# SUB-ADDITIVE PRESSURE FOR IFS WITH TRIANGULAR MAPS

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Abstract. We investigate properties of the zero of the subadditive pressure which is a most important tool to estimate the Hausdorff dimension of the attractor of a non-conformal iterated function system (IFS). Our result is a generalization of the main results of Miao, Falconer [6] and Manning, Simon [8].

## 1. Introduction

Since the main goal of this paper is to improve a tool which is used to estimate the Hausdorff dimension, first we define the Hausdorff measure and Hausdorff dimension of a bounded set $A \subset \mathbb{R}^n$. Let

$$\mathcal{H}^s_\delta = \inf \left\{ \sum_i \mid U_i \mid^s : A \subset \bigcup_i U_i \quad \mid U_i \mid < \delta \right\} \tag{1.1}$$

where $\mid U \mid$ is the diameter of $U$. Now we define the $s$-dimensional Hausdorff measure of $A$ by

$$\mathcal{H}^s(A) = \lim_{\delta \to \infty} \mathcal{H}^s_\delta(A). \tag{1.2}$$

We call $\dim_H A$ the Hausdorff dimension of $A$ and

$$\dim_H A = \inf \left\{ s : \mathcal{H}^s(A) = 0 \right\}. \tag{1.3}$$

We consider the Hausdorff dimension of the attractors of iterated function systems (IFS) which are non-conformal. (We say that a map is conformal if the derivative is a similarity in every point.) The dimension theory of non-conformal IFS is very difficult and there are only very few results. The most important tool of

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this field is the sub-additive pressure, which was defined by K. Falconer [4] and L. Barreira [1]. Unfortunately, we know very little about sub-additive pressure itself.

In the conformal case, the sub-additive pressure coincides with the usual topological pressure, see for example [10, Chapter 9].

The simplest non-conformal situation is the case of self-affine IFS. To study the dimension of a self-affine attractor we consider the $k$-th approximation of the attractor with the so called $k$-th cylinders which are naturally defined by the $k$ fold application of the functions of the IFS. To measure the contribution of such a $k$ cylinder to the covering sum which appears in the definition of the Hausdorff measure (see (1.1) and (1.2) for each of these $k$-th cylinders we consider the singular value function. These are non-negative valued functions defined in a neighborhood of the attractor. The dimension of the attractor is related to the exponential growth rate of the sum of the values of these exponentially many singular value functions in the self affine case (see [2]). To verify this it was essential that this exponential growth rate is the same wherever we evaluate these singular value functions, since the singular value functions are constant in the self-affine case.

Falconer [4], Barreira [1] considered the more general situation when the IFS is no longer self-affine. In this case, using a similar method, it turns out that under a technical condition (which was named by Barreira as 1-bunched property) the exponential growth rate of the sum of the value of the singular value functions does not depend on wherever they are evaluated. We express this phenomenon as the "insensitivity property holds".

This is a very important property of the sub-additive pressure and in general we do not know if it holds or not. The main goal of this paper is to verify this property in a special case when the 1-bunched property does not hold but the IFS consists of maps with lower triangular derivative matrices. This paper is a generalization of the result of K. Simon and A. Manning [8]. They proved the same assertion in two dimension.

Even the 1-bunched condition is not satisfied, Zhang [11] found that the zero of the sub-additive pressure is an upper bound for the Hausdorff dimension. As an application we supply two examples of such IFS which we are able to calculate the Hausdorff dimension using that the insensibility property holds.

The main theorem is also a generalization of a recent paper by K. Falconer and J. Miao [6]. They gave a formula to estimate the Hausdorff dimension of self-affine fractals generated by upper-triangular matrices. We will show a formula to estimate the sub-additive pressure in non-conformal case and we will prove that the sub-additive pressure depends only on the diagonal elements of the derivative matrices in the case when the derivative matrices are triangular. In this paper we use the method of K. Falconer's and J. Miao's article [6].

## 2. Definitions

In this section we define our iterated function system and the subadditive pressure.

Throughout this paper we will always assume the following, let $M \subset \mathbb{R}^n$ be non-empty, open and bounded set, and let $F_i : M \mapsto M$ contractive maps for every $i = 1, ..., l$. For an $\mathbf{i} = i_1 i_2 ... i_k$, $i_j \in \{1, ..., l\}$, we write $F_{\mathbf{i}}(\underline{x}) = F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_n}(\underline{x})$. Our principal assumption about the maps $F_i$, $i = 1, ..., l$ is that

$$F_i(x_1, ..., x_n) = \left(f_i^1(x_1), f_i^2(x_1, x_2), ..., f_i^n(x_1, ..., x_n)\right), \tag{2.1}$$

and $F_i(x_1, ..., x_n) \in C^{1+\varepsilon}(\overline{M})$ for every $i = 1, ..., l$. Moreover we require that $D_{\underline{x}} F_i$ is regular (non-singular matrix) for every $\underline{x} \in \overline{M}$ and every $i \in \{1, ..., l\}$. Denote the elements of $D_{\underline{x}} F_{\mathbf{i}}$ by $x_{ij}(\mathbf{i}, \underline{x})$.

**Proposition 2.1.** *There exists a real constant $0 < C < \infty$ such that*

$$C^{-1} < \frac{|x_{ii}(\mathbf{i}, \underline{x})|}{|x_{ii}(\mathbf{i}, \underline{y})|} < C \tag{2.2}$$

*for every $\underline{x}, \underline{y} \in \overline{M}$ and for every $\mathbf{i} \in \{1, ..., l\}^*$.*

*Proof.* Let $G_i^{(m)} : \mathbb{R}^m \mapsto \mathbb{R}^m$ for every integer $m$ between 1 and $n$, be the restriction of $F_i$ to the first $m$ component, i.e.:

$$G_i^{(m)}(x_1, ..., x_m) := \left(f_i^1(x_1), f_i^2(x_1, x_2), ..., f_i^m(x_1, ..., x_m)\right).$$

From [9, Page 198; Propostion 20.1 (3)] it follows that for every $\underline{x}, \underline{y} \in \overline{M}$, for every $\mathbf{i} \in \{1, ..., l\}^*$ finite sequence, and for $1 \leq m \leq n$ there exists a real $0 < C_m < \infty$ constant that

$$C_m^{-1} < \frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})} < C_m.$$

Since for every $m$, the matrix $D_{\underline{x}}G_{\mathbf{i}}^{(m)}$ is lower triangular matrix, the Jacobian is the following

$$\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x}) = |x_{11}(\mathbf{i},\underline{x})\cdots x_{mm}(\mathbf{i},\underline{x})|\,.$$

Therefore for every integer $1 \leq m < n$ and for every $\underline{x},\underline{y} \in M$

$$\frac{C_m^{-1}}{C_{m+1}} < \frac{\frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})}}{\frac{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{y})}} < \frac{C_m}{C_{m+1}^{-1}}$$

and

$$\frac{\frac{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m)}(\underline{y})}}{\frac{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{x})}{\text{Jac } G_{\mathbf{i}}^{(m+1)}(\underline{y})}} = \frac{\left|x_{m+1m+1}\left(\mathbf{i},\underline{y}\right)\right|}{\left|x_{m+1m+1}\left(\mathbf{i},\underline{x}\right)\right|}.$$

Then $C := \max_{1\leq m<n-1}\left\{\frac{C_m}{C_{m+1}^{-1}}, C_1\right\}$ choice completes the proof of the proposition.

□

The singular values of a linear contraction $T$ are the positive square roots of the eigenvalues of $TT^*$, where $T^*$ is the transpose of $T$. Let $\alpha_k(D_{\underline{x}}F_{\mathbf{i}})$ be the $k$-th greatest singular value of the $D_{\underline{x}}F_{\mathbf{i}}$ matrix. The singular value function $\phi^s$ is defined for $0 \leq s \leq n$ as

$$\phi^s(D_{\underline{x}}F_{\mathbf{i}}) := \alpha_1(D_{\underline{x}}F_{\mathbf{i}})...\alpha_{k-1}(D_{\underline{x}}F_{\mathbf{i}})\alpha_k(D_{\underline{x}}F_{\mathbf{i}})^{s-k+1} \tag{2.3}$$

where $k-1 < s \leq k$ and $k$ is positive integer. We define the maximum and the minimum of the singular value function as

$$\overline{\phi}^s(\mathbf{i}) := \max_{\underline{x}\in\overline{M}} \phi^s(D_{\underline{x}}F_{\mathbf{i}})\;,\; \underline{\phi}^s(\mathbf{i}) := \min_{\underline{x}\in\overline{M}} \phi^s(D_{\underline{x}}F_{\mathbf{i}}).$$

We define the sub-additive pressure after K. Falconer 1994 and L. Barreira 1996:

$$P(s) := \lim_{k\to\infty}\frac{1}{k}\log\sum_{|\mathbf{i}|=k}\overline{\phi}^s(\mathbf{i}) \tag{2.4}$$

and define the lower pressure:

$$\underline{P}(s) := \liminf_{k\to\infty}\frac{1}{k}\log\sum_{|\mathbf{i}|=k}\underline{\phi}^s(\mathbf{i}). \tag{2.5}$$

## 3. Sub-additive pressure for triangular maps

In this section we are going to state and prove the main theorem of the paper. Namely, the sub-additive pressure is equal to the lower pressure, which implies the insensitivity property. More precisely, it implies that the exponential growth rate of the sum of the value of the singular value functions does not depend on wherever they are evaluated. (see (2.4), (2.5))

**Theorem 3.1.** *Let $0 \leq s \leq n$. If $F_1, ..., F_l$ contractive maps in form (2.1) and $F_i \in C^{1+\varepsilon}$ for every $1 \leq i \leq l$ then*

$$P(s) = \underline{P}(s).$$

In the following we state some linear algebra definitions and lemmas, the proofs of which can be found in article [6].

The m-dimensional exterior algebra $\Phi^m$ is a vector space spanned by formal elements $v_1 \wedge ... \wedge v_m$ with $v_i \in \mathbb{R}^n$ such that $v_1 \wedge ... \wedge v_m = 0$ if $v_i = v_j$ for some $i \neq j$, and such that interchanging two different elements reverses the sign, i.e. $v_1 \wedge ...v_i...v_j... \wedge v_m = -v_1 \wedge ...v_j...v_i... \wedge v_m$, if $i \neq j$. Then $\Phi^m$ has dimension $\binom{n}{m}$ with basis $\{e_{j_1} \wedge ... \wedge e_{j_m} : 1 \leq j_1 < ... < j_m \leq n\}$ where $e_1, ...e_n$ are a given set of orthonormal vectors in $\mathbb{R}^n$.

Let us define a scalar product on $\Phi^m$ in the following way. Let

$$< v_1 \wedge \cdots \wedge v_m, u_1 \wedge \cdots \wedge u_m >_{\Phi^m} = \det\left((< v_i, u_j >)_{i,j=1...m}\right),$$

where $< ., . >$ is the usual scalar product on $\mathbb{R}^n$. One can extend $< ., . >_{\Phi^m}$ to every element of $\Phi^m$ by the natural way. Then $\Phi^m$ becomes a Hilbert-space. Let us define the norm $\|.\|$ on $\Phi^m$ by $< ., . >_{\Phi^m}$ by the usual way. Then it is easy to see that $\|v_1 \wedge ... \wedge v_m\|$ is equal to the absolute m-dimensional volume of the parallelepiped spanned by $v_1, ...v_m$, for every $v_1 \wedge ... \wedge v_m$, see [7, p. 44].

We may also define an other norm $\|.\|_\infty$ on $\Phi^m$ by

$$\left\| \sum_{1 \leq i_1 < ... < i_m \leq m} \lambda_{i_1...i_m} (e_{i_1} \wedge ... \wedge e_{i_m}) \right\|_\infty := \max |\lambda_{i_1...i_m}|.$$

If $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is linear then there is an induced linear mapping $\widetilde{T} : \Phi^m \mapsto \Phi^m$ given by

$$\widetilde{T}(v_1 \wedge ... \wedge v_m) := (Tv_1) \wedge ... \wedge (Tv_m).$$

The norms on $\Phi^m$ induce norms on the space of linear mappings $\mathfrak{L}(\Phi^m, \Phi^m)$ in the usual way by

$$\left\|\widetilde{T}\right\| = \sup_{w\in\Phi^m, w\neq 0} \frac{\left\|\widetilde{T}w\right\|}{\|w\|}.$$

Then with respect to the norm $\|.\|$

$$\left\|\widetilde{T}\right\| = \phi^m(T) \tag{3.1}$$

and with respect to the $\|.\|_\infty$

$$\left\|\widetilde{T}\right\|_\infty = \max\left\{\left|T^{(m)}\right| : T^{(m)} \text{ is an } m\times m \text{ minor of } T\right\}, \tag{3.2}$$

where $T^{(m)} = T\binom{r_1,\dots r_m}{s_1,\dots,s_m}$ is the determinant of that $m\times m$ minor of $n\times n$ matrix $T$ which is determined by the elements of $T$ in the rows $1 \leq r_1 < \dots < r_m \leq n$ and columns $1 \leq s_1 < \dots < s_m \leq n$. The space of linear mappings $\mathfrak{L}(\Phi^m, \Phi^m)$ is of finite dimension $\binom{n}{m}^2$. Since any two norms on a finite dimensional normed space are equivalent, there are constants $0 < c_1 < c_2 < \infty$ depending only on $n$ and $m$ such that

$$c_1\left\|\widetilde{T}\right\|_\infty \leq \left\|\widetilde{T}\right\| \leq c_2\left\|\widetilde{T}\right\|_\infty. \tag{3.3}$$

Now we notice several lemmas relating to minors of matrices. We will need some well-known lemmas.

**Lemma 3.2.** *Let $x_i \geq 0$, $i = 1, \dots, m$ and $p \in \mathbb{R}^+$.*

(1) *If $p > 1$, then $(x_1^p + \dots + x_m^p) \leq (x_1 + \dots + x_m)^p \leq m^{p-1}(x_1^p + \dots + x_m^p)$*
(2) *If $0 < p \leq 1$, then $m^{p-1}(x_1^p + \dots + x_m^p) \leq (x_1 + \dots + x_m)^p \leq (x_1^p + \dots + x_m^p)$.*

**Lemma 3.3.** *Let $a_n$ a sequence of real numbers such that $a_{n+m} \leq a_n + a_m$. Then there exists $\lim_{n\to\infty} \frac{a_n}{n}$ and it equals to $\inf_n \frac{a_n}{n}$.*

We first look at the expansion of $m\times m$ minors of the product of $k$ matrices $A = A_1A_2\cdots A_k$, where for $i = 1, \dots, k$

$$A_i = \begin{bmatrix} a_{11}^i & a_{12}^i & \dots & a_{1n}^i \\ a_{21}^i & a_{22}^i & \dots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^i & a_{n2}^i & \dots & a_{nn}^i \end{bmatrix}.$$

**Lemma 3.4.** *For $1 \le m \le n$, the $m \times m$ minors of $A = A_1 \cdots A_k$ have formal expansions in terms of the entries of the $A_i$ of the form*

$$A\binom{r_1, ...r_m}{s_1, ..., s_m} = \sum_{c_1,...,c_k} \pm a^1_{1(c_1)} \cdots a^1_{m(c_1)} a^2_{1(c_2)} \cdots a^2_{m(c_2)} \cdots a^k_{1(c_k)} \cdots a^k_{m(c_k)}$$

*such that for each $i = 1, ..., k$, the $a^i_{1(c_i)} \cdots a^i_{m(c_i)}$ are distinct entries $a^i_{rs}$ of $A_i$. In particular, for each $i$, $1(c_i), ..., m(c_i)$ denote pairs $(r, s)$ corresponding to entries in $m$ different rows and columns of the $i$th matrix $A_i$, and the sum is over all such entry combinations $(c_1, ..., c_k)$ with appropriate sign $\pm$.*

The proof of this Lemma can be found on [6, Lemmma 2.2]. Now we consider lower triangular matrices. For $i = 1, ..., k$, let

$$U_i = \begin{bmatrix} u^i_1 & 0 & ... & 0 \\ u^i_{21} & u^i_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u^i_{n1} & u^i_{n2} & ... & u^i_n \end{bmatrix}.$$

We consider the product

$$U = U_1 \cdots U_k = \begin{bmatrix} u_1 & 0 & ... & 0 \\ u_{21} & u_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & ... & u_n \end{bmatrix}.$$

We note that

$$u_{rs} = \sum_{r \ge r_1 \ge ... \ge r_{k-1} \ge s} u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s} \quad 1 \le r \le s \le n \tag{3.4}$$

since all other products are 0.

**Lemma 3.5.** *With notations as in above, let $U_1, ..., U_k$ be lower triangular matrices and $U = U_1 \cdots U_k$. Then*

(1) *If $r < s$, $u_{rs} = 0$*
(2) *If $r = s$, $u_{rs} \equiv u_r = u^1_r \cdots u^k_r$*
(3) *If $r > s$, then the sum (3.4) for $u_{rs}$ has at most $k^{r-s} \le k^{n-1}$ non-zero terms. Moreover, each non-zero summand $u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}$ has at most $n-1$ non-diagonal terms in the product, i.e. terms with $r \ne r_1$ or $r_i \ne r_{i+1}$ or $r_{k-1} \ne s$.*

The proof can also be found in [6, Lemma 2.3] for upper-triangular matrices. Now we extend the estimate of Lemma 3.5 to minors.

**Lemma 3.6.** *Let $U_1, ..., U_k$ and $U$ be lower triangular matrices as in above. Then each $m \times m$ minor of $U$ has an expansion of the form*

$$U\binom{r_1, ...r_m}{s_1, ..., s_m} = \sum_{c_1,...,c_k} \pm u^1_{1(c_1)} u^2_{1(c_2)} \cdots u^k_{1(c_k)} \cdots u^1_{m(c_1)} u^2_{m(c_2)} \cdots u^k_{m(c_k)}$$

*where $1(c_i), ..., m(c_i)$ are as in Lemma 3.4 and*

(1) *there are at most $m!k^{m(n-1)}$ terms in the sum which are non-zero,*
(2) *each summand contains at most $(n-1)^m$ non-diagonal elements in the product.*

The proof is equivalent to the proof of [6, Lemma 2.4]. Before we prove the Theorem 3.1, we define two sums.

$$H(s,r) = \max_{\substack{j_1,...,j_{m-1}\\ j'_1,...,j'_m}} \sum_{|\mathbf{i}|=r} (d_{j_1j_1}(\mathbf{i}) \cdots d_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s} (d_{j'_1j'_1}(\mathbf{i}) \cdots d_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \tag{3.5}$$

where $m-1 < s \leq m$ and $d_{jj}(\mathbf{i}) = \inf_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$. Moreover

$$T(s,r) = \max_{\substack{j_1,...,j_{m-1}\\ j'_1,...,j'_m}} \sum_{|\mathbf{i}|=r} (t_{j_1j_1}(\mathbf{i}) \cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s} (t_{j'_1j'_1}(\mathbf{i}) \cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \tag{3.6}$$

where $m-1 < s \leq m$ and $t_{jj}(\mathbf{i}) = \sup_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$. It is easy to see from Proposition 2.1 and the definition of the two sums that

$$H(s,r) \leq T(s,r) \leq C^s H(s,r). \tag{3.7}$$

**Lemma 3.7.** *For every positive integers $r, z$, $T(s, r+z) \leq T(s,r)T(s,z)$. Moreover $\lim_{r\to\infty} \frac{\log T(s,r)}{r}$ exists and equal with $\inf_r \frac{\log T(s,r)}{r}$.*

*Proof of Lemma 3.7.* From the definition $T(s,r)$ it follows

$$T(s,r+z)=\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}}\sum_{|\mathbf{i}|=r+z}(t_{j_1j_1}(\mathbf{i})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s}(t_{j'_1j'_1}(\mathbf{i})\cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1}\leq$$

$$\leq\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}}(\sum_{|\mathbf{i}|=r}\sum_{|\mathbf{h}|=z}((t_{j_1j_1}(\mathbf{i})t_{j_1j_1}(\mathbf{h})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i})t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}\times$$

$$\times(t_{j'_1j'_1}(\mathbf{i})t_{j'_1j'_1}(\mathbf{h})\cdots t_{j'_mj'_m}(\mathbf{i})t_{j'_mj'_m}(\mathbf{h}))^{s-m+1})=$$

$$=\max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}}(\sum_{|\mathbf{i}|=r}(t_{j_1j_1}(\mathbf{i})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s}(t_{j'_1j'_1}(\mathbf{i})\cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1}\times$$

$$\times\sum_{|\mathbf{h}|=z}(t_{j_1j_1}(\mathbf{h})\cdots t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}(t_{j'_1j'_1}(\mathbf{h})\cdots t_{j'_mj'_m}(\mathbf{h}))^{s-m+1}))\leq$$

$$\leq T(s,r)T(s,z).$$

The existence of the limit follows from Lemma 3.3. □

The proof of Theorem 3.1 follows the line of the proof of [6, Theorem 2.5], but our Theorem is not a consequence of [6, Theorem 2.5]. The most important alteration is that some of the functions in [6] are affine. So the derivatives in our case are not constant matrices. To control the consequences of this phenomenon in our proof, we have to state a Lemma.

**Lemma 3.8.** *Let $X$ compact subset of $\mathbb{R}^n$ and let $\{f_i\}$ be finitely many continuous, real valued functions. Then*

$$\sup_{\underline{x}\in X}\max_i f_i(\underline{x})=\max_i\sup_{\underline{x}\in X}f_i(\underline{x}).$$

*Proof of Lemma 3.8.* Since $X$ is compact, we have $\underline{x}_i\in X$ such that $f_i(\underline{x}_i)=\sup_{\underline{x}}f_i(\underline{x})$. Therefore

$$\sup_{\underline{x}}\max_i f_i(\underline{x})\leq\max_i\sup_{\underline{x}}f_i(\underline{x})=\max_i f_i(\underline{x}_i)=\max_{i,j}f_i(\underline{x}_j)=\max_j\max_i f_i(\underline{x}_j)$$
$$\leq\sup_{\underline{x}}\max_i f_i(\underline{x}),$$

which was to be proved. □

Moreover, in the proof of [6, Theorem 2.5], the singular value functions and the minors of the derivative matrices were compared. During the proof of Theorem 3.1 we will do this as well, however, we have to introduce in the proof a new IFS, which will be the $r$-th iteration of the original IFS, to take separation between the growth rate of the non-zero and the non-diagonal terms of the minors of the derivative matrices.

*Proof of Theorem 3.1.* Let

$$\{G_h\}_{h=1}^{l^r} = \{F_{i_1\dots i_r}\}_{i_1=1,\dots,i_r=1}^{l,\dots,l}. \tag{3.8}$$

In this case a $h$ index is suit a $\mathbf{i} \in \{1, ..., l\}^r$ finite sequence, length $r$. Let us define

$$\overline{\phi'}^s(\mathbf{h}) = \sup_{\underline{x}} \phi^s(D_{\underline{x}}G_{\mathbf{h}}),$$
$$\underline{\phi'}^s(\mathbf{h}) = \inf_{\underline{x}} \phi^s(D_{\underline{x}}G_{\mathbf{h}})$$

for $\mathbf{h} \in \{1, ..., l^r\}^*$, corresponding to IFS $\{G_h\}_{h=1}^{l^r}$, see (2.3).

It is easy to see that

$$\sum_{|\mathbf{i}|=kr} \phi^s(D_{\underline{x}}F_{\mathbf{i}}) = \sum_{|\mathbf{h}|=k} \phi^s(D_{\underline{x}}G_{\mathbf{h}}). \tag{3.9}$$

where $\mathbf{i} \in \{1, ..., l\}^{kr}$ and $\mathbf{h} \in \{1, ..., l^r\}^k$. The elements of $D_{\underline{x}}G_h$, denoted by $y_{ij}(h, \underline{x})$, are equal with $x_{ij}(\mathbf{i}, \underline{x})$ for a suit finite sequence $\mathbf{i}$, length $r$. It is very simple to see that
$\phi^s(D_{\underline{x}}G_{\mathbf{h}}) = (\phi^{m-1}(D_{\underline{x}}G_{\mathbf{h}}))^{m-s}(\phi^m(D_{\underline{x}}G_{\mathbf{h}}))^{s-m+1}$, where $m-1 < s \leq m$. By using relations (3.1), (3.2) and (3.3) it follows that

$$\phi^m(D_{\underline{x}}G_{\mathbf{h}}) \geq c_2 \max\left\{\left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right| : D_{\underline{x}}G_{\mathbf{h}}^{(m)} \text{ is an } m \times m \text{ minor of } D_{\underline{x}}G_{\mathbf{h}}\right\}.$$

The maximum $m \times m$ minor of $D_{\underline{x}}G_{\mathbf{h}}$ is at least the largest product of $m$ distinct diagonal elements of $D_{\underline{x}}G_{\mathbf{h}}$, since such products are themselves minors of triangular matrices. Therefore

$$\underline{\phi'}^s(\mathbf{h}) \geq$$
$$c_2^s\left(\inf_{\underline{x}}\left|y_{j_1j_1}(\mathbf{h}, \underline{x}) \cdots y_{j_{m-1}j_{m-1}}(\mathbf{h}, \underline{x})\right|\right)^{m-s}\left(\inf_{\underline{x}}\left|y_{j'_1j'_1}(\mathbf{h}, \underline{x}) \cdots y_{j'_mj'_m}(\mathbf{h}, \underline{x})\right|\right)^{s-m+1}$$

for every $j_1, ..., j_{m-1}, j'_1, ..., j'_m$.

By the chain rule $D_{\underline{x}}G_{\mathbf{h}} = D_{G_{h_2\dots h_k}(\underline{x})}G_{h_1}D_{G_{h_3\dots h_k}(\underline{x})}G_{h_2}\cdots D_{\underline{x}}G_{h_k}$,
$y_{jj}(\mathbf{h}, \underline{x}) = y_{jj}(h_1, G_{h_2\dots h_k}(\underline{x}))\, y_{jj}(h_2, G_{h_3\dots h_k}(\underline{x}))\cdots y_{jj}(h_k, \underline{x})$. It follows with the

notation $\inf_{\underline{x}} |y_{jj}(h,\underline{x})| = d'_{jj}(h)$ that

$$\inf_{\underline{x}} \left|y_{j_1j_1}(\mathbf{h},\underline{x})\cdots y_{j_{m-1}j_{m-1}}(\mathbf{h},\underline{x})\right|^{m-s} \inf_{\underline{x}} \left|y_{j'_1j'_1}(\mathbf{h},\underline{x})\cdots y_{j'_mj'_m}(\mathbf{h},\underline{x})\right|^{s-m+1} \geq$$

$$\geq (d'_{j_1j_1}(h_1)\cdots d'_{j_1j_1}(h_k)d'_{j_2j_2}(h_1)\cdots d'_{j_{m-1}j_{m-1}}(h_1)\cdots d'_{j_{m-1}j_{m-1}}(h_k))^{m-s}\times$$

$$\times (d'_{j'_1j'_1}(h_1)\cdots d'_{j'_1j'_1}(h_k)d'_{j'_2j'_2}(h_1)\cdots d'_{j'_mj'_m}(h_1)\cdots d'_{j'_mj'_m}(h_k))^{s-m+1}.$$

The next inequality follows from the rearrangement of the product

$$\sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \geq$$

$$c_2^s \sum_{|\mathbf{h}|=k} (d'_{j_1j_1}(h_1)\cdots d'_{j_{m-1}j_{m-1}}(h_1))^{m-s}(d'_{j'_1j'_1}(h_1)\cdots d'_{j'_mj'_m}(h_1))^{s-m+1}\cdots$$

$$\cdots(d'_{j_1j_1}(h_k)\cdots d'_{j_{m-1}j_{m-1}}(h_k))^{m-s}(d'_{j'_1j'_1}(h_k)\cdots d'_{j'_mj'_m}(h_k))^{s-m+1} =$$

$$= c_2^s((d'_{j_1j_1}(1)\cdots d'_{j_{m-1}j_{m-1}}(1))^{m-s}(d'_{j'_1j'_1}(1)\cdots d'_{j'_mj'_m}(1))^{s-m+1} + \cdots$$

$$\cdots + (d'_{j_1j_1}(l^r)\cdots d'_{j_{m-1}j_{m-1}}(l^r))^{m-s}(d'_{j'_1j'_1}(l^r)\cdots d'_{j'_mj'_m}(l^r))^{s-m+1})^k.$$

The inequality above is true for every $j_1,...,j_{m-1},j'_1,...,j'_m$, therefore we obtain the maximum. From the definition of $\{G_h\}_{h=1}^{l^r}$ and $H(s,r)$, see (3.5) and (3.8), it follows

$$\sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \geq c_2^s H(s,r)^k. \tag{3.10}$$

By using relations (3.1), (3.2) and (3.3) it follows similarly that

$$\phi^m(D_{\underline{x}}G_{\mathbf{h}}) \leq c_1 \max\left\{\left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right| : D_{\underline{x}}G_{\mathbf{h}}^{(m)} \text{ is an } m\times m \text{ minor of } D_{\underline{x}}G_{\mathbf{h}}\right\}.$$

Therefore

$$\sum_{|\mathbf{h}|=k} \overline{\phi'}^s(\mathbf{i}) \leq$$

$$c_1^2 \sum_{|\mathbf{h}|=k} \left(\sup_{\underline{x}} \max_{m-1\times m-1 \text{ minor}} \left|D_{\underline{x}}G_{\mathbf{h}}^{(m-1)}\right|\right)^{m-s} \left(\sup_{\underline{x}} \max_{m\times m \text{ minor}} \left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right|\right)^{s-m+1}.$$

By Lemma 3.8, the order of the supremum and the maximum can be changed in this situation and we can estimate the sum with

$$C \max_{\left\{\substack{r_1,\dots,r_{m-1}\\ s_1,\dots,s_{m-1}}\right\}} \max_{\left\{\substack{r'_1,\dots,r'_m\\ s'_1,\dots,s'_m}\right\}} \sum_{|\mathbf{h}|=k} \left(\sup_{\underline{x}} \left|D_{\underline{x}}G_{\mathbf{h}}^{(m-1)}\right|\right)^{m-s} \left(\sup_{\underline{x}} \left|D_{\underline{x}}G_{\mathbf{h}}^{(m)}\right|\right)^{s-m+1}$$

where $r_1,...,r_{m-1}$ are the rows and $s_1,...,s_{m-1}$ are the columns of the $(m-1)\times(m-1)$ minor, and $r'_1,...,r'_m$ are the rows and $s'_1,...,s'_m$ are the columns of $m\times m$ minor, moreover $C = c_1^2\binom{n}{m}^2\binom{n}{m-1}^2$. By the chain rule
$D_{\underline{x}}G_{\mathbf{h}} = D_{G_{h_2...h_k}(\underline{x})}G_{h_1}D_{G_{h_3...h_k}(\underline{x})}G_{h_2}...D_{\underline{x}}G_{h_k}$, we obtain

$$\begin{aligned}
&D_{\underline{x}}G_{\mathbf{h}}\binom{r_1,...,r_m}{s_1,...,s_m} = \\
&\sum_{c_1,...,c_k} \pm y_{1(c_1)}(h_1, G_{h_2...h_k}(\underline{x}))...y_{1(c_k)}(h_k,\underline{x})...y_{m(c_1)}(h_1, G_{h_2...h_k}(\underline{x}))\times \\
&\times y_{m(c_2)}(h_2, G_{h_3...h_k}(\underline{x}))...y_{m(c_k)}(h_k,\underline{x}).
\end{aligned} \tag{3.11}$$

Therefore

$$\begin{aligned}
&\sup_{\underline{x}}\left|D_{\underline{x}}G^{(m)}_{\mathbf{h}}\right| \leq \sum_{c_1,...,c_k}\sup_{\underline{x}}\left|y_{1(c_1)}(h_1,\underline{x})\right|...\sup_{\underline{x}}\left|y_{1(c_k)}(h_k,\underline{x})\right|...\sup_{\underline{x}}\left|y_{m(c_1)}(h_1,\underline{x})\right|\times \\
&\times\sup_{\underline{x}}\left|y_{m(c_2)}(h_2,\underline{x})\right|...\sup_{\underline{x}}\left|y_{m(c_k)}(h_k,\underline{x})\right|.
\end{aligned} \tag{3.12}$$

Denote by $t'_{kl}(h) := \sup_{\underline{x}}|y_{kl}(h,\underline{x})|$ the suprema. It follows from the inequality (3.12) and the Lemma 3.2

$$\begin{aligned}
&\sum_{|\mathbf{h}|=k}\sup_{\underline{x}}\left|D_{\underline{x}}G^{(m-1)}_{\mathbf{h}}\right|^{m-s}\sup_{\underline{x}}\left|D_{\underline{x}}G^{(m)}_{\mathbf{h}}\right|^{s-m+1} \leq \\
&\sum_{\substack{c_1,...,c_k\\ c'_1,...,c'_k}}((t'_{1(c_1)}(1)...t'_{m-1(c_1)}(1))^{m-s}(t'_{1(c'_1)}(1)...t'_{m(c'_1)}(1))^{s-m+1}+ \\
&...+(t'_{1(c_1)}(l^r)...t'_{m-1(c_1)}(l^r))^{m-s}(t'_{1(c'_1)}(l^r)...t'_{m(c'_1)}(l^r))^{s-m+1})\times \\
&...\times((t'_{1(c_k)}(1)...t'_{m-1(c_k)}(1))^{m-s}(t'_{1(c'_k)}(1)...t'_{m(c'_k)}(1))^{s-m+1}+ \\
&...+(t'_{1(c_k)}(l^r)...t'_{m-1(c_k)}(l^r))^{m-s}(t'_{1(c'_k)}(l^r)...t'_{m(c'_k)}(l^r))^{s-m+1}).
\end{aligned} \tag{3.13}$$

Lemma 3.6 implies that each non-zero term of the sum above has at most $2(n-1)^m = b$ of the indices $1(c_1),...,m-1(c_1),...,1(c_k),...,m-1(c_k)$,
$1(c'_1),...,m(c'_1),...,1(c'_k),...,m(c'_k)$ that are non-diagonal terms. Thus, for each set of indices $(c_1,...,c_k,c'_1,...,c'_k)$, we have at least $k-b$ of these indices such that

$1(c_r), ..., m-1(c_r), 1(c'_r), ..., m(c'_r)$ are all diagonal entries. For such $c_r$ and $c'_r$

$$((t'_{1(c_r)}(1)...t'_{m-1(c_r)}(1))^{m-s}(t'_{1(c'_r)}(1)...t'_{m(c'_r)}(1))^{s-m+1} + ...$$

$$... + (t'_{1(c_r)}(l^r)...t'_{m-1(c_1)}(l))^{m-s}(t'_{1(c'_r)}(l^r)...t'_{m(c'_r)}(l^r))^{s-m+1}) \leq$$

$$\leq \max_{\{j_1,...,j_{m-1}\},\{j'_1,...,j'_m\}} ((t'_{j_1 j_1}(1)...t'_{j_{m-1}j_{m-1}}(1))^{m-s}(t'_{j'_1}(1)...t'_{j'_m}(1))^{s-m+1} + ...$$

$$... + (t'_{j_1 j_1}(l^r)...t'_{j_{m-1}j_{m-1}}(l^r))^{m-s}(t'_{j'_1}(l^r)...t'_{j'_m j'_m}(l^r))^{s-m+1}) = T(s,r).$$

The last equality follows from definition of $\{G_h\}_{h=1}^{l^r}$ and $T(s,r)$. Hence from (3.13)

$$\begin{aligned} &\sum_{|\mathbf{h}|=k} \sup_{\underline{x}} \left| D_{\underline{x}} G_{\mathbf{h}}^{(m-1)} \right|^{m-s} \sup_{\underline{x}} \left| D_{\underline{x}} G_{\mathbf{h}}^{(m)} \right|^{s-m+1} \leq \\ &\leq \sum_{\substack{c_1,...,c_k \\ c'_1,...,c'_k}} \left( T(s,r)^{k-b} (l^r)^b \right) \leq c'' k^q l^{rb} T(s,r)^{k-b}, \end{aligned} \tag{3.14}$$

where, using Lemma 3.6, $c'' = m!(m-1)!$ and $q = (2m-1)(n-1)$.

By using (3.7), (3.9), (3.10) and (3.14)

$$\begin{aligned} &\sum_{|\mathbf{i}|=kr} \overline{\phi}^s(\mathbf{i}) = \sum_{|\mathbf{h}|=k} \overline{\phi'}^s(\mathbf{h}) \leq c'' k^q l^{rb} T(s,r)^{k-b} \leq c''(C^s)^k k^q l^{rb} T(s,r)^{-b} H(s,r)^k \leq \\ &\leq c'''(C^s)^k k^q l^{rb} T(s,r)^{-b} \sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) = c''' k^q l^{rb} T(s,r)^{-b} \sum_{|\mathbf{i}|=kr} \underline{\phi}^s(\mathbf{i}). \end{aligned} \tag{3.15}$$

We take the logarithm of both sides of the inequality and we divide by $kr$, then

$$\begin{aligned} &\frac{\log \sum_{|\mathbf{i}|=kr} \overline{\phi}^s(\mathbf{i})}{kr} \leq \\ &\leq \frac{\log c'''}{kr} + \frac{q \log k}{kr} + \frac{rb \log l}{kr} + \frac{(kb)\log(C^s)}{kr} + \frac{-b \log T(s,r)}{kr} + \frac{\log \sum_{|\mathbf{i}|=kr} \underline{\phi}^s(\mathbf{i})}{kr} \end{aligned} \tag{3.16}$$

is true for every positive $k, r$ integer. We take limit inferior of both sides. The limit exists in the left-hand side of the inequality and in the right-hand side the limit of every term exists and equals zero except the last term. Therefore

$$P(s) \leq \underline{P}(s)$$

While the opposite relation is trivial this completes the proof. $\square$

The next corollary is a consequence of the previous proof.

**Corollary 3.9.** *For $0 \leq s \leq n$. If $F_1, ..., F_l$ contractive maps in form (2.1) and $F_i \in C^{1+\varepsilon}$ for every $1 \leq i \leq l$ then*

$$P(s) = \lim_{r\to\infty} \frac{1}{r} \log\Big( \max_{\substack{j_1,\dots,j_{m-1} \\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} \left(|x_{j_1 j_1}(\mathbf{i},\underline{x})| \dots |x_{j_{m-1}j_{m-1}}(\mathbf{i},\underline{x})|\right)^{m-s} \times \\ \times \left(\left|x_{j'_1 j'_1}(\mathbf{i},\underline{x})\right| \dots \left|x_{j'_m j'_m}(\mathbf{i},\underline{x})\right|\right)^{s-m+1} \Big) \tag{3.17}$$

*for every $\underline{x} \in M$.*

*Proof.* It follows from inequality (3.7) that the $\lim_{r\to\infty} \frac{\log H(s,r)}{r}$ exists and

$$\lim_{r\to\infty} \frac{\log H(s,r)}{r} = \lim_{r\to\infty} \frac{\log T(s,r)}{r}.$$

It is clear by (3.15) that $\lim_{r\to\infty} \frac{\log T(s,r)}{r} = P(s)$. Because of the definition $H(s,r), T(s,r)$, this is exactly what we want to prove. □

## 4. Some applications

In this section we compute the Hausdorff dimension of some non-conformal IFS by using Corollary 3.9. It follows from [11] that the Hausdorff dimension is less than or equal to $s_0$ where $P(s_0) = 0$. We will show some examples where the root is exactly the dimension.

### 4.1. **Example 1.**

The easiest example is the non-linear modified Sierpinski-triangular. Let

$$T = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

and $T_i\underline{x} = T\underline{x} + \underline{v}_i$ for $i = 1, 2, 3$, where $v_1 = \binom{0}{0}, v_2 = \binom{\frac{2}{3}}{0}, v_3 = \binom{\frac{1}{3}}{\frac{2}{3}}$. We call the attractor of this IFS as modified Sierpinski-triangular. Clearly, the Hausdorff and box dimension is $\frac{\ln 3}{\ln 3} = 1$.

Let $f_i : [0, 1] \mapsto [0, 1]$ functions for $i = 1, 2, 3$ in $C^{1+\varepsilon}$ such that

$$F_i \binom{x}{y} = \begin{pmatrix} \frac{x}{3} + v_i \\ y/3 + f_i(x) + w_i \end{pmatrix}$$

are contractions where $\binom{v_1}{w_1} = \binom{0}{0}, \binom{v_2}{w_2} = \binom{\frac{2}{3}}{0}, \binom{v_3}{w_3} = \binom{\frac{1}{3}}{\frac{1}{2}}$. We can consider the attractor as a non-linear Sierpinski-triangular.

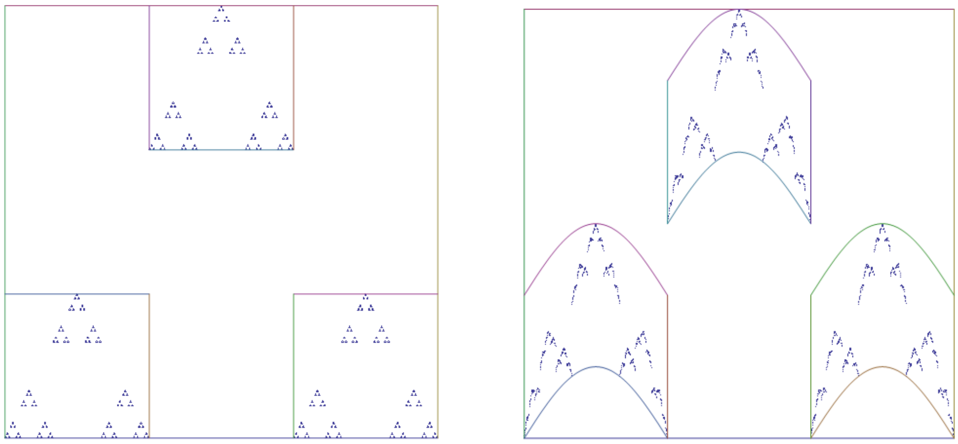

FIGURE 1. The image of the modified and the non-linear modified Sierpinski-triangular for $f_i(x) = \sin(\pi x)/6$ for every $i$.

We prove that the Hausdorff dimension of the non-linear modified Sierpinski-triangular is equal to 1, assuming that for $i = 1, 2, 3$ we have $f_i \in C^{1+\varepsilon}$ and

$$(f_i'(x))^2 + \mid f_i'(x) \mid \sqrt{(f_i'(x))^2 + \frac{4}{9}} < \frac{16}{9}.$$

We need this assumption to provide that the $\{F_1, F_2, F_3\}$ is contracting.

From the definition in this case it is easy to see that $x_{11}(\mathbf{i}, \underline{x}) = x_{22}(\mathbf{i}, \underline{x}) = \frac{1}{3}^{|\mathbf{i}|}$. We can suppose that $1 \leq s < 2$. Then by using Corollary 3.9

$$P(s) = \lim_{r \to \infty} \frac{1}{r} \log \left( \max_{\substack{j_1, \\ j_1', j_2'}} \sum_{|\mathbf{i}|=r} (|x_{j_1 j_1}(\mathbf{i}, \underline{x})|)^{2-s} \times \left( \left| x_{j_1' j_1'}(\mathbf{i}, \underline{x}) \right| \left| x_{j_2' j_2'}(\mathbf{i}, \underline{x}) \right| \right)^{s-2+1} \right) =$$

$$\lim_{r \to \infty} \frac{1}{r} \log \left( \sum_{|\mathbf{i}|=r} \left( \frac{1}{3}^{|\mathbf{i}|} \right)^{2-s} \left( \frac{1}{3}^{|\mathbf{i}|} \frac{1}{3}^{|\mathbf{i}|} \right)^{s-1} \right) = \lim_{r \to \infty} \frac{1}{r} \log \left( 3^r \frac{1}{3}^{sr} \right) = \log 3 - s \log 3.$$

It is easy to see that $P(s) = 0$ if and only if $s = 1$, which is the upper bound of the Hausdorff dimension of the modified non-linear attractor, this follows from [11]. To get a lower bound it is enough to project it onto the $x$ axis and we get the $[0, 1]$ interval.

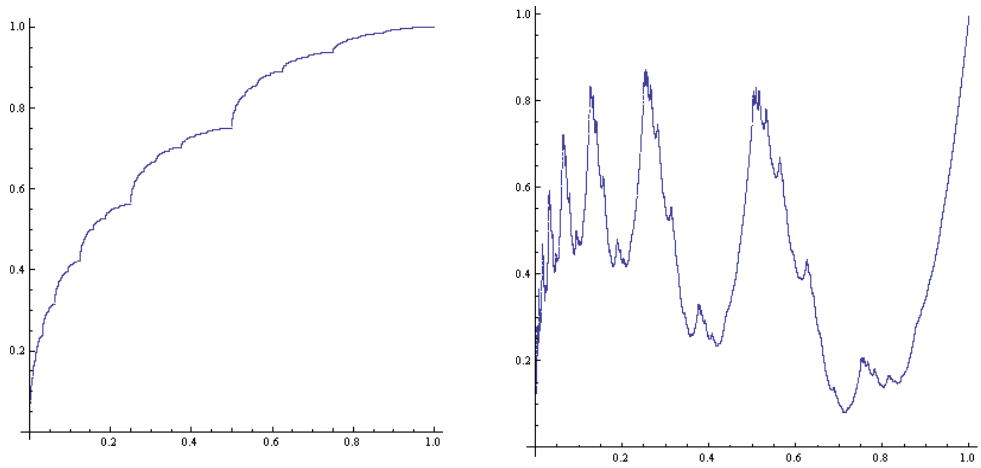


FIGURE 2. The images of the attractors in case $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{4}$, $f_0(x) = (1-c_2)\sin(\pi x)$, $f_1(x) = -c_2\sin(\pi x)$

4.2. **Example 2.** The next example is a non-linear perturbation of a self-affine IFS. Let $c_1, c_2 \in (0,1)$. Consider the following self-affine IFS

$$g_0(\underline{x}) = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \underline{x}, \quad g_1(\underline{x}) = \begin{bmatrix} 1-c_1 & 0 \\ 0 & 1-c_2 \end{bmatrix} \underline{x} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

It is easy to see that the attractor of this IFS has Hausdorff dimension 1 since it is a graph of a strictly monotone function. We perturb this IFS as follows, let $\{\widetilde{g}_0, \widetilde{g}_1\}$ the following

$$\widetilde{g}_0(x,y) = \begin{bmatrix} c_1 x \\ c_2 y + f_0(x) \end{bmatrix}, \quad \widetilde{g}_1(x,y) = \begin{bmatrix} (1-c_1)x + c_1 \\ (1-c_2)y + c_2 + f_1(x) \end{bmatrix}.$$

where $f_0, f_1 \in C^{1+\varepsilon}$ and $f_i$ are periodic with period 1. Moreover we suppose that $\widetilde{g}_0, \widetilde{g}_1$ are contractions, namely the following inequalities hold

$$\begin{aligned} &c_1^2 + (f_0'(x))^2 + c_2^2 + \sqrt{(c_1^2 + (f_0'(x))^2 + c_2^2)^2 - 4c_1^2 c_2^2} < 2 \\ &(1-c_1)^2 + (f_1'(x))^2 + (1-c_2)^2 \\ &+ \sqrt{((1-c_1)^2 + (f_1'(x))^2 + (1-c_2)^2)^2 - 4(1-c_1)^2(1-c_2)^2} < 2. \end{aligned}$$

In this case the Hausdorff dimension of the modified attractor is greater than or equal to 1 since the projection to the $x$ axis is the $[0,1]$ interval. To get an upper bound we have to use the sub-additive pressure and Corollary 3.9. For every

$\mathbf{i} \in \{0,1\}^*$ we have $x_{11}(\mathbf{i}, \underline{x}) = c_1^{\sharp_0 \mathbf{i}}(1-c_1)^{\sharp_1 \mathbf{i}}$ and $x_{22}(\mathbf{i}, \underline{x}) = c_2^{\sharp_0 \mathbf{i}}(1-c_2)^{\sharp_1 \mathbf{i}}$ where $\sharp_j \mathbf{i}$ is the number of $j$s in $\mathbf{i}$. Then

$$\max_j \sum_{|\mathbf{i}|=r} x_{jj}(\mathbf{i}, \underline{x})^{2-s} \left(x_{11}(\mathbf{i}, \underline{x}) x_{22}(\mathbf{i}, \underline{x})\right)^{s-2+1} =$$

$$\max_j \sum_{|\mathbf{i}|=r} c_j^{(2-s)\sharp_0 \mathbf{i}}(1-c_j)^{(2-s)\sharp_1 \mathbf{i}} c_1^{(s-1)\sharp_0 \mathbf{i}}(1-c_1)^{(s-1)\sharp_1 \mathbf{i}} c_2^{(s-1)\sharp_0 \mathbf{i}}(1-c_2)^{(s-1)\sharp_1 \mathbf{i}} =$$

$$\max\left\{\left(c_1 c_2^{s-1} + (1-c_1)(1-c_2)^{s-1}\right)^r, \left(c_2 c_1^{s-1} + (1-c_2)(1-c_1)^{s-1}\right)^r\right\}.$$

Therefore by formula (3.17) we have $P(1) = 0$, and by [11] 1 is an upper bound for Hausdorff dimension, so the Hausdorff dimension is exactly 1.

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