NOTES: **Probability 1 Practice** **WEEK 2** 09.11

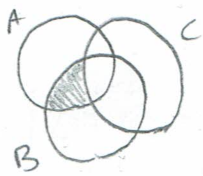
2.1 a) $r=0$: $A^c \cap B^c \cap C^c$

$r=1$: $\underbrace{(A \cap B^c \cap C^c)}_{\text{only „A" happens}} \cup \underbrace{(A^c \cap B \cap C^c)}_{\text{or only „B" happens} \dots} \cup (A^c \cap B^c \cap C)$

$r=2$: $(A \cap B \cap C^c) \cup (A^c \cap B \cap C) \cup (A \cap B^c \cap C)$

b) $r=1$: $A \cup B \cup C$

$r=2$: $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

2.2 a) 

$(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$

$(A \setminus B = A \cap B^c)$

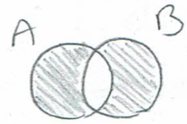
i) $x \in LHS \Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \notin C \Rightarrow x \in A \setminus C \text{ and } x \in B \setminus C$

$\Rightarrow x \in RHS \Rightarrow (A \cap B) \setminus C \subseteq (A \setminus C) \cap (B \setminus C)$

ii) $x \in RHS \Rightarrow (x \in A \text{ and } x \notin C) \text{ and } (x \in B \text{ and } x \notin C) \Rightarrow x \in A \cap B \text{ and } x \notin C$

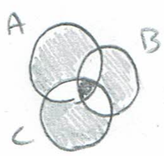
$\Rightarrow x \in LHS \Rightarrow (A \setminus C) \cap (B \setminus C) \subseteq (A \cap B) \setminus C$

using i) and ii), $LHS = RHS$

b) def: $A \circ B = (A \cup B) \setminus (A \cap B)$ 

If $x \in A \circ (B \circ C) \Rightarrow$
- $x \in A,\ x \notin (B \circ C)$ → $x \in B,\ x \in C$ / $x \notin B,\ x \notin C$
- $x \notin A,\ x \in (B \circ C)$ → $x \in B,\ x \notin C$ / $x \notin B,\ x \in C$

If $x \in (A \circ B) \circ C \Rightarrow$
- $x \in C,\ x \notin (A \circ B)$ → $x \in A,\ x \in B$ / $x \notin A,\ x \notin B$
- $x \notin C,\ x \in (A \circ B)$ → $x \in A,\ x \notin B$ / $x \notin A,\ x \in B$



$\Rightarrow LHS = RHS$

2.3 a) $\Omega = \{\{c_1, c_2, c_3, \{D_1, D_2\}\} \mid c_i \in \{H, T\},\ D_i \in \{1, 2 \dots 6\}\}$

e.g. $\{H, T, H, \{3, 5\}\} \in \Omega$

answer: $2 \cdot 2 \cdot 2 \cdot \underbrace{\binom{6+2-1}{2}}_{\text{combination with repetition}} = 168$

b) $\Omega = \{\{\{B_1, B_2, B_3\}, \{W_1, W_2\}\} \mid B_i, W_i \in \{1, \dots 6\}\}$

e.g. $\{\{2, 2, 5\}, \{4, 3\}\} \in \Omega$

answer: $\binom{6+3-1}{3} \cdot \binom{6+2-1}{2} = 56 \cdot 21 = 1176$

2.4 $A :=$ { Throw a die 4 times, and roll 6 at least once }

$B :=$ { Throw two dice 24 times, and roll (6,6) at least once }

$\mathbb{P}(A) = 1 - \underbrace{\mathbb{P}(A^c)}_{\text{don't roll 6 at all}} = 1 - \underbrace{\left(5/6\right)}_{\mathbb{P}(\text{roll } 1,2,3,4 \text{ or } 5) = 5/6}^4 = 0.5177$

$\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - \underbrace{\left(\frac{35}{36}\right)}_{}^{24} = 0.4914$

$\mathbb{P}(\text{roll } (1,1), (1,2), \dots (5,5), (5,6) \text{ or } (6,5)) = 35/36$

answer:

$\mathbb{P}(A) > \mathbb{P}(B)$,

2.5 $p = \mathbb{P}(\text{Adam wins a match}) = 0.5$

$\mathbb{P}(A) = \binom{4}{3} \cdot p^3 \cdot (1-p)^{4-3} = 0.25$

$\mathbb{P}(B) = \binom{8}{5} \cdot (1-p)^5 \cdot p^{8-5} = 0.21875$

answer: „A" has a higher $\mathbb{P}$.

2.6 $\Omega = \{TT, HH, HTT, THH, THTT, HTHH, \dots\}$

generally: $\underbrace{\begin{matrix} HTHT \cdots HTT \\ THTH \cdots THH \end{matrix}}_{}$

n long, 2^{-n} probability for both

$$\sum_{r=2}^{5} 2 \cdot 2^{-r} = \mathbb{P}(A) \text{ because } \begin{matrix} TT \\ HH \end{matrix} \Big\} 2 \cdot 2^{-2}$$

$$\begin{matrix} HTT \\ THH \end{matrix} \Big\} 2 \cdot 2^{-3}$$

$$\begin{matrix} THTT \\ HTHH \end{matrix} \Big\} 2 \cdot 2^{-4}$$

$$\ldots\ldots \Big\} 2 \cdot 2^{-5}$$

$$\Rightarrow \underline{\underline{\mathbb{P}(A) = 0.9375}}$$

$$\underline{\underline{\mathbb{P}(B)}} = \sum_{r=1}^{\infty} 2 \cdot 2^{-2r} = 2 \cdot \sum_{r=1}^{\infty} \left(\frac{1}{4}\right)^r = 2 \cdot \left(\frac{1}{1-1/4} - 1\right)$$

only even throws

because r starts at 1, not 0, so we subtract $(1/4)^0$.

$$= 2\left(\frac{4}{3} - 1\right) = \underline{\underline{2/3}}$$

note: $\sum_{r=2}^{\infty} 2 \cdot 2^{-r} = 2\left(\frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2}\right) = 1 = \mathbb{P}(\Omega)$

2.7 $P(A)P(B) - P(A \cap B) = (P(A \setminus B) + P(A \cap B)) \cdot (P(B \setminus A) + P(A \cap B)) - P(A \cap B)$

1) $\mathbb{P}(\circ) \geq 0 \longrightarrow \geq \mathbb{P}(A \cap B)^2 - \mathbb{P}(A \cap B) \geq -\frac{1}{4}$

$f(x) = x^2 - x \qquad f'(x) = 2x - 1 \qquad \left.\begin{matrix} 2x - 1 = 0 \\ x = 1/2 \end{matrix}\right\}$ f takes minimum value here

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4} \Rightarrow f(x) \geq -\frac{1}{4}$

2) $P(A)P(B) - P(A \cap B) = P(A)P(B) - P(A) - P(B) - P(A \cup B)$

$= P(A)(P(B) - 1) + P(A \cup B) - P(B)$

$= -P(A)P(B^c) + P(A \setminus B)$

$= -P(A)P(B^c) + P(A \cap B^c) \leq \frac{1}{4}$

↳ using 1)

2.8 a) $P(A) \geq 0.8 \Rightarrow P(A^c) \leq 0.2 \qquad (\mathbb{P}(A^c) = 1 - \mathbb{P}(A))$

$P(B) \geq 0.5 \Rightarrow P(B^c) \leq 0.5$

$P(A \cap B) = P((A^c \cup B^c)^c) = 1 - \mathbb{P}(A^c \cup B^c)$

$= 1 - \mathbb{P}(A^c) - \mathbb{P}(B^c) + \mathbb{P}(A^c \cap B^c)$

$\geq 1 - \mathbb{P}(A^c) - \mathbb{P}(B^c) \geq 0.3$

b) Mathematical induction

$n=1$: $\mathbb{P}(A_1) \geq \mathbb{P}(A_1) - (1-1)$ ✓

Assume it holds for n

$$\mathbb{P}(A_1 \cap A_2 \cdots \cap A_{n+1}) = \mathbb{P}((\underbrace{A_1 \cap \cdots \cap A_n}_{B_1}) \cap (\underbrace{A_{n+1}}_{B_2}))$$

$$\geq P(A_1 \cap \cdots \cap A_r) + \mathbb{P}(A_{r+1}) - (2-1)$$

$$\geq P(A_1) + \cdots P(A_r) - (r-1) + \mathbb{P}(A_{r+1}) - (2-1)$$

$$= P(A_1) + \cdots + P(A_{r+1}) - r \quad ✓$$

2.10 a) $\Omega = \{(D_1, \ldots, D_6) \mid D_i \in \{1, 2, \ldots 6\}\}$

$|\Omega| = 6^6$ (all events)

good events: $6!$

$\mathbb{P}(a) = 6!/6^6 = 5/324$

b) INCLUSION - EXCLUSION IDENTITY: If $A_1, A_2 \cdots$ are events

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1, \ldots i_r} \mathbb{P}\left(\bigcap_{r=1}^{r} A_{i_r}\right)$$

$E_i = \{$ „i" is rolled at least once" $\}$ $\quad (E_i^c = F_i)$

$F_i = \{$ There is no „i" rolled $\}$

$$\mathbb{P}(\text{one of everything}) = \mathbb{P}\left(\bigcap_{i=1}^{6} E_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{6} F_i\right)$$

$$= 1 - \sum_{r=1}^{5} (-1)^{r+1} \underbrace{\sum_{1 \leq i_1, \ldots i_r \leq 6}'}_{\binom{6}{r}} \underbrace{\mathbb{P}(F_{i_1} \cap \cdots \cap F_{i_r})}_{\frac{(6-r)^{10}}{(6)^{10}}} = \frac{38045}{139968}$$

≈ 0.271812

note: $\mathbb{P}(F_i) = (5/6)^{10}$

$\mathbb{P}(F_i \cap F_j) = (4/6)^{10}$ etc

11) $$\underbrace{\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)}_{1} = \sum_{r=1}^{n} (-1)^{r+1} \underbrace{\sum_{1_1 < \dots < i_r}}_{\binom{n}{r}} \underbrace{\mathbb{P}\left(\bigcap_{r=1}^{r} A_{ir}\right)}_{1}$$

If we choose $A_i = \Omega, \forall i$

$$1 = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \Rightarrow 0 = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} + \underbrace{(-1)^{0+1}}_{-1} \binom{n}{0}$$

$$= \sum_{r=0}^{n} (-1)^{r+1} \binom{n}{r}$$

2.12) $R = \{\text{we draw red}\}$
$W = \{\text{we draw white}\}$
$B = \{\text{we draw blue}\}$

$$\underline{\mathbb{P}(R \cap W \cap B)} = 1 - \mathbb{P}(R^c \cup W^c \cup B^c)$$

$$= 1 - \Big(\underbrace{\mathbb{P}(R^c)}_{\frac{\binom{13}{5}}{\binom{19}{5}}} + \underbrace{\mathbb{P}(W^c)}_{\frac{\binom{13}{5}}{\binom{19}{5}}} + \underbrace{\mathbb{P}(B^c)}_{\frac{\binom{12}{5}}{\binom{19}{5}}} - \underbrace{\mathbb{P}(R^c \cap W^c)}_{\frac{\binom{7}{5}}{\binom{19}{5}}} - \underbrace{\mathbb{P}(R^c \cap B^c)}_{\frac{\binom{6}{5}}{\binom{19}{5}}} - \underbrace{\mathbb{P}(W^c \cap B^c)}_{\frac{\binom{6}{5}}{\binom{19}{5}}} + \underbrace{\mathbb{P}(W^c \cap B^c \cap R^c)}_{0}$$

$$= 1 - 0.2866 = \underline{0.71336}$$