Advanced mathematics for civil engineers

Exercises in Partial differential equations

1. Describe the sine Fourier series on the interval $[0, 2\pi]$ of the function

$$h(x) = \begin{cases} 1 & \text{if } x \in [0, \pi) \\ 0 & \text{if } x\pi \end{cases}$$

(Hint:$h(x) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right)$ if $0 \le x < \pi$.) DO NOT USE THE CHEATING SHEET!

2. Using the cheating sheet below, give the sine Fourier series of the function

$$G(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 2 - x & \text{if } 1 \le x \le 2. \end{cases}$$

3. Using the cheating sheet below, give the sine Fourier series of the function $\Phi(x) = x(2 - x)$ on the interval $[0, 2]$.

4. Using the previous exercises, solve the following problems:

(a)

$$\begin{cases} u''_{tt} = 3.1^2 u''_{xx} & 0 < t, 0 \le x \le 2 \\ u(x, 0) = \frac{1}{40}(1 - |x - 1|) & 0 \le x \le 2 \\ u'_t(x, 0) = \frac{1}{80} x(2 - x) & 0 \le x \le 2 \\ u(0, t) = u(2, t) = 0 & 0 < t \end{cases}$$

(b)

$$\begin{cases} u'_t = u''_{xx} & 0 < t, 0 \le x \le 2 \\ u(x, 0) = f(x) & 0 \le x \le 2 \\ u(0, t) = u(2, t) = 0 & 0 < t \end{cases} \quad \text{where } f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 \le x \le 2. \end{cases}$$

5. On the interval $[0, 2]$, there is a string. The initial form is given by $f \equiv 0$ function and the movement is described by the equation $u''_{tt} = u''_{xx}$. We hit this string with a hammer on the interval $[1/2, 3/2]$ from below with

velocity 1. Give the shape of the string at time $t = 3.5$ with the method of D'Alambert.

(Hint: the general formula $u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$, where $u''_{tt} = c^2 u''_{xx}$, $u(x,0) = f(x)$, $u'_t(x,0) = g(x)$.)

6. Let us consider a string with infinite length. Hold down on the points $(-1,0)$, $(-1/2,1)$, $(1/2,1)$ and $(1,0)$ and release it. The movement of the string is described by the equation $u''_{tt} = u''_{xx}$. Describe the shape of the string at time $t = 20$!

Cheating sheet (which you are allowed to use on the exam)

1. The sine Fourier series of the function $f\colon [0,L] \mapsto \mathbb{R}$, $\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$, where $b_n = \frac{2}{L}\int_0^L \sin(\frac{n\pi}{L}t)dt$.
2. $\int x\sin(ax)dx = \frac{\sin(ax) - ax\cos(ax)}{a^2} + C$ and
$\int x^2 \sin(ax)dx = -\frac{a^2x^2\cos(ax) - 2\cos(ax) - 2ax\sin(ax)}{a^3} + C$.
3. Wibrating string:

$$\begin{cases} u''_{tt} = c^2 u''_{xx} & 0 < t, 0 \le x \le L \\ u(x,0) = f(x) & 0 \le x \le L \\ u'_t(x,0) = g(x) & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & 0 < t \end{cases}$$

Solution: $u(x,t) = \sum_{k=1}^{\infty} \sin(\frac{k\pi}{L}x)\left(A_k\cos(\frac{kc\pi}{L}t) + B_k\sin(\frac{kc\pi}{L}t)\right)$, where $\{A_k\}$ are the coefficients of the sine Fourier series of $f(x)$ and $\{B_k\}$ are the coefficients of the sine Fourier series of $g(x)$.

4. Heat conduction in a finite rod:

$$\begin{cases} u'_t = \alpha^2 u''_{xx} & 0 < t, 0 \le x \le L \\ u(x,0) = f(x) & 0 \le x \le L \\ u(0,t) = u(2,t) = 0 & 0 < t \end{cases}$$

Solution: $u(x,t) = \sum_{n=1}^{\infty} A_n e^{\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \sin(\frac{n\pi}{L}x)$, where $\{A_n\}$ are the coefficients of the Fourier sine series of $f(x)$.

5. The sine Fourier series of some functions:
   (a) $f(x) = x$ if $0 \le x < \pi$: $f(x) = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots\right)$ for $0 \le x < \pi$
   (b) $g(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}$: $g(x) = \frac{4}{\pi}\left(\sin(x) - \frac{\sin(3x)}{3^2} + \frac{\sin(5x)}{5^2} - \frac{\sin(7x)}{7^2} + \cdots\right)$ for $0 \le x \le \pi$.
   (c) $h(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & x = 0, \pi \end{cases}$: $h(x) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \cdots\right)$ for $0 \le x \le \pi$
   (d) $\varphi(x) = x(\pi - x)$: $\varphi(x) = \frac{8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \frac{\sin(7x)}{7^3} + \cdots\right)$ for $0 \le x \le \pi$