NOTES: Probability Practice - Week 1 09/04

1.1 a) $\underset{\substack{\uparrow\\ \text{1st}\\ \text{instrument}}}{4} \cdot \underset{\substack{\uparrow\\ \text{2nd}\\ \dots}}{3} \cdot \underset{\substack{\uparrow\\ \text{3rd}\\ \dots}}{2} \cdot \underset{\substack{\uparrow\\ \text{4th}\\ \text{instrument}}}{1} = 4!$

answer: $4! = 24$

b) Jay and Jack can only play piano and drums, so either
1) Jay: Piano, Jack: Drums
or
2) Jay: Drums, Jack: Piano

Jim and John must play the other 2.

answer: $2! \cdot 2! = 4$

1.2 a) $\underset{\substack{\downarrow\\ \underbrace{2,3,4 \dots \text{or } 9}_{(8 \text{ possible})}}}{\underline{?}} \quad \underset{\substack{\downarrow\\ \underbrace{0 \text{ or } 1}_{2}}}{\underline{?}} \quad \underset{\substack{\searrow\\ \underbrace{1,2,\dots \text{or } 9}_{9}}}{\underline{?}}$ e.g 309

answer: $8 \cdot 2 \cdot 9 = 144$

b) $\underset{\substack{\downarrow\\ 1}}{\underline{4}} \ \underset{\substack{\downarrow\\ 2}}{\underline{?}} \ \underset{\substack{\downarrow\\ 9}}{\underline{?}}$ $\Rightarrow$ answer: $1 \cdot 2 \cdot 9 = 18$

1.3 a) $8!$ from def of permutation

b) $\underset{1}{\circ} \ \underset{2}{\circ} \ \underset{3}{\boxed{Ⓐ Ⓑ}} \ \underset{4}{\circ} \ \underset{5}{\circ} \ \underset{6}{\circ} \ \underset{7}{\circ}$

Trick: combine A and B into one unit, which gives $7!$ arrangements, but then each arrangement can have $\underbrace{\text{AB or BA}}_{2}$

$\Rightarrow$ answer: $7! \cdot 2 = 10080$

c) 2 seating types (M: male, F: female)

$\Leftrightarrow$ 1) MFMFMFMF
2) FMFMFMFM

$4!$ ways to choose order of men, and $4!$ for women.

answer: $4! \cdot 4! \cdot 2 = 1152$

d) 1) (MMMMMM) F F F

$\quad$ 1 $\quad$ 2 3 4

answer: $4! \cdot 5! = 2880$

$5!$: arrange the men inside the circle

e) (MF) (MF) (MF) (MF)

$\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4

answer: $4! \cdot 2^4 = 384$

$4!$: arrange the circles

2^4: arrange the couples inside the circles

(1.4) Each handshake is an (unordered) subset of size 2 of the set $\{1, 2, \dots 20\}$ => answer: $\binom{20}{2} = 190$

(1.5) Ace, 2 3 4 5 6 7 8 9 10 Jack Queen King } 13 types

Each can be ♣ ♦ ♥ or ♠ } times 4

=> 52 distinct cards $(4 \cdot 13 = 52)$, we draw 5 cards from 52 => answer: $\binom{52}{5} = 2{,}598{,}960$

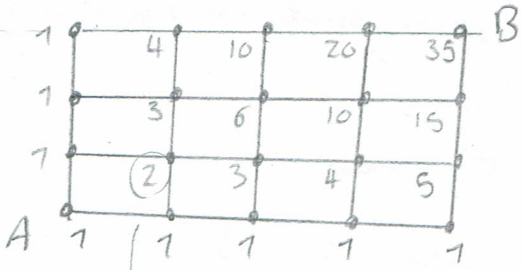
(1.6) $\binom{10}{5} \cdot \binom{12}{5} \cdot 5!$ ← pair 5 men and 5 women

answer: $\binom{10}{5}$ choose women, $\binom{12}{5}$ choose men $= 23{,}950{,}080$

(1.7) $\underbrace{\text{R R}}_{\text{2 from 5}}$ $\quad$ $\underbrace{\text{D D}}_{\text{2 from 6}}$ $\quad$ $\underbrace{\text{I I I}}_{\text{3 from 4}}$

answer: $\binom{5}{2} \cdot \binom{6}{2} \cdot \binom{4}{3} = 10 \cdot 15 \cdot 4 = 600$

(1.8)



Example: There are 2 paths from A to there: 

answer: 35, but notice that the paths are a pascal pyramid. We go 4 to the right and 3 up to get to B, so its location in the pyramid is $\binom{7}{3} = 35$

1.11

$\downarrow$ $\binom{n}{1}$ ways to choose empty urn

$\underbrace{\qquad\qquad\qquad}$ distribute n balls into the remaining $n-1$ urns, with at least 1 in each urn.

- 2 balls have to be in the same urn. $\binom{n}{2}$ choices
- Which urn has 2 balls? $\binom{n-1}{1}$ choices
- arrange the remaining $n-2$ balls: $(n-2)!$

good events: $\binom{n}{1}\cdot\binom{n}{2}\cdot\binom{n-1}{1}\cdot(n-2)!$

all events: n^n (n unique balls into n unique urns)

answer: $$\frac{\binom{n}{1}\binom{n}{2}\cdot\binom{n-1}{1}\cdot(n-2)!}{n^n}$$

1.12 all events: $\binom{64}{8}$ → 64 squares on the board, → 8 rooks

If none knock each other out, then there can only be 1 in each row and column.

column 1: 8 possibilities
column 2: 7 possibilities
⋮
column 8 1 possibility

$\Big\}$ good events: $8!$

answer: $8! \,/\, \binom{64}{8}$

1.13 blue: $n-N$, red: N

a) good: $\binom{n-N}{r}$, all: $\binom{n}{r}$ answer: $\dfrac{\binom{n-N}{r}}{\binom{n}{r}}$

b) good: $(n-N)^r$, all n^r answer: $\left(\dfrac{n-N}{n}\right)^r$

1.14 a) all: $\binom{2n}{2r}$

good: $\binom{n}{2r}\cdot 2^{2r}$ → from each pair, we can only pick 1 shoe, 2 possibilities (left, right)

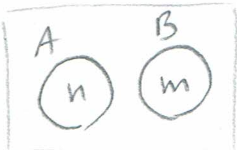
↖ we choose 2r pairs

answer: $\binom{n}{2r} \cdot 2^{2r} \Big/ \binom{2n}{2r}$

b) $\underbrace{\binom{n}{1}}_{\text{One pair chosen}} \cdot \underbrace{\binom{n-1}{2r-2} \cdot 2^{2r-2}}_{\text{using a), no pairs are chosen from the remaining } n-1 \text{ pairs}}$: good events, answer: $\dfrac{\binom{n}{1}\binom{n-1}{2r-2}2^{2r-2}}{\binom{2n}{2r}}$

c) $\binom{n}{2} \cdot \binom{n-2}{2r-4} \cdot 2^{2r-4}$: good events, answer: $\dfrac{\binom{n}{2}\binom{n-2}{2r-4}2^{2r-4}}{\binom{2n}{2r}}$

1.15



$|A| = n$

$|B| = m$

$$\underbrace{\binom{n+m}{r}}_{r \text{ from } A \cup B} = \underbrace{\binom{n}{0}\binom{m}{r}}_{\substack{0 \text{ from } A \\ r \text{ from } B}} + \underbrace{\binom{n}{1}\binom{m}{r-1}}_{\substack{1 \text{ from } A \\ r-1 \text{ from } B}} + \dots + \underbrace{\binom{n}{r}\binom{m}{0}}_{\substack{r \text{ from } A \\ 0 \text{ from } B}}$$

or ... or

1.16 Let $m = n$, $r = n$ then use 1.15

$$\binom{n+n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \dots + \binom{n}{n}\binom{n}{0} = \sum_{r=0}^{n} \binom{n}{r}\binom{n}{n-r}$$

rule: $\binom{n}{r} = \binom{n}{n-r}$, so $\binom{2n}{n} = \sum_{r=0}^{n} \binom{n}{r}^2$