Megoldókulcs
05.31 gyak.

Javított

1) a) zárt, irreducibilis, aperiodikus.

$$(\pi_0, \pi_1, \pi_2)\begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.2 & 0.8 \end{pmatrix}$$

$0.8\pi_R + 0.3\pi_D = \pi_R$ $\quad \frac{3}{2}\pi_D = \pi_R$

$0.2\pi_D + 0.8\pi_Z = \pi_Z$

$\pi_Z = \pi_D \Rightarrow \left(\frac{3}{2}+1+1\right)\pi_D = 1$

$\pi_D = \frac{2}{7}$

$\pi_R = \frac{3}{7}$, $\pi_Z = \frac{2}{7}$

b) $T_D := \inf\{n \geq 1 : X_n = D\}$

$\mathbb{E}_D T_D = \frac{1}{\pi_D} = \frac{7}{2}$

c) $\mathbb{E}_R T_Z = ?$

(1) $\mathbb{E}_R T_Z = 1 + 0.8\,\mathbb{E}_R T_Z + 0.2\,\mathbb{E}_D T_Z$

(2) $\mathbb{E}_D T_Z = 1 + 0.3\,\mathbb{E}_R T_Z + 0.5\,\mathbb{E}_D T_Z$

$(1) \Rightarrow \mathbb{E}_R T_Z = 5 + \mathbb{E}_D T_Z \overset{(2)}{\Rightarrow} \mathbb{E}_D T_Z = 1 + 0.3(5 + \mathbb{E}_D T_Z) + 0.5\,\mathbb{E}_D T_Z$

$0.2\,\mathbb{E}_D T_Z = 2.5 \Rightarrow \mathbb{E}_D T_Z = \frac{25}{2}$

$\Rightarrow \mathbb{E}_R T_Z = \frac{35}{2}$

d) $\mathbb{E}_Z\left(\sum_{k=1}^{T_R} \mathbb{1}\{X_k = D\}\right) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \mathbb{P}(X_k = D \mid X_0 = Z, T_R = n)\cdot \mathbb{P}(T_R = n)$

$= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \mathbb{P}(X_k = D \mid X_0 = Z, T_R = n)\cdot\mathbb{P}(T_R = n) = \sum_{k=1}^{\infty} \underbrace{\mathbb{P}(T_R > k, X_k = D \mid X_0 = Z)}_{Q^{(k)}_{Z,D}}$

$$Q := \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$(0.5-\lambda)(0.8-\lambda)-0.2^2=0$$

$$\lambda^2-\cancel{0}1.3\lambda+0.4-0.04=0$$

$$\lambda=\frac{1.3\pm\sqrt{1.3^2-4\cdot 0.36}}{2}=\frac{1.3\pm 0.5}{2}=\begin{cases}\underline{\underline{0.9}}\\ \underline{\underline{0.4}}\end{cases}$$

$$\begin{bmatrix}0.5 & 0.2\\ 0.2 & 0.8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{pmatrix}0.9x\\ 0.9y\end{pmatrix}$$

$$0.5x+0.2y=0.9x \qquad \boxed{y=2x}$$

$$\begin{bmatrix}0.5 & 0.2\\ 0.2 & 0.8\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{pmatrix}0.4x\\ 0.4y\end{pmatrix} \qquad 0.5x+0.2y=0.4x$$

$$\boxed{y=-\frac{1}{2}x}$$

$$U=\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix} \qquad U^{-1}=\frac{1}{5}\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix} \qquad D=\begin{bmatrix}0.9 & 0\\ 0 & 0.4\end{bmatrix}$$

$$Q=U\cdot DU^{-1} \Rightarrow \sum_{k=1}^{\infty}Q^{(k)}=U\sum_{k=1}^{\infty}D^{(k)}U^{-1}=$$

~~vagy~~

$$\sum_{l=1}^{\infty}Q^l=(I-Q)^{-1}-I=(I-Q)^{-1}Q$$

$$\Rightarrow U\begin{bmatrix}9 & 0\\ 0 & \frac{4}{\cancel{5}6}\end{bmatrix}U^{-1}=\begin{bmatrix}\cancel{\frac{61}{25}}\ \frac{10}{3} & \cancel{\frac{82}{25}}\ \frac{10}{3}\\ \cancel{\frac{82}{25}}\ \frac{10}{3} & \cancel{\frac{184}{25}}\ \frac{25}{3}\end{bmatrix}$$

$$\Rightarrow \mathbb{E}_z\left(\sum_{k=1}^{T_R}\mathbb{I}\{X_k=D\}\right)=\underline{\underline{\cancel{\frac{82}{25}}\ \frac{10}{3}}}$$

2) Autók érkezése: $N(t)$ ~~$\mathbb{E}N(5)$~~ λ param poi

$$5\lambda=\mathbb{E}(N(5))=1 \Rightarrow \lambda=\frac{1}{5}$$

! $N_1(t)$ 1 ~~utas~~ utas autó; $N_2(t)$ 2 utas autó; $N_3(t)$ 3 utas autó

$N_1(t)\sim\frac{1}{10}$ Poi $\quad N_2(t)\sim\frac{1}{15}$ Poi $\quad N_3(t)\sim\frac{1}{30}$ Poi.

$Y_i \sim$ vezér által vett i-edik oldal

$$\mathbb{E}\left(\sum_{k=1}^{N_1(60)} Y_k\right) + \mathbb{E}\left(\sum_{k=1}^{N_2(60)} Y_k^1 + Y_k^2\right) + \mathbb{E}\left(\sum_{k=1}^{N_3(60)} Y_k^1 + Y_k^2 + Y_k^3\right)$$

$$= \frac{60}{10}\cdot 10 + \frac{60}{15}\cdot 20 + \frac{60}{30}\cdot 30 = \$200$$

$$\mathrm{Var}\left(\sum_{k=1}^{N_1(60)} Y_k\right) = \mathbb{E}N_1(60)\cdot \mathrm{Var}\,Y_k + (\mathbb{E}Y_k)^2\cdot \mathrm{Var}\,N_1(10) = \frac{60}{10}\cdot 4 + 100\cdot\frac{60}{10}$$

$$= 624$$

$$\mathrm{Var}\left(\sum_{k=1}^{N_2(60)} (Y_k^1 + Y_k^2)\right) = \mathbb{E}(N_2(60))\cdot \overbrace{\mathrm{Var}(Y_k^1 + Y_k^2)}^{\mathrm{Var}\,Y_k^1 + \mathrm{Var}\,Y_k^2} + (\mathbb{E}(Y_k^1 + Y_2^2))^2 \cdot \mathrm{Var}\,N_2(60) =$$

$$= \frac{60}{15}\cdot 8 + 400\cdot\frac{60}{15} = 1632$$

$$\mathrm{Var}\left(\sum_{k=1}^{N_3(60)} Y_k^1 + Y_k^2 + Y_k^3\right) = \frac{60}{30}\cdot 12 + 900\cdot\frac{60}{30} = 1824$$

$$\Rightarrow \mathrm{Var} = 4080 \qquad D = \$63.87$$

3) a) $g_Y(x) = \frac{x^3}{8} + \frac{x^2}{8} + \frac{x}{2} + \frac{1}{4}$

$$\mathbb{E}Y = g_Y'(1) \Rightarrow \mathbb{E}X_2 = (g_Y'(1))^2 = \frac{81}{64}$$

$$\frac{3}{8} + \frac{2}{8} + \frac{1}{2} = \frac{9}{8} \qquad g_Y''(1) = \frac{6}{8} + \frac{2}{8} = 1$$

$$\mathbb{E}(X_2(X_2-1)) = \mathbb{E}X_2^2 - \mathbb{E}X_2 = g_Y''(1)((g_Y'(1))^2 + g_Y'(1)) - \cancel{(g_Y'(1))^2} - g_Y'(1)$$

$$g_{X_2}''(1) = g_Y''(g_Y(1))\cdot(g_Y'(1))^2 + g_Y'(g_Y(1))\cdot g_Y''(1) = g_Y''(1)(g_Y'(1)^2 + g_Y'(1))$$

$$\Rightarrow \mathbb{E}X_2^2 = 1\left(\frac{81}{64} + \frac{9}{8}\right) + \left(\frac{9}{8}\right)^2 = \frac{\cancel{225}}{\cancel{64}}\ \frac{117}{32} \Rightarrow \mathrm{Var}\,X_2 = \frac{\cancel{117}}{\cancel{64}\ 32} - \left(\frac{81}{64}\right)^2 = \cancel{\frac{18}{8}}\ \frac{9}{4}$$

$$\cancel{D = \ldots} \qquad D = \sqrt{\frac{8415}{4096}}$$

b) $\mathbb{P}(\text{kihal}) = q \qquad g_Y(q) = q$

$$\frac{x^3}{8}+\frac{x^2}{8}+\frac{x}{2}+\frac{1}{4}=x$$

$$\frac{x^3}{8}+\frac{x^2}{8}-\frac{x}{2}+\frac{1}{4}=0 \qquad x_0=1 \ \underline{\text{gyök}}$$

$$\left(\frac{x^3}{8}+\frac{x^2}{8}-\frac{x}{2}+\frac{1}{4}\right):x-1=\frac{x^2}{8}+\frac{x}{4}-\frac{1}{4} \Rightarrow$$

$$\frac{x^3}{8}-\frac{x^2}{8}$$

$$\frac{x^2}{4}-\frac{x}{2}+\frac{1}{4}$$

$$\frac{x^2}{4}-\frac{x}{4}$$

$$-\frac{x}{4}+\frac{1}{4}$$

$$x_{1,2}=\frac{-\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{4}{32}}}{2\cdot\frac{1}{8}}=\cancel{\frac{-\frac{1}{4}\pm\frac{3}{4}}{\frac{1}{4}}}=$$

$$\frac{-\frac{1}{4}\pm\frac{\sqrt{3}}{4}}{\frac{1}{4}}=\begin{cases}\sqrt{3}-1\\ -\sqrt{3}-1\end{cases}$$

$$\Rightarrow \mathbb{P}(\text{nem hal ki}) = \underline{\underline{2-\sqrt{3}}}$$

4) a)

$$Q_{i,j}=\begin{cases}\lambda & i=0,\ j=1\\ i\cdot\lambda & i\geq 1\ \ j=i+1\\ i\mu & i\geq 1\ \ j=i-1\\ \cancel{0} & \cancel{\text{else}}\\ -\lambda & i=0,\ j=0\\ -i(\lambda+\mu) & i\geq 1\ \ i=j\\ 0 & \text{else}\end{cases}$$

$$Q=\begin{bmatrix}-\lambda & \lambda & 0 & 0 & \dots\\ \mu & -(\mu+\lambda) & \lambda & 0 & \dots\\ 0 & 2\mu & -2(\mu+\lambda) & 2\lambda & 0 \ \dots\\ 0 & 0 & 3\mu & -3(\mu+\lambda) & 3\lambda \dots\\ \vdots & & & & \ddots\end{bmatrix}$$

b) π_i detailed balance

$$\pi_i\cdot Q_{i,i+1}=\pi_{i+1}Q_{i+1,i}$$

$$\pi_0\cdot\lambda=\mu\cdot\pi_1 \qquad \pi_{i+1}=\frac{i\cdot\lambda}{(i+1)\mu}\pi_i \qquad i\geq 1$$

$$\cancel{\pi_1\cdot\lambda=2\mu\,\pi_2}$$

$\Rightarrow \pi_1 = \frac{\lambda}{\mu} \cdot \pi_0 \qquad i\; \pi_2 = \frac{\lambda}{2\mu}\pi_1 = \frac{\lambda^2}{2\mu^2}\pi_0$

$\underline{\pi_i = \frac{\lambda^i}{(i)\cdot\mu^i}\pi_0}$ induktiv: $\pi_{i+1} = \frac{i\lambda}{(i+1)\mu}\cdot\pi_i = \frac{i\lambda}{(i+1)\mu}\cdot\frac{\lambda^i}{i\mu^i} = \frac{\lambda^{i+1}}{(i+1)\mu^{i+1}}$

$\Rightarrow \pi_0 \cdot \left(1 + \sum_{i=1}^{\infty} \frac{\lambda^i}{i\mu^i}\right) = 1 \Rightarrow \underline{\underline{\pi_0 = \frac{1}{1 - \lg(1-\frac{\lambda}{\mu})}}}$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ $\qquad \pi_i = \frac{\frac{\lambda^i}{i\mu^i}}{1-\lg(1-\frac{\lambda}{\mu})}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\lg(1-x) \qquad \Rightarrow$ d.h.! $\lambda < \mu$

Transient: $\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\mu_1 \cdots \mu_k}{\lambda_1 \cdots \lambda_k} = \sum_{n=1}^{\infty} \frac{n!\,\mu^n}{(n-1)!\,\lambda^n} < \infty \Leftrightarrow \mu < \lambda$.

Megoldások

05. 31. ellenőrző

1) i) $\forall\, 0 \le s_0 < t_0 \le s_1 < t_1 \le \dots \le s_n < t_n$

$N(t_n) - N(s_n), \dots, N(t_0) - N(s_0)$ független (független növekményű)

$$\mathbb{P}(N(t+\Delta t) = N(t)) = 1 - \lambda \Delta t + o(\Delta t)$$

$$\mathbb{P}(N(t+\Delta t) = N(t)+1) = \lambda \Delta t + o(\Delta t)$$

$$\mathbb{P}(N(t+\Delta t) \geq N(t)+2) = o(\Delta t), \text{ ahol } \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0.$$

ii) ! $\tau_i \sim \exp(\lambda)$ függetlenek. $S_n := \sum_{k=1}^{n} \tau_k \quad S_0 := 0$

$$N(t) = \max\{n \geq 0 : S_n \leq t\}$$

2) Osszuk fel $[a,b]$-t $\frac{b-a}{n}$ hosszú intervallumra, $I_i^n :=$ i-dik intervallum.

$E_i :=$ i-dik int legalább 2 esemény $\quad F_i :=$ i-ben pontosan 1

~~$\mathbb{P}(N(b) - N(a) = k) = \mathbb{P}($~~

$$\mathbb{P}\Big(\underbrace{\bigcup_{i=1}^{n} E_i}_{=: Z_n}\Big) \leq \sum_{i=1}^{n} o\left(\frac{b-a}{n}\right) = n \cdot o\left(\frac{b-a}{n}\right)$$

$$\Rightarrow \mathbb{P}(N(b) - N(a) = k) = \mathbb{P}(N(b) - N(a) = k, Z) + \mathbb{P}(N(b) - N(a) = k \mid Z^c)\,\mathbb{P}(Z^c)$$

$$p_n := \mathbb{P}(F_j \mid Z^c) = \frac{\mathbb{P}(F_j)}{\mathbb{P}(E_j^c)} = \frac{\lambda \cdot \frac{b-a}{n} + o\left(\frac{b-a}{n}\right)}{1 - o\left(\frac{b-a}{n}\right)} \Rightarrow n \cdot p_n \to \lambda(b-a)$$

$$\Rightarrow \mathbb{P}(N(b) - N(a) = k \mid Z_n^c) = \binom{n}{k} \cdot p_n^k \cdot (1-p_n)^{n-k} \longrightarrow \frac{(\lambda(b-a))^k}{k!} e^{-\lambda(b-a)}$$

$\mathbb{P}(Z_n^c) \to 1 \quad n \to \infty.$

3) a) $\mathbb{N}$ idő S állapottér

~~$\forall n \geq 1\ \forall\ t_0 \leq t_1 \leq t_2 \dots \leq t_n$~~ $\forall n \geq 1$ $\forall\ x_0, \dots x_n \in S$

$$\cancel{\mathbb{P}(X_n = x}\quad \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1} \dots X_0 = x_0) = \mathbb{P}(X_n = x_n \mid X_n = x_{n-1})$$

b) $y \in S$ $T_y := \inf\{n \geq 1 : X_n = y\}$

$\mathbb{P}_y(T_y < \infty) = 1$ rek. $\mathbb{P}_y(T_y < \infty) < 1$ tranziens

c) $y \in S$ rekurrens $\Leftrightarrow \sum_{n=1}^{\infty} \mathbb{P}(X_n = y \mid X_0 = y) < \infty$

d) ! ~~x rek. x → y ⇒ y rek~~

$\rho_{xy} := \mathbb{P}_x(T_y < \infty)$

Lem: $\rho_{xy} > 0$ & $\rho_{yx} < 1 \Rightarrow x$ tranziens.

$$\mathbb{P}_x(T_x = \infty) = \mathbb{P}(T_x = \infty \mid X_1^{k} = y) \cdot \mathbb{P}(X_1^{k} = y \mid X_0 = x) = \overset{1-\rho_{yx}}{} \cdot \underset{>0}{\rho_{yx}}$$

$\rho_{xy} > 0 \Rightarrow \mathbb{P}(X_0 = x, X_1 = x_1, \dots X_k = y) > 0$

Lem: x rek. & $x \to y \Rightarrow y$ rekurrens.

$$\Rightarrow \rho_{xy} > 0 \Rightarrow \rho_{yx} = 1 \Rightarrow \exists (n), m \geq 1\ P_{xy}^{(n)} > 0\ P_{yx}^{(m)} > 0$$

$$\sum_{k=1}^{\infty} P_{yy}^{(k)} \geq P_{yx}^{(n)} \left(\sum_{k=1}^{\infty} P_{xx}^{(k)}\right) P_{xy}^{(m)} = \infty$$

$$\mathbb{E}_x N(y) = \sum_{n=1}^{\infty} P_{xy}^{(n)}$$

Lem: $\exists C$ véges & zárt ⇒ $\exists$ rekurrens áll.

Tfh. e. $\mathbb{E}_x(N(y)) < \infty$, ahol $N(y) = \#\{n \geq 1 : X_n = y\}$

$$\sum_{x \in C} \mathbb{E}_x N(y) = \sum_{x \in C} \sum_{n=1}^{\infty} P_{xy}^{(n)} = \sum_{n \geq 1}^{\infty} \underbrace{\sum_{x \in C} P_{xy}^{(n)}}_{1} = \infty.$$

5) a) $\int_{\Omega} |X(\omega)| \, d\mathbb{P}(\omega) < \infty \Rightarrow$ ~~$\forall Z \in \mathcal{F}$~~

$\mathbb{E}(X|\mathcal{F})$ az a $\mathcal{F}$ mérhető Z v.v.-inf. -t, hogy

$\forall A \in \mathcal{F}$ esetén $\int_A X \, d\mathbb{P} = \int_A Z \, d\mathbb{P}$.

b) $$\mathbb{E}(X|\mathcal{F}) = \sum_i \frac{\mathbb{E}(X \cdot \mathbb{1}_{\Omega_i})}{\mathbb{P}(\Omega_i)} \cdot \mathbb{1}_{\Omega_i}$$

6) ! $X_1, \ldots X_n$ ... f.a.e. $N(t) = \max\{n \geq 0 : S_n \leq t\}$

$$S_n = \sum_{k=1}^{n} X_k \qquad \frac{N(t)}{t} \to \frac{1}{\mathbb{E}X_i} \text{ m.b.}$$

7) ~~$Q_{i,i+1} \pi_{i+1} = Q_{i+1,i} \pi_i$, akkor~~

$Q_{i,j} \pi_j = Q_{j,i} \pi_i$, akkor Q a ... gen.

$\pi^T P_t \equiv \pi^T \;\; \forall t \geq 0$ -ra

8) $\pi^T P_t = \pi^T \;\; \forall t \geq 0 \Rightarrow \Big/ \frac{d}{dt}$ Kolmogorov - ~~előre~~ hátra

$\pi^T Q = \pi^T P_t Q = 0^T \Rightarrow t=0$ -ra $\pi^T Q = 0^T$.

$$\pi^T Q = 0^T \Rightarrow \pi^T Q P_t = 0 \Rightarrow \pi^T \frac{d}{dt} P_t = 0^T$$

$$\frac{d}{dt} \pi^T P_t = 0^T$$

$$\Rightarrow \pi^T P_t = \text{const}^T$$

$$\pi^T P_0 = \pi^T \Rightarrow \square.$$