claim follows immediately, and the second claim follows by the Cauchy-Schwarz inequality. □

The *L^1-distance* between probability vectors $(p_i)_{i\in\mathcal{I}}$ and $(q_i)_{i\in\mathcal{I}}$ is defined as

$$\mathrm{dist}_1((p_i)_{i\in\mathcal{I}}, (q_i)_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} |p_i - q_i|.$$

For any $\mu, \nu \in \mathcal{P}([0,1]^d)$, the following implication holds for all $\ell \in \mathbb{N}$:

$$\operatorname{dist}_1((\mu(I))_{I\in\mathcal{D}_\ell}, (\nu(I))_{I\in\mathcal{D}_\ell}) \leqslant 2^{-\ell} \quad \Rightarrow \quad d_{\mathrm{LP}}(\mu,\nu) \leqslant \sqrt{d}2^{-\ell}. \tag{3.1}$$

Additionally, the L^1-distance satisfies

$$\begin{aligned} &\operatorname{dist}_1((\mu(I))_{I\in\mathcal{D}_\ell}, (\nu(I))_{I\in\mathcal{D}_\ell}) \\ &\quad\leqslant \sum_{m=0}^{\ell-1} \int \operatorname{dist}_1((\mu^{\mathcal{D}_m(x)}(I))_{I\in\mathcal{D}_1}, (\nu^{\mathcal{D}_m(x)}(I))_{I\in\mathcal{D}_1})\,\mathrm{d}\mu(x), \end{aligned} \tag{3.2}$$

where the dyadic magnification $\mu^{\mathcal{D}_m(x)}(I)$ is defined in (2.1). To see this, observe that

$$\begin{aligned} \operatorname{dist}_1((\mu(I))_{I\in\mathcal{D}_\ell}, (\nu(I))_{I\in\mathcal{D}_\ell}) &= \sum_{I\in\mathcal{D}_\ell} |\mu(I)-\nu(I)| \\ &= \sum_{J\in\mathcal{D}_{\ell-1}} \sum_{\substack{I\in\mathcal{D}_\ell\\ I\subseteq J}} \left|\frac{\mu(I)}{\mu(J)}\mu(J) - \frac{\nu(I)}{\nu(J)}\nu(J)\right| \\ &\leqslant \sum_{J\in\mathcal{D}_{\ell-1}} \sum_{\substack{I\in\mathcal{D}_\ell\\ I\subseteq J}} \left|\frac{\mu(I)}{\mu(J)} - \frac{\nu(I)}{\nu(J)}\right|\mu(J) + |\mu(J)-\nu(J)|\frac{\nu(I)}{\nu(J)} \\ &= \int \operatorname{dist}_1((\mu^{\mathcal{D}_{\ell-1}(x)}(I))_{I\in\mathcal{D}_1}, (\nu^{\mathcal{D}_{\ell-1}(x)}(I))_{I\in\mathcal{D}_1})\,\mathrm{d}\mu(x) \\ &\qquad + \operatorname{dist}_1((\mu(I))_{I\in\mathcal{D}_{\ell-1}}, (\nu(I))_{I\in\mathcal{D}_{\ell-1}}), \end{aligned}$$

where we adopt the convention that if $\mu(J) = 0$ (or $\nu(J) = 0$) for $J \in \mathcal{D}_{\ell-1}$, then $\mu(I)/\mu(J) = 0$ (or $\nu(I)/\nu(J) = 0$) for all $I \in \mathcal{D}_\ell$ with $I \subseteq J$. The proof of (3.2) is completed by induction on ℓ. We next state two technical lemmas.

Lemma 3.2. *Let η be a Radon measure on $\mathbb{R}^d$ with $\underline{\dim}_{\mathrm{H}}\,\eta > 0$. Then, for every $\varepsilon > 0$, there is $\rho > 0$ such that for η-almost every x,*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sup_{\rho\leqslant r\leqslant 1} \eta_{x,k}(B(0,r+\rho)\setminus B(0,r-\rho)) < \varepsilon. \tag{3.3}$$

Proof. We proceed by contradiction. Suppose there exists $\varepsilon > 0$ such that for every $\rho > 0$, there is a set $A_\rho \subseteq \mathbb{R}^d$ with $\eta(A_\rho) > 0$ such that, for each $x \in A_\rho$, the assertion (3.3) fails. Thus, for each $x \in A_\rho$, there exists a sequence $(N_k^\rho(x))_{k\in\mathbb{N}}$ in $\mathbb{N}$ satisfying

$$\liminf_{k\to\infty} \frac{1}{N_k^\rho(x)} \sum_{j=0}^{N_k^\rho(x)-1} \sup_{\rho\leqslant r\leqslant 1} \eta_{x,j}(B(0,r+\rho)\setminus B(0,r-\rho)) \geqslant \varepsilon. \tag{3.4}$$

By Theorem 2.3, for η-almost every $x \in A_\rho$, any accumulation point of the sequence

$$\left(\frac{1}{N_k^\rho(x)} \int_0^{N_k^\rho(x)} \delta_{\eta_{x,t}}\,\mathrm{d}t\right)_{k\in\mathbb{N}}$$

is a fractal distribution. Let P_ϱ denote such an accumulation point. Since $\underline{\dim}_{\mathrm{H}}\,\eta > 0$, Proposition 2.7 ensures that $\dim P_\varrho \geqslant \underline{\dim}_{\mathrm{H}}\,\eta > 0$.

Define

$$\alpha_k = \#\Big\{ j \in \{0, \dots, N_k^\rho(x) - 1\} : \sup_{\rho \leqslant r \leqslant 1} \eta_{x,j}(B(0, r+\rho) \setminus B(0, r-\rho)) \geqslant \frac{\varepsilon}{2} \Big\}.$$

and $\alpha = \liminf_{k\to\infty} \alpha_k / N_k^\rho(x)$. By (3.4), the reverse Markov inequality yields $\alpha \geqslant (\varepsilon - \varepsilon/2)/(1 - \varepsilon/2) > \varepsilon/2$. For any $k \geqslant 0$, if $\sup_{\rho \leqslant r \leqslant 1} \eta_{x,k}(B(0, r+\rho) \setminus B(0, r-\rho)) \geqslant \varepsilon/2$, then the distribution $\int_k^{k+1} \delta_{\eta_{x,t}} \, \mathrm{d}t$ is supported on the set

$$E_\rho = \Big\{ \mu \in \mathcal{P}(B(0,1)) : \mu(B(0, r+\rho) \setminus B(0, r-\rho)) \geqslant \frac{\varepsilon}{2} \text{ for some } \frac{\rho}{e} \leqslant r \leqslant 1 \Big\}.$$

Thus, by the definition of P_ρ, the fractal distribution P_ρ assigns mass at least $\alpha \geqslant \varepsilon/2$ to E_ρ. Let P be an accumulation point of $(P_\rho)_{\rho>0}$ as $\rho \to 0$. By Theorem 2.1, P is a fractal distribution. We claim that P assigns mass at least $\varepsilon/2$ to the set

$$E_0 = \Big\{ \mu \in \mathcal{P}(B(0,1)) : \mu(\partial B(0, r)) \geqslant \frac{\varepsilon}{2} \text{ for some } 0 \leqslant r \leqslant 1 \Big\}.$$

To verify this, consider the sets

$$E_\rho^0 = \Big\{ \mu \in \mathcal{P}(B(0,1)) : \mu(\{x : r - \rho \leqslant \|x\| \leqslant r + \rho\}) \geqslant \frac{\varepsilon}{2} \text{ for some } 0 \leqslant r \leqslant 1 \Big\}.$$

Since $E_\rho \subseteq E_\rho^0$, and E_ρ^0 is closed and decreases to E_0 as $\rho \to 0$, it follows that $P(E_\rho^0) \geqslant \varepsilon/2$ for each $\rho > 0$. Consequently, $P(E_0) \geqslant \varepsilon/2$. Since $\dim P_\varrho \geqslant \underline{\dim}_{\mathrm{H}} \eta > 0$ for every $\varrho > 0$, Lemma 2.6 implies that $\dim P > 0$. However, this contradicts Theorem 2.9, as $P(E_0) \geqslant \varepsilon/2 > 0$. □

Lemma 3.3. *For any $r, \tau > 0$ and for all $\tau/6 \geqslant \varepsilon \geqslant \rho > 0$ the following holds: If η and λ are Borel probability measures on $[-1,1]^d$ and $B = B(x, r)$ is a closed ball such that*

$$\begin{aligned} d_{\mathrm{LP}}(\eta, \lambda) &< \rho, \\ \eta(B(x, r+\rho)) &\leqslant \eta(B(x, r-\rho)) + \varepsilon, \\ \min\{\eta(B), \lambda(B)\} &\geqslant \tau, \end{aligned}$$

then

$$d_{\mathrm{LP}}(\eta_B, \lambda_B) < \frac{6\varepsilon}{\tau}.$$

Proof. For any measurable set A, the assumptions on η and λ imply

$$\begin{aligned} \lambda_B(A) &= \lambda(B)^{-1} \lambda(B \cap A) \\ &\leqslant (\eta(B(x, r-\rho)) - \rho)^{-1} (\eta(B^\rho \cap A^\rho) + \rho) \\ &\leqslant (\eta(B) - 2\varepsilon)^{-1} \eta(B \cap A^\rho) + 2\varepsilon(\eta(B) - 2\varepsilon)^{-1} \\ &\leqslant \frac{1}{1 - 2\varepsilon/\tau} \eta(B)^{-1} \eta(B \cap A^\rho) + \frac{3\varepsilon}{\tau} \\ &\leqslant \eta_B(A^\rho) + \frac{6\varepsilon}{\tau} \leqslant \eta_B(A^{6\varepsilon/\tau}) + \frac{6\varepsilon}{\tau}, \end{aligned}$$

which verifies the claim. □

For $\nu \in \mathcal{P}(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))$ and $\mu \in \mathcal{P}([-1,1]^d)$, the *generalized convolution* is defined as

$$\nu \cdot \mu = \int h_*\mu \,\mathrm{d}\nu(h).$$

We observe that this definition naturally extends classical convolution. Indeed, when ν concentrates on translations, that is, when $\nu = \int \delta_{T_{-y}} \,\mathrm{d}\mu'(y)$ for some $\mu' \in \mathcal{P}(\mathbb{R}^d)$, we recover the classical convolution $\nu \cdot \mu = \mu' * \mu$ of probability measures.

The map $\Delta\colon \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d) \times \mathcal{P}([-1,1]^d) \to \mathcal{P}(\mathbb{R}^d)$ defined by $\Delta(h,\mu) = h_*\mu$ is measurable with respect to the product σ-algebra $\mathcal{B}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)) \otimes \mathcal{B}(\mathcal{P}([-1,1]^d))$ and the Borel σ-algebra $\mathcal{B}(\mathcal{P}(\mathbb{R}^d))$. The measurability follows from the continuity of the push-forward operation. For any $h_n \to h$ in $\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$ and $\mu_n \to \mu$ in the weak*-topology in $\mathcal{P}([-1,1]^d)$, we have $(h_n)_*\mu_n \to h_*\mu$ in the weak*-topology in $\mathcal{P}(\mathbb{R}^d)$. This follows from the uniform convergence $h_n \to h$ on compact sets and the bounded convergence theorem.

This makes the generalized convolution $\nu \cdot \mu$ a well-defined probability measure on $\mathbb{R}^d$. Moreover, for ν-almost every $h \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$ and $h_*\mu$-almost every $y \in \mathbb{R}^d$, we have $y \in \mathrm{spt}(\nu \cdot \mu)$, ensuring that the magnification $(\nu \cdot \mu)_{y,t}$ is well-defined. The $\mathcal{C}^{1+\alpha}_{\mathrm{conf}}$-topology provides sufficient regularity for the conformal maps while preserving the measurability properties required for integration. The Hölder condition with exponent $\alpha > 0$ ensures compactness properties on bounded sets, which are essential for convergence arguments; see the proof of Lemma 3.6.

The proofs of Theorems 1.1–1.4 rely on a key result established in Proposition 3.5. We begin by presenting and proving its dyadic version below. Notably, these results do not require assumptions on the entropy or dimension of the measure ν.

Proposition 3.4. *Let $\nu \in \mathcal{P}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d))$ and $\mu \in \mathcal{P}([-1,1]^d)$. For any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $\delta > 0$ such that the following holds for all $n \geqslant N$: If for some $\xi \geqslant 0$,*

(1) *for ν-almost every $h \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$, the measure $h_*\mu$ satisfies*

$$h_*\mu\left(\left\{x \in \mathbb{R}^d : \frac{1}{n}\sum_{k=1}^{n-1} H((h_*\mu)^{\mathcal{D}_k(x)}, \mathcal{D}_1) \geqslant \xi\right\}\right) \geqslant 1 - \delta,$$

(2) *the Shannon entropy of the generalized convolution satisfies*

$$H(\nu \cdot \mu, \mathcal{D}_n) \leqslant n(\xi + \delta),$$

then

$$\iint \frac{1}{n}\sum_{k=0}^{n-1} d_{\mathrm{LP}}((\nu \cdot \mu)^{\mathcal{D}_k(y)}, (h_*\mu)^{\mathcal{D}_k(y)}) \,\mathrm{d}(h_*\mu)(y) \,\mathrm{d}\nu(h) < \varepsilon. \tag{3.5}$$

Proof. By standard properties of Shannon entropy, for any $\eta \in \mathcal{P}(\mathbb{R}^d)$, the entropy with respect to the partition $\mathcal{D}_n$ can be expressed as

$$H(\eta, \mathcal{D}_n) = -\sum_{k=0}^{n-1} \sum_{I_k \in \mathcal{D}_k} \sum_{\substack{I_{k+1} \in \mathcal{D}_{k+1} \\ I_{k+1} \subset I_k}} \eta(I_{k+1}) \log \frac{\eta(I_{k+1})}{\eta(I_k)}. \tag{3.6}$$

Applying (3.6) to the generalized convolution $\nu \cdot \mu$, we obtain

$$\begin{aligned}
H(\nu \cdot \mu, \mathcal{D}_n) &= -\sum_{k=0}^{n-1} \sum_{I_k \in \mathcal{D}_k} \sum_{\substack{I_{k+1} \in \mathcal{D}_{k+1} \\ I_{k+1} \subset I_k}} \int \left(h_*\mu(I_{k+1}) \log \frac{\nu \cdot \mu(I_{k+1})}{\nu \cdot \mu(I_k)} \right) \mathrm{d}\nu(h) \\
&= -\sum_{k=0}^{n-1} \sum_{I_k \in \mathcal{D}_k} \int \left(h_*\mu(I_k) \sum_{\substack{I_{k+1} \in \mathcal{D}_{k+1} \\ I_{k+1} \subset I_k}} \frac{h_*\mu(I_{k+1})}{h_*\mu(I_k)} \log \frac{\nu \cdot \mu(I_{k+1})}{\nu \cdot \mu(I_k)} \right) \mathrm{d}\nu(h) \\
&= -\sum_{k=0}^{n-1} \iint \left(\sum_{I \in \mathcal{D}_1} (h_*\mu)^{\mathcal{D}_k(x)}(I) \log(\nu \cdot \mu)^{\mathcal{D}_k(x)}(I) \right) \mathrm{d}(h_*\mu)(x) \, \mathrm{d}\nu(h) \\
&= -\iint \sum_{k=0}^{n-1} \left(\sum_{I \in \mathcal{D}_1} (h_*\mu)^{\mathcal{D}_k(x)}(I) \log(\nu \cdot \mu)^{\mathcal{D}_k(x)}(I) \right) \mathrm{d}(h_*\mu)(x) \, \mathrm{d}\nu(h).
\end{aligned}$$

To simplify notation, write

$$\begin{aligned}
p^h_{k,\ell}(x) &= ((h_*\mu)^{\mathcal{D}_k(x)}(I))_{I \in \mathcal{D}_\ell}, \\
q_{k,\ell}(x) &= ((\nu \cdot \mu)^{\mathcal{D}_k(x)}(I))_{I \in \mathcal{D}_\ell},
\end{aligned}$$

in which case

$$\begin{aligned}
H(p^h_{k,1}(x)) &= -\sum_{I \in \mathcal{D}_1} (h_*\mu)^{\mathcal{D}_k(x)}(I) \log (h_*\mu)^{\mathcal{D}_k(x)}(I), \\
D_{\mathrm{KL}}(p^h_{k,1}(x) \,\|\, q_{k,1}(x)) &= -\sum_{I \in \mathcal{D}_1} (h_*\mu)^{\mathcal{D}_k(x)}(I) \log \frac{(\nu \cdot \mu)^{\mathcal{D}_k(x)}(I)}{(h_*\mu)^{\mathcal{D}_k(x)}(I)}.
\end{aligned}$$

The above quantities are defined for ν-almost every h and $h_*\mu$-almost every x, but they are not necessarily measurable with respect to the product σ-algebra $\mathcal{B}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)) \otimes \mathcal{B}(\mathcal{P}([-1,1]^d))$. Nevertheless, the Shannon entropy of the generalized convolution $\nu \cdot \mu$ can be expressed as

$$\begin{aligned}
H(\nu \cdot \mu, \mathcal{D}_n) = &\iint \sum_{k=0}^{n-1} H(p^h_{k,1}(x)) \, \mathrm{d}(h_*\mu)(x) \, \mathrm{d}\nu(h) \\
&+ \iint \sum_{k=0}^{n-1} D_{\mathrm{KL}}(p^h_{k,1}(x) \,\|\, q_{k,1}(x)) \, \mathrm{d}(h_*\mu)(x) \, \mathrm{d}\nu(h).
\end{aligned} \tag{3.7}$$

By Gibbs' inequality Lemma 3.1(1), we have

$$D_{\mathrm{KL}}(p^h_{k,1}(x) \,\|\, q_{k,1}(x)) \geqslant 0.$$

From assumption (1), it follows that

$$\int \sum_{k=0}^{n-1} H(p^h_{k,1}(x)) \, \mathrm{d}(h_*\mu)(x) \geqslant n\xi(1-\delta). \tag{3.8}$$

Thus, combining (3.7) and (3.8) with assumption (2), we derive

$$\begin{aligned}\iint \sum_{k=0}^{n-1} D_{\mathrm{KL}}(p_{k,1}^h(x) \,\|\, q_{k,1}(x)) \,\mathrm{d}(h_*\mu)(x) \,\mathrm{d}\nu(h) \\ &= H(\nu\cdot\mu, \mathcal{D}_n) - \iint \sum_{k=0}^{n-1} H(p_{k,1}^h(x)) \,\mathrm{d}(h_*\mu)(x) \,\mathrm{d}\nu(h) \\ &\leqslant n(\xi+\delta) - n\xi(1-\delta) \leqslant \delta n(1+\xi).\end{aligned} \tag{3.9}$$

To simplify notation, define the expectation

$$\mathbb{E}(\,\cdot\,) = \iint \frac{1}{n} \sum_{k=0}^{n-1} \cdot \,\mathrm{d}(h_*\mu)(x) \,\mathrm{d}\nu(h).$$

The expected value can be decomposed as

$$\begin{aligned}&\mathbb{E}(\mathrm{dist}_1(p_{k,1}^h(x), q_{k,1}(x))) \\ &\quad = \mathbb{E}(\mathrm{dist}_1(p_{k,1}^h(x), q_{k,1}(x)) \mathbb{1}\{D_{\mathrm{KL}}(p_{k,1}^h(x) \,\|\, q_{k,1}(x)) > \delta^{2/3}\}) \\ &\qquad + \mathbb{E}(\mathrm{dist}_1(p_{k,1}^h(x), q_{k,1}(x)) \mathbb{1}\{D_{\mathrm{KL}}(p_{k,1}^h(x) \,\|\, q_{k,1}(x)) \leqslant \delta^{2/3}\}),\end{aligned}$$

where $\mathbb{1}A$ denotes the indicator function of a set $A \subseteq \mathbb{R}^d$. Using Gibbs' inequality (Lemma 3.1(2)) and the fact that dist_1 is bounded by 2 for probability vectors, we obtain

$$\mathbb{E}(\mathrm{dist}_1(p_{k,1}^h(x), q_{k,1}(x))) \leqslant 2\mathbb{E}(\mathbb{1}\{D_{\mathrm{KL}}(p_{k,1}^h(x) \,\|\, q_{k,1}(x)) > \delta^{2/3}\}) + \delta^{1/3} 2^d \sqrt{2}.$$

Applying Markov's inequality, this becomes

$$\mathbb{E}(\mathrm{dist}_1(p_{k,1}^h(x), q_{k,1}(x))) \leqslant \frac{2\mathbb{E}(D_{\mathrm{KL}}(p_{k,1}^h(x) \,\|\, q_{k,1}(x)))}{\delta^{2/3}} + \delta^{1/3} 2^d \sqrt{2}.$$

Finally, by (3.9), we have

$$\begin{aligned}\mathbb{E}(\mathrm{dist}_1(p_{k,1}^h(x), q_{k,1}(x))) &\leqslant \frac{2\delta(1+\xi)}{\delta^{2/3}} + \delta^{1/3} 2^d \sqrt{2} \\ &= \delta^{1/3}(2(1+\xi) + 2^d\sqrt{2}) = C\delta^{1/3},\end{aligned} \tag{3.10}$$

where $C = 2(1+\xi) + 2^d\sqrt{2}$. Using (3.2) and the identity $\mathrm{d}(h_*\mu)^{\mathcal{D}_m(x)}(y) \,\mathrm{d}(h_*\mu)(x) = \mathrm{d}(h_*\mu)(y)$, we obtain

$$\begin{aligned}\iint \mathrm{dist}_1(p_{k,\ell}^h(x), q_{k,\ell}(x)) \,\mathrm{d}(h_*\mu)(x) \,\mathrm{d}\nu(h) \\ &\leqslant \sum_{m=0}^{\ell-1} \iint \mathrm{dist}_1(p_{k+m,1}^h(x), q_{k+m,1}(x)) \,\mathrm{d}(h_*\mu)(x) \,\mathrm{d}\nu(h)\end{aligned}$$

for all $\ell \in \mathbb{N}$. Thus, for all sufficiently large $n \geqslant \ell$, (3.10) implies

$$\sum_{k=0}^{n-1} \iint \operatorname{dist}_1(p^h_{k,\ell}(x), q_{k,\ell}(x)) \,\mathrm{d}(h_*\mu)(x) \,\mathrm{d}\nu(h) \leqslant \ell \sum_{k=0}^{n+\ell-1} \iint \operatorname{dist}_1(p^h_{k,1}(x), q_{k,1}(x)) \,\mathrm{d}(h_*\mu)(x) \,\mathrm{d}\nu(h) \leqslant 2C\ell n\delta^{1/3}.$$

Choose $\ell \in \mathbb{N}$ such that $2^{-(\ell+1)} < \delta^{1/6} \leqslant 2^{-\ell}$. Combining the above inequality with (3.1) and Markov's inequality yields

$$\begin{aligned}
&\mathbb{E}(d_{\mathrm{LP}}((h_*\mu)^{\mathcal{D}_k(y)}, (\nu\cdot\mu)^{\mathcal{D}_k(y)})) \\
&\quad = \mathbb{E}(d_{\mathrm{LP}}((h_*\mu)^{\mathcal{D}_k(y)}, (\nu\cdot\mu)^{\mathcal{D}_k(y)})\mathbb{1}\{\operatorname{dist}_1(p^h_{k,\ell}(x), q_{k,\ell}(x)) > 2^{-\ell}\}) \\
&\qquad\quad + \mathbb{E}(d_{\mathrm{LP}}((h_*\mu)^{\mathcal{D}_k(y)}, (\nu\cdot\mu)^{\mathcal{D}_k(y)})\mathbb{1}\{\operatorname{dist}_1(p^h_{k,\ell}(x), q_{k,\ell}(x)) \leqslant 2^{-\ell}\}) \\
&\quad \leqslant 2^{\ell}\mathbb{E}(\operatorname{dist}_1(p^h_{k,\ell}(x), q_{k,\ell}(x))) + \sqrt{d}2^{-\ell} \leqslant 2^{\ell+1}C\ell\delta^{1/3} + \sqrt{d}2^{-\ell} \\
&\quad \leqslant -\frac{C}{3}\delta^{1/6}\log\delta + 2\sqrt{d}\delta^{1/6}.
\end{aligned}$$

In particular, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\iint \frac{1}{n}\sum_{k=0}^{n-1} d_{\mathrm{LP}}((\nu\cdot\mu)^{\mathcal{D}_k(y)}, (h_*\mu)^{\mathcal{D}_k(y)}) \,\mathrm{d}(h_*\mu)(y) \,\mathrm{d}\nu(h) < \varepsilon,$$

as required. □

We next transform the previous proposition into a result on ball magnifications, as needed to analyze the tangent measures of a quasi-Bernoulli measure.

Proposition 3.5. *Let $\nu \in \mathcal{P}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d))$ and $\mu \in \mathcal{P}([-1,1]^d)$ be such that $\underline{\dim}_{\mathrm{H}}\nu\cdot\mu > 0$. For any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and $\delta > 0$ such that the following holds for all $n \geqslant N$: If for some $\xi \geqslant 0$,*

(1) *for ν-almost every $h \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$, the measure $h_*\mu$ satisfies*

$$h_*\mu\left(\left\{x \in \mathbb{R}^d : \frac{1}{n}\sum_{k=1}^{n-1} H((h_*\mu)^{\mathcal{D}_k(x)}, \mathcal{D}_1) \geqslant \xi\right\}\right) \geqslant 1-\delta,$$

(2) *the Shannon entropy of the generalized convolution satisfies*

$$H(\nu\cdot\mu, \mathcal{D}_n) \leqslant n(\xi+\delta),$$

then

$$\iint \frac{1}{n}\int_1^n d_{\mathrm{LP}}((\nu\cdot\mu)_{y,t}, (h_*\mu)_{y,t}) \,\mathrm{d}t \,\mathrm{d}(h_*\mu)(y) \,\mathrm{d}\nu(h) < \varepsilon. \tag{3.11}$$

Proof. Building on Proposition 3.4, we aim to transition from dyadic magnifications to ball magnifications. Specifically, we prove that (3.5) implies (3.11). Fix $\delta > 0$ and $m \in \mathbb{N}$. By translating the dyadic grid randomly with respect to the Lebesgue measure, we may assume that $\nu\cdot\mu$-almost every y equidistributes under the map

$$F\colon \mathbb{R}^d \to \mathbb{R}^d, \quad F(x_1,\ldots,x_d) = (2x_1 \bmod 1, \ldots, 2x_d \bmod 1).$$

Note that this translation does not affect the assumptions or conclusions. Thus, for $\nu \cdot \mu$-almost every y, there exists a set $\mathcal{N}_\delta \subseteq \mathbb{N}$ with $\liminf_{n\to\infty} \frac{1}{n}\#(\mathcal{N}_\delta \cap [0,n]) \geqslant 1-\delta$ such that $B(F^k y, 2^{-m+1}) \subseteq [0,1]^d$, or equivalently,

$$B(y, 2^{-k-m+1}) \subseteq \mathcal{D}_k(y) \tag{3.12}$$

for all $k \in \mathcal{N}_\delta$. Observe that for any finite Borel measure η on $\mathbb{R}^d$ and $y \in \operatorname{spt}\eta$,

$$-\frac{1}{n}\sum_{k=0}^{n-1} \frac{\log \eta_{y,k}(B(0,2^{-m}))}{m} = \frac{1}{nm}\biggl(\log \prod_{j=0}^{m-1} \eta(y, 2^{-j}) - \log \prod_{j=0}^{m-1} \eta(y, 2^{-n-j})\biggr).$$

The $\limsup_{n\to\infty}$ of this quantity is bounded above by $\overline{\dim}_{\mathrm{loc}}(\eta, y)$. Applying this, for ν-almost every h and $h_*\mu$-almost every y, there exist $c > 0$ and a set $\mathcal{N}'_\delta \subseteq \mathbb{N}$ with $\liminf_{n\to\infty} \frac{1}{n}\#(\mathcal{N}'_\delta \cap [0,n]) \geqslant 1-\delta$ such that

$$c \leqslant \frac{\eta(B(y, 2^{-k-m}))}{\eta(B(y, 2^{-k}))} \leqslant 1$$

for all $k \in \mathcal{N}'_\delta$ and $\eta \in \{\nu \cdot \mu, h_*\mu\}$. In particular, for each $k \in \mathcal{N}_\delta \cap \mathcal{N}'_\delta$, we have

$$c \leqslant \frac{\eta(B(y, 2^{-k-m}))}{\eta(\mathcal{D}_k(y))} \leqslant 1 \tag{3.13}$$

for both $\eta \in \{\nu \cdot \mu, h_*\mu\}$.

Let $\varepsilon' > 0$. By the assumption $\underline{\dim}_{\mathrm{H}} \nu \cdot \mu > 0$ and Lemma 3.2, there exists $\rho > 0$ such that for $\nu \cdot \mu$-almost every y, there is a set $\mathcal{N}''_\delta \subseteq \mathbb{N}$ with $\liminf_{n\to\infty} \frac{1}{n}\#(\mathcal{N}''_\delta \cap [0,n]) \geqslant 1-\delta$ and

$$\sup_{\rho \leqslant r \leqslant 1} (\nu \cdot \mu)_{y,k}(B(0, r+\rho) \setminus B(0, r-\rho)) < \varepsilon' \tag{3.14}$$

for all $k \in \mathcal{N}''_\delta$. Replacing ρ by a smaller value if necessary and applying Lemma 3.3, we conclude that for every y and k satisfying (3.12), (3.13), (3.14), and

$$d_{\mathrm{LP}}((\nu \cdot \mu)^{\mathcal{D}_k(y)}, (h_*\mu)^{\mathcal{D}_k(y)}) < \rho,$$

we have

$$\int_0^1 d_{\mathrm{LP}}((\nu \cdot \mu)_{y,k+m-1+t}, (h_*\mu)_{y,k+m-1+t})\,\mathrm{d}t < \frac{6\varepsilon'}{c}$$

for all $0 \leqslant t \leqslant 1$. By choosing ε' sufficiently small, the left-hand side can be made arbitrarily small, independently of ε. Thus, for sufficiently small $\delta > 0$, the implication

$$\frac{1}{n}\sum_{k=1}^{n} d_{\mathrm{LP}}((\nu \cdot \mu)^{\mathcal{D}_k(y)}, (h_*\mu)^{\mathcal{D}_k(y)}) < \rho^2 \quad \Rightarrow \quad \frac{1}{n}\int_m^{n+m} d_{\mathrm{LP}}((\nu \cdot \mu)_{y,t}, (h_*\mu)_{y,t})\,\mathrm{d}t < \varepsilon$$

holds for all large n. Since this implication is valid for ν-almost every h and $h_*\mu$-almost every y, the claim (3.11) follows from (3.5). □

We utilize Proposition 3.5 to demonstrate that a quasi-Bernoulli measure μ exhibits the same tangential behavior as its tangent measures, that is, its weak*-accumulation points of the scenery $(\mu_{x,t})_{t\geqslant 0}$. To that end, we prove in the following lemma that the tangent measures of a quasi-Bernoulli measure μ are equivalent to convex combinations of conformal images of μ.

Lemma 3.6. *Let Φ be a conformal iterated function system on $\mathbb{R}^d$ with attractor K, and let μ be a quasi-Bernoulli measure on K. For μ-almost every x and every tangent distribution P at x, for P-almost every η, there exists a measure $\nu \in \mathcal{P}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d))$ such that $\eta \sim_C (\nu \cdot \mu)_{B(0,1)}$, where $C \geqslant 1$ is the constant from* (1.3).

Proof. Write $\Phi = (f_i)_{i\in\Lambda}$ and let $\bar{\mu}$ be a quasi-Bernoulli measure on $\Lambda^{\mathbb{N}}$ with $\Pi_*\bar{\mu} = \mu$. For $x \in \operatorname{spt}\mu$ and $t > 0$, define

$$\Lambda(x,t) = \{\mathtt{i} \in \Lambda^* : \operatorname{diam}(f_{\mathtt{i}}(K)) \leqslant 2^{-t} < \operatorname{diam}(f_{\mathtt{i}^-}(K)) \\ \text{and } f_{\mathtt{i}}(K) \cap B(x, 2^{-t}) \neq \emptyset\}.$$

The bounded distortion property of Φ (see e.g. [23, Lemma 2.1]) gives $L \geqslant 1$ such that for all $\mathtt{i} \in \Lambda^*$,

$$L^{-1} \leqslant \frac{\|Df_{\mathtt{i}}|_x\|}{\|Df_{\mathtt{i}}|_y\|} \leqslant L \tag{3.15}$$

for all $x, y \in U$. Let $g_{x,t}$ be the affine map $y \mapsto 2^t(y-x)$. Clearly, there exists $M \geqslant 1$ depending only on Φ such that $g_{x,t} \circ f_{\mathtt{i}} \in \mathcal{C}^{1+\alpha}_M([-1,1]^d)$ for all $x \in \operatorname{spt}\mu$, $t > 0$, and $\mathtt{i} \in \Lambda(x,t)$. Now, $\mu|_{B(x,2^{-t})} = \sum_{\mathtt{i}\in\Lambda(x,t)} \Pi_*(\bar{\mu}|_{[\mathtt{i}]})|_{B(x,2^{-t})}$. Using the quasi-Bernoulli property (1.3), we have $\Pi_*(\bar{\mu}|_{[\mathtt{i}]}) \sim_C \bar{\mu}([\mathtt{i}])(f_{\mathtt{i}})_*\mu$, where $C \geqslant 1$ is from (1.3). Thus,

$$\mu|_{B(x,2^{-t})} \sim_C \sum_{\mathtt{i}\in\Lambda(x,t)} \bar{\mu}([\mathtt{i}])((f_{\mathtt{i}})_*\mu)|_{B(x,2^{-t})}. \tag{3.16}$$

Define

$$\nu(x,t) = C(x,t)^{-1} \sum_{\mathtt{i}\in\Lambda(x,t)} \bar{\mu}([\mathtt{i}])\delta_{g_{x,t}\circ f_{\mathtt{i}}} \in \mathcal{P}(\mathcal{C}^{1+\alpha}_M(\mathbb{R}^d)),$$

where $C(x,t) = \sum_{\mathtt{i}\in\Lambda(x,t)} \bar{\mu}([\mathtt{i}])$. By (3.16), $\mu_{x,t} \sim_C (\nu(x,t)\cdot\mu)_{B(0,1)}$. Define

$$E = \{\eta \in \mathcal{P}(B(0,1)) : \eta \sim_C (\nu\cdot\mu)_{B(0,1)} \text{ for some } \nu \in \mathcal{P}(\mathcal{C}^{1+\alpha}_M([-1,1]^d))\},$$
$$E_r = \{\eta \in \mathcal{P}(B(0,1)) : \nu\cdot\mu(B(0,1)) \geqslant r \text{ and } \\ \eta \sim_C (\nu\cdot\mu)_{B(0,1)} \text{ for some } \nu \in \mathcal{P}(\mathcal{C}^{1+\alpha}_M([-1,1]^d))\}$$

for all $r > 0$. Note that $\mu_{x,t} \in E$ for all $x \in \operatorname{spt}\mu$ and $t \geqslant 0$. Our aim is to show that for μ-almost every x and any tangent distribution P at x, $P(E) = 1$.

We claim that the set E_r is closed for all $r > 0$. Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in E_r converging to η. By compactness of $\mathcal{P}(\mathcal{C}^{1+\alpha}_M([-1,1]^d))$ in the weak*-topology, by possibly passing to a subsequence, we find a sequence $(\nu_n)_n$ in $\mathcal{P}(\mathcal{C}^{1+\alpha}_M([-1,1]^d))$ converging to ν such that $\eta_n \sim_C (\nu_n\cdot\mu)_{B(0,1)}$ and $\nu_n\cdot\mu(B(0,1)) \geqslant r$. By Lemma 3.3 and continuity of $\nu \mapsto \nu\cdot\mu$, we have $(\nu_n\cdot\mu)_{B(0,1)} \to (\nu\cdot\mu)_{B(0,1)}$. Thus, $\nu\cdot\mu(B(0,1)) \geqslant r$ and $\eta \sim_C (\nu\cdot\mu)_{B(0,1)}$, so $\eta \in E_r$ and E_r is closed.

By (3.16), we have

$$C^{-1}\frac{\mu(B(x,2^{-t}))}{C(x,t)} \leqslant (\nu(x,t)\cdot\mu)(B(0,1)) \leqslant C\frac{\mu(B(x,2^{-t}))}{C(x,t)}. \tag{3.17}$$

By the definition of $\Lambda(x,t)$ and bounded distortion (3.15), there exists $k_0 \in \mathbb{N}$ depending only on Φ such that $f_{\mathtt{i}}(K) \subset B(x, 2^{-t+k_0})$ for all $x \in \operatorname{spt}\mu$, $t > 0$, and $\mathtt{i} \in \Lambda(x,t)$, and

consequently

$$C(x,t) \leqslant \mu(B(x,2^{-t+k_0})). \tag{3.18}$$

Assume $k_0 \geqslant 2$. For μ-almost every x, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} -\log \frac{\mu(B(x,2^{-k-1}))}{\mu(B(x,2^{-k+k_0}))} &= \limsup_{n\to\infty} \frac{1}{n}\biggl(\log \prod_{j=0}^{k_0} \mu(B(x,2^j)) - \log \prod_{j=0}^{k_0+2} \mu(B(y,2^{-n+j}))\biggr)\\ &\leqslant (k_0+3)d.\end{aligned}$$

Thus, for μ-almost every x and any $\delta > 0$, there exist $0 < r < -C^{-1}\log\frac{(k_0+3)d}{\delta}$ and $I'_\delta \subseteq \mathbb{N}$ with $\liminf_{n\to\infty} \frac{1}{n}\#(I'_\delta \cap [0,n]) \geqslant 1-\delta$ such that

$$\frac{\mu(B(x,2^{-k-1}))}{\mu(B(x,2^{-k+k_0}))} \geqslant Cr \tag{3.19}$$

for all $k \in I'_\delta$. Indeed, let $J_\delta = \{k \in \mathbb{N} : \frac{\mu(B(x,2^{-k}))}{\mu(B(x,2^{-k+k_0-1}))} < Cr\}$. If $\limsup_{n\to\infty} \frac{1}{n}\#(J_\delta \cap [0,n]) \geqslant \delta$, then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} -\log \frac{\mu(B(x,2^{-k-1}))}{\mu(B(x,2^{-k+k_0}))} \geqslant \delta e^{-Cr} > (k_0+3)d,$$

which is a contradiction.

Hence, for any $k \in I'_\delta$ and $t \in [k,k+1]$,

$$\begin{aligned}(\nu(x,t)\cdot\mu)(B(0,1)) &\geqslant C^{-1}\frac{\mu(B(x,2^{-t}))}{C(x,t)} \geqslant C^{-1}\frac{\mu(B(x,2^{-t}))}{\mu(B(x,2^{-t+k_0}))}\\ &\geqslant C^{-1}\frac{\mu(B(x,2^{-k-1}))}{\mu(B(x,2^{-k+k_0}))} \geqslant r\end{aligned}$$

by (3.17), (3.18), and (3.19). Let $I_\delta = \bigcup_{k\in I'_\delta}[k,k+1]$. Then $\liminf_{n\to\infty}\frac{1}{n}\mathcal{L}^1(I_\delta\cap[0,n]) = \liminf_{n\to\infty}\frac{1}{n}\#(I'_\delta\cap[0,n]) \geqslant 1-\delta$, where $\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$. For μ-almost every x and any tangent distribution P at x,

$$\begin{aligned}P(E) \geqslant P(E_r) &\geqslant \limsup_{n\to\infty}\frac{1}{n}\int_0^n \delta_{\mu_{x,t}}(E_r)\,\mathrm{d}t\\ &\geqslant \liminf_{n\to\infty}\frac{1}{n}\int_{I_\delta\cap[0,n]} \delta_{\mu_{x,t}}(E_r)\,\mathrm{d}t \geqslant 1-\delta\end{aligned}$$

as E_r is closed and $\mu_{x,t} \in E_r$ for each $t \in I_\delta$. Letting $\delta \to 0$ completes the proof. □

Remark 3.7. The measure ν found in Lemma 3.6 satisfies the following properties:

(1) If $\Phi = (f_i)_{i\in\Lambda}$ satisfies the open set condition or, more generally, the weak separation condition, then ν is a finite sum of weighted Dirac measures. In contrast, without such separation conditions, ν may be non-atomic. To understand why this holds, recall that the weak separation condition ensures that the set $\Lambda(x,t)$ is finite with cardinality uniformly bounded across x and t. As a result, $\nu(x,t)$ is a

finite convex combination of Dirac measures $\delta_{g_{x,t}\circ f_{\mathtt{i}}}$, with weights $\bar{\mu}([\mathtt{i}])/C(x,t)$. In the limit, as $t\to\infty$, ν inherits this structure, becoming a finite sum of weighted Dirac measures supported on maps of the form $g_{x,t}\circ f_{\mathtt{i}}$. Without separation conditions, overlaps between $f_{\mathtt{i}}(K)$ for different $\mathtt{i}$ may accumulate, leading to unbounded cardinality of $\Lambda(x,t)$ as $t\to\infty$. This can produce a continuous distribution of maps $g_{x,t}\circ f_{\mathtt{i}}$ in the weak limit, resulting in a non-atomic measure ν.

(2) For ν-almost every $h\in\mathcal{C}^{1+\alpha}_M([-1,1]^d)$ and every $x\in B(0,1)$, we have

$$\|Dh\|^{-1}Dh\in\overline{\{\|Df_{\mathtt{i}}\|^{-1}Df_{\mathtt{i}}:\mathtt{i}\in\Lambda^*\}},\tag{3.20}$$

where the closure is taken in the topology induced by the supremum norm. To see why this holds, recall from the paragraph following (3.15) that for any h in the support of ν, there exists a sequence $(g_{y_n,t_n}\circ f_{\mathtt{i}_n})_{n\in\mathbb{N}}$ with $y_n\in\operatorname{spt}\mu$, $t_n>0$, and $\mathtt{i}_n\in\Lambda^*$ such that

$$\|h-g_{y_n,t_n}\circ f_{\mathtt{i}_n}\|_{\mathcal{C}^1}\to 0.$$

In particular, this implies $\sup_{x\in[-1,1]^d}\|Dh|_x-D(g_{y_n,t_n}\circ f_{\mathtt{i}_n})|_x\|\to 0$. Since $M^{-1}\leqslant\|D(g_{y_n,t_n}\circ f_{\mathtt{i}_n})\|\leqslant M$ for every $n\in\mathbb{N}$, it follows that

$$\sup_{x\in[-1,1]^d}\left\|\frac{Dh|_x}{\|Dh|_x\|}-\frac{D(g_{y_n,t_n}\circ f_{\mathtt{i}_n})|_x}{\|D(g_{y_n,t_n}\circ f_{\mathtt{i}_n})|_x\|}\right\|\to 0.$$

Noting that $D(g_{y_n,t_n}\circ f_{\mathtt{i}_n})|_x=Dg_{y_n,t_n}|_{f_{\mathtt{i}_n}(x)}Df_{\mathtt{i}_n}|_x=2^tDf_{\mathtt{i}_n}|_x$, we obtain

$$\frac{D(g_{y_n,t_n}\circ f_{\mathtt{i}_n})|_x}{\|D(g_{y_n,t_n}\circ f_{\mathtt{i}_n})|_x\|}=\frac{Df_{\mathtt{i}_n}|_x}{\|Df_{\mathtt{i}_n}|_x\|}$$

for all $x\in\operatorname{spt}\mu$. This completes the proof of (3.20).

(3) If $\Phi=(f_i\colon\mathbb{R}^d\to\mathbb{R}^d,\ f_i(x)=r_iO_ix+a_i)_{i\in\Lambda}$ is a similarity iterated function system, then ν is supported on maps of the form $x\mapsto rOx+a$, where $r>0$ is a scalar, $O\in\overline{\langle O_1,\dots,O_N\rangle}$ is an orthogonal matrix, and $a\in\mathbb{R}^d$ is a translation vector. Here the closure of the group generated by the orthogonal matrices $\{O_i\}_{i\in\Lambda}$ is in the operator norm. To understand why this holds, note that, as the maps $f_{\mathtt{i}}$ are similarities with contraction coefficient $r_{\mathtt{i}}$ and orthogonal matrix $O_{\mathtt{i}}\in\langle O_1,\dots,O_N\rangle$, the composite $g_{x,t}\circ f_{\mathtt{i}}$ has derivative $2^tr_{\mathtt{i}}O_{\mathtt{i}}$ with $M^{-1}\leqslant 2^tr_{\mathtt{i}}\leqslant M$. Therefore, the claim follows from (2).

Theorems 2.2 and 2.3 imply that the tangent measures of a quasi-Bernoulli measure μ exhibit more regular local statistics than μ itself. Leveraging Lemma 3.6, which characterizes these tangents as equivalent to convex combinations of conformal images of μ, and Proposition 3.5, we can transfer this regularity to μ. Specifically, combining Theorems 2.2, 2.3, Lemma 3.6, and Proposition 2.5, there exists $\eta\in\mathcal{P}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}(\mathbb{R}^d))$ such that $\eta\cdot\mu$ is uniformly scaling and generates an ergodic fractal distribution. To apply Proposition 3.5 and conclude that μ is also uniformly scaling, we require that the convolution with η does not increase the entropy of μ. This is ensured by Proposition 2.7, given the exact-dimensionality of μ.

We are ready to prove Theorems 1.4, 1.3, 1.2, and 1.1.

Proof of Theorem 1.4. Without loss of generality, we may assume that Φ consists maps of $\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$. Our objective is to establish the existence of a function $h \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$ such that $h_*\mu$, where μ is the quasi-Bernoulli measure on K, is uniformly scaling along a subsequence. We assume that the dimension of μ satisfies $\dim \mu > 0$; otherwise, by [15, Corollary 6.6], μ is uniformly scaling and generates the Dirac distribution δ_{δ_0}. By Theorem 2.3, we restrict our attention to tangent distributions of μ that are fractal distributions. Let P be an ergodic component of a tangent distribution of μ satisfying the conclusion of Lemma 3.6. Applying Theorem 2.2 and Lemma 3.6 to P, we identify a measure $\nu \in \mathcal{P}(C^{1+\alpha}_{\mathrm{conf}}([-1,1]^d))$ such that $(\nu \cdot \mu)_{B(0,1)}$ is uniformly scaling and generates P. This yields a structured representation of the tangent distributions: each tangent distribution is expressed as a convex combination of ergodic fractal distributions, with their associated tangent measures being convex combinations of conformal images of μ, each generating the corresponding ergodic component. Consequently, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \iint_{h^{-1}(B(0,1))} d_{\mathrm{LP}}\left(P, \frac{1}{n}\int_0^n \delta_{(\nu\cdot\mu)_{h(x),t}}\,\mathrm{d}t\right)\mathrm{d}\mu(x)\,\mathrm{d}\nu(h) = 0. \tag{3.21}$$

By Proposition 2.7, the measure ν can be chosen such that $(\nu\cdot\mu)_{B(0,1)}$ is exact-dimensional. Thus, by (2.2),

$$\lim_{n\to\infty} \frac{1}{n} H((\nu\cdot\mu)_{B(0,1)}, \mathcal{D}_n) = \dim \mu.$$

Moreover, by Lemma 2.8, for ν-almost every $h \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$ and $h_*\mu$-almost every x, we have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} H((h_*\mu)^{\mathcal{D}_k(x)}, \mathcal{D}_1) = \dim \mu.$$

Hence, the measures ν and μ satisfy the conditions of Proposition 3.5 with $\xi = \dim\mu$ for any $\varepsilon > 0$. Applying this proposition, for any $\varepsilon > 0$ and all sufficiently large n, we have

$$\iint d_{\mathrm{LP}}\left(\frac{1}{n}\int_0^n \delta_{(\nu\cdot\mu)_{h(x),t}}\,\mathrm{d}t, \frac{1}{n}\int_0^n \delta_{(h_*\mu)_{h(x),t}}\,\mathrm{d}t\right)\mathrm{d}\mu(x)\,\mathrm{d}\nu(h) < \varepsilon. \tag{3.22}$$

Combining (3.21) and (3.22), applying the triangle inequality, and letting $\varepsilon \to 0$, we obtain

$$\lim_{n\to\infty} \iint_{h^{-1}(B(0,1))} d_{\mathrm{LP}}\left(P, \frac{1}{n}\int_0^n \delta_{(h_*\mu)_{h(x),t}}\,\mathrm{d}t\right)\mathrm{d}\mu(x)\,\mathrm{d}\nu(h) = 0.$$

Consequently, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{1}{n_k}\int_0^{n_k} \delta_{(h_*\mu)_{h(x),t}}\,\mathrm{d}t = P$$

in the weak*-topology, for ν-almost every h and μ-almost every $x \in h^{-1}(B(0,1))$. Selecting h such that $h(B(0,1)) \subseteq B(0,1)$ completes the proof. □

The proof of Theorem 1.3 essentially continues where the proof of Theorem 1.4 left.

Proof of Theorem 1.3. The proof builds on the arguments from the proof of Theorem 1.4, aiming to show that any two ergodic components of tangent distributions of the quasi-Bernoulli measure μ are related by an orthogonal transformation, reinforcing the uniform scaling property across different tangent measures. As before, we restrict our

attention to tangent distributions of μ that are fractal distributions by Theorem 2.3, and we assume that $\dim \mu > 0$. Let P and P' be ergodic components of tangent distributions of μ, both satisfying the conclusion of Lemma 3.6. Following the proof of Theorem 1.4, we identify maps $h, g \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$ and a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{(h_*\mu)_{h(x),t}} \,\mathrm{d}t = P, \\ \lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{(g_*\mu)_{g(x),t}} \,\mathrm{d}t = P', \end{aligned} \tag{3.23}$$

in the weak*-topology for μ-almost every x. Next, applying Proposition 2.4 to the measures $h_*\mu$ and $g_*\mu$ with the inverse maps h^{-1} and g^{-1}, and combining with (3.23), we find that, for μ-almost every x and all sufficiently large $t_0(x), t_0'(x) \geqslant 0$,

$$S_{t_0(x)}(Dh|_x^{-1})_{**}P = \lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{\mu_{x,t}} \,\mathrm{d}t = S_{t_0'(x)}(Dg|_x^{-1})_{**}P'.$$

Since S_t commutes with orthogonal matrices and P and P' are $(S_t)_{t\geqslant 0}$-invariant, this implies

$$\left(\frac{Dh|_x^{-1}}{\|Dh|_x^{-1}\|}\right)_{**} P = \left(\frac{Dg|_x^{-1}}{\|Dg|_x^{-1}\|}\right)_{**} P'$$

or, denoting by P_0 the fractal distribution $(\|Dh|_x^{-1}\|^{-1} Dh|_x^{-1})_{**}P$,

$$P' = \left(\frac{Dg|_x^{-1}}{\|Dg|_x^{-1}\|}\right)_{**}^{-1} P_0 = \left(\frac{Dg|_x}{\|Dg|_x\|}\right)_{**} P_0. \tag{3.24}$$

Consider a tangent distribution Q of the measure μ and let $Q = \int Q_\omega \,\mathrm{d}Q(\omega)$ denote its ergodic decomposition, where Q_ω are the ergodic components. Applying (3.24) with Q_ω in place of P' and recalling Remark 3.7(2), we conclude that, for Q-almost every ω, there exists an orthogonal matrix $U_\omega \in \mathcal{O}_\Phi$ such that

$$Q_\omega = (U_\omega)_{**} P_0.$$

Since every ergodic component of a tangent distribution of μ is an orthogonal transformation of P_0, this completes the proof. □

The proof of Theorem 1.2 relies on the arguments presented above.

Proof of Theorem 1.2. We aim to show that all ergodic components of the tangent distributions of the quasi-Bernoulli measure μ are identical, implying that μ has a unique tangent distribution. Recall from the proof of Theorem 1.3 that if P and P' are ergodic components of the tangent distributions of μ, there exist maps $h, g \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]^d)$ and a sequence $(n_k)_{k\in\mathbb{N}}$ such that (3.23) holds, i.e., $\lim_{k\to\infty}\langle h_*\mu\rangle_{h(x),n_k} = P$ and $\lim_{k\to\infty}\langle g_*\mu\rangle_{g(x),n_k} = P'$ in the weak*-topology for μ-almost every $x \in K$.

Fix $\varepsilon > 0$. The matrices $\|Dh|_x\|^{-1} Dh|_x$ and $\|Dg|_x\|^{-1} Dg|_x$ are orthogonal and, by Remark 3.7(2), there exist $\mathtt{i}_h, \mathtt{i}_g \in \Lambda^*$ such that

$$\left\| \frac{Dh|_x}{\|Dh|_x\|} - \frac{Df_{\mathtt{i}_h}|_x}{\|Df_{\mathtt{i}_h}|_x\|} \right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \frac{Dg|_x}{\|Dg|_x\|} - \frac{Df_{\mathtt{i}_g}|_x}{\|Df_{\mathtt{i}_g}|_x\|} \right\| < \frac{\varepsilon}{2} \tag{3.25}$$

for all $x \in K$. Since $\mathrm{Id} \in \mathcal{O}_\Phi(\mathtt{j})$ for all $\mathtt{j} \in \Lambda^*$, there exist $\mathtt{i}, \mathtt{i}' \in \Lambda^*$ and points $x_0, x_0' \in K$ such that

$$\left\| \frac{D(f_{\mathtt{i}_h} \circ f_{\mathtt{i}})|_{x_0}}{\|D(f_{\mathtt{i}_h} \circ f_{\mathtt{i}})|_{x_0}\|} - \mathrm{Id} \right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \frac{D(f_{\mathtt{i}_g} \circ f_{\mathtt{i}'})|_{x_0'}}{\|D(f_{\mathtt{i}_g} \circ f_{\mathtt{i}'})|_{x_0'}\|} - \mathrm{Id} \right\| < \frac{\varepsilon}{2}. \tag{3.26}$$

Combining (3.25) and (3.26), and applying the triangle inequality, we see that

$$\left\| \frac{D(h \circ f_{\mathtt{i}})|_{x_0}}{\|D(h \circ f_{\mathtt{i}})|_{x_0}\|} - \mathrm{Id} \right\| < \varepsilon \quad \text{and} \quad \left\| \frac{D(g \circ f_{\mathtt{i}'})|_{x_0'}}{\|D(g \circ f_{\mathtt{i}'})|_{x_0'}\|} - \mathrm{Id} \right\| < \varepsilon. \tag{3.27}$$

The quasi-Bernoulli property ensures that $(f_{\mathtt{j}})_*\mu \ll \mu$ for all $\mathtt{j} \in \Lambda^*$. Thus, $(h \circ f_{\mathtt{i}})_*\mu \ll h_*\mu$ and $(g \circ f_{\mathtt{i}'})_*\mu \ll g_*\mu$. By (3.23) and Proposition 2.5, the measures $(h \circ f_{\mathtt{i}})_*\mu$ and $(g \circ f_{\mathtt{i}'})_*\mu$ generate P and P', respectively, along the sequence $(n_k)_{k\in\mathbb{N}}$ for μ-almost all x. Applying Proposition 2.4 to the measures h_μ and g_μ with the inverse maps h^{-1} and g^{-1}, and using (3.23) and the $(S_t)_{t\geqslant 0}$-invariance of P and P', we obtain

$$\left(\frac{(D(h \circ f_{\mathtt{i}})|_x^{-1})}{\|D(h \circ f_{\mathtt{i}})|_x^{-1}\|} \right)_{**} P = \lim_{k\to\infty} \langle \mu \rangle_{x,n_k} = \left(\frac{(D(g \circ f_{\mathtt{i}'})|_x^{-1})}{\|D(g \circ f_{\mathtt{i}'})|_x^{-1}\|} \right)_{**} P'$$

in the weak*-topology for μ-almost every x. Since μ is fully supported on K, we can select x in a set of positive μ-measure arbitrarily close to x_0 or x_0'. Taking the limits as $x \to x_0$ and $x \to x_0'$ in this manner, the continuity of the derivatives and (3.27) imply that, as $\varepsilon \to 0$, the linear transformations approach the identity. Thus, $P = \lim_{k\to\infty}\langle\mu\rangle_{x,n_k} = P'$, proving that all ergodic components are equal. □

Finally, we show that conformal iterated function systems on $\mathbb{R}$ and similarity iterated function systems on $\mathbb{R}^d$ satisfy the conditions of Theorem 1.2.

Proof of Theorem 1.1. The case of a conformal iterated function system on $\mathbb{R}$ is straightforward, since $\|Df_{\mathtt{ji}}|_x\|^{-1} Df_{\mathtt{ji}}|_x \in \{-1, 1\}$. Thus, it suffices to show that for every similarity iterated function system $\Phi = (f_i(x) = r_i O_i x + a_i)_{i\in\Lambda}$ on $\mathbb{R}^d$,

$$\mathrm{Id} \in \mathcal{O}_\Phi(\mathtt{j}) = \overline{\{O_{\mathtt{ji}} : \mathtt{i} \in \Lambda^*\}}$$

for every $\mathtt{j} \in \Lambda^*$. Let $\mathtt{j}_n$ denote the n-fold concatenation of the finite word $\mathtt{j}$. Since the group of orthogonal matrices is compact in the operator norm, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $O_{\mathtt{j}_{n_k}} = (O_{\mathtt{j}})^{n_k}$ converges to some orthogonal matrix U. By passing to a subsequence if necessary, we may assume that $n_{k+1} - n_k \to \infty$. Hence,

$$\|O_{\mathtt{j}_{n_{k+1}}} - O_{\mathtt{j}_{n_k}}\| = \|O_{\mathtt{j}} O_{\mathtt{j}_{n_{k+1}-n_k-1}} - \mathrm{Id}\| \to 0$$

as $k \to \infty$, which completes the proof. □

4. Normal numbers in self-conformal sets

In this section, we apply Theorem 1.1 to prove Theorem 1.5, following the approach of Hochman and Shmerkin [17]. We first recall key definitions and results from [17] concerning the spectral properties of generated distributions. As in [17, Theorem 1.4], the proof of Theorem 1.5 combines [17, Theorem 1.1] with an analysis of the pure-point spectrum of the generated distribution. Let P be a distribution invariant under the semi-flow $(S_t)_{t\geqslant 0}$. For $x \in \mathbb{R}$, define $e(x) = e^{2\pi i x} \in \mathbb{C}$. A number $\alpha \geqslant 0$ is an *eigenvalue* of

P if there exists a non-trivial measurable function $\varphi\colon \mathcal{P}(\mathbb{R}) \to \mathbb{C}$, called an *eigenfunction*, such that

$$\varphi(S_t\mu) = e(t\alpha)\varphi(\mu)$$

for all $t \geqslant 0$ and P-almost every μ. The set of all eigenvalues of P is its *pure-point spectrum*. The following result, due to Hochman and Shmerkin [17, Theorems 1.1 and 1.2], is central to our proof.

Theorem 4.1. *Let μ be a Radon measure on $\mathbb{R}$ generating a non-trivial ergodic tangent distribution P, and let $\beta > 1$ be a Pisot number. If the pure-point spectrum of P contains no non-zero integer multiple of $1/\log\beta$, then μ-almost every x is normal.*

Given a Radon measure μ and $t_0 > 0$, we say that μ *t_0-generates* a distribution Q at a point x if

$$Q = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{\mu_{x,kt_0}}.$$

We recall the following property of fractal distributions from [17, Lemma 4.9 and Proposition 4.15].

Lemma 4.2. *If P is a fractal distribution, then for any $t_0 > 0$, P-almost every η t_0-generates an S_{t_0}-ergodic distribution P_x at η-almost every x. Furthermore, if φ is an eigenfunction of P for some eigenvalue k/t_0 with $k \in \mathbb{Z}$, then there exists $c \in \mathbb{C}$ such that $\varphi(\zeta) = c$ for η-almost every x and P_x-almost all ζ.*

With this foundation, we proceed to prove Theorem 1.5. The proof follows the strategy of [17, Theorem 1.4], utilizing Proposition 3.5 to connect the sceneries of μ to its tangent measures.

Proof of Theorem 1.5. Let P be the tangent distribution generated by μ-almost every x, as guaranteed by Theorem 1.1. Since P is scaling invariant, by Proposition 2.4, for any diffeomorphism g, P is also the tangent distribution generated by $g_*\mu$-almost every x. Let $\beta > 1$ be a Pisot number such that Φ is arithmetically independent of β. By Theorem 4.1, it suffices to show that $\frac{k}{\log\beta}$ is not an eigenvalue of P for any integer $k \neq 0$.

Suppose, for contradiction, that $t_0 = \frac{k_0}{\log\beta}$ is an eigenvalue of P for some $k_0 \neq 0$, with an associated eigenfunction $\varphi\colon \mathcal{P}(\mathbb{R}) \to \mathbb{C}$. By Lemma 4.2, there exists a constant $c \in \mathbb{C}$ such that, for P-almost every η and η-almost every x, η t_0-generates an S_{t_0}-ergodic distribution P_x such that $\varphi(\zeta) = c$ for P_x-almost every ζ. By Lemma 3.6, we can choose $\eta \sim_C (\nu \cdot \mu)_{B(0,1)}$ for some $\nu \in \mathcal{P}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1]))$. Following the proof of Theorem 1.4, there exists $h \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1,1])$ and a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{k=1}^{n_k} \delta_{(h_*\mu)_{h(x),kt_0}} = P_{h(x)}$$

for μ-almost all x. In particular, along the sequence $(n_k)_{k\in\mathbb{N}}$, $h_*\mu$ t_0-generates almost everywhere a distribution for which $\varphi \equiv c$ almost surely. The measure $h_*\mu$ is quasi-Bernoulli for the conjugated conformal iterated function system $h\Phi = \{h \circ f_i \circ h^{-1} : i \in \Lambda\}$,

where $\lambda(h \circ f_i \circ h^{-1}) = \lambda(f_i)$. To simplify notation, we assume that μ satisfies

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{k=1}^{n_k} \delta_{\mu_{x,kt_0}} = P_x$$

for μ-almost all x, and that $\varphi \equiv c$ holds P_x-almost surely. This assumption entails no loss of generality, as our goal is to derive a contradiction from the existence of a self-conformal measure μ with this property.

Since Φ is arithmetically independent of β, there exist $n \in \mathbb{N}$ and $\mathtt{i} \in \Lambda^*$ such that $0 < |\log \lambda(f_\mathtt{i}) + n \log \beta| < \frac{1}{t_0}$, or equivalently, $0 < |t_0 \log \lambda(f_\mathtt{i}) + k_0 n| < 1$. Thus, $t_0 \log \lambda(f_\mathtt{i})$ is not an integer, so

$$e(-t_0 \log \lambda(f_\mathtt{i})) \neq 1. \tag{4.1}$$

Let x_0 be the fixed point of f_1, and let U be a small interval centered at x_0. Since $\mu_U \ll \mu$ and $(f_\mathtt{i})_*\mu_U \ll \mu$, Proposition 2.5 implies that both μ_U and $(f_\mathtt{i})_*\mu_U$ t_0-generate the distribution P_x along the sequence $(n_k)_{k\in\mathbb{N}}$ for μ-almost every x. By [17, Lemma 4.16], the distribution t_0-generated by $(f_\mathtt{i})_*\mu_U$ at $f_\mathtt{i}(y)$ along the sequence $(n_k)_{k\in\mathbb{N}}$ is $(S_{-\log f_\mathtt{i}'(y)})P_y$. Thus,

$$\begin{aligned} c &= \frac{1}{\mu(U)} \int_{f_\mathtt{i}(U)} \int \varphi(\zeta) \,\mathrm{d}P_y(\zeta) \,\mathrm{d}(f_\mathtt{i})_*\mu(y) \\ &= \frac{1}{\mu(U)} \int_U \int \varphi(S_{-\log f_\mathtt{i}'(y)}\zeta) \,\mathrm{d}P_y(\zeta) \,\mathrm{d}\mu(y). \end{aligned}$$

Since φ is an eigenfunction for t_0, we have $\varphi(S_{-\log f_\mathtt{i}'(y)}\zeta) = e(-t_0 \log f_\mathtt{i}'(y))\varphi(\zeta) = e(-t_0 \log f_\mathtt{i}'(y))c$. As $\operatorname{diam}(U) \to 0$, we have $|f_\mathtt{i}'(y)| \to \lambda(f_\mathtt{i})$, so the integral approaches

$$\frac{1}{\mu(U)} \int_U \int e(-t_0 \log f_\mathtt{i}'(y))c \,\mathrm{d}P_y(\zeta) \,\mathrm{d}\mu(y) \to e(-t_0 \log \lambda(f_\mathtt{i}))c.$$

By (4.1), we see that $c = e(-t_0 \log \lambda(f_\mathtt{i}))c$ is a contradiction unless $c = 0$. Thus, $\varphi \equiv 0$, contradicting the non-triviality of φ. Hence, $\frac{k_0}{\log \beta}$ is not an eigenvalue, and the pure-point spectrum of P contains no non-zero integer multiples of $\frac{1}{\log \beta}$.

For the case where Φ is totally non-linear, the proof follows analogously, with modifications as in [17, Theorem 1.5]. □

5. Dissonance of quasi-Bernoulli measures

In this section, we prove Theorem 1.6. Let $\Pi_{d,k}$ denote the set of linear maps from $\mathbb{R}^d$ to $\mathbb{R}^k$. By [15, Theorem 1.22], if P is a fractal distribution, then for every $\pi \in \Pi_{d,k}$ and P-almost every μ, the pushforward measure $\pi_*\mu$ is exact-dimensional. This allows us to define the *expected dimension*

$$E_P(\pi) = \int \dim \pi_*\mu \,\mathrm{d}P(\mu)$$

for all fractal distributions P and linear maps $\pi \in \Pi_{d,k}$. We denote by $\mathrm{GL}_k(\mathbb{R})$ the general linear group of degree k. The group $\mathrm{GL}_d(\mathbb{R})$ acts on $\Pi_{d,k}$ via $U \cdot \pi = \pi \circ U^{-1}$, and $\mathrm{GL}_k(\mathbb{R})$ acts via $V \cdot \pi = V \circ \pi$. These actions commute, inducing a joint action of $\mathrm{GL}_k(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R})$ on $\Pi_{d,k}$. If $A \subseteq \mathrm{GL}_d(\mathbb{R})$ is a group of invertible linear maps, it induces

an action on measures and distributions, which we keep denoting by A for simplicity. A fractal distribution P is *non-singular* with respect to A if $a_{**}P \sim P$ for all $a \in A$.

For product measures, we equip $\mathbb{R}^d \times \mathbb{R}^k$ with the maximum metric, under which the magnification of a product measure $\mu \times \nu$ at (x, y) by e^r satisfies $(\mu \times \nu)(x, y), r = \mu x, r \times \nu_{y,r}$. For $t, s \geqslant 0$, define the scaling flow (S_t, S_s) by $(x, y) \mapsto (2^t x, 2^s y)$ and $\mu \times \nu \mapsto S_t\mu \times S_s\nu$ on measures. We now recall a key result from [15, Proposition 1.38] concerning the push-forward of fractal distributions under linear maps.

Proposition 5.1. *Let P be an ergodic fractal distribution non-singular with respect to a group $A \subseteq \mathrm{GL}_d(\mathbb{R})$. Then $E_P(\,\cdot\,)$ is constant on the $\overline{A}$-orbits in $\Pi_{d,k}$, where $\overline{A}$ is the topological closure of A. In particular, if an orbit $O \subseteq \Pi_{d,k}$ under the action of $\mathrm{GL}_k(\mathbb{R}) \times \overline{A}$ has non-empty interior, then*

$$E_P(\pi) = \min\{k, \dim P\}$$

for all $\pi \in O$.

The following theorem is a consequence of [16, Theorem 1.10] and is stated in the language of fractal distributions as [15, Theorem 1.23].

Theorem 5.2. *Let μ be a Radon measure on $\mathbb{R}^d$ that generates an ergodic fractal distribution P along a subsequence. Then*

$$\dim \pi_*\mu \geqslant E_P(\pi),$$

for all $\pi \in \Pi_{d,k}$.

Before proving Theorem 1.6, we establish that for quasi-Bernoulli measures μ and ν on $\mathbb{R}$, their product $\mu \times \nu$ generates an ergodic fractal distribution almost everywhere along a subsequence.

Lemma 5.3. *Let Φ and Ψ be conformal iterated function systems on $\mathbb{R}$, with associated quasi-Bernoulli measures μ and ν, respectively. There exist a sequence $(n_k)_{k\in\mathbb{N}}$ and an ergodic fractal distribution P with $\dim P \geqslant \dim(\mu \times \nu)$ such that*

$$\lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{(\mu\times\nu)_{(x,y),t}} \,\mathrm{d}t = P$$

for $\mu \times \nu$-almost all (x, y).

Proof. Since $(\mu \times \nu)_{(x,y),t} = \mu_{x,t} \times \nu_{y,t}$ under the maximum metric, we apply Proposition 2.7 and Lemma 3.6. These imply the existence of an ergodic fractal distribution P with $\dim P \geqslant \dim(\mu \times \nu)$ and measures $\zeta_1, \zeta_2 \in \mathcal{P}(\mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1, 1]))$ such that the measure $(\zeta_1 \cdot \mu)_{B(0,1)} \times (\zeta_2 \cdot \nu)_{B(0,1)}$ is uniformly scaling and generates P. Following the argument in Theorem 1.3, there exist $h, g \in \mathcal{C}^{1+\alpha}_{\mathrm{conf}}([-1, 1])$ and a sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{(h_*\mu\times g_*\nu)_{(h(x),g(y)),t}} \,\mathrm{d}t = P$$

for $\mu \times \nu$-almost every (x, y). By Proposition 2.4, for sufficiently large t_0,

$$\lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{(\mu\times\nu)_{(x,y),t}} \,\mathrm{d}t = (S_{t_0+\log h'(x)}, S_{t_0+\log g'(y)})P$$

for $\mu \times \nu$-almost every (x, y). Since linear maps preserve the ergodicity of fractal distributions by [15, Proposition 1.8], the distribution $(S_{t_0+\log h'(x)}, S_{t_0+\log g'(y)})P$ is ergodic, completing the proof. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let P be the ergodic fractal distribution given by Lemma 5.3, satisfying $\dim P \geqslant \dim(\mu\times\nu)$ and generated by $\mu\times\nu$ along a subsequence for $\mu\times\nu$-almost every (x, y). For $\mathtt{i} \in \Gamma^*$ and $\mathtt{j} \in \Lambda^*$, let U and V be open intervals with $\mu(U) > 0$ and $\nu(V) > 0$. By the quasi-Bernoulli property, $(f_\mathtt{i})_*\mu_U \times (g_\mathtt{j})_*\nu_V \ll \mu \times \nu$. Thus, by Proposition 2.5, $(f_\mathtt{i})_*\mu_U \times (g_\mathtt{j})_*\nu_V$ generates P. Since $\mu_U \times \nu_V = ((f_\mathtt{i})^{-1}, (g_\mathtt{j})^{-1})_*((f_\mathtt{i})_*\mu_U \times (g_\mathtt{j})_*\nu_V)$, Proposition 2.4 implies that, for sufficiently large t_0, $\mu_U \times \nu_V$ generates the distribution

$$(S_{t_0+\log(f_\mathtt{i})'(x)}, S_{t_0+\log(g_\mathtt{j})'(y)})P$$

at $\mu \times \nu$-almost every (x, y). Using the $(S_t, S_t)_{t\geqslant 0}$-invariance of P, we deduce that P is invariant under $(S_{|\log g_\mathtt{j}'(y)-\log f_\mathtt{i}'(x)|}, \mathrm{Id})$ for all $\mathtt{i} \in \Gamma^*$, $\mathtt{j} \in \Lambda^*$, and $(x, y) \in f_\mathtt{i}(U) \times g_\mathtt{j}(V)$.

Now, let U and V be small open intervals centered at the fixed points of $f_\mathtt{i}$ and $g_\mathtt{j}$, respectively. As $\mathrm{diam}(U), \mathrm{diam}(V) \to 0$, the continuity of $(t, s) \mapsto (S_t, S_s)P$ implies that P is invariant under $(S_{|\log\lambda(f_\mathtt{i})-\log\lambda(g_\mathtt{j})|}, \mathrm{Id})$ for all $\mathtt{i} \in \Gamma^*$ and $\mathtt{j} \in \Lambda^*$. Write $D = \{|\log\lambda(f_\mathtt{i}) - \log\lambda(g_\mathtt{j})| : \mathtt{i} \in \Gamma^* \text{ and } \mathtt{j} \in \Lambda^*\}$. Recall our assumption that for any $\varepsilon > 0$, there exist $\mathtt{i} \in \Gamma^*, j \in \Lambda^*$ such that $0 < |\log\lambda(f_\mathtt{i}) - \log\lambda(g_\mathtt{j})| < \varepsilon$. Using this, and the relation $\lambda(f \circ f) = \lambda(f)\lambda(f)$, we deduce that the set D is dense in $[0, +\infty)$. Let $A = \{(S_t, \mathrm{Id}) : t \in D\}$ and recall it induces an action on distributions. Consider the map π defined as $(x, y) \mapsto x + y$. The orbit of π under $\mathrm{GL}_1(\mathbb{R}) \times \overline{A}$ is $\{(x, y) \mapsto \alpha(2^t x + y) : \alpha > 0 \text{ and } t \geqslant 0\}$, which has non-empty interior and contains π. By Proposition 5.1, $E_P(\pi) = \min\{1, \dim P\}$. Since $\dim P \geqslant \dim(\mu \times \nu) = \dim\mu + \dim\nu$ by Lemma 5.3 and the exact-dimensionality of μ and ν, and since $\dim(\mu * \nu) = \dim\pi_*(\mu \times \nu) \geqslant E_P(\pi)$ by Theorem 5.2, the result follows. □

6. Self-affine measures

In this section, we prove Theorem 1.7. Consider a diagonal iterated function system $\Phi = (f_i)_{i\in\Lambda}$ on $\mathbb{R}^2$, where each map $f_i \colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $f_i(x) = A_i x + a_i$, is affine with

$$A_i = \begin{pmatrix} \rho_i & 0 \\ 0 & \lambda_i \end{pmatrix}$$

for all $i \in \Lambda$. Let $\bar{\mu}$ be a Bernoulli measure on $\Lambda^{\mathbb{N}}$. By Birkhoff's ergodic theorem, there exist *Lyapunov exponents* $\chi_1(\mu)$ and $\chi_2(\mu)$ for the associated self-affine measure μ, defined as

$$\chi_1(\mu) = \lim_{n\to\infty} \frac{\log \rho_{\mathtt{i}|_n}}{n} \quad \text{and} \quad \chi_2(\mu) = \lim_{n\to\infty} \frac{\log \lambda_{\mathtt{i}|_n}}{n}$$

for $\bar{\mu}$-almost every $\mathtt{i} \in \Lambda^{\mathbb{N}}$. We say that μ has a *simple Lyapunov spectrum* if $\chi_1(\mu) \neq \chi_2(\mu)$.

Proposition 6.1. *Let Φ be a proximal, diagonal iterated function system on $\mathbb{R}^2$ with attractor K satisfying the rectangular strong separation condition. Then any self-affine measure μ on K is uniformly scaling and generates an ergodic fractal distribution P. Furthermore, $P = (S_{-\log\rho_\mathtt{i}}, S_{-\log\lambda_\mathtt{j}})P$ for all $\mathtt{i}, \mathtt{j} \in \Lambda^*$.*

Assuming Proposition 6.1 holds, we prove Theorem 1.7 immediately by adapting the strategy from the final paragraph of the proof of Theorem 1.6.

Proof of Theorem 1.7. Let μ be a self-affine measure on the attractor of a diagonal iterated function system Φ, with a simple Lyapunov spectrum. By Proposition 6.1, μ is uniformly scaling and generates an ergodic fractal distribution P such that $P = (S_{-\log \rho_{\mathtt{i}}}, S_{-\log \lambda_{\mathtt{j}}})P$ for all $\mathtt{i}, \mathtt{j} \in \Lambda^*$.

Consider a non-principal orthogonal projection $\pi \in \Pi_{2,1}$, given by $\pi(x, y) = ax + by$ with $a, b > 0$ (the case $a, b < 0$ is analogous). Choose $N \in \mathbb{N}$ such that $2^{-N} \leqslant a, b \leqslant 2^N$. By assumption, for any $\varepsilon > 0$, there exist $\mathtt{i}, \mathtt{j} \in \Lambda^*$ such that $0 < |\log \rho_{\mathtt{i}} - \log \lambda_{\mathtt{j}}| < \varepsilon$. Thus, the set $D = \{-\log \rho_{\mathtt{i}} + \log \lambda_{\mathtt{j}} : \mathtt{i}, \mathtt{j} \in \Lambda^*\}$ is dense in either $[0, N]$ or $[-N, 0]$. Without loss of generality, assume D is dense in $[0, N]$.

Define the group $A = \{(S_t, \mathrm{Id}) : t \in D \cap [0, N]\}$. By Proposition 6.1, P is invariant under every map in A. The orbit of π under $\mathrm{GL}_1(\mathbb{R}) \times \overline{A}$ is $\{(x, y) \mapsto \alpha(2^t x + y) : \alpha > 0 \text{ and } t \in [0, N]\}$, which contains π and has non-empty interior. Theorem 5.2 implies $\dim \pi_* \mu \geqslant E_P(\pi)$ and, by Proposition 5.1, $E_P(\pi) = \min\{1, \dim P\}$. Since $\dim P = \dim \mu$ by Proposition 2.7, we conclude that $\dim \pi_* \mu \geqslant \min\{1, \dim \mu\}$ as claimed. □

The remainder of this section is dedicated to the proof of Proposition 6.1. We first assume that μ has a simple Lyapunov spectrum, then address the adjustments needed to remove this assumption.

6.1. **Simple Lyapunov spectrum.** Without loss of generality, we assume the y-axis is the major asymptotic contracting direction, i.e., $\chi_1(\mu) > \chi_2(\mu)$. By the definitions of the Lyapunov exponents $\chi_1(\mu)$ and $\chi_2(\mu)$, for $\bar{\mu}$-almost every $\mathtt{i} \in \Lambda^{\mathbb{N}}$ and sufficiently large n (depending on $\mathtt{i}$), we have $\lambda_{\mathtt{i}|_n} < \rho_{\mathtt{i}|_n}$.

For a rectangle $R = [a, b] \times [c, d]$ with $0 < \mu(R) < \infty$, define the normalized measure $\mu_R = \mu(R)^{-1} \mu|_R$. Let $H \colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $H(x, y) = (\frac{2x-(b+a)}{b-a}, \frac{2y-(c+d)}{d-c})$, be the linear map that translates and rescales R to $[-1, 1]^2$. Generalizing the definition (2.1), set $\mu^R = H_* \mu_R$. Let $\pi \in \Pi_{2,1}$ be the orthogonal projection onto the x-axis. The disintegration of μ with respect to π is given by $\mu = \int \delta_x \times \mu_x \, \mathrm{d}\pi_* \mu(x)$, where μ_x is a measure on $\mathbb{R}$ for each x. For $\mathtt{i} \in \Lambda^{\mathbb{N}}$, we denote $\mu_{\mathtt{i}} = \mu_{\pi(\Pi(\mathtt{i}))}$.

Lemma 6.2. *Let μ be a Borel probability measure on $\mathbb{R}^2$, and let $\pi \in \Pi_{2,1}$ be the orthogonal projection onto the x-axis. Then, for $\pi_* \mu$-almost every x,*

$$d_{\mathrm{LP}}\big(\mu^{\pi^{-1}(B(x, 2^{-r}))}, (\pi_* \mu)_{x,r} \times \mu_x\big) \to 0$$

as $r \to \infty$.

Proof. By Lusin's theorem, for every $n \in \mathbb{N}$, there exists a set $E_n \subseteq \mathbb{R}$ with $\pi_* \mu(E_n) \geqslant 1 - 1/n$ such that the map $x \mapsto \mu_x$ is continuous on E_n. By the Besicovitch density point theorem, there exists a subset $E_n' \subseteq E_n$ with $\pi_* \mu(E_n') = \pi_* \mu(E_n)$ such that, for all $x \in E_n'$,

$$\frac{\pi_* \mu(E_n' \cap B(x, 2^{-r}))}{\pi_* \mu(B(x, 2^{-r}))} \to 1$$

as $r \to \infty$. Fix $n \in \mathbb{N}$ and $x \in E'_n$. Let $H_r \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map that translates and rescales $\pi^{-1}(B(x, 2^{-r})) \cap (\mathbb{R} \times [-1,1])$ onto $[-1,1]^2$. The Lévy-Prokhorov distance is

$$d_{\mathrm{LP}}\big(\mu^{\pi^{-1}(B(x,2^{-r}))}, (\pi_*\mu)_{x,r} \times \mu_x\big) = d_{\mathrm{LP}}\bigg(\int_{B(x,2^{-r})} H_r(\delta_y \times \mu_y)\,\mathrm{d}(\pi_*\mu)_{x,r}(y), \int_{B(x,2^{-r})} H_r(\delta_y \times \mu_x)\,\mathrm{d}(\pi_*\mu)_{x,r}(y)\bigg).$$

Since $x \mapsto \mu_x$ is continuous on E'_n, for $y \in E'_n \cap B(x, 2^{-r})$, we have $d_{\mathrm{LP}}(\mu_y, \mu_x) \to 0$ as $r \to \infty$. Moreover, as E'_n has density 1 at x, the contribution of points outside E'_n is negligible. Thus, the integrands converge, and the distance tends to 0,

$$d_{\mathrm{LP}}(\mu^{\pi^{-1}(B(x,2^{-r}))}, (\pi_*\mu)_{x,r} \times \mu_x) \to 0$$

as $r \to \infty$. Since $\pi_*\mu(\bigcup_{n=1}^{\infty} E'_n) = 1$, the result holds for $\pi_*\mu$-almost every x. □

By combining Lemma 6.2 with the self-affinity of μ, we deduce that, outside a negligible set of scales, magnifications of μ exhibit a product structure.

Lemma 6.3. *Let Φ be a diagonal iterated function system on $\mathbb{R}^2$ with attractor K satisfying the rectangular strong separation condition, let μ be a self-affine measure on K with $\chi_1(\mu) > \chi_2(\mu)$, and let $\pi \in \Pi_{2,1}$ be the orthogonal projection onto the x-axis. There exists $c > 0$ such that, for $\bar{\mu}$-almost every $\mathtt{i} \in \Lambda^{\mathbb{N}}$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} d_{\mathrm{LP}}\big(\mu_{\Pi(\mathtt{i}),-\log\lambda_{\mathtt{i}|_k}+c}, S_c(T_{\Pi(\sigma^k\mathtt{i})})_*((\pi_*\mu)_{\pi(\Pi(\sigma^k\mathtt{i})),\log\rho_{\mathtt{i}|_k}-\log\lambda_{\mathtt{i}|_k}} \times \mu_{\sigma^k\mathtt{i}})\big) \to 0$$

as $n \to \infty$.

Proof. Define $c = \max\{-\log \operatorname{dist}(\varphi_i([-1,1]^2), \varphi_j([-1,1]^2)) : i, j \in \Lambda \text{ such that } i \neq j\}$. By the assumption $\chi_1(\mu) > \chi_2(\mu)$, for $\bar{\mu}$-almost every $\mathtt{i} \in \Lambda^{\mathbb{N}}$ and sufficiently large k, we have $\lambda_{\mathtt{i}|_k} < \rho_{\mathtt{i}|_k}$. The definition of c ensures that the ball $B(\Pi(\mathtt{i}), 2^{\log\lambda_{\mathtt{i}|_k}-c})$ intersects only $\varphi_{\mathtt{i}|_k}([-1,1]^2)$. Define the rectangle

$$R_{\mathtt{i},n} = \varphi_{\mathtt{i}|_n}^{-1}(B(\Pi(\mathtt{i}), 2^{\log\lambda_{\mathtt{i}|_n}-c})),$$

which is $[-2^{\log\lambda_{\mathtt{i}|_k}-\log\rho_{\mathtt{i}|_k}-c}, 2^{\log\lambda_{\mathtt{i}|_k}-\log\rho_{\mathtt{i}|_k}-c}] \times [-2^{-c}, 2^{-c}]$ translated to have center $\Pi(\sigma^k\mathtt{i})$. By self-affinity,

$$\mu_{\Pi(\mathtt{i}),-\log\lambda_{\mathtt{i}|_n}+c} = (\varphi_{\mathtt{i}|_n}\mu)^{B(\Pi(\mathtt{i}),2^{\log\lambda_{\mathtt{i}|_n}-c})} = \mu^{R_{\mathtt{i},n}}.$$

Fix $\varepsilon > 0$. By Egorov's theorem and Lemma 6.2, there exist a set $A \subset \Lambda^{\mathbb{N}}$ with $\bar{\mu}(A) > 1 - \varepsilon$ and $N \in \mathbb{N}$ such that, for all $\mathtt{i} \in A$ and $r \geqslant N$,

$$d_{\mathrm{LP}}(\mu^{\pi^{-1}(B(\Pi(\mathtt{i}),2^{-r}))}, (\pi_*\mu)_{\Pi(\mathtt{i}),r} \times \mu_{\mathtt{i}}) < \varepsilon.$$

For $\mathtt{i} \in A$ and k sufficiently large such that $\sigma^k\mathtt{i} \in A$, set $r_k = \log\rho_{\mathtt{i}|_k}$ and $s_k = \log\lambda_{\mathtt{i}|_k}$. Then,

$$d_{\mathrm{LP}}(\mu_{\Pi(\mathtt{i}),-s_k+c}, S_c(T_{\Pi(\sigma^k\mathtt{i})})_*((\pi_*\mu)_{\pi(\Pi(\sigma^k\mathtt{i})),r_k-s_k} \times \mu_{\sigma^k\mathtt{i}})) < \varepsilon \tag{6.1}$$

and

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} d_{\mathrm{LP}}(\mu_{\Pi(\mathtt{i}),-s_k+c}, S_c(T_{\Pi(\sigma^k\mathtt{i})})_*((\pi_*\mu)_{\pi(\Pi(\sigma^k\mathtt{i})),r_k-s_k}\times\mu_{\sigma^k\mathtt{i}}))\\ \leqslant \frac{1}{n}\sum_{k=0}^{n-1}(\mathbb{1}\{\sigma^k\mathtt{i}\in A\}\varepsilon+\mathbb{1}\{\sigma^k\mathtt{i}\notin A\})+\frac{2N}{n}\\ \leqslant \varepsilon+\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}\{\sigma^k\mathtt{i}\notin A\}+\frac{2N}{n}.\end{aligned}$$

As $n\to\infty$, the second term converges to $\bar{\mu}(\Lambda^{\mathbb{N}}\setminus A)\leqslant\varepsilon$, and the third term vanishes. Hence, the sum is bounded by 2ε. Since $\varepsilon>0$ is arbitrary, the result follows. □

To prove Proposition 6.1, we first show that the product measures $\pi_*\mu\times\mu_{\mathtt{i}}$ are uniformly scaling along a subsequence for $\bar{\mu}$-almost every $\mathtt{i}\in\Lambda^{\mathbb{N}}$, then apply Lemma 6.3 to extend this property to μ. Notably, this result does not require assumptions on the Lyapunov exponents $\chi_1(\mu)$ and $\chi_2(\mu)$.

Lemma 6.4. *Let Φ be a diagonal iterated function system on $\mathbb{R}^2$ with attractor K satisfying the rectangular strong separation condition, let μ be a self-affine measure on K with $\chi_1(\mu)>\chi_2(\mu)$, and let $\pi\in\Pi_{2,1}$ be the orthogonal projection onto the x-axis. There exist an ergodic fractal distribution P with $\dim P\geqslant\dim\mu$ and a sequence $(n_k)_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty}\frac{1}{n_k}\int_0^{n_k}\delta_{(\pi_*\mu\times\mu_{\mathtt{i}})_{(x,y),t}}\,\mathrm{d}t=P \tag{6.2}$$

for $\bar{\mu}$-almost every $\mathtt{i}\in\Lambda^{\mathbb{N}}$ and $\pi_*\mu\times\mu_{\mathtt{i}}$-almost every (x,y). Furthermore,

$$P=(S_{-\log\rho_{\mathtt{i}}},S_{-\log\lambda_{\mathtt{j}}})P$$

for all $\mathtt{i},\mathtt{j}\in\Lambda^*$.

Proof. Let $\pi'\in\Pi_{2,1}$ be the orthogonal projection onto the y-axis. By [20, Theorem 4.2], for $\bar{\mu}$-almost every $\mathtt{i}\in\Lambda^{\mathbb{N}}$ and sufficiently large t, the measure $\mu_{\mathtt{i}}$ satisfies

$$(\mu_{\mathtt{i}})_{\pi'(\Pi(\mathtt{i})),t}=(\mu_{\sigma(\mathtt{i})})_{\pi'(\Pi(\sigma(\mathtt{i}))),t+\log\lambda_{\mathtt{i}|_1}} \tag{6.3}$$

implying that $\mu_{\mathtt{i}}$ is uniformly scaling and generates the ergodic fractal distribution

$$Q'=\iint_0^{-\log\lambda_{\mathtt{i}|_1}} S_t(T_{\pi'(\Pi(\mathtt{i}))})_*\mu_{\mathtt{i}}\,\mathrm{d}t\,\mathrm{d}\bar{\mu}(\mathtt{i}).$$

Since $\pi_*\mu$ is a self-similar measure on $\mathbb{R}$ and $\dim(\pi_*\mu\times\mu_{\mathtt{i}})=\dim\mu$ for $\bar{\mu}$-almost every $\mathtt{i}$ by [5, Theorem 2.7] (see also [6]), we proceed as in the proof of Lemma 5.3. Let Q be an ergodic component of a tangent distribution of μ. Since Q is supported on product measures with second marginal projection Q', P-almost every measure has the form $\eta\times S_t(T_x)_*\mu_{\mathtt{i}}$, where η is a tangent measure of $\pi_*\mu$, $t\geqslant 0$, $x\in\mathbb{R}^2$, and $\mathtt{i}\in\Lambda^{\mathbb{N}}$. By Lemma 3.6, $\eta=\int((h\circ\pi)_*\mu)_{B(0,1)}\,\mathrm{d}\nu(h)$ for some probability measure ν on affine functions from $\mathbb{R}^2$ to $\mathbb{R}$. Thus, arguing as in the proof of Lemma 5.3, there exists an affine map $h\colon\mathbb{R}^2\to\mathbb{R}^2$ such that $h_*(\pi_*\mu\times\mu_{\mathtt{i}})$ is uniformly scaling and generates Q. Applying Proposition 2.4, we obtain (6.2) with P as a restriction of an affine image of Q.

The invariance $P = (S_{-\log \rho_{\mathtt{i}}}, S_{-\log \lambda_{\mathtt{j}}})P$ for all $\mathtt{i}, \mathtt{j} \in \Lambda^*$ follows from (6.3), the self-similarity of $\pi_*\mu$, and Proposition 2.5. □

We now prove Proposition 6.1 in the case where μ has a simple Lyapunov spectrum. As the proof follows ideas from Theorem 1.3, we provide a concise sketch.

Proof of Proposition 6.1. Let Q be an ergodic component of a tangent distribution of μ. For μ-almost every point, Q-almost every measure is uniformly scaling and generates Q. By Lemma 6.3, Q-almost every measure has the form

$$(\eta \times \mu_{\mathtt{i}})_{(x,y),t}, \tag{6.4}$$

where η is a tangent measure of $\pi_*\mu$, $\mathtt{i} \in \Lambda^{\mathbb{N}}$, $(x, y) \in \mathbb{R}^2$, and $t \geqslant 0$. It follows from the proof of Lemma 3.6 that $\eta = ((\nu \cdot \pi_*\mu)_{B(0,1)})$ for a probability measure ν supported on affine maps $h\colon \mathbb{R} \to \mathbb{R}$, $h(y) = r_h y + a_h$, with r_h in the closure of $\{\rho_{\mathtt{i}}\lambda_{\mathtt{j}}^{-1} : \mathtt{i}, \mathtt{j} \in \Lambda^*\}$. Following the approach of Theorem 1.3, there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and an affine map $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{k\to\infty} \frac{1}{n_k} \int_0^{n_k} \delta_{((h\circ\pi)_*\mu\times\mu_{\mathtt{i}})_{(x,y),t}} \,\mathrm{d}t = Q$$

for $\pi_*\mu \times \mu_{\mathtt{i}}$-almost every (x, y). Let P be the ergodic fractal distribution from Lemma 6.4. By Proposition 2.4, for some $t_0 \geqslant 0$,

$$Q = (S_{t_0-\log r_h}, S_{t_0})P = P,$$

since $r_h \in \overline{\{\rho_{\mathtt{i}}\lambda_{\mathtt{j}}^{-1} : \mathtt{i}, \mathtt{j} \in \Lambda^*\}}$ and P is invariant under $(S_{-\log \rho_{\mathtt{i}}}, S_{-\log \lambda_{\mathtt{j}}})P$ by Lemma 6.4. □

6.2. **Equal Lyapunov exponents.** Without assuming $\chi_1(\mu) > \chi_2(\mu)$, the condition $\lambda_{\mathtt{i}|_n} < \rho_{\mathtt{i}|_n}$ for $\bar{\mu}$-almost every $\mathtt{i} \in \Lambda^{\mathbb{N}}$ and all large n, which was critical in Lemma 6.3, no longer holds. However, by [28, Theorem 1], for a proximal iterated function system Φ, we have

$$\lim_{n\to\infty} \frac{1}{n} \#\Big\{k \in \{1, \ldots, n\} : \Big|\log \frac{\rho_{\mathtt{i}|_k}}{\lambda_{\mathtt{i}|_k}}\Big| \leqslant M\Big\} = 0$$

for $\bar{\mu}$-almost every $\mathtt{i} \in \Lambda^{\mathbb{N}}$ and for all $M > 0$. For each $\varepsilon > 0$ and sufficiently large $M > 0$, define

$$A_\lambda = \{k \in \{1, \ldots, n\} : \log \lambda_{\mathtt{i}|_k} \leqslant \log \rho_{\mathtt{i}|_k} - M\},$$
$$A_\rho = \{k \in \{1, \ldots, n\} : \log \rho_{\mathtt{i}|_k} \leqslant \log \lambda_{\mathtt{i}|_k} - M\}.$$

Then, $\lim_{n\to\infty} \frac{1}{n}\#((A_\lambda \cup A_\rho) \cap [1, n]) = 1$. For $k \in A_\lambda$, the inequality (6.1) from Lemma 6.3 holds as stated. For $k \in A_\rho$, (6.1) holds after swapping $\lambda_{\mathtt{i}|_k}$ and $\rho_{\mathtt{i}|_k}$ and replacing π, the projection onto the x-axis, with the projection onto the y-axis. Thus, Lemma 6.3 holds with these modifications for $k \in A_\rho$.

In the proof of Proposition 6.1, this adjustment implies that the measure in (6.4) may take the form $(\mu_{\mathtt{i}} \times \eta)_{(x,y),t}$, where $\mu_{\mathtt{i}}$ is a conditional measure of μ supported on a horizontal line segment, and η is a tangent measure of $\pi'_*\mu$, the projection of μ onto the y-axis. With the roles of the x- and y-axes interchanged as needed, the proof proceeds analogously.

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