! $a_1 \dots a_n \in \mathbb{R}$; $b \in \mathbb{R}$ or $\in \mathbb{C}$

$\Rightarrow$ the equation of the form $a_1 \cdot x_1 + \dots a_n x_n = b$ where x_i are the "unknown" linear equations

Linear equation system:

$a_{11} x_1 + \dots a_{1n} x_n = b_1$

$a_{21} x_1 + \dots a_{2n} x_n = b_2$

$a_{m1} x_1 + \dots a_{mn} x_n = b_m$

n - variable - unknown, m - equations.

Main question: How to solve this equation?

⊗ Soto Gauss - Elimination — page 6 - 7.

Vector space: Roughly: ~~a vectorspace~~ we call V a vectorspace

if 1) $x, y \in V \Rightarrow \underline{x} + \underline{y} \in V$

2) $\forall a \in \mathbb{R}$ $a \cdot \underline{x} \in V$ $a \cdot (\underline{x} + \underline{y}) = a\underline{x} + b\underline{y}$

3) $(a+b) \cdot \underline{x} = a \cdot \underline{x} + b\underline{y}$

4) $\exists! \, \underline{0}$ s.t. $\underline{v} + \underline{0} = \underline{0} + \underline{v} = \underline{v}$

5) $1 \cdot \underline{v} = \underline{v}$ 7) $\forall v \, \exists! w$ $v + w = ~~w~~ + ~~v~~ = \underline{0}$

Example: ~~ordered~~ $\mathbb{R}^n \rightarrow$ ~~ordered number~~

$\mathbb{R}^n = \{(x_1, x_2, \dots x_n) : x_i \in \mathbb{R} \text{ \& } 1 \leq i \leq n\}$

sum of vectors $\underline{x} = (x_1, \dots, x_n)$ $\Rightarrow$ $\underline{x} + \underline{y} = (x_1 + y_1, \dots, x_n + y_n)$.

$\underline{y} = (y_1 \dots y_n)$

$a \cdot \underline{x} = (ax_1, \dots ax_n)$.

Linear combination

! $\underline{x}_1, \dots, \underline{x}_n \in V$; $a_1, \dots, a_n \in \mathbb{R}$

we call the vector ~~$a_1\underline{x}_1 + \dots + \underline{x}_n$~~ $a_1\underline{x}_1 + \dots + a_n\underline{x}_n$ the linear combination of vectors.

$a_1, \dots, a_n$ are called the coefficients.

Example

$$x_1 - x_2 = -1$$
$$5x_1 + 2x_2 = 16$$

$\Rightarrow$ $\underline{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} -1 \\ 16 \end{pmatrix}$

$$\Leftrightarrow \quad x_1 \cdot \underline{v}_1 + x_2 \underline{v}_2 = \underline{b}$$

$$x_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 16 \end{pmatrix}.$$

$\Rightarrow$ Solving a linear equation system $\Leftrightarrow$ finding linear combination which ~~sets~~ equals to the desired vector.

~~Matrix form: Useful tool: matrices.~~

~~$\underline{\underline{A}} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$ We call $\underline{\underline{A}}$ an $n \times m$ matrix if it is a table of entries with containing n-rows & m-colons.~~

~~Sum: $\underline{\underline{A}} + \underline{\underline{B}} = (a_{ij} + b_{ij})$ $\Rightarrow$ $n \times m + n \times m$ valid.~~

~~$\underline{\underline{A}} \cdot \underline{\underline{B}} = \left(\sum_{j=1}^{m} a_{ij} b_{jk} \right)$ $n \times m \cdot m \times k$ valid. ("row times colon-" ")~~

Leave it in this form!

$\Rightarrow$ $a_{11}x_1 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + \dots + a_{2n}x_n = b_2$

$\vdots$

$a_{m1}x_1 + \dots \; a_{mn}x_n = b_m$

$$\underline{u}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \dots, \underline{u}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{K}^m \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{K}^m$$

$\Rightarrow x_1 \cdot \underline{u}_1 + \dots + x_n \cdot \underline{u}_m = \underline{b}$

When can it be solved?

<u>def</u>: We call W a <u>subspace</u> of V if.

$\forall \underline{a}_1 \dots a_n \in W \; \forall \lambda_1 \dots \lambda_n \in \mathbb{K}$

$\Rightarrow \lambda_1 \underline{a}_1 + \dots + \lambda_n \underline{a}_n \in W$.

That is, all linear combinations of every vector in W contained in W.

<u>th</u>: $W \subset V$ is subspace $\Leftrightarrow \forall \underline{v}, \underline{w} \in W \Rightarrow \underline{v} + \underline{w} \in W$

$\forall \underline{v} \in W$ & $a \in \mathbb{K} \Rightarrow a \cdot \underline{v} \in W$.

<u>Ex</u>: $\{\underline{0}\}$ $\quad \mathbb{K}^n \quad \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{K} \right\} \subset \mathbb{K}^3$

~~Observe~~

<u>def</u>: Let $\underline{u}_1 \dots \underline{u}_n \in V$. We call the subspace W ~~span$\{\underline{u}_1 \dots \underline{u}_n\}$~~

<u>spanned by</u> $\underline{u}_1 \dots \underline{u}_n$ if ~~$\forall \lambda_1 \dots \lambda_n$~~ if it is the smallest subspace

containing every linear combination of $\underline{u}_1 \dots \underline{u}_n$.

ie. $\forall W'$ s.t. contains it, $W \subseteq W'$ subspace.

$\Rightarrow W = \{\lambda_1 \underline{u}_1 + \dots \lambda_n \underline{u}_n : \lambda_1 \dots \lambda_n \in \mathbb{K}\}$

notation: $\text{span}(\underline{u}_1 \dots u_n)$.

$$2x_1 - 3x_2 + 4x_3 + 5x_4 = 4$$
$$x_1 + x_3 - x_4 = 1$$
$$x_2 - x_3 = 5$$

What can we do?
- sum equations
- multiply with numbers (non zero!)

linear eq.

Coefficient matrix:

$$\begin{bmatrix} 2 & -3 & 4 & 5 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Extended coeff. matrix:
$$\left[\begin{array}{cccc|c} 2 & -3 & 4 & 5 & 4 \\ 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 5 \end{array}\right]$$

change 1 & 2.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & -3 & 2 & 7 & 2 \\ 0 & 1 & -1 & 0 & 5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 5 \\ 0 & -3 & 2 & 7 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & -1 & 7 & 17 \end{array}\right] \Rightarrow +x_3 = \underline{17 + 7x_4}$$

$$x_2 = x_3 + 5 = \underline{\underline{7x_4 + 22}}$$

$$x_1 = -x_3 + x_4 + 1 = \underline{\underline{-6x_4 - 16}}$$

$\Rightarrow x_4$ free

not unique solution.

def: Elementary row manipulations:

a) multiply a row with non-zero value
b) change two rows
c) add a scalar multiplier to another row.

def: We say that the extended coefficient matrix is in a row-echelon form

if) a) the rows containing of 0 are the last rows of the matrix.
b) if a row contains non-zero element then the first non-zero element is 1.
c) in two consequtive non-zero rows, the 1 in the first one comes earlier, ie the left.

Example:

$$\begin{bmatrix} 1 & 4 & 3 & 2 & 9 & 11 \\ 0 & 1 & 2 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Algorithm: a) If the first column contains of 0 ⇒ leave it (it is a free variable)

b) go to first non-zero column.
1) if the first element is 0 then change with a row which has non-zero element. ($a_{11} \neq 0$)
2) divide the row with the first element.
3) substract from the other rows the multiplier of the first row to get 0. $(\cdot a_{i1} \cdot)$
4) goto 2 row and column.

~~t~~ — exp.   lin. alg. — 6 — | Lecture |

→ basis, basis trafo, eigenvalues, eigenvectors, determinant.

Let V be a vectorspace & $\underline{u}_1, \dots \underline{u}_n \in V$

$\Rightarrow \lambda_1 \dots \lambda_n \in \mathbb{R} \Rightarrow \lambda_1 \underline{u}_1 + \dots + \lambda_n \underline{u}_n$ lin. combination

$\text{span}\{\underline{u}_1, \dots \underline{u}_n\} = \{\lambda_1 \underline{u}_1 + \dots + \lambda_n \underline{u}_n \,\&\, \lambda_i \in \mathbb{R}\ 1 \le i \le n\}$.

<u>def</u>: We call $\underline{u}_1, \dots \underline{u}_n$ <u>lin. ~~dep~~ independent</u>, if.

$\lambda_1 \underline{u}_1 + \dots + \lambda_n \underline{u}_n = \underline{0} \Leftrightarrow \lambda_1 = \dots = \lambda_n = 0$.

<u>dependent</u>: otherwise, so if lin. dependent $\Rightarrow$

$\exists i$ s.t. $\underline{u}_i$ ~~te~~ is a lin. combination of the others.

<u>def</u>: We call $\{\underline{u}_1, \dots \underline{u}_n\}$ <u>generator</u> of V if

$\forall \underline{v} \in V \Rightarrow \exists \lambda_1 \dots \lambda_n$ s.t. $\lambda_1 \underline{u}_1 + \dots + \lambda_n \underline{u}_n = \underline{v}$.

<u>def</u>: We call $\{\underline{u}_1, \dots \underline{u}_n\}$ basis of V if.

→ lin. independent

→ generator of V.

<u>Th</u>: ~~$\forall \underline{v} \in V$~~ if $\{\underline{u}_1, \dots \underline{u}_n\}$ a basis of V

& $\{\underline{v}_1, \dots \underline{v}_m\}$ are

$\Rightarrow n = m$. We call it the dim of V. notation dim V.

<u>Th</u>: if $B = \{\underline{u}_1, \dots \underline{u}_n\}$ basis of V, then

for any $\underline{v} \in V\ \exists!\ \lambda_1 \dots \lambda_n$ s.t. $\underline{v} = \lambda_1 \underline{u}_1 + \dots + \lambda_n \underline{u}_n$

<u>Notation</u>: $[\underline{v}]_B = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$.

Change basis: let $B = \{u_1 \dots u_n\}$ & $B' = \{v_1 \dots v_n\}$.

$$[u]_B = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \quad [u]_{B'} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad [v_i]_B = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{in} \end{bmatrix}.$$

$$u = \alpha_1 v_1 + \dots \alpha_n v_n = \alpha_1(a_{11} u_1 + \dots a_{1n} u_n)$$
$$\alpha_2(a_{21} u_1 + \dots + a_{2n} u_n) + \dots + \alpha_n(a_{n1} u_1 + \dots + a_{nn} u_n)$$
$$= (\alpha_1 a_{11} + \alpha_2 a_{21} + \dots \alpha_n a_{n1}) u_1 + \dots + (\alpha_1 a_{1n} + \dots \alpha_n a_{nn}) u_n$$
$$\Rightarrow \beta_i = \alpha_1 a_{1i} + \dots \alpha_n a_{ni}$$

B' element in basis B

$$\Rightarrow [u]_B = \big[[v_1]_B \dots [v_n]_B \big] [u]_{B'}$$

matrix multiplication: $A = (a_{ij})_{i,j=1}^{n,m}$ $B = (b_{jk})_{j,k=1}^{m,l} = AB = \left(\sum_{j=1}^{m} a_{ij} b_{jk} \right)_{i,k=1}^{n,l}$

Notation: $P_{B',B} = \big[[v_1]_B \dots [v_n]_B \big]$
from B' to B. Prop.: $P_{B',B} = \left(P_{B,B'}\right)^{-1}$

Linear transformations: let U, V be vector spaces.

$F: U \to V$ linear transformation if
$\forall v, w \in U \Rightarrow F(v + w) = F(v) + F(w)$
$\forall \alpha \in \mathbb{R}, v \in U \Rightarrow F(\alpha v) = \alpha F(v)$.

Ex: ⇒ Rotation 30° with plane.
⇒ translation is not.
⇒ multiply with Mx in $\mathbb{R}^n$.

Lin. transformation on basis

Goal: write the lin. trans. as a matrix

$F: V \to W$ lin. trans.

$B = \{\underline{u}_1, \dots \underline{u}_n\} \subset V$ $B' = \{\underline{v}_1, \dots \underline{v}_n\} \subset W$ basis.

$$[F(\underline{b})]_{B'} = [F(\alpha_1 \underline{u}_1 + \dots \alpha_n \underline{u}_n)]_{B} =$$
$$= \alpha_1 [F(\underline{u}_1)]_B + \dots + \alpha_n [F(\underline{u}_n)]_B$$
$$\Rightarrow [F(\underline{b})]_{B'} = [[F(\underline{u}_1)]_{B'} \dots F(\underline{u}_n)]_{B'}] [\underline{b}]_B.$$

Change basis: $F: V \to V$ lin. tr.

$B = \{\underline{u}_1, \dots \underline{u}_n\}$ $B' = \{\underline{v}_1, \dots \underline{v}_n\}$

$[F]_B = [[F(\underline{u}_1)]_B \dots [F(\underline{u}_n)]_B]$ known.

$[F]_{B'} = ?$

$$[F(\underline{b})]_B = [F]_B [\underline{b}]_B = [F]_B \underbrace{[[\underline{v}_1]_B \dots [\underline{v}_n]_B]}_{P_{B',B}} [\underline{b}]_{B'}$$

$$\| $$

$$P_{B',B} [F(\underline{b})]_{B'} \Rightarrow$$

$$[F]_{B'} = (P_{B',B})^{-1} [F_B] P_{B',B}.$$

Linear equations revisited:

$\alpha_1 \underline{u}_1 + \dots \alpha_n \underline{u}_n = \underline{b}$

$\Rightarrow \underline{\underline{A}} = [\underline{u}_1 \dots \underline{u}_n] \Rightarrow \underline{\underline{A}}\, \underline{x} = \underline{b}$

i.e. a lin. equation is looking for $\underline{x}$ s.t. its image is $\underline{b}$

i.e.: "$\underline{x} = A^{-1}\underline{b}$" So when does the inverse exists? [inverse → x]

Determinant: $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ Is it definable? note of $n \times n$ & $n \neq m$

the concept of speaking about determinant $n<m$ ⇒ not compute / is over
$n > m$ ⇒ there are everything.

M_{ij} = matrix $(n-1)\times(n-1)$ by removing row i & column j.

$C_{ij} = (-1)^{i+j} \det M_{ij}$

⇒ $\det A = a_{i1}C_{i1} + \dots a_{in}C_{in}$ $\forall i \leq n$.

Th: expansion Thm. (we use it to define).

Th (cross expansion thm) if $a_{i1}C_{j1} + \dots a_{in}C_{jn} = 0$ for $i \neq j$.

for 2×2 $\det A = ad - bc$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so [3×3 det = \\\ − ///]

Thm: Properties: $\det(AB) = \det(A)\cdot\det(B)$.
$\det(A^T) = \det(A)$.

def: inverse: $BA = AB = I$. (since $IA = AI = A$).
$\det A \neq 0 \Leftrightarrow \exists! A^{-1}$ $\det A = \frac{1}{\det A^{-1}}$.

Eigenvalues:

def: ! A $n \times n$ real/~~complex~~ m.
! $\underline{x} \neq \underline{0}$ we call $\underline{x}$ ~~the~~ an eigenvector of A
if $\exists \lambda \in \mathbb{R}$ st. $\underline{A}\underline{x} = \lambda \underline{x}$. We call λ eigenvalue of A.

Suppose that $A \in \mathbb{R}^{n \times n}$ has n indep. eigenvects.

$\Rightarrow$ $\mathcal{B} = \{\underline{x}_1 \dots \underline{x}_n\}$ basis of $\mathbb{R}^n$.

$\Rightarrow$ $[A]$ def. lin. transform F on $\mathbb{R}^n$

& $[F]_T = A = [\underline{u}_1 \dots \underline{u}_n]$

but $F\underline{x}_i = \lambda_i \underline{x}_i$ $\qquad F(\underline{e}_i) = \underline{u}_i$

$$[F]_{\mathcal{B}} = P_{\mathcal{B},T}^{-1} [F]_T P_{\mathcal{B},T}$$ we know.

but. $[F]_{\mathcal{B}} [\underline{x}_i]_{\mathcal{B}} = [F]_{\mathcal{B}} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$= [F\underline{x}_i]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow$ $[F]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ diagonal.

$\Rightarrow$ $$A = P_{\mathcal{B},T} \, D \, (P_{\mathcal{B},T})^{-1}$$

~~Geo~~ <u>Geometric meaning of determinant.</u>

$\det(A) = \det(P D P^{-1}) = \det(P)\det(D)\det(P)^{-1}$

$= \det(D) = \lambda_1 \cdots \lambda_n$.

<u>multiply all the eigenvalues.</u>

Let $K = \{(x_1 \dots x_n) : 0 \le x_i \le 1\}$ unit cube.

$\Rightarrow$ $A(K)$ is a parallelepiped.

If A is already diagonal.

$\Rightarrow$ ~~area~~ volume $(A(K)) = |\lambda_1 \cdots \lambda_n|$ volume ~~area~~ (K).

↑ each side is multiplied

Ex. det: This was true in general. i.e.

volume $(A(K)) = |\det(A)|$ volume (K)

K any set of $\mathbb{R}^n$.

How to find eigenvalues & eigenvectors?

if $\underline{A}\underline{x} = \lambda \underline{x} \Rightarrow (A - \lambda I)\underline{x} = \underline{0}$

since $\underline{x} \neq \underline{0}$ it is possible $\Leftrightarrow \det(A - \lambda I) = 0$.

$\Rightarrow$ after solving this eq. finding x is a lin. eq.

Observe: $\underline{x}$ eigenvector $\Rightarrow$ $2\underline{x}$ & $3\underline{x}$, $c \cdot \underline{x}$ is eigenvector

$\Rightarrow \infty$ many solutions. $\Longrightarrow$ -10+1-

Scalar product: We want to say what is length and what is angle.

$\exists!\ \langle \cdot , \cdot \rangle : V \times V \mapsto \mathbb{R}$ s.t.

a) $\langle \underline{u}, \underline{v} \rangle = \overline{\langle \underline{v}, \underline{u} \rangle}$

b) $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$

c) $\langle \alpha \underline{u}, \underline{v} \rangle = \alpha \cdot \langle \underline{u}, \underline{v} \rangle$

d) $\langle \underline{u}, \underline{u} \rangle \geq 0$ & $\langle \underline{u}, \underline{u} \rangle = 0 \Leftrightarrow \underline{u} = \underline{0}$

Then: $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$ length

$$\measuredangle (\underline{u}, \underline{v}) = \arccos\left(\frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \cdot \|\underline{v}\|}\right).$$

If $\langle \underline{u}, \underline{v} \rangle = 0$ then $\underline{u} \perp \underline{v}$

Ex: $\underline{u} = (u_1, \dots u_n) \in \mathbb{R}^n \Rightarrow \langle \underline{u}, \underline{v} \rangle = \underline{u}^T \underline{v} = u_1 v_1 + \dots u_n v_n$.

Orthonormal matrices: We call $Q \in \mathbb{R}^{n\times n}$ orthonormal if $Q^TQ = I$.

That is, $Q^T = Q^{-1}$. $\iff$ for every column $q_i \perp q_j$ & $\|q_i\|=1$.

Thm: 1) $\|Q\underline{x}\| = \|\underline{x}\|$ (length preserving)

2) $\langle Q\underline{x}, Q\underline{y}\rangle = \langle \underline{x}, \underline{y}\rangle$ (angle preserving)

3) $\det(Q) = 1$ (area preserving)

Symmetric matrices: def: $\boxed{A^T = A}$ $\in \mathbb{C}^{n\times n}$

Thm If A is symmetric then any eigenvalue is real.

$$\lambda\langle \underline{x}, \underline{x}\rangle = \langle A\underline{x}, \underline{x}\rangle = \langle \underline{x}, A^T\underline{x}\rangle = \langle \underline{x}, A\underline{x}\rangle = \bar{\lambda}\langle \underline{x}, \underline{x}\rangle$$

$\langle \underline{x}, \underline{x}\rangle > 0$. $\square$

Thm: If $\underline{x}_i, \underline{x}_j$ eigenvectors for $\lambda_i \neq \lambda_j \Rightarrow \underline{x}_i \perp \underline{x}_j$

$$\lambda_i\langle \underline{x}_i, \underline{x}_j\rangle = \langle A\underline{x}_i, \underline{x}_j\rangle = \langle \underline{x}_i, A\underline{x}_j\rangle = \lambda_j\langle \underline{x}_i, \underline{x}_j\rangle$$

$\Rightarrow \langle \underline{x}_i, \underline{x}_j\rangle = 0$. $\square$

Gram-Schmidt orthogonalization: If $B = \{\underline{u}_1 \dots \underline{u}_n\}$ basis.

$\Rightarrow \exists\ B' = \{\underline{v}_1 \dots \underline{v}_n\}$ orthogonal basis.

That is, $\|\underline{v}_i\| = 1$ & $\langle \underline{v}_i, \underline{v}_j\rangle = 0$ $\underline{v}_i \neq \underline{v}_j$.

Algorithm. $\underline{v}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|}$ $\quad \underline{v}_2' = \underline{u}_2 + c_{12}\underline{v}_1 \Rightarrow \langle \underline{v}_2', \underline{v}_1\rangle = 0$

$c_{12} = -\langle \underline{u}_2, \underline{v}_1\rangle$

$\underline{v}_2 = \frac{\underline{v}_2'}{\|\underline{v}_2'\|}$

$\underline{v}_3' = \underline{u}_3 + c_{21}\underline{v}_1 + c_{22}\underline{v}_2 \Rightarrow c_{2i} = -\langle \underline{u}_3, \underline{v}_i\rangle$

$\underline{v}_3 = \frac{\underline{v}_3'}{\|\underline{v}_3'\|}$.

~~Self fact:~~

~~$A =$~~

i.e.: $\exists$ Q orthogonal matrix & D diagonal matrix $A = QDQ^T$

s.t. ~~$A = Q^T D Q$~~ $Q^T A Q_1 = D$

i.e. A defines a linear action on $\mathbb{R}^n$ so that $\exists$ orthogonal basis in which it is coordinate multiplication.

~~Proof about ... be precise:~~

~~geometric ... of the ...~~

Trace of matrices:

! A $n \times n$ linear or complex matrix. $A = (a_{ij})_{i,j=1}^{d,d}$

$\Rightarrow \operatorname{tr}(A) = \sum_{i=1}^{d} a_{ii}$ (sum of diagonal elements

Properties:
- $\operatorname{tr}(A^T) = \operatorname{tr}(A)$
- $\operatorname{tr}(c \cdot A) = c \operatorname{tr}(A)$
- $\operatorname{tr}(A \pm B) = \operatorname{tr}(A) \pm \operatorname{tr}(B)$
- $\operatorname{tr}(A \cdot B) = \operatorname{tr}(BA)$
- $\operatorname{tr}(ABC) = \operatorname{tr}(BCA) = \operatorname{tr}(CBA)$ but $\neq \operatorname{tr}(BAC)$.
- $\operatorname{tr}(A) = \operatorname{tr}(PDP^{-1}) = \operatorname{tr}(D) = \lambda_1 + \dots + \lambda_d$

Eigenvalues.

Scalar product for matrices, i.e. "two-dot" product:

! $A, B \in \mathbb{R}^{d\times d} \Rightarrow A:B = tr(A^T B) = \sum_{i,j=1}^{d} a_{ij} b_{ij}$

it implies

$\Rightarrow tr(A^T B) = tr(B^T A)$

$\rightarrow tr((A+C)^T B) = tr(A^T B) + tr(C^T B)$

$\rightarrow tr((cA)^T B) = c \cdot tr(A^T B) \quad c \in \mathbb{R}$

$\rightarrow tr(A^T A) = \sum_{i,j=1}^{d} a_{ij}^2 \geq 0 \; \& \; = 0 \; a_{ij} = 0 \; \forall i,j$ too.

Skew-symmetric matrices:

def. We call A skew-symmetric if $A = -A^T$

Th: for any $A \in \mathbb{R}^{d\times d}$ $\exists$ S & R (symmetric, skew-symmetric matrix)

s.t. $A = S + R$

Proof: $S := \frac{A + A^T}{2}$ & $R = \frac{A - A^T}{2}$.

Gauss-Jordan elimination:

Recall: row-echelon form if
- all zero rows are the last.
- if a row is non-zero ⇒ first non-zero element is 1 (leader-one)
- if two consecutive non-zero rows, the leader-one of the first starts earlier than the second.

(that is, below a leader-one, everything is 0.)

Now: goal: create zeros above the leader 1-s.

! (8) Kvadratikus egyenletek

Ex:
$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Gauss elim.

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Where to use it? Finding basis of spanned subspace; determine the dimension,

recall: $S = \{\underline{v}_1, \dots \underline{v}_n\} \in \mathbb{R}^m$

⊗ $V = \text{Span}\{\underline{v}_1, \dots \underline{v}_n\}$

Th: If we write $A = \begin{bmatrix} \underline{v}_1^T \\ \vdots \\ \underline{v}_n^T \end{bmatrix}$, after doing elementary row transformations $\Rightarrow$ the rows span the same space V.

proof

Ex $\underline{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ $\underline{v}_4 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ -7 \end{bmatrix}$ $\underline{v}_5 = \begin{bmatrix} 5 \\ -8 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -5 & -3 & 6 \\ 0 & 1 & 3 & 0 \\ 2 & -1 & 4 & -7 \\ 5 & -8 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -13 \\ 0 & 0 & -5 & -13 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{13}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{cases} \dim V = 3 \\ B = \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{5} \end{pmatrix} \right\} \\ B = \{\underline{v}_1, \underline{v}_2, \underline{v}_4\} \end{cases}$$

Fundamental subspaces of matrices

$A \in \mathbb{R}^{k \times s}$ $\quad A = [\underline{c}_1 \dots \underline{c}_s] = \begin{bmatrix} \underline{r}_1 \\ \vdots \\ \underline{r}_k \end{bmatrix}$

$\underline{c}_i \in \mathbb{R}^k$ $\quad \underline{r}_j \in \mathbb{R}^s$

def: $\mathrm{col}(A) = \mathrm{span}\{\underline{c}_1 \dots \underline{c}_s\} \subseteq \mathbb{R}^k$

$\mathrm{row}(A) = \mathrm{span}\{\underline{r}_1 \dots \underline{r}_k\} \subseteq \mathbb{R}^s$

Th: $\dim \mathrm{col}(A) = \dim \mathrm{row}(A) =: \mathrm{rank}(A)$ (8) =) yolo p.14

$\mathrm{null}(A) = \{\underline{x} \in \mathbb{R}^s : A\underline{x} = \underline{0}\}$ $\quad \mathrm{nullity}(A) = \dim\{\mathrm{null}(A)\}$

Observe: $\mathrm{row}(A^T) = \mathrm{col}(A)$ & $\mathrm{col}(A^T) = \mathrm{row}(A)$.

These subspaces ($\mathrm{row}(A)$, $\mathrm{col}(A)$, $\mathrm{null}(A)$, $\mathrm{null}(A^T)$) are the fundamental subspaces.

Th For any $A \in \mathbb{R}^{k \times s}$ $\quad \mathrm{rank}(A) + \mathrm{nullity}(A) = s$

Proof: Since the Gauss-Jordan elimination does not change the subspaces $\mathrm{row}(A)$ & $\mathrm{null}(A)$, so ($A x = 0$ system) we can assume that it is in row-echelon form. =) no. of leading one's + free parameters = s in the solution of $A\underline{x} = \underline{0}$. but no. of leading one's = dim row(A) · no. of free param. = dim null(A).

Th: Let $W \subseteq \mathbb{R}^s$ be a subspace =) $\dim W + \dim W^\perp = s$ where $W^\perp = \{\underline{w} \in \mathbb{R}^s : \forall \underline{v} \in W \ \langle \underline{v}, \underline{w} \rangle = 0\}$.

(D) =)

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Proof: Let W & let $B = \{u_1 \ldots u_k\}$ a basis of W

~~$A = [u_1 \ldots u_k]$~~

$A = \begin{bmatrix} u_1^T \\ \vdots \\ u_k^T \end{bmatrix}$ of $Ax = 0 \iff u_i^T x = 0 \ \forall i$

$\Rightarrow w^T x = 0 \ \forall w \in W$.

$\square$

~~Pro~~ ~~Prove~~

Prop: $col(A)^\perp = null(A^T)$ & $row(A)^\perp = null(A)$.

$Ax \in col(A)$ $0 = y^T A x \ \forall y \Rightarrow (A^T y)^T x \ \forall x \Rightarrow A^T y = 0$

Th: Let $A \in \mathbb{R}^{n \times n}$ & $T_A : \mathbb{R}^n \to \mathbb{R}^n$ $x \mapsto Ax$.

The following are equivalent:

(i) A ~~[...]~~ Gauss-Jordan elimination A to the form I

(ii) $\exists A^{-1}$

(iii) $Ax = 0 \iff x = 0$

(iv) $\exists! x$ s.t. $Ax = b$ for each $b \in \mathbb{R}^n$

(v) $\det(A) \neq 0$

(vi) $\lambda = 0$ is not an eigenvalue.

(vii) T_A is one-to-one

(viii) row vectors of A are lin. indep. and

(ix) column vectors of A are lin. indep.

(x) row vectors of A form a basis

(xi) column vectors form a basis

(xii) $rank(A) = n$ (xiii) $null(A) = 0$.

Th: If $\dim W = n-1$ & $W \subseteq \mathbb{R}^n \Rightarrow \exists a \in \mathbb{R}^n$ s.t.

$W^\perp = \{ c\,a : c \in \mathbb{R} \}$

Th: $A \in \mathbb{R}^{m \times n} \Rightarrow rank(A) = rank(A^T A)$.

Proof: Enough: $nullity(A) = nullity(A^T A)$

So: $v \in null(A) \Leftrightarrow v \in null(A^T A)$ trivially.

If $\underline{v} \in null(A^TA)$.

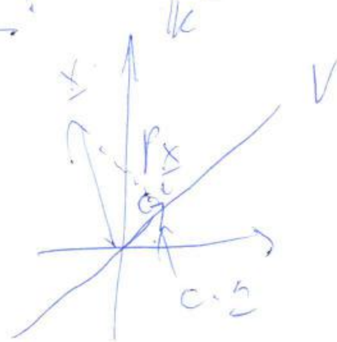
Observe: $(A^TA)^T = A^TA$

thus. $\underline{v}^T(A^TA) = \underline{0}^T$

$\Rightarrow 0 = \underline{v}^T(A^TA)\underline{v} = (A\underline{v})^T(A\underline{v}) \Leftrightarrow A\underline{v} = \underline{0} \Rightarrow \underline{v} \in null(A)$. (since it is scalar product)

□

Orthogonal projections:



In dim 2: Let $V \subseteq \mathbb{R}^2$ space & let $\underline{a} \in V$ non-zero vector. $\underline{a}$

we know. $\langle c\cdot\underline{a}, \underline{v} - c\cdot\underline{a}\rangle = 0$

$$\Rightarrow c\,\underline{a}^T\underline{v} = c^2\,\underline{a}^T\underline{a}$$

$$c = \frac{\underline{a}^T\underline{v}}{\underline{a}^T\underline{a}}$$

$$\Rightarrow P(\underline{v}) = \frac{(\underline{a}^T\underline{v})\underline{a}}{\underline{a}^T\underline{a}} \Rightarrow \frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}}\underline{v} \qquad \underline{a}\underline{a}^T \in \mathbb{R}^{2\times 2},\ \underline{a}^T\underline{a} \in \mathbb{R}.$$

Projections in general:

Let $W \subseteq \mathbb{R}^n$ subspace. Let $P_W : \mathbb{R}^n \to W$ orthogonal project.

i.e. $\forall \underline{x} \in \mathbb{R}^n$ $P_W\underline{x} \in W$ & $\langle \underline{x} - P_W\underline{x}, P_W\underline{x}\rangle = 0$.

Let $\{\underline{u}_1, \ldots, \underline{u}_k\}$ a basis of W & $M = [\underline{u}_1 \ldots \underline{u}_k]$

$$P = M(M^TM)^{-1}M^T.$$

Ex: $x - 4y + 2z = 0$ plane in $\mathbb{R}^3$

~~$u_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}$~~

$u_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ a basis. $\begin{bmatrix} -2 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$M = \begin{bmatrix} -2 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $M^T M = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & -2 \end{bmatrix}$ $\begin{matrix} 4 & 1 \\ -4 & 4 \end{matrix}$ $(M^T M)^{-1} = \dfrac{\begin{pmatrix} 4 & -1 \\ 4 & 4 \end{pmatrix}}{20}$

$\begin{bmatrix} 4 & -1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & -2 \end{bmatrix}$

$\begin{bmatrix} -2 & 4 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & 18 \\ 4 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 18 & 8 & -4 \\ 4 & 4 & 8 \\ -4 & 4 & 18 \end{bmatrix} \frac{1}{20}.$

Proof recall: $M = [u_1, \dots u_k]$ $\Rightarrow W = col(M)$ & $W^\perp = null(M^T)$.

$\underline{x} = \underline{v}_x + \underline{a}_x$, where $\underline{v}_x \in col(M)$ & $\underline{a}_x \in null(M^T)$

if $\underline{v}_x \in col(M_x) \Rightarrow \exists \underline{v}$ $\underline{v}_x = M\underline{v}$ & $M^T \underline{a}_x = \underline{0}$

$\Rightarrow M^T(\underline{x} - M\underline{v}) = \underline{0}$

$\Rightarrow M^T \underline{x} = M^T M \underline{v}$

Thus for every $\underline{x}$ $\exists!$ solution $\underline{v} \in \mathbb{R}^k$.

rank $(M^T M) = $ null$(M^T) = k$

$\Rightarrow \underline{v} = (M^T M)^{-1} M^T \underline{x}$.

$\Rightarrow \underline{v}_x = M(M^T M)^{-1} M^T \underline{x}$ $\square$.

Prop: If P is an orthogonal projection $\Rightarrow P^2 = P$ & $P^T = P$
Moreover if $P^2 = P$ & $P^T = P$ is a orthogonal projection ~~onto basis &~~ & $W = col(P)$.

(Smallest squares)

Application: If $A\underline{x} = \underline{b}$ has no solution then
find $\underline{x}$ st. $\|\underline{b} - A\underline{x}\| = \min_{\underline{y}} \|A\underline{y} - \underline{b}\|$

$A\underline{y} \in col(A)$ & $\forall \underline{b}^* \in col(A)$ $\exists \underline{y}$ $A\underline{y} = \underline{b}^*$

! $\underline{b}^*$ the orthogonal proj. of $\underline{b}$ to $col(A)$.

$\Rightarrow$ ~~Ax~~ solving $A\underline{x} = \underline{b}^*$

$\Leftrightarrow \quad \underline{b} - A\underline{x} = \underline{b} - \underline{b}^*$

smallest squares.

$\overset{(*)}{\Rightarrow} \quad A^T(\underline{b} - A\underline{x}) = A^T(\underline{b} - \underline{b}^*) = \underline{0}$

because $\underline{b} - \underline{b}^* \perp \text{col}(A)$

if $\|A\underline{x} - \underline{b}\|$ - minimal $\Leftrightarrow A^T\underline{b} = A^TA\underline{x}$ ~~$x = (A^TA)^{-1}$~~

(note: A^TA not nec. invertible!)

$(x_1, y_1), \ldots (x_n, y_n) \in \mathbb{R}^2$. best line. $y_i = a + bx_i$

~~or the~~ with the smallest error $\sum_i (y_i - a - bx_i)^2$

$$\Rightarrow \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$A \quad \underline{x} \quad \underline{b}$

$$A^T\underline{b} = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\Rightarrow A^TA = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \end{bmatrix}.$$

Positive definite matrices:

def: $A \in \mathbb{R}^{n\times n}$ ~~symmetric~~

pos. def.: $\underline{x}^TA\underline{x} > 0 \quad \forall \underline{x} \neq 0 \in \mathbb{R}^n$

pos. semidef.: $\underline{x}^TA\underline{x} \geq 0 \quad \forall \underline{x} \in \mathbb{R}^n$

neg. def.: $\underline{x}^TA\underline{x} < 0 \quad \forall \underline{x} \neq \underline{0} \in \mathbb{R}^n$

neg. semidef.: $\underline{x}^TA\underline{x} \leq 0 \quad \forall \underline{x} \in \mathbb{R}^n$

indefinite: $\exists \underline{y} \neq 0, \underline{x} \neq \underline{0} \quad \underline{y}^TA\underline{y} > 0 \quad \underline{x}^TA\underline{x} < 0$.

Thm: A ~~symmetric~~ ~~mx.~~ pos. def $\Leftrightarrow$ each eigenvalue is positive.

pos. semidef. $\Leftrightarrow$ ~~—~~ // — is non-negative

neg. def. $\Leftrightarrow$ each eigenvalue is neg.

neg. semi. $\Leftrightarrow$ — // — is non-pos. (indef $\Leftrightarrow$ pos. & neg.)

! $\lambda_1 \dots \lambda_n$ the eigenvalues of A with multiplicity

$\Rightarrow A = PDP^T$

$\underline{x}^T A \underline{x} = (P^T\underline{x})^T D P^T \underline{x} = y^T D y = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 \geq 0$

$y := P^T \underline{x}$

Lemma: ! $A \in \mathbb{R}^{n \times n} \Rightarrow A^T A$ is symmetric & pos. semidefinite.

Lemma: If A symmetric & pos. def. $\exists B$ s.t. $A = B^2$

$A = PDP^T \Rightarrow D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \lambda_i > 0 \quad \forall i$

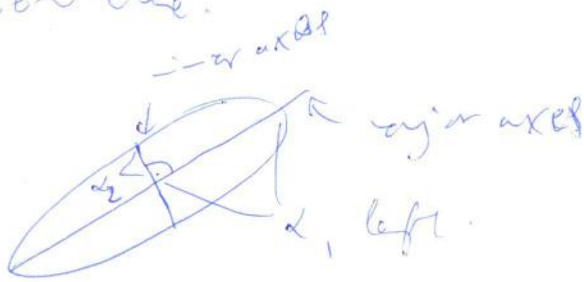
$\tilde{D} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix} \Rightarrow P\tilde{D}P^T P\tilde{D}P^T \ \& \ P\tilde{D}P^T = B, \bigcirc.$

Singular value decomposition: ! $A \in \mathbb{R}^{2 \times 2}$

! $T_A : \underline{x} \mapsto A\underline{x}$! B be the unit ball centered at the origin.

What does $T_A(B)$ look like?

It is an ellipse.



Take A^TA & ! α_1^2, α_2^2 eigenvalues & $\underline{v}_1, \underline{v}_2$ eigenvectors.

$\underline{v}_1 \perp \underline{v}_2$ (in this case) $\Rightarrow A^TA = V \Sigma V^T \Rightarrow V^TA^TAV = \Sigma$

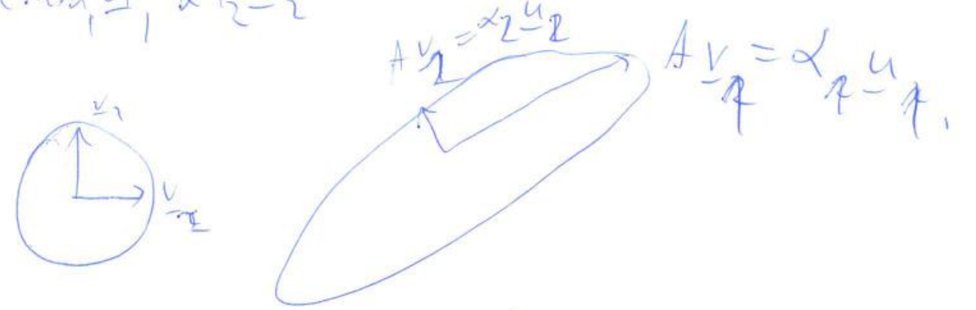
$V = [\underline{v}_1 \ \underline{v}_2]$ $\quad (AV)^TAV = \Sigma$

$\stackrel{(*)}{\Rightarrow} A\underline{v}_i \perp A\underline{v}_j \ \& \ \|A\underline{v}_i\| = \alpha_i^2$

$\underline{u}_i := \frac{1}{\alpha_i} A\underline{v}_i \Rightarrow U = AV \cdot D^{-1} \Rightarrow$ ~~$A = V^T$~~ $\boxed{A = UDV^T}$

$U^TU = I.$

Hence $T_A(\mathbb{1})$ is an ellipse with axis $\alpha_1 \underline{u}_1$ & $\alpha_2 \underline{u}_2$



Polar decomposition:

Th: $A \in \mathbb{R}^{n \times n}$ s.t. $\text{rank}(A) = k \Rightarrow \exists P$ sym. pos. semidef. s.t. $\text{rank}(P) = k$ & Q orthonormal mx s.t.

$$A = PQ$$

Moreover, if $\text{rank}(A) = n \Rightarrow P$ pos. def.

$$A = UDV^T = \underbrace{UDU^T}_{P}\underbrace{UDV^T}_{Q} \qquad U\underbrace{V^TV}_{I}U^T = I.$$

Spectral decomposition of symmetric mx

$A = QDQ^T$ s.t. $\Rightarrow A = \sum_i \lambda_i u_i \underline{u}_i^T$ This form is called spectral decomp.

$Q = [\underline{u}_1 \dots \underline{u}_n]$. $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

! $f(x)$ function with Taylor series around 0. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

$A = QDQ^T \Rightarrow A^k = QD^kQ^T$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n = Q\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} D^n\right) Q^T = Q\begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} Q^T$$

$\Rightarrow f(A) := Q\begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix} Q^T$. In particular

$$e^{tA} = Q\begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} Q^T$$