A2 vizsga jan. 10.

(1.) a) $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ skaláris szorzat, ha:

- $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle \quad \forall \underline{u}, \underline{v} \in V$ 1
- $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle \quad \forall \underline{u}, \underline{v}, \underline{w} \in V$ 1
- $\langle \alpha \underline{u}, \underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle \quad \forall \alpha \in \mathbb{R}, \underline{u}, \underline{v} \in V$ 1
- $\langle \underline{u}, \underline{u} \rangle \geq 0$ és $\langle \underline{u}, \underline{u} \rangle = 0$ akkor és csak akkor, ha $\underline{u} = \underline{0}$ 1

b) $\underline{v} \in V$ hossza: $\| \underline{v} \| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$ 1

c) $\cos \alpha = \dfrac{\langle \underline{u}, \underline{v} \rangle}{\| \underline{u} \| \cdot \| \underline{v} \|}$ 1

d) $| \langle \underline{v}, \underline{w} \rangle | \leq \| \underline{v} \| \, \| \underline{w} \|$

ha $\underline{w} = \underline{0}$, akkor mindkét oldal 0. 0,5

tfh. $\underline{w} \neq 0$, akkor $\forall t \in \mathbb{R}$ esetén

$0 \leq \| \underline{v} - t\underline{w} \|^2 = \langle \underline{v} - t\underline{w}, \underline{v} - t\underline{w} \rangle =$ 1

$= \langle v, v \rangle - 2t \langle \underline{w}, \underline{v} \rangle + t^2 \langle \underline{w}, \underline{w} \rangle$ 1

mivel ez minden $t \in \mathbb{R}$ esetén igaz, ezért a diszkrimináns nem pozitív:

$4 \langle \underline{u}, \underline{w} \rangle^2 - 4 \| \underline{u} \|^2 \cdot \| \underline{w} \|^2 \leq 0$ 1

$\langle \underline{u}, \underline{w} \rangle^2 \leq \| \underline{u} \|^2 \cdot \| \underline{v} \|^2$

$\langle \underline{u}, \underline{w} \rangle \leq \| \underline{u} \| \cdot \| \underline{v} \|$ 0,5

(2.) $f_x'(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$ 1

$f_y'(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$ 1

Young - tétel

Ha $f(x, y)$ és parc. deriváltjai f_x', f_y', f_{yx}'', f_{xy}'' léteznek 1 egy olyan nyílt tartományon, ami tartalmazza az (x_0, y_0) pontot és itt valamennyi folytonos, 1 akkor $f_{yx}''(x_0, y_0) = f_{xy}''(x_0, y_0)$ 1

(vagy elsőrendű parc. deriváltak léteznek (x_0, y_0) környezetében és (x_0, y_0)-ban totálisan diffható.

(3.) Parseval - formula

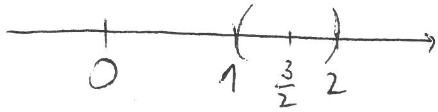
Ha a 2π szerint periodikus $f(x)$ Riemann-integrálható a $[0, 2\pi]$ intervallumban, akkor

$$\int_0^{2\pi} (f(x))^2 \, dx = 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$ 3

$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \, dx$ 2

(4.) $\sum \frac{(2x-3)^n}{n\lg n} = \sum \frac{2^n\left(x-\frac{3}{2}\right)^n}{n\lg n}$ 1

$R = \frac{1}{\lim \sqrt[n]{\frac{2^n}{n\lg n}}} = \lim \sqrt[n]{\frac{n\lg n}{2^n}} = \frac{1}{2}$ 1 1



$\boxed{x=1}$ $\sum_{k=2}^{\infty} \frac{(-1)^n}{n\lg n} \Rightarrow$ Leibniz $\left|\frac{1}{n\lg n}\right| \underset{\infty}{\overset{n}{\searrow}} 0$ konv. 2

$\boxed{x=2}$ $\sum_{k=2}^{\infty} \frac{1}{n\lg n} \Rightarrow$ int krit $\int_2^{\infty} \frac{1}{x\lg x}\,dx = \cancel{\ln(x\ln x)}\ \left[\ln\ln x\right]_2^{\infty} = \infty$ 2

div

konv int : $[1, 2)$

(5.) $a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx = \frac{1}{2\pi}\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{2\pi} = \frac{1}{\pi}$ 2 $\qquad \forall k\ b_k = 0$ mert páros 1

$a_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\cos(kx)\,dx = (*)$

1 1

$\int \underset{g'}{\cos x}\ \underset{f}{\cos kx} = \cos kx\cdot\sin x \mp k\int \underset{g'}{\sin x}\cdot\underset{f}{\sin(kx)} = \cos kx\,\sin x \mp k(-\cos x)\sin kx \pm k^2\int \cos x\cos kx$

$g=\sin x \quad f' = \pm\sin(x)\,k \qquad g=-\cos x \quad f' = -\cos kx\cdot k$

$\int \cos x\cos kx = \frac{1}{1 \mp k^2}\left(\cos kx\ \sin x + k\cos x\ \sin kx\right)$

$(*) = \frac{1}{\pi}\frac{1}{1+k^2}\left[\cos kx\sin x + \underbrace{k\cos x}_{\substack{\uparrow \\ 0\text{ lesz}}}\ \sin kx\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\pi}\frac{1}{1+k^2}\left(\cos\frac{k\pi}{2} + \cos\frac{k\pi}{2}\right) = \frac{2\cos\frac{k\pi}{2}}{\pi(k^2 \mp 1)} = a_k$ 1

Fourier sor: $\frac{1}{\pi} + \sum_{k=1}^{\infty} \frac{2\cos\left(\frac{k\pi}{2}\right)}{\pi(k^2 \mp 1)}\cos(kx)$

6.

a) $\left(\begin{array}{ccc|ccc} 2 & 4 & 2 & 1 & 0 & 0 \\ -6 & 3 & 0 & 0 & 1 & 0 \\ 3 & -3 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 2 & 4 & 2 & 1 & 0 & 0 \\ 0 & 15 & 4 & 3 & 1 & 0 \\ 0 & -5 & 2 & -\frac{1}{2} & 0 & 1 \end{array}\right) \sim$ 1

$②+3①$
$③-\frac{4}{2}①$

~~②~~ $③+\frac{1}{3}②$

$\sim \left(\begin{array}{ccc|ccc} & & & & & \\ 0 & 15 & 4 & 3 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{3} & 1 \end{array}\right) /\cdot\frac{3}{2} \sim \left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 1 & \frac{3}{4} & \frac{1}{2} & \frac{3}{2} \end{array}\right) \ldots$

$\frac{1}{108}\begin{pmatrix} -9 & +18 & 27 \\ 18 & 0 & +18 \\ 6 & -12 & 18 \end{pmatrix}^T = \frac{1}{108}\begin{pmatrix} -9 & -18 & 6 \\ +18 & 0 & -12 \\ 27 & +18 & 18 \end{pmatrix}$ 3

b) 3 1

c) det számos: $-18+2\cdot18-(-18-4\cdot18)=18+5\cdot18=6\cdot18=108$ 2

f) $f(x,y)=x\cdot e^{-2(x^2+y^2)}$

$\frac{\partial}{\partial x}f=(1-4x^2)e^{-2(x^2+y^2)}$ 1

$\frac{\partial^2}{\partial x^2}f=\underbrace{\left(-8x-4x(1-4x^2)\right)}_{16x^3-12x}e^{-2(x^2+y^2)}$ 0,5

$\frac{\partial}{\partial y}f=-4xye^{-2(x^2+y^2)}$ 1

$\frac{\partial^2}{\partial y^2}f=\underbrace{\left(-4x-4y(-4xy)\right)}_{-4x+16xy^2}e^{-2(x^2+y^2)}$ 0,5

krit. pontok: $(1-4x^2)=0 \Rightarrow x=\pm\frac{1}{2}$

1 $P_1\left(\frac{1}{2},0\right)$ és $P_2\left(-\frac{1}{2},0\right)$

$\frac{\partial^2}{\partial x\partial y}f=\cancel{(-4y+16x^2y)e^{-2(x^2+y^2)}}$ 0,5

$(-4y+16x^2y)e^{-2(x^2+y^2)}$

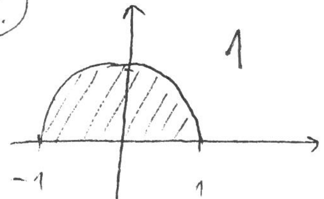
$\begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} \\ \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} -4e^{-\frac{1}{2}} & 0 \\ 0 & -2e^{-\frac{1}{2}} \end{pmatrix} \rightarrow \det = 8e^{-\frac{1}{2}} > 0 \Rightarrow$ szélső ért. $\frac{\partial^2}{\partial x^2}<0$ ezért max. hely

0,5 0,5

értéke: $\frac{1}{2\sqrt{e}}$

$P_2\begin{pmatrix} 4e^{-\frac{1}{2}} & 0 \\ 0 & 8e^{-\frac{1}{2}} \end{pmatrix} \rightarrow \det>0 \Rightarrow \frac{\partial^2}{\partial x^2}>0 \Rightarrow$ min hely. értéke: $-\frac{1}{2\sqrt{e}}$ 0,5

~~7~~ (8.)



1

$r \in [0, 1]$

$\varphi \in [0, \pi]$

$f = \sqrt{1+r^2}$

Jacobi!!

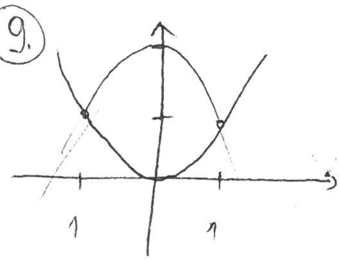
$$\int_0^1 \int_0^\pi r \cdot \sqrt{1+r^2}\, d\varphi\, dr = \pi \cdot \frac{1}{2} \int_0^1 2r(1+r^2)^{\frac{1}{2}} dr = \pi \cdot \frac{1}{2} \left[\frac{(1+r)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 =$$

2 1 1 2

$$= \left[\frac{\pi}{3} (1+r^2)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{3} \left(2^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) = \frac{\pi}{3}(\sqrt{8} - 1)$$

1

(9.)



henger koordináták

$r \in [0, 1]$ $\quad z \in [r^2, 2-r^2]$

$\varphi \in [0, 2\pi]$

2

$$\int_0^1 \int_0^{2\pi} \int_{r^2}^{2-r^2} r\, dz\, d\varphi\, dr = \int_0^1 \int_0^{2\pi} [r \cdot z]_{r^2}^{2-r^2} d\varphi\, dr = \int_0^1 \int_0^{2\pi} r(2-r^2-r^2)\, d\varphi\, dr =$$

3

$$= 2\pi \int_0^1 2r - 2r^3\, dr = 4\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = 4\pi \cdot \frac{1}{4} = \pi$$

súlypont: homogenitás és szimmetria miatt $(0, 0, 1)$

2