Matematika A2

1. vizsga - megoldások

①
$\sum_{n=0}^{\infty} (-1)^n a_n \quad (a_n \geq 0)$ alternáló sor konvergens, ha a_n monoton csökkenő és $\lim_{n\to\infty} a_n = 0$.

Biz: lásd jegyzet

②
a, lin. függetlenség: $\lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2 + \ldots + \lambda_n \underline{v}_n = \underline{0}$

$\Rightarrow \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$

b, generátorrendszer: $\text{lin}\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\} = V$, azaz

$\forall \underline{v} \in V$ előáll $\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_n \underline{v}_n$ alakban

c, bázis: $\forall v \in V$ egyértelműen áll elő $\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \ldots + \alpha_n \underline{v}_n$ alakban

vagy: lineárisan független és generátorrendszer

③
$f'_x(x_0, y_0) = \lim_{x \to y_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$

$f'_y(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$

$\text{grad} f(x_0, y_0) = \begin{bmatrix} f'_x(x_0, y_0) \\ f'_y(x_0, y_0) \end{bmatrix}$

$f'_v(x_0, y_0) = \text{grad} f(x_0, y_0) \cdot \underline{v} \quad (\text{ha } \|\underline{v}\| = 1)$

max növekedés iránya: $\text{grad} f(x_0, y_0)$, hiszen ha $\underline{v} \upuparrows \text{grad} f$ irányú

$f'_v(x_0, y_0) = |\text{grad} f| \cdot \underbrace{|v|}_{=1} \cdot \cos\gamma$; $-1 \leq \cos\gamma \leq 1$

$\Rightarrow$ maximális, ha $\cos\gamma = 1 \Rightarrow \gamma = 0$

4.

$$A = \begin{pmatrix} 3 & -1 & 5 & 0 \\ 4 & -1 & 6 & 1 \\ 0 & 1 & -2 & 3 \end{pmatrix} \overset{S_1 \cdot \frac{1}{3}}{\sim} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{5}{3} & 0 \\ 4 & -1 & 6 & 1 \\ 0 & 1 & -2 & 3 \end{pmatrix} \overset{S_2 - 4S_1}{\sim}$$

$$\begin{pmatrix} 1 & -\frac{1}{3} & \frac{5}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \\ 0 & 1 & -2 & 3 \end{pmatrix} \overset{S_2 \cdot 3}{\sim} \begin{pmatrix} 1 & -\frac{1}{3} & \frac{5}{3} & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{pmatrix} \overset{S_3 - S_2}{\sim}$$

$$\begin{pmatrix} 1 & -\frac{1}{3} & \frac{5}{3} & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rang}(A) = 2$$

$\underline{v}_1 = (3, 4, 0) \qquad \underline{v}_2 = (-1, -1, 1)$

$\underline{u}_1 = \underline{v}_1 = (3, 4, 0)$

$$\underline{u}_2 = \underline{v}_2 - \frac{\langle \underline{v}_2, \underline{u}_1 \rangle}{\|\underline{u}_1\|^2} \cdot \underline{u}_1 = (-1, -1, 1) - \frac{-7}{25} \cdot (3, 4, 0)$$

$$= \left(-\frac{4}{25}, \frac{3}{25}, 1\right)$$

$$\underline{u}_1^* = \frac{\underline{u}_1}{\|\underline{u}_1\|} = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$

$$\underline{u}_2^* = \frac{\underline{u}_2}{\|\underline{u}_2\|} = \ldots \approx (0.156, 0.117, 0.98)$$

$$\|\underline{u}_2\| = \sqrt{\left(\frac{-4}{25}\right)^2 + \left(\frac{3}{25}\right)^2 + 1^2} \approx 1.02$$

5.

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ -1 & 2 & 0 & 5 \\ 2 & -1 & a & b \end{array}\right] \underset{S_3-2S_1}{\overset{S_2+S_1}{\sim}} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & a+4 & b+4 \end{array}\right] \overset{S_3-S_2}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & a+6 & b+1 \end{array}\right]$$

- nincs megoldás, ha $a+6=0$ és $b+1 \neq 0$
  $a=-6$ és $b \neq -1$
- végtelen sok megoldás, ha $a=-6$ és $b=-1$
- pontosan egy megoldás, ha $a \neq -6$.

6.

$$\begin{vmatrix} 7-\lambda & 4 & 0 \\ -9 & -3-\lambda & 5 \\ 3 & 0 & -\lambda-4 \end{vmatrix} = \dots = \lambda(\lambda^2-1) = 0$$

$\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 1$

$$v_1 = \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 10 \\ -15 \\ 6 \end{bmatrix}$$

$$C = \left[\begin{array}{c|c|c} v_1 & v_2 & v_3 \end{array}\right]$$

$$A^{100} = C \cdot \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \cdot C^{-1} = \dots$$

7.

$f(x,y) = (x^2 - x - 2)(y^2 - 4) - 3$

$f'_x = (2x-1)(y^2-4) = 0 \quad \Rightarrow x = \frac{1}{2} \vee y = \pm 2$

$f'_y = (x^2 - x - 2)(2y) = 0 \quad \Rightarrow y = 0 \vee x = -1 \vee x = 2$

$\Rightarrow$ lehetséges szélsőértékhelyek:

$\left(\frac{1}{2}, 0\right), (-1, 2) \; (-1,-2), (2,2) \; (2,-2)$

$$D(x,y) = \begin{vmatrix} 2(y^2-4) & (2x-1)2y \\ (2x-1)2y & 2(x^2-x-2 \end{vmatrix}$$

$D\left(\frac{1}{2}, 0\right) = \begin{vmatrix} -8 & 0 \\ 0 & -4{,}5 \end{vmatrix} = 36 > 0 \Rightarrow$ sz. é.

és $-8 < 0 \Rightarrow$ max

$D(-1,2) = D(-1,-2) = D(2,2) = D(2,-2) = -144 < 0$

$\Rightarrow$ nyeregpontok

8.

$$\int_0^1 \int_{y^3}^1 y^2 \cos y^3 \, dx \, dy = \int_0^1 \int_0^{x^{1/3}} y^2 \cos y^3 \, dy \, dx =$$

$$= \int_0^1 \left[\frac{1}{3} \sin y^3\right]_0^{x^{1/3}} dx = \int_0^1 \frac{1}{3} \sin x - \frac{1}{3} \sin 0 \, dx$$

$$= \left[-\frac{1}{3} \cos x\right]_0^1 = -\frac{1}{3}(\cos 1 - 1)$$

9. hengerkoordinátákkal:

$$D = \{(r, l, z) \mid 0 \le r \le 1,\ 0 \le l \le 2\pi,\ 0 \le z \le 1\}$$

$$\int_0^1 \int_0^{2\pi} \int_0^1 4z \sin(r^2+1)\, r \, dr \, dl \, dz = \int_0^1 \int_0^{2\pi} \left[2z(-\cos(r^2+1)) \right]_{r=0}^{1} dl\, dz =$$

$$= \int_0^1 \int_0^{2\pi} -2z\cos 2 + 2z \cos 1 = \int_0^1 2\pi (2z\cos 1 - 2z\cos 2)\, dz =$$

$$= \left[\frac{z^2}{2} (4\pi(\cos 1 - \cos 2)) \right]_0^1 = 2\pi(\cos 1 - \cos 2).$$