ON THE DIMENSION OF THE GRAPH OF THE CLASSICAL WEIERSTRASS FUNCTION

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Abstract. This paper examines dimension of the graph of the famous Weierstrass non-differentiable function
\[ W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x) \]
for an integer \( b \) larger than 1 and \( 1/b < \lambda < 1 \). We prove that for every \( b \) there exists (explicitly given) \( \lambda_b \in (1/b, 1) \) such that the Hausdorff dimension of the graph is equal to \( D = 2 + \frac{\log \lambda}{\log 2} \) for every \( \lambda \in (\lambda_b, 1) \). We also show that the dimension is equal to \( D \) for almost every \( \lambda \) on some larger interval. This partially solves a well-known thirty-year-old conjecture.

1. Introduction and statements

This paper is devoted to the study of dimension of the graph of the famous function
\[ W_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi b^n x) \]
for \( x \in \mathbb{R} \), where \( 0 < \lambda < 1 < b \) and \( b\lambda > 1 \), introduced by Weierstrass in 1872 as one of the first examples of a continuous nowhere differentiable function on the real line. In fact, Weierstrass proved the non-differentiability for some values of the parameters, while the complete proof was given by Hardy [11] in 1916. Later, starting from the work of Besicovitch and Ursell [5], the graphs of functions of the form
\[ f(x) = \sum_{n=0}^{\infty} b^{-2} \phi(b_n x + \theta_n) \quad (1.1) \]
for non-constant Lipschitz, piecewise \( C^1 \), \( \mathbb{Z} \)-periodic functions \( \phi : \mathbb{R} \to \mathbb{R} \) and \( 1 < D < 2 \), \( b_{n+1}/b_n > b > 1 \), \( \theta_n \in \mathbb{R} \) were studied from a geometric point of view as fractal curves in the plane. Much attention was paid to the classical case \( b_n = b^n \) for an integer \( b \) larger than 1 and \( \theta_n = 0 \). Then the graph of \( f \) is an invariant repeller for the expanding dynamical system
\[ \Phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}, \]
\[ \Phi(x,y) = \left( bx \mod 1, \frac{y - \phi(x)}{\lambda} \right) \quad (1.2) \]
with Lyapunov exponents \( \log 2 \), \( \log \lambda \) for \( \lambda = b^{D-2} \), which allows to use the methods of ergodic theory of smooth dynamical systems.

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The case of the Weierstrass function \( W_{\lambda,b} \) for integer \( b \) is of particular interest, because then it is the real part of the lacunar (Hadamard gaps) power series

\[
W(z) = \sum_{n=0}^{\infty} \lambda^n z^{b^n}, \quad z \in \mathbb{C}, \ |z| \leq 1
\]
on the unit circle, which relates the problem to harmonic analysis and boundary behaviour of analytic maps. For instance, it was proved by Salem and Zygmund [26] and Kahane, Weiss and Weiss [15] that for \( \lambda \) sufficiently close to 1, the image of the unit circle under \( W \) is a Peano curve, i.e. it covers an open subset of the plane. Moreover, Belov [3] and Barański [2] showed that in this case the map \( W \) does not preserve (forwardly) Borel sets on the unit circle. The complicated topological boundary behaviour of \( W \) was also studied recently by Dong, Lau and Liu in [8].

The graph of a function \( f \) of the form (1.1) is approximately self-affine with scales \( \lambda \) and \( 1/b \), which suggests that its dimension should be equal to

\[
D = 2 + \frac{\log \lambda}{\log b}.
\]
Indeed, Kaplan, Mallet-Paret and Yorke [14] proved that the box dimension of the graph of \( f \) is equal to \( D \). However, the question of determining the Hausdorff dimension turned out to be much more complicated. The conjecture that it is equal to \( D \) for the classical Weierstrass case \( f = W_{\lambda,b} \) was formulated by Mandelbrot in 1977 [18] and then repeated in many papers, see e.g. [4, 9, 13, 16, 20, 23] and the references therein.

In 1986, Mauldin and Williams [20] proved that if a function \( f \) has the form (1.1) with \( b_n = b^n \) for an integer \( b \) larger than 1, then for given \( D \) there exists a constant \( C > 0 \) such that the Hausdorff dimension of the graph is larger than \( D - C/\log b \) for large \( b \). Shortly after, Przytycki and Urbański showed in [23] that if \( f = W_{\lambda,b} \) for any integer \( b \) larger than 1, then the Hausdorff dimension of the graph is larger than 1. Rezakhanlou [25] proved that the packing dimension of the graph of \( W_{\lambda,b} \) is equal to \( D \) and in [12], Hu and Lau showed the same for the so-called \( K \)-dimension (both are not smaller than the Hausdorff dimension).

In 1992, Ledrappier [16] proved that if \( f \) has the form (1.1) with \( b_n = 2^n \), \( \phi(x) = \text{dist}(x, \mathbb{Z}) \) and \( \theta = 0 \), then the Hausdorff dimension of the graph is equal to \( D \) provided the infinite Bernoulli convolutions \( \sum_{n=0}^{\infty} \pm 2^{(1-D)n} \), with \( \pm \) chosen independently with probability \((1/2, 1/2)\), have absolutely continuous distribution (by the result of Solomyak [29], this holds for almost all \( D \in (1,2) \) with respect to the Lebesgue measure). Analogous result for other functions \( \phi \) was showed by Solomyak in [28].

In 1998, Hunt [13] proved that in the case \( b_n = b^n \) for an integer \( b \) larger than 1, if one considers the numbers \( \theta_n \) in (1.1) as independent random variables with uniform distribution on \([0,1]\), then for many functions \( \phi \), including \( \phi(x) = \cos(2\pi x) \), the Hausdorff dimension of the graph is equal to \( D \) almost surely.

It is interesting to notice that in the case \( b_{n+1}/b_n \to \infty \) the question of determining the Hausdorff and box dimension of graphs of functions (1.1) can be solved completely, as proved recently by Carvalho [7] and Barański [1]. In this case the Hausdorff and upper box dimension need not coincide.

Recently, Biacino [6] and Fu [10] solved partially the question of determining the Hausdorff dimension of the graph of the classical Weierstrass function \( W_{\lambda,b} \), showing that it is equal to \( D \) for sufficiently large integers \( b \).
In this paper we make a further step, proving the conjecture for every integer \( b \) larger than 1, provided \( \lambda \) is sufficiently close to 1. The proof uses methods of ergodic theory of smooth dynamical systems. In fact, we show that the measure \( \mu_{\lambda,b} \) has dimension \( D \), where

\[
\mu_{\lambda,b} = ((\text{Id}, W_{\lambda,b}))[0,1] \ast \mathcal{L}|[0,1]
\]

is the lift of the Lebesgue measure \( \mathcal{L} \) on \([0,1]\) to the graph of \( W_{\lambda,b} \).

**Definition.** We say that a Borel measure \( \mu \) in a metric space \( X \) has local dimension \( d \) at a point \( x \in X \), if

\[
\lim_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} = d,
\]

where \( B_r(x) \) denotes the ball of radius \( r \) centered at \( x \). If the local dimension of \( \mu \) exists and is equal to \( d \) at \( \mu \)-almost every \( x \), then we write \( \dim \mu = d \).

If \( \dim \mu = d \), then every set of positive measure \( \mu \) has Hausdorff dimension at least \( d \).

Denote by \( G_{\lambda,b} \subset \mathbb{R}^2 \) the graph of the function \( W_{\lambda,b} \) on the interval \([0,1]\), i.e.

\[
G_{\lambda,b} = \{(x, W_{\lambda,b}(x)) : x \in [0,1]\}.
\]

Let \( \dim_H \) and \( \dim_B \) denote, respectively, the Hausdorff and box dimension (for the definition and basic properties of the Hausdorff and box dimension we refer to [9, 19]). As mentioned above, it is well-known that \( \dim_B G_{\lambda,b} = D \). Since \( \dim_H G_{\lambda,b} \leq \dim_B G_{\lambda,b} \), to determine the Hausdorff dimension of \( G_{\lambda,b} \) it is sufficient to prove \( \dim \mu_{\lambda,b} = D \).

The first of the paper is the following.

**Theorem A.** For every positive integer \( b \) larger than 1,

\[
\dim \mu_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}
\]

for every \( \lambda \in (\lambda_b, 1) \), where \( \lambda_b \) is equal to the unique zero of the function

\[
h_b(\lambda) = \begin{cases}
\frac{1}{16 \lambda^2 (4 \lambda - 1)^2} + \frac{1}{64 \lambda^2 (4 \lambda - 1)^2} - \frac{5}{64 \lambda^2} + \frac{\sqrt{2}}{2 \lambda} - 1 & \text{for } b = 2 \\
\frac{1}{(4 \lambda - 1)^2} + \frac{1}{(4 \lambda - 1)^2} - \sin^2 \frac{\pi}{b} & \text{for } b \geq 3
\end{cases}
\]

on the interval \((1/b, 1)\). In particular,

\[
\dim_H G_{\lambda,b} = \dim_B G_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}
\]

for every \( \lambda \in (\lambda_b, 1) \).

Using Peres–Solomyak transversality methods, we can extend the result for almost every \( \lambda \) on some larger interval. To state the next theorem, we need to recall some definitions related to so-called \((\ast)\)-functions considered in the study of infinite Bernoulli convolutions (see e.g. [21, 22, 28]). For \( \beta \geq 1 \) let

\[
\mathcal{G}_\beta = \left\{ g(t) = 1 + \sum_{n=1}^{\infty} g_n t^n, g_n \in [-\beta, \beta] \text{ for } n \geq 1 \right\}.
\]

Let \( y(\beta) \) be the smallest possible value of positive double roots of functions in \( \mathcal{G}_\beta \), i.e.

\[
y(\beta) = \inf \left\{ t > 0 : \text{there exists } g \in \mathcal{G}_\beta \text{ such that } g(t) = g'(t) = 0 \right\}.
\]
Theorem B. For every positive integer $b$ larger than 1,
\[
\dim \mu_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}
\]
for Lebesgue almost every $\lambda \in (\tilde{\lambda}_b, 1)$, where $\tilde{\lambda}_b$ is equal to the unique root of the equation
\[
y \left( \frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2\lambda - 1)^2}} \right) = \frac{1}{b\lambda}
\]
on the interval $(1/b, 1)$. In particular,
\[
\dim_H G_{\lambda,b} = \dim_B G_{\lambda,b} = 2 + \frac{\log \lambda}{\log b}
\]
for Lebesgue almost every $\lambda \in (\tilde{\lambda}_b, 1)$.

Estimating the numbers $\lambda_b$ and $\tilde{\lambda}_b$ in the above theorems, we obtain the following.

Corollary C.
\[
\begin{align*}
\dim_H G_{\lambda,2} &= 2 + \frac{\log \lambda}{\log 2} & \text{for every } \lambda \in (0.9531, 1) \text{ and almost every } \lambda \in (0.81, 1), \\
\dim_H G_{\lambda,3} &= 2 + \frac{\log \lambda}{\log 3} & \text{for every } \lambda \in (0.7269, 1) \text{ and almost every } \lambda \in (0.55, 1), \\
\dim_H G_{\lambda,4} &= 2 + \frac{\log \lambda}{\log 4} & \text{for every } \lambda \in (0.6083, 1) \text{ and almost every } \lambda \in (0.44, 1).
\end{align*}
\]
For every $b \geq 5$,
\[
\dim_H G_{\lambda,b} = 2 + \frac{\log \lambda}{\log b} \quad \text{for every } \lambda \in (0.5448, 1) \text{ and almost every } \lambda \in (1.04/\sqrt{b}, 1).
\]

Obviously, using Theorem A and B, one can get better estimates for $b \geq 5$ (for large $b$, the numbers $\lambda_b$ tend to $1/\pi$ and $\tilde{\lambda}_b \sqrt{b}$ tends to $1/\sqrt{\pi}$, see Lemmas 3.4 and 4.1).

2. Background

We consider $G_{\lambda,b}$ as an invariant repeller of the dynamical system (1.2) for $\phi(x) = \cos(2\pi x)$ and use the results of ergodic theory of non-uniformly hyperbolic smooth dynamical systems on manifolds (Pesin theory) developed by Ledrappier and Young in [17] and applied by Ledrappier in [16] to study the graphs of the Weierstrass-type functions. The theory in [17] is valid for smooth diffeomorphisms, so to apply it for $\Phi$ one considers the inverse limit (alternatively, it is possible to use analogous theory for smooth endomorphisms developed by Qian, Xie and Zhu in [24]).

For the reader’s convenience, let us recall the results of Ledrappier–Young theory from [16, 17] applied for the graph of $W_{\lambda,b}$. (Note that the quoted results are formulated in [16] for $b = 2$. However, the theory is valid for any integer $b$ larger than 1.) Consider the symbolic space
\[
\Sigma = \{0, \ldots, b - 1\}^{\mathbb{Z}^+}
\]
and let
\[
\Sigma^* = \bigcup_{n=0}^{\infty} \{0, \ldots, b - 1\}^n
\]
be the set of finite length words of symbols. For a finite length word \((i_1, \ldots, i_n) \in \Sigma^*\) let 
\([i_1, \ldots, i_n] = \{ (j_1, j_2, \ldots) \in \Sigma : j_1 = i_1, \ldots, j_n = i_n \}.\)
Define for \(x \in [0,1]\) and \(\gamma \in (1/b, 1)\) a mapping \(Y_{x,\gamma}\) from \(\Sigma\) to the real line as follows:
\[
Y_{x,\gamma}(i) = 2\pi \sum_{n=1}^{\infty} \gamma^n \sin \left( 2\pi \left( \frac{x}{b^n} + \frac{i_1}{b^n} + \cdots + \frac{i_n}{b^n} \right) \right),
\]
(2.1)
where \(i = (i_1, i_2, \ldots)\) and 
\[
\gamma = \frac{1}{b\lambda}.
\]
The latter formula will be used throughout the paper.
Define the inverse of the map \(\Phi\) from (1.2) as the map \(F: [0,1] \times \mathbb{R} \times \Sigma \rightarrow [0,1] \times \mathbb{R} \times \Sigma,\)
\[
F(x, y, i) = \left( \frac{x}{b} + \frac{i}{b}, \lambda y + \phi \left( \frac{x}{b} + \frac{i}{b} \right), \sigma(i) \right),
\]
where \(\phi(x) = \cos(2\pi x), i = (i_1, i_2, \ldots)\) and \(\sigma\) is the left-side shift on \(\Sigma\). We have 
\[
F(G_{\lambda,b} \times \Sigma) = G_{\lambda,b} \times \Sigma, \quad F_*(\mu_{\lambda,b} \times \mathbb{P}) = \mu_{\lambda,b} \times \mathbb{P}.
\]
Defining 
\[
F_i(x, y) = \left( \frac{x}{b} + \frac{i}{b}, \lambda y + \phi \left( \frac{x}{b} + \frac{i}{b} \right) \right)
\]
for \(i \in \{0, \ldots, b-1\}\), we have 
\[
DF_i(x, y) = \begin{bmatrix} 1/b & 0 \\ \phi'(x/b + i/b)/b & \lambda \end{bmatrix}
\]
Consider the products of these matrices which arise by composing the maps \(F_1, F_2, \ldots\) for given \(i = (i_1, i_2, \ldots)\). By the Oseledets multiplicative ergodic theorem, the Lyapunov exponents of the system are equal to \(-\log 2, \log \lambda\) and there is exactly one strong stable direction in \(\mathbb{R}^2\) (corresponding to the exponent \(-\log 2\)), given by 
\[
J_{x,i} = \begin{bmatrix} 1 \\ -\sum_{n=1}^{\infty} \gamma^n \phi'(x/b^n + i_1/b^n + \cdots + i_n/b^n) \end{bmatrix} = \begin{bmatrix} 1 \\ Y_{x,\gamma}(i) \end{bmatrix}
\]
for \(\gamma = 1/(b\lambda)\). In fact, 
\[
DF_i(x, y)(J_{x,i}) = \frac{1}{b} J_{x/b+i_1/b, \sigma(i)}.
\]
Note that \(J_{x,i}\) does not depend on \(y\). For given \(i\), the vector field \(J_{x,i}\) defines a foliation of \((0,1) \times \mathbb{R}\) into strong stable manifolds, given by parallel smooth curves \(\Gamma_{x,y,i}\) (graphs of functions of the first coordinate).
For the measure \(\mu = \mu_{\lambda,b}\) there exists a system of conditional measures \(\mu_{x,y,i}\) on \(\Gamma_{x,y,i}\) associated to this foliation treated as a measurable partition. Take a vertical line \(\ell\) and let \(\nu_{x,i}\) (called transversal measure) be the projection of \(\mu\) to \(\ell\) along the curves \(\Gamma_{x,y,i}, y \in \mathbb{R}\). The following result is a part of the Ledrappier–Young theory from [17] (see also [16, Proposition 2]).
Theorem 2.1 (Ledrappier–Young). The local dimension of the measure $\mu$ exists and is constant $\mu$-almost everywhere. The local dimension of the measure $\mu_{x,y,1}$ exists, is constant $\mu_{x,y,1}$-almost everywhere, and is constant for $(\mu \times P)$-almost every $(x, y, 1)$. The local dimension of the measure $\nu_{x,1}$ exists, is constant $\nu_{x,1}$-almost everywhere, and is constant for $(L \times P)$-almost every $(x, 1)$, where $L$ is the Lebesgue measure. Moreover,

$$\dim \mu = \dim \mu_{x,y,1} + \dim \nu_{x,1}$$

and

$$\log b \dim \mu_{x,y,1} - \log \lambda \dim \nu_{x,1} = \log b.$$

The latter is a “conditional” version of the Pesin entropy formula. As a corollary, one gets

$$\dim \mu_{\lambda,b} = 1 + \left(1 + \frac{\log \lambda}{\log b}\right) \dim \nu_{x,1}.$$  \hfill (2.2)

In [16], Ledrappier proved a kind of the Marstrand-type projection theorem, showing that if the distribution of angles of directions $J_{x,1}$ has dimension 1, then the dimension of the transversal measure is also equal to 1. More precisely, he proved the following.

Let $P = \{\frac{1}{b}, \ldots, \frac{1}{b}\} \mathbb{Z}^+$ be the uniform Bernoulli measure on $\Sigma$ and let

$$m_{x,\gamma} = (Y_{x,\gamma})_x P.$$

Theorem 2.2 (Ledrappier, [16]). Let $\gamma \in (1/b, 1)$. If $\dim m_{x,\gamma} = 1$ for Lebesgue almost every $x \in (0, 1)$, then $\dim \nu_{x,1} = 1$.

In view of (2.2), this implies that to have $\dim \mu_{\lambda,b} = 2 + \log \lambda/\log b$, it is enough to prove that $\dim m_{x,\gamma} = 1$ for Lebesgue almost every $x \in (0, 1)$. In fact, we will show that $m_{x,\gamma}$ is absolutely continuous with respect to the Lebesgue measure for Lebesgue almost every $x \in (0, 1)$, which is a stronger property.

3. Proof of Theorem A

In the proof of Theorem A we use a result due to Tsujii [30]. He considered the SBR measure $\nu$ for a skew product $T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ of the form

$$T(x, y) = (bx, \gamma y + \varphi(x))$$

for an integer $b$ larger than 1, a real number $\gamma \in (0, 1)$ and a $C^2$ function $\varphi$ on $S^1 = \mathbb{R}/\mathbb{Z}$. We apply here his results for $\varphi(x) = \sin(2\pi x)$.

Definition 3.1 (Tsujii, [30]). Let $\varepsilon, \delta > 0$, $i, j \in \Sigma$, $m \in \mathbb{N}$, $k \in \{1, \ldots, b^m\}$. The functions $Y_{x,\gamma}(i)$ and $Y_{x,\gamma}(j)$ are called $(\varepsilon, \delta)$-transversal on the interval $I_{m,k} = [(k-1)/b^m, k/b^m]$ if for every $x \in I_{m,k}$,

$$|Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| > \varepsilon \quad \text{or} \quad \left|\frac{d}{dx} Y_{x,\gamma}(i) - \frac{d}{dx} Y_{x,\gamma}(j)\right| > \delta.$$

Otherwise they are called $(\varepsilon, \delta)$-tangent on $I_{m,k}$.

Let $e(n, m; \varepsilon, \delta)$ be the maximum over $k \in \{1, \ldots, b^m\}$ and $(i_1, \ldots, i_n) \in \Sigma^*$ of the maximal number of finite words $(j_1, \ldots, j_n) \in \Sigma^*$ for which there exist $i \in [i_1, \ldots, i_n]$ and $j \in [j_1, \ldots, j_n]$ such that the functions $Y_{x,\gamma}(i)$ and $Y_{x,\gamma}(j)$ are $(\varepsilon, \delta)$-tangent on $I_{m,k}$.

Remark. The above definition is suited to the case $\varphi(x) = \sin(2\pi x)$. In general, instead of $Y_{x,\gamma}(i)$ one should take $\sum_{n=1}^{\infty} \gamma^n \varphi(x/b^n + i_1/b^n + \cdots + i_n/b)$. 


In [30], Tsujii proved the following result.

**Theorem 3.2** (Tsujii, [30, Proposition 8]). If $e(n,m; \varepsilon, \delta) < \gamma^n h^n$ for some $\varepsilon, \delta > 0$ and positive integers $n, m$, then the SBR measure $\nu$ for $T$ is absolutely continuous with respect to the Lebesgue measure on $S^1 \times \mathbb{R}$.

There is a direct relation between the SBR measure $\nu$ for $\varphi(x) = \sin(2\pi x)$ and the measure $m_{x,\gamma}$. More precisely, we have

$$\nu = \Psi_* (\mathcal{L}|_{S^1} \times \mathbb{P}),$$

where $\Psi : S^1 \times \Sigma \to S^1 \times \mathbb{R}$,

$$\Psi(x, i) = \left( x, \frac{Y_{x,\gamma}(i)}{2\pi \gamma} \right)$$

and $\mathcal{L}$ is the Lebesgue measure (for details, see [30]). Hence, for a measurable $A \subset S^1 \times \mathbb{R}$, we have

$$\nu(A) = (\mathcal{L}|_{S^1} \times \mathbb{P}) \left( \left\{ (x, i) : \left( x, \frac{Y_{x,\gamma}(i)}{2\pi \gamma} \right) \in A \right\} \right) = \int_{S^1} m_{x,\gamma}(\{2\pi \gamma y : (x, y) \in A\}) dx.$$

This easily implies the following lemma.

**Lemma 3.3.** If the SBR measure $\nu$ for $T(x, y) = (bx, \gamma y + \sin(2\pi x))$ is absolutely continuous, then the measure $m_{x,\gamma}$ is absolutely continuous for Lebesgue almost every $x \in (0, 1)$, in particular $\dim m_{x,\gamma} = 1$ for Lebesgue almost every $x \in (0, 1)$.

Now we will find conditions under which the measure $\nu$ is absolutely continuous. To use Theorem 3.2, we check the transversality condition for the functions $Y_{x,\gamma}$. First, we prove the existence of the numbers $\lambda_b$ defined in Theorem A.

**Lemma 3.4.** For every integer $b$ larger than 1, the function $h_b$ is strictly decreasing on the interval $(1/b, 1)$ and has a unique zero $\lambda_b \in (1/b, 1)$. In particular, $\lambda_2 < 0.9531$, $\lambda_3 < 0.7269$, $\lambda_4 < 0.6083$ and $\lambda_b < 0.5448$ for $b \geq 5$. Moreover, $\lambda_b \to 1/\pi$ as $b \to \infty$.

**Proof.** Consider first the case $b = 2$. We easily check

$$\frac{d}{d\lambda} \left( -\frac{5}{64\lambda^2} + \frac{\sqrt{2}}{2\lambda} \right) < 0$$

for $\lambda \in (1/2, 1)$, which immediately implies that the function $h_2$ is strictly decreasing on the interval $(1/2, 1)$. Moreover, $h_2(\lambda) \to +\infty$ as $\lambda \to (1/2)^+$ and $h_2(1) < 0$. Hence, $h_2$ has a unique zero $\lambda_2 \in (1/2, 1)$.

Consider now the case $b \geq 3$. It is obvious that $h_b$ is strictly decreasing on the interval $(1/b, 1]$ and tends to $+\infty$ as $\lambda \to (1/b)^+$. Using the inequality $\sin x > x - x^3/6$ for $x > 0$, we get

$$h_b(\lambda) < \frac{1}{(b\lambda - 1)^2} + \frac{1}{(b^2\lambda - 1)^2} + \frac{\pi^4}{3b^4} - \frac{\pi^2}{b^2} = \frac{H_b(\lambda)}{b^2}$$

for

$$H_b(\lambda) = \frac{1}{(\lambda - 1/b)^2} + \frac{1}{(b\lambda - 1/b)^2} + \frac{\pi^4}{3b^2} - \pi^2.$$

For $\lambda \in (1/b, 1]$, the function $b \mapsto H_b(\lambda)$ is strictly decreasing. Moreover, $H_3(1) < 0$, so $h_b(1) < 0$ for $b \geq 3$. This proves the existence of the unique zero $\lambda_b \in (1/b, 1)$ of the function $h_b$. 


One can directly check that \( h_2(0.9531), h_3(0.7269), h_4(0.6083) < 0 \), which shows \( \lambda_2 < 0.9531, \lambda_3 < 0.7269, \lambda_4 < 0.6083 \). Moreover, \( H_5(0.5448) < 0 \), so \( H_6(0.5448) < 0 \) for every \( b \geq 5 \), which implies \( \lambda_6 < 0.5448 \) for \( b \geq 5 \). The last assertion of the lemma follows easily from the definition of the function \( h_b \) and the fact \( \lim_{x \to 0} \sin \frac{x}{x} = 1 \).

Now we prove the transversality condition for the functions \( Y_{\cdot, \gamma} \).

**Proposition 3.5.** If \( \gamma \in (1/b, 1/(b\lambda_6)) \), then there exists \( \delta > 0 \) such that for every \( i = (i_1, i_2, \ldots), j = (j_1, j_2, \ldots) \in \Sigma \) with \( i_1 \neq j_1 \) and every \( x \in [0, 1] \),

\[
|Y_{x, \gamma}(i) - Y_{x, \gamma}(j)| > \delta \quad \text{or} \quad \left| \frac{d}{dx} Y_{x, \gamma}(i) - \frac{d}{dx} Y_{x, \gamma}(j) \right| > \delta.
\]

**Proof.** Fix \( \gamma \in (1/b, 1/(b\lambda_6)) \). Suppose the assertion does not hold. Then for every \( \delta > 0 \) there exist \( i = (i_1, i_2, \ldots), j = (j_1, j_2, \ldots) \in \Sigma \) with \( i_1 \neq j_1 \) and every \( x \in [0, 1] \), such that

\[
|Y_{x, \gamma}(i) - Y_{x, \gamma}(j)| \leq \delta, \quad \left| \frac{d}{dx} Y_{x, \gamma}(i) - \frac{d}{dx} Y_{x, \gamma}(j) \right| \leq \delta. \quad (3.1)
\]

First, consider the case \( b \geq 3 \). By the definition of \( Y_{x, \gamma} \) (see (2.1)),

\[
|Y_{x, \gamma}(i) - Y_{x, \gamma}(j)| \geq 2\pi\gamma \left| \sin \left( \frac{2\pi x + i_1}{b} \right) - \sin \left( \frac{2\pi x + j_1}{b} \right) \right| - 4\pi \sum_{n=2}^{\infty} \gamma^n = 4\pi\gamma \left| \sin \left( \frac{2\pi |i_1 - j_1|}{2b} \right) \right| \left| \cos \left( \frac{2\pi (2x + i_1 + j_1)}{2b} \right) - \frac{4\pi\gamma^2}{1 - \gamma} \right| \geq 4\pi\gamma \left| \cos \left( \frac{2\pi (2x + i_1 + j_1)}{2b} \right) \right| - \frac{4\pi\gamma^2}{1 - \gamma},
\]

as \( 1 \leq |i_1 - j_1| \leq b - 1 \). Similarly, since

\[
\frac{d}{dx} Y_{x, \gamma}(i) = 4\pi^2 \sum_{n=1}^{\infty} \left( \frac{\gamma}{b} \right)^n \cos \left( 2\pi \left( \frac{x}{b^n} + \frac{i_1}{b^n} + \cdots + \frac{i_n}{b^n} \right) \right),
\]

we obtain

\[
\left| \frac{d}{dx} Y_{x, \gamma}(i) - \frac{d}{dx} Y_{x, \gamma}(j) \right| \geq \frac{4\pi^2\gamma}{b} \left| \cos \left( \frac{2\pi x + i_1}{b} \right) - \cos \left( \frac{2\pi x + j_1}{b} \right) \right| - \frac{8\pi^2 \gamma^2}{b} \sum_{n=2}^{\infty} \left( \frac{\gamma}{b} \right)^n \geq \frac{8\pi^2\gamma}{b} \left| \sin \left( \frac{2\pi |i_1 - j_1|}{2b} \right) \right| \left| \sin \left( \frac{2\pi (2x + i_1 + j_1)}{2b} \right) \right| - \frac{8\pi^2\gamma^2}{b(b - \gamma)}, \quad (3.3)
\]

By (3.1), (3.2) and (3.3),

\[
\sin \frac{\pi}{b} \left| \cos \left( \frac{2\pi (2x + i_1 + j_1)}{2b} \right) \right| \leq \frac{\gamma}{1 - \gamma} + \frac{\delta}{4\pi\gamma},
\]

\[
\sin \frac{\pi}{b} \left| \sin \left( \frac{2\pi (2x + i_1 + j_1)}{2b} \right) \right| \leq \frac{\gamma}{b - \gamma} + \frac{\delta b}{8\pi^2\gamma}.
\]

Taking the sum of the squares of the two inequalities, we get

\[
\sin^2 \frac{\pi}{b} \leq \left( \frac{\gamma}{1 - \gamma} + \frac{\delta}{4\pi\gamma} \right)^2 + \left( \frac{\gamma}{b - \gamma} + \frac{\delta b}{8\pi^2\gamma} \right)^2.
\]
Since \( \delta \) is arbitrarily small, in fact this implies
\[
0 \leq \frac{\gamma^2}{(1 - \gamma)^2} + \frac{\gamma^2}{(b - \gamma)^2} - \sin^2 \frac{\pi}{b} = h_b(\lambda)
\]
for \( \lambda = 1/(b\gamma) > \lambda_b \), which contradicts Lemma 3.4. This ends the proof in the case \( b \geq 3 \).
Consider now the case \( b = 2 \). We improve the estimates made by Tsuji in [30, Appendix]. In this case we need to consider also the second term of \( Y_{x,\gamma} \). Since \( i_1 \neq j_1 \), we can assume \( i_1 = 1, j_1 = 0 \). Then
\[
|Y_{x,\gamma}(i) - Y_{x,\gamma}(j)|
\]
\[
\geq 2\pi \gamma \left| \sin(\pi(x + 1)) - \sin(\pi x) + \gamma \left( \sin \left( \frac{\pi x + 1 + 2i_2}{2} \right) - \sin \left( \frac{\pi x + 2j_2}{2} \right) \right) \right| - 4\pi \sum_{n=3}^{\infty} \gamma^n
\]
\[
= 4\pi \gamma \left| \sin(\pi x) - \gamma \left( \sin \left( \frac{1 + 2(i_2 - j_2)}{4} \right) \cos \left( \frac{2x + 1 + 2(i_2 + j_2)}{4} \right) \right) \right| - \frac{4\pi \gamma^3}{1 - \gamma}
\]
and
\[
\left| \frac{d}{dx} Y_{x,\gamma}(i) - \frac{d}{dx} Y_{x,\gamma}(j) \right|
\]
\[
\geq 2\pi \gamma \left| \cos(\pi(x + 1)) - \cos(\pi x) + \frac{\gamma}{2} \left( \cos \left( \frac{\pi x + 1 + 2i_2}{2} \right) - \cos \left( \frac{\pi x + 2j_2}{2} \right) \right) \right| - 8\pi \sum_{n=3}^{\infty} \left( \frac{\gamma}{2} \right)^n
\]
\[
= 4\pi \gamma \left| \cos(\pi x) + \frac{\gamma}{2} \left( \sin \left( \frac{1 + 2(i_2 - j_2)}{4} \right) \sin \left( \frac{2x + 1 + 2(i_2 + j_2)}{4} \right) \right) \right| - \frac{2\pi \gamma^3}{2 - \gamma}
\]
which together with (3.1) implies
\[
\left| \sin(\pi x) - \gamma \left( \sin \left( \frac{1 + 2(i_2 - j_2)}{4} \right) \cos \left( \frac{2x + 1 + 2(i_2 + j_2)}{4} \right) \right) \right| \leq \frac{\gamma^2}{1 - \gamma} + \frac{\delta}{4\pi \gamma},
\]
\[
\left| \cos(\pi x) + \frac{\gamma}{2} \left( \sin \left( \frac{1 + 2(i_2 - j_2)}{4} \right) \sin \left( \frac{2x + 1 + 2(i_2 + j_2)}{4} \right) \right) \right| \leq \frac{\gamma^2}{2(2 - \gamma)} + \frac{\delta}{4\pi^2 \gamma}.
\]
Recall that \( i_2, j_2, x \) depend on \( \delta \). Taking a sequence of \( \delta \)-s tending to \( 0 \) we can choose a subsequence such that \( i_2, j_2, x \) converge, so by continuity we can assume
\[
\left| \sin(\pi x) - \gamma \left( \sin \left( \frac{1 + 2(i_2 - j_2)}{4} \right) \cos \left( \frac{2x + 1 + 2(i_2 + j_2)}{4} \right) \right) \right| \leq \frac{\gamma^2}{1 - \gamma},
\]
\[
\left| \cos(\pi x) + \frac{\gamma}{2} \left( \sin \left( \frac{1 + 2(i_2 - j_2)}{4} \right) \sin \left( \frac{2x + 1 + 2(i_2 + j_2)}{4} \right) \right) \right| \leq \frac{\gamma^2}{2(2 - \gamma)}.
\]
for some \( i_2, j_2 \in \{0, 1\} \) and \( x \in [0, 1] \). Taking the sum of the squares of the two inequalities and noting that \( \sin^2(\pi(1 + 2(i_2 - j_2))/4) = 1/2 \), we obtain
\[
g(x) \geq 0,
\]
where
\[
g(t) = \tilde{g}(t) - \frac{3\gamma^2}{8} \cos^2 \left( \frac{2t + 1 + 2(i_2 + j_2)}{4} \right)
\]
for
\[
\tilde{g}(t) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{\gamma^2}{8} - 1 + 2\gamma \sin \left(\frac{\pi}{4} \frac{1 + 2(i_2 - j_2)}{4}\right) \sin(\pi t) \cos \left(\frac{\pi}{4} \frac{2t + 1 + 2(i_2 + j_2)}{4}\right)
\]
\[
- \gamma \sin \left(\frac{\pi}{4} \frac{1 + 2(i_2 - j_2)}{4}\right) \cos(\pi t) \sin \left(\frac{\pi}{4} \frac{2t + 1 + 2(i_2 + j_2)}{4}\right).
\]

We have
\[
g'(t) = \frac{3\pi \gamma}{8} \cos \left(\frac{\pi}{4} \frac{2t + 1 + 2(i_2 + j_2)}{4}\right)
\]
\[
4 \sin \left(\frac{\pi}{4} \frac{1 + 2(i_2 - j_2)}{4}\right) \cos(\pi t) + \gamma \sin \left(\frac{\pi}{4} \frac{2t + 1 + 2(i_2 + j_2)}{4}\right)
\]
and
\[
\tilde{g}'(t) = \frac{3\pi \gamma}{2} \sin \left(\frac{\pi}{4} \frac{1 + 2(i_2 - j_2)}{4}\right) \cos(\pi t) \cos \left(\frac{\pi}{4} \frac{2t + 1 + 2(i_2 + j_2)}{4}\right). \]

Now we consider four cases, depending on the values of \(i_2, j_2\).

First, let \(i_2 = j_2 = 0\). Then
\[
\tilde{g}'(t) = \frac{3\sqrt{2}\pi \gamma}{4} \cos(\pi t) \cos \left(\frac{\pi}{4} \frac{2t + 1}{4}\right) \geq 0
\]
for \(t \in [0, 1]\). Hence,
\[
g(x) \leq \tilde{g}(x) \leq \tilde{g}(1) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{\gamma^2}{8} + \frac{\gamma}{2} - 1. \tag{3.5}
\]

Let now \(i_2 = j_2 = 1\). Then
\[
\tilde{g}'(t) = -\frac{3\sqrt{2}\pi \gamma}{4} \cos(\pi t) \cos \left(\frac{\pi}{4} \frac{2t + 1}{4}\right) \leq 0
\]
for \(t \in [0, 1]\), so
\[
g(x) \leq \tilde{g}(x) \leq \tilde{g}(0) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{\gamma^2}{8} + \frac{\gamma}{2} - 1. \tag{3.6}
\]

The third case is \(i_2 = 1, j_2 = 0\). Then
\[
g'(t) = -\frac{3\pi \gamma}{8} \sin \left(\frac{\pi}{4} \frac{2t + 1}{4}\right) \left(2\sqrt{2} \cos(\pi t) + \gamma \cos \left(\frac{\pi}{4} \frac{2t + 1}{4}\right)\right) \begin{cases} \leq 0 & \text{for } t \in [0, 1/2] \\ > 0 & \text{for } t \in (1/2, 1], \end{cases}
\]
which implies
\[
g(x) \leq \max(g(0), g(1)) = \frac{\gamma^4}{(1-\gamma)^2} + \frac{\gamma^4}{4(2-\gamma)^2} - \frac{5\gamma^2}{16} - \frac{\gamma}{2} - 1. \tag{3.7}
\]
The last case is $i_2 = 0, j_2 = 1$. Then

$$g'(t) = -\frac{3\pi\gamma}{8} \sin \left( \frac{\pi}{4} \left( \frac{2t + 1}{4} \right) \right) \left( -2\sqrt{2} \cos(\pi t) + \gamma \cos \left( \frac{\pi}{4} \left( \frac{2t + 1}{4} \right) \right) \right)$$

$$= -\frac{3\sqrt{2}\pi\gamma}{16} \sin \left( \frac{\pi}{4} \left( \frac{2t + 1}{4} \right) \right) \left( \cos \frac{\pi t}{2} - \sin \frac{\pi t}{2} \right) \left( \gamma - 4 \left( \cos \frac{\pi t}{2} + \sin \frac{\pi t}{2} \right) \right)$$

$$\begin{cases} 
\geq 0 & \text{for } t \in [0, 1/2] \\
< 0 & \text{for } t \in (1/2, 1],
\end{cases}$$

since $\gamma - 4(\cos(\pi t/2) + \sin(\pi t/2)) \leq \gamma - 4 < 0$ for $t \in [0, 1]$. Hence,

$$g(x) \leq g(1/2) = \frac{\gamma^4}{(1 - \gamma)^2} + \frac{\gamma^4}{4(2 - \gamma)^2} - \frac{5\gamma^2}{16} + \sqrt{2}\gamma - 1. \quad (3.8)$$

Considering the conditions (3.5)–(3.8) we easily conclude that the largest upper estimate for $g(x)$ appears in (3.8). Therefore, by (3.4), in all cases we have

$$0 \leq \frac{\gamma^4}{(1 - \gamma)^2} + \frac{\gamma^4}{4(2 - \gamma)^2} - \frac{5\gamma^2}{16} + \sqrt{2}\gamma - 1 = h_2(\lambda)$$

for $\lambda = 1/(2\gamma) > \lambda_2$, which contradicts Lemma 3.4. This ends the proof in the case $b = 2$. \qed

To conclude the proof of Theorem A, it is enough to notice that by Proposition 3.5, for $\lambda \in (\lambda_b, 1)$ we have $e(1, 1; \delta, \delta) = 1 < \gamma b$ and use Theorem 3.2, Lemma 3.3, Theorem 2.2 and (2.2). The estimates for $\lambda_2, \lambda_3$ and $\lambda_4$ in Corollary C follow from Lemma 3.4.

### 4. Proof of Theorem B

Using the transversality method developed by Peres andSolomyak in the study of infinite Bernoulli convolutions (see [21, 22]), with a minor modification on the standard argument, we will show that $m_{x, \gamma}$ is absolutely continuous for Lebesgue almost every $(x, \gamma) \in (0, 1) \times (1/b, 1/(b\lambda_b))$. The statement will follow from the Fubini theorem.

First, we prove the existence of the numbers $\tilde{\lambda}_b$ defined in Theorem B.

**Lemma 4.1.** For every integer $b$ larger than 1 there exists a unique number $\tilde{\lambda}_b \in (1/b, 1)$ such that

$$y \left( \frac{1}{\sqrt{\sin^2(\pi/b) - 1/((b^2\lambda_b - 1)^2)} \right) = \frac{1}{b\lambda_b}$$

and for $\lambda \in (1/b, 1),$

$$y \left( \frac{1}{\sqrt{\sin^2(\pi/b) - 1/((b^2\lambda - 1)^2)} \right) < \frac{1}{b\lambda} \quad \iff \quad \lambda \in (1/b, \tilde{\lambda}_b).$$

Moreover, $\hat{\lambda}_b < \lambda_b$ for every $b \geq 2$, $\tilde{\lambda}_b < 1.04/\sqrt{b}$ for every $b \geq 5$ and $\tilde{\lambda}_b\sqrt{b} \to 1/\sqrt{\pi}$ as $b \to \infty$.

**Proof.** First, note that

$$\sin \frac{\pi}{b} > \frac{1}{b^2\lambda - 1}$$
for every \( \lambda \in (1/b, 1) \). Indeed, for \( b = 2 \) it is obvious and for \( b \geq 3 \),
\[
\sin \frac{\pi}{b} - \frac{1}{b^2 \lambda - 1} > \sin \frac{\pi}{b} - \frac{1}{b - 1} > 0
\]
since \( h_b(1) < 0 \) (see the proof of Lemma 3.4). This implies that
\[
\beta = \beta(\lambda) = \frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2 \lambda - 1)^2}}
\]
is well-defined for \( \lambda \in (1/b, 1) \). Obviously, \( \beta > 1 \).

It is known (see [22]) that for \( \beta \geq 1 \) the function \( \beta \mapsto y(\beta) \) is strictly decreasing, continuous and satisfies
\[
1 > y(\beta) \geq \frac{1}{1 + \sqrt{\beta}}. \tag{4.1}
\]
Moreover,
\[
y(\beta) = \frac{1}{1 + \sqrt{\beta}} \quad \text{for } \beta \geq 3 + \sqrt{8}. \tag{4.2}
\]
This implies that \( y(\beta) - 1/(b \lambda) \) strictly increases with respect to \( \lambda \in (1/b, 1) \), moreover \( y(\beta) - 1/(b \lambda) < 0 \) for \( \lambda \) sufficiently close to \( 1/b \) and
\[
y(\beta) - \frac{1}{(b \lambda)} > \frac{1}{1 + \sqrt{\beta}} - \frac{1}{b \lambda} \tag{4.3}
\]
for \( \lambda \in (1/b, 1) \). By the definition of \( \beta \), the inequality
\[
\frac{1}{1 + \sqrt{\beta}} - \frac{1}{(b \lambda)} > 0 \tag{4.4}
\]
is equivalent to \( \tilde{h}_b(\lambda) < 0 \) for
\[
\tilde{h}_b(\lambda) = \frac{1}{(b \lambda - 1)^4 + \frac{1}{(b^2 \lambda - 1)^2}} - \sin^2 \frac{\pi}{b}.
\]
We have \( \tilde{h}_b(\lambda) < h_b(\lambda) \), so by Lemma 3.4, the inequality (4.4) holds for \( \lambda \) sufficiently close to 1. By (4.3), \( y(\beta) - 1/(b \lambda) > 0 \) for \( \lambda \) sufficiently close to 1. This implies that there exists a unique number \( \lambda_b \in (1/b, 1) \) such that \( \lambda_b < \lambda_\beta \) and \( y(\beta) = 1/(b \lambda) \).

Like in the proof of Lemma 3.4, using the inequality \( \sin x - x^3/6 \) for \( x > 0 \), we obtain
\[
\tilde{h}_b(\lambda) < \frac{1}{(b \lambda - 1)^4 + \frac{1}{(b^2 \lambda - 1)^2}} + \frac{\pi^4}{3b^4} - \frac{\pi^2}{b^2} = \tilde{H}_b(\lambda)
\]
for
\[
\tilde{H}_b(\lambda) = \frac{1}{(\sqrt{b} \lambda - 1/\sqrt{b})^4} + \frac{1}{(b \lambda - 1/b)^2} + \frac{\pi^4}{3b^2} - \pi^2.
\]
Substituting \( \lambda = c/\sqrt{b} \) for \( c > 0 \), we get
\[
\tilde{H}_b(c/\sqrt{b}) = \frac{1}{(c - 1/\sqrt{b})^4} + \frac{1}{(c \sqrt{b} - 1/b)^2} + \frac{\pi^4}{3b^2} - \pi^2.
\]
The function \( \tilde{H}_b(c/\sqrt{b}) \) is strictly decreasing with respect to \( c \) and \( b \) and one can directly check \( \tilde{H}_5(1.04/\sqrt{5}) < 0 \). This implies that \( \lambda_b < 1.04/\sqrt{5} \) for every \( b \geq 5 \).

For \( \beta \geq 19 \),
\[
\beta > \frac{1}{\sin(\pi/19)} > \frac{19}{\pi} > 3 + \sqrt{8},
\]
so by (4.2), the number \( \tilde{\lambda}_b \) is equal to the unique zero of the function \( \tilde{h}_b \) on the interval \((1/b, 1)\). This easily implies that \( \lambda_b \sqrt{b} \to 1/\sqrt{\pi} \) as \( b \to \infty \) (the details are left to the reader).

Let

\[
\tilde{\gamma}_b = \frac{1}{b \lambda_b}.
\]

Now we prove a modified transversality condition for the functions \( Y_\gamma(i) \). The trick we use is to consider transversality with respect to two variables \( x, \gamma \).

**Proposition 4.2.** For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( i = (i_1, i_2, \ldots), \ j = (j_1, j_2, \ldots) \in \Sigma \) with \( i_1 \neq j_1 \),

\[
|Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| > \delta \quad \text{or} \quad \left| \frac{d}{dx}Y_{x,\gamma}(i) - \frac{d}{dx}Y_{x,\gamma}(j) \right| > \delta
\]

for every \( x \in (0, 1) \) and \( \gamma \in (1/b, \tilde{\gamma}_b - \varepsilon) \).

**Proof.** The proof is similar to the proof of Proposition 3.5. Suppose that the statement does not hold. Then for every \( \delta > 0 \) there exist \( i = (i_1, i_2, \ldots), \ j = (j_1, j_2, \ldots) \in \Sigma \) with \( i_1 \neq j_1 \),

where \( \delta \) is to consider transversality with respect to two variables \( x, \gamma \).


d\gamma
\leq \sum_{i \in \Sigma} \gamma_i 

Using the fact \( i_1 \neq j_1 \) and (4.7), we obtain

\[
|y_1| \geq 2 \sin \frac{\pi}{b} \left| \cos \left( 2\pi \frac{2x + i_1 + j_1}{2b} \right) \right|
\]

and \( |y_n| \leq 2 \) for \( n \geq 2 \). Using the fact \( i_1 \neq j_1 \) and (4.7), we obtain

\[
\left| y_1 \right| \geq 2 \sin \frac{\pi}{b} \left| \cos \left( 2\pi \frac{2x + i_1 + j_1}{2b} \right) \right|
\]

or

\[
\left| y_1 \right| > 2 \sin^2 \frac{\pi}{b} - \left( \frac{\gamma}{b - \gamma} + \frac{\delta b^2}{8\pi^2} \right)^2
\]

and

\[
\left| y_1 \right| > 2 \sin^2 \frac{\pi}{b} - \left( \frac{1}{b - 1} + \frac{\delta b^2}{8\pi^2} \right)^2
\]


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in particular $y_1 \neq 0$ for sufficiently small $\delta$ (because $h_b(1) < 0$, see the proof of Lemma 3.4). Hence, for the function

$$g(t) = \frac{Y_{x,t}(i) - Y_{x,t}(j)}{2\pi y_1 t}$$

we have

$$g(t) = 1 + \sum_{n=1}^{\infty} g_n t^n,$$

where

$$|g_n| = \left|\frac{y_{n+1}}{y_1}\right| < \frac{1}{\sqrt{\sin^2(\pi/b) - (\gamma/(b - \gamma) + \delta b/(8\pi^2 \gamma))^2}},$$

This implies that $g \in G_\beta$ for

$$\beta = \frac{1}{\sqrt{\sin^2(\pi/b) - (\gamma/(b - \gamma) + \delta b/(8\pi^2 \gamma))^2}}.$$ 

On the other hand, by (4.5) and (4.8),

$$|g(\gamma)| \leq \frac{\delta}{2\pi |y_1| \gamma} < \frac{\delta b}{4\pi \sqrt{\sin^2(\pi/b) - (1/(b - 1) + \delta b^2/(8\pi^2))^2}} \quad \text{(4.9)}$$

and

$$|g'(\gamma)| \leq \frac{(\gamma + 1)\delta}{2\pi |y_1| \gamma^2} < \frac{\delta b^2}{2\pi \sqrt{\sin^2(\pi/b) - (1/(b - 1) + \delta b^2/(8\pi^2))^2}} \quad \text{(4.10)}$$

Note that $g$, $\gamma$ and $\beta$ depend on $\delta$. Take a sequence of $\delta$-s tending to 0. Then we can choose a subsequence such that $\gamma \to \gamma_* \in [1/b, \tilde{\gamma} b - \varepsilon]$, $\beta \to \beta_*$ for

$$\beta_* = \frac{1}{\sqrt{\sin^2(\pi/b) - (\gamma_*/(b - \gamma_*))^2}} < \frac{1}{\sqrt{\sin^2(\pi/b) - (\tilde{\gamma} b_*/(b - \tilde{\gamma} b_*))^2}}$$

and $g$ converges uniformly in $[1/b, \tilde{\gamma} b]$ to a function $g_* \in G_{\beta_*}$. Since the right-hand sides of (4.9) and (4.10) tend to 0 as $\delta \to 0$, we obtain

$$g_*(\gamma_*) = g_*'(\gamma_*) = 0,$$

so $y(\beta_*) \leq \gamma_*$. This is a contradiction, because by Lemma 4.1,

$$y(\beta_*) = y\left(\frac{1}{\sqrt{\sin^2(\pi/b) - 1/(b^2 \lambda_* - 1)^2}}\right) > \frac{1}{b \lambda_*} = \gamma_*$$

for $\lambda_* = 1/(b \gamma_*) > 1/(b \tilde{\gamma} b) = \tilde{\lambda}$. This ends the proof. □

As a simple consequence of the previous proposition one can prove the following statement (for the proof we refer to [27, Lemma 7.3]).

**Lemma 4.3.** For every $\varepsilon > 0$ there exists a constant $C > 0$ such that for every $i = (i_1, i_2, \ldots)$, $j = (j_1, j_2, \ldots) \in \Sigma$ with $i_1 \neq j_1$,

$$L_2 \left(\{ (x, \gamma) \in (0, 1) \times (1/b, \tilde{\gamma} b - \varepsilon) : |Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| < r \} \right) \leq Cr$$

for every $r > 0$, where $L_2$ is the Lebesgue measure on the plane.
To state next results, we need to introduce some notation. For \( i = (i_1, i_2, \ldots) \in \Sigma \) let \( i|_n = (i_1, \ldots, i_n) \). For \( i = (i_1, i_2, \ldots), \ j = (j_1, j_2, \ldots) \in \Sigma \) let
\[
i \land j = \min \{ n \geq 0 : i_{n+1} \neq j_{n+1} \}.
\]
For a finite length word \((l_1, \ldots, l_n) \in \Sigma^* \) let
\[
A_{(l_1, \ldots, l_n)} = \{(i, j) \in \Sigma^2 : i \land j = n\}.
\]
We note that for the empty word we have \( A_\emptyset = \{(i, j) \in \Sigma^2 : i_1 \neq j_1\} \). We will write
\[
A_{(l_1, \ldots, l_n)}|_N = \{(i|_N, j|_N) : (i, j) \in A_{(l_1, \ldots, l_n)}\}
\]
for \( N \geq 1 \). For a finite length word \( \tilde{i} = (i_1, \ldots, i_n) \in \Sigma^* \) let
\[
v_{\tilde{i}}(x) = \frac{x}{b^n} + i_1 \frac{1}{b^n} + \cdots + i_n \frac{1}{b^n}.
\]
Let us observe that for any \( i, j \in A_{\tilde{i}} \),
\[
|Y_{x, \gamma}(i) - Y_{x, \gamma}(j)| = \gamma^n |Y_{\tilde{v}_\gamma(x), \gamma}(\sigma^n i) - Y_{\tilde{v}_\gamma(x), \gamma}(\sigma^n j)|,
\]
where \( \sigma \) denotes the left-side shift on \( \Sigma \) and \( n \) is the length of \( \tilde{i} \).

Unfortunately, because of the structure of the measure \( m_{x, \gamma} \), it is not possible to apply directly the transversality method and Lemma 4.3. To avoid this difficulty, we introduce the following lemma.

**Lemma 4.4.** Let \( i = (i_1, i_2, \ldots) \), \( j = (j_1, j_2, \ldots) \in \Sigma \) with \( i_1 \neq j_1 \). Then for every \( r > 0 \) there exists \( N = N(r) \) such that
\[
|Y_{x, \gamma}(i) - Y_{x, \gamma}(j)| < r \quad \Rightarrow \quad |Y_{x, \gamma}(i|_N 0) - Y_{x, \gamma}(j|_N 0)| < 2r
\]
for every \( x \in (0, 1) \) and \( \gamma \in (1/b, \bar{\gamma}_b) \), where \( 0 = (0, 0, \ldots) \).

**Proof.** We have
\[
\begin{align*}
&\|Y_{x, \gamma}(i) - Y_{x, \gamma}(j)| - |Y_{x, \gamma}(i|_N 0) - Y_{x, \gamma}(j|_N 0)|\| \\
&\leq |(Y_{x, \gamma}(i) - Y_{x, \gamma}(i|_N 0)) - (Y_{x, \gamma}(j) - Y_{x, \gamma}(j|_N 0))| \\
&\leq \gamma^n |Y_{\tilde{v}_{i|_N}(x), \gamma}(\sigma^n i) - Y_{\tilde{v}_{i|_N}(x), \gamma}(0)| + \gamma^n |Y_{\tilde{v}_{j|_N}(x), \gamma}(\sigma^n j) - Y_{\tilde{v}_{j|_N}(x), \gamma}(0)| \\
&\leq \gamma^n \frac{8 \pi \gamma}{1 - \gamma} < \frac{8 \pi \bar{\gamma}_b}{1 - \gamma} \leq r,
\end{align*}
\]
which implies the inequality (4.12) for sufficiently large \( N = N(r) \). 

**Proposition 4.5.** For Lebesgue almost every \( \gamma \in (1/b, \bar{\gamma}_b) \) the measure \( m_{x, \gamma} \) is absolutely continuous (in particular, \( \dim m_{x, \gamma} = 1 \)) for Lebesgue almost every \( x \in (0, 1) \).

**Proof.** Take \( \varepsilon > 0 \). We will prove that \( m_{x, \gamma} \) is absolutely continuous with respect to the Lebesgue measure, with density in \( L^2 \), for Lebesgue almost every \( (x, \gamma) \in R_\varepsilon \), where
\[
R_\varepsilon = (0, 1) \times (1/b + \varepsilon, \bar{\gamma}_b - \varepsilon).
\]
Since \( \varepsilon > 0 \) is arbitrarily small, this will imply the statement. Denote by
\[
D(m_{x, \gamma}, y) = \liminf_{r \to 0} \frac{m_{x, \gamma}(B_r(y))}{2r}
\]
the lower density of the measure $m_{x,\gamma}$ at the point $y$, where $B_r(y)$ denotes the ball with radius $r$ centered at $y$. By [19, Theorem 2.12], if $D(m_{x,\gamma}, y) < \infty$ for $m_{x,\gamma}$-almost every $y$, then the measure $m_{x,\gamma}$ is absolutely continuous. It is enough to show that

$$I := \iint_{R_\varepsilon} \int_{\mathbb{R}} D(m_{x,\gamma}, y) \, dm_{x,\gamma}(y) \, dL_2(x, \gamma) < \infty.$$ 

The statement follows from the Fubini theorem. By standard manipulations we have

$$I \leq \liminf_{r \to 0} \frac{1}{2r} \iint_{\Sigma \times \Sigma} L_2 \left( \{(x, \gamma) \in R_\varepsilon : |Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| < r \} \right) \, d\mathbb{P}(i) \, d\mathbb{P}(j).$$

Then

$$\iint_{\Sigma \times \Sigma} L_2 \left( \{(x, \gamma) \in R_\varepsilon : |Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| < r \} \right) \, d\mathbb{P}(i) \, d\mathbb{P}(j) = \sum_{n=0}^{\infty} \sum_{i \in \{0, \ldots, b-1\}^n} \iint_{A_\sigma} L_2 \left( \{(x, \gamma) \in R_\varepsilon : |Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| < r \} \right) \, d\mathbb{P}(i) \, d\mathbb{P}(j).$$

By (4.11), for any $i, j \in A_\sigma$,

$$L_2 \left( \{(x, \gamma) \in R_\varepsilon : |Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| < r \} \right) = L_2 \left( \left\{ (x, \gamma) \in R_\varepsilon : \left| Y_{x,\gamma}(\sigma^n i) - Y_{x,\gamma}(\sigma^n j) \right| < \gamma^{-n} r \right\} \right) = b^n L_2 \left( \left\{ (x, \gamma) \in R_\varepsilon : |Y_{x,\gamma}(\sigma^n i) - Y_{x,\gamma}(\sigma^n j)| < \gamma^{-n} r \right\} \right) \leq b^n L_2 \left( \left\{ (x, \gamma) \in R_\varepsilon : |Y_{x,\gamma}(\sigma^n i) - Y_{x,\gamma}(\sigma^n j)| < \left( \frac{1}{b} + \varepsilon \right)^{-n} r \right\} \right),$$

where $R_{\varepsilon, \sigma} = (v_\sigma(0), v_\sigma(1)) \times (1/b + \varepsilon, \gamma b)$. Applying Lemma 4.4, we get

$$b^n L_2 \left( \left\{ (x, \gamma) \in R_{\varepsilon, \sigma} : |Y_{x,\gamma}(\sigma^n i) - Y_{x,\gamma}(\sigma^n j)| < \left( \frac{1}{b} + \varepsilon \right)^{-n} r \right\} \right) \leq b^n L_2 \left( \left\{ (x, \gamma) \in R_{\varepsilon, \sigma} : |Y_{x,\gamma}(\sigma^n i) - Y_{x,\gamma}(\sigma^n j)| < 2 \left( \frac{1}{b} + \varepsilon \right)^{-n} r \right\} \right),$$

where $N$ depends on $n, r$. Hence,

$$\sum_{i \in \{0, \ldots, b-1\}^n} \iint_{A_\sigma} L_2 \left( \{(x, \gamma) \in R_\varepsilon : |Y_{x,\gamma}(i) - Y_{x,\gamma}(j)| < r \} \right) \, d\mathbb{P}(i) \, d\mathbb{P}(j) \leq \sum_{i \in \{0, \ldots, b-1\}^n} \sum_{(\bar{\kappa}, \bar{l}) \in A_{\bar{\kappa}}|_N} \frac{b^n}{b^{2n+2N}} L_2 \left( \left\{ (x, \gamma) \in R_{\bar{\kappa}, \phi} : |Y_{x,\gamma}(\bar{\kappa}0) - Y_{x,\gamma}(\bar{\phi}0)| < 2 \left( \frac{1}{b} + \varepsilon \right)^{-n} r \right\} \right) = \sum_{(\bar{\kappa}, \bar{l}) \in A_{\bar{\kappa}}|_N} \frac{b^n}{b^{2n+2N}} L_2 \left( \left\{ (x, \gamma) \in R_{\bar{\kappa}, \phi} : |Y_{x,\gamma}(\bar{\kappa}0) - Y_{x,\gamma}(\bar{\phi}0)| < 2 \left( \frac{1}{b} + \varepsilon \right)^{-n} r \right\} \right).$$
where in the last inequality we used that $R_{r}=\bigcup_{i=0}^{b-1}R_{r_{i}}$. Using Lemma 4.3 we get

$$I \leq \lim \inf_{r \to 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{(x, t) \in A_{0}[N]} \frac{b^{n}}{b^{2n+2N}} L_{2} \left( \{ (x, \gamma) \in R_{r} : |Y_{x, \gamma}(Y0) - Y_{x, \gamma}(0)| < 2 \left( \frac{1}{b} + \varepsilon \right)^{-n} \} \right)$$

$$\leq \lim \inf_{r \to 0} \frac{1}{2r} \sum_{n=0}^{\infty} \sum_{(x, t) \in A_{0}[N]} \frac{b^{n}}{b^{2n+2N}} 2C^{r} \left( \frac{1}{b} + \varepsilon \right)^{-n} \leq C \sum_{n=0}^{\infty} (1 + b\varepsilon)^{-n},$$

which is finite since $\varepsilon > 0$. \hfill \Box

**Proof of Theorem B.** The result is a consequence of Proposition 4.5, Proposition 2.2 and (2.2). \hfill \Box

To obtain more precise estimates of $\tilde{\lambda}_{2}, \tilde{\lambda}_{3}, \tilde{\lambda}_{4}$ presented in Corollary C, one needs to find suitable $(\ast)$-functions. To do it, we use the following result.

**Lemma 4.6** (Peres, Solomyak [22, Lemma 5.1]). Let $\beta \geq 1$. Suppose that for some positive integer $k = k(\beta)$ and a real number $\eta = \eta(\beta)$ there exists a function $g_{\beta} : \mathbb{R} \to \mathbb{R}$,

$$g_{\beta}(t) = 1 - \beta \sum_{n=1}^{k-1} t^{n} + \eta t^{k} + \beta \sum_{n=k+1}^{\infty} t^{n}$$

such that for some $t_{\beta} \in (0, 1)$,

$$g_{\beta}(t) > 0 \quad \text{and} \quad g'_{\beta}(t_{\beta}) < 0.$$

Then $y(\beta) > t_{\beta}$. More precisely, there exists $\varepsilon > 0$ such that for every $g \in \mathcal{G}_{\beta}$ and every $t \in (0, t_{\beta})$,

$$g(t) < \varepsilon \quad \Rightarrow \quad g'(t) < -\varepsilon.$$

Let

$$\beta = \frac{1}{\sqrt{\sin^{2}(\pi/b) - 1/(b^{2}e - 1)^{2}}}$$

and consider functions $g_{\beta}$ defined in Lemma 4.6.

For $b = 2$ take $k = 4$, $\eta = 0.81$, $\lambda = 0.81$. Then $g_{\beta}(0.62) > 0$ and $g'_{\beta}(0.62) < 0$, so $y(\beta) > 0.62$. On the other hand, $1/(2\lambda) = 1/1.62 < 0.62$. By Lemma 4.1, $\tilde{\lambda}_{2} < 0.81$.

For $b = 3$ take $k = 4$, $\eta = 1.43398$, $\lambda = 0.55$. Then $g_{\beta}(0.6061) > 0$ and $g'_{\beta}(0.6061) < 0$, so $y(\beta) > 0.6061$. On the other hand, $1/(3\lambda) = 1/1.65 < 0.6061$. By Lemma 4.1, $\tilde{\lambda}_{3} < 0.55$.

For $b = 4$ take $k = 3$, $\eta = -0.298$, $\lambda = 0.44$. Then $g_{\beta}(0.569) > 0$ and $g'_{\beta}(0.569) < 0$, so $y(\beta) > 0.569$. On the other hand, $1/(4\lambda) = 1/1.76 < 0.569$. By Lemma 4.1, $\tilde{\lambda}_{4} < 0.44$.

**References**


