



BME

LECTURE NOTES FOR THE SUBJECT

Mathematics MSc for Civil Engineers

Levente David

lectured by
Balázs Bárány

October 24, 2025

Preface

This is an extended lecture note for the subjects "Mathematics MSc for Civil Engineers" and "Construction Information Technology Mathematics" taught at the Budapest University of Technology and Economics in English language.

The project supported by the Doctoral Excellence Fellowship Programme (KCEP) is funded by the National Research, Development and Innovation Fund of the Ministry of Culture and Innovation and the Budapest University of Technology and Economics.

Contents

1	Partial differential equations	3
1.1	Fourier series	3
1.1.1	Fourier-sine series	4
1.1.2	Exercises	8
1.2	Vibrating string	8
1.2.1	Bernoulli's solution	10
1.2.2	D'Alembert's solution	14
1.2.3	Exercises	17
1.3	Heat transport	18
1.3.1	Exercises	20
1.4	Mixed exercises in Partial Differential Equations	21

Chapter 1

Partial differential equations

In this chapter, we will study the time change of certain physical phenomena like the shape of a string or the temperature of a rod. First, we will recall some definitions and statements for Fourier series, which helps us describing these functions.

1.1 Fourier series

Definition 1.1.1 (Fourier series). Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a $2p$ -periodic, integrable function on $[-p, p]$. Its **Fourier coefficients** are

$$a_0 = \frac{1}{2p} \int_{-p}^p f(x) dx, \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos\left(\frac{\pi n}{p} x\right) dx, \quad b_n = \frac{1}{p} \int_{-p}^p f(x) \sin\left(\frac{\pi n}{p} x\right) dx.$$

The **Fourier series** of f is

$$S[f](x) := a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n}{p} x\right) + b_n \sin\left(\frac{\pi n}{p} x\right) \right).$$

Theorem 1.1.2. Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ is $2p$ -periodic, piecewise continuously differentiable with only finitely many discontinuities on $[-p, p]$, and integrable on $[-p, p]$. Then for every $x \in \mathbb{R}$ the Fourier series $S[f](x)$ converges, and

$$S[f](x) = \frac{\lim_{y \rightarrow x^-} f(y) + \lim_{y \rightarrow x^+} f(y)}{2}.$$

In particular:

- If f is continuous at x , then $S[f](x) = f(x)$.
- If f has a jump discontinuity at x , then $S[f](x)$ equals the average of the left and right limits.

Theorem 1.1.2 particularly states that every wave can be described as (countably infinite) linear combination of basic waves of sin and cos. Note that the Fourier-series is unique.

If a $2p$ -periodic function f is odd, that is, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, then

$$\begin{aligned}
 a_n &= \frac{1}{p} \int_{-p}^p f(x) \cos\left(\frac{\pi n}{p}x\right) dx \\
 &= \frac{1}{p} \int_{-p}^0 f(x) \cos\left(\frac{\pi n}{p}x\right) dx + \frac{1}{p} \int_0^p f(x) \cos\left(\frac{\pi n}{p}x\right) dx \\
 &\stackrel{\text{substitute } y=-x}{=} \frac{1}{p} \int_p^0 f(-y) \cos\left(-\frac{\pi n}{p}y\right)(-1) dy + \frac{1}{p} \int_0^p f(x) \cos\left(\frac{\pi n}{p}x\right) dx \\
 &= -\frac{1}{p} \int_0^p f(y) \cos\left(\frac{\pi n}{p}y\right) dy + \frac{1}{p} \int_0^p f(x) \cos\left(\frac{\pi n}{p}x\right) dx = 0
 \end{aligned} \tag{1.1.1}$$

and

$$\begin{aligned}
 b_n &= \frac{1}{p} \int_{-p}^p f(x) \sin\left(\frac{\pi n}{p}x\right) dx \\
 &= \frac{1}{p} \int_{-p}^0 f(x) \sin\left(\frac{\pi n}{p}x\right) dx + \frac{1}{p} \int_0^p f(x) \sin\left(\frac{\pi n}{p}x\right) dx \\
 &\stackrel{\text{substitute } y=-x}{=} \frac{1}{p} \int_p^0 f(-y) \sin\left(-\frac{\pi n}{p}y\right)(-1) dy + \frac{1}{p} \int_0^p f(x) \sin\left(\frac{\pi n}{p}x\right) dx \\
 &= \frac{2}{p} \int_0^p f(y) \sin\left(\frac{\pi n}{p}y\right) dy.
 \end{aligned} \tag{1.1.2}$$

1.1.1 Fourier-sine series

Using the previous, our goal now is to describe continuous functions $f: [0, p] \rightarrow \mathbb{R}$ with finitely many discontinuity points defined only on an interval.

Definition 1.1.3. Let $p > 0$ and let $f: [0, p] \rightarrow \mathbb{R}$ be given such that f has only finitely many discontinuity points, and $0 = f(0) = f(p)$. We define the $2p$ -periodic, odd extension $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ of f as follows: First, we define the odd extension $\bar{f}$ of f on $[-p, p]$ by

$$\bar{f}(x) := \begin{cases} f(x), & 0 \leq x \leq p, \\ -f(-x), & -p \leq x < 0, \end{cases}$$

and then, we extend further it to $\hat{f}$ defined on $\mathbb{R}$ $2p$ -periodically by

$$\hat{f}(x) := \bar{f}(x - kp) \text{ for } x \in [(2k-1)p, (2k+1)p], \text{ where } k \in \mathbb{Z}$$

Clearly, $f(x) = \hat{f}(x)$ for every $x \in [0, p]$. Thus, using Theorem 1.1.2, (1.1.1) and (1.1.2) for the $2p$ -periodic odd extension, we get the following corollary:

Corollary 1.1.4. Let $p > 0$ and suppose that $f: [0, p] \rightarrow \mathbb{R}$ is a piecewise continuously differentiable function with only finitely many discontinuities. Then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{p}x\right), \text{ where } b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{\pi n}{p}x\right) dx$$

for every $x \in [0, p]$ for which f is continuous.

Example 1.1.5. Consider the function $f: [0, 2] \rightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 2 - x & \text{if } 1 \leq x \leq 2. \end{cases}$$

Determine the Fourier-sine series of f .

By Corollary 1.1.4, we only need to determine the b_n coefficients. Using the piecewise definition of f we split the integral defining the Fourier series

$$b_n = \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2 - x) \sin\left(\frac{n\pi x}{2}\right) dx.$$

Compute the first and second integral by parts (take $u = x$, $v' = \sin(\frac{n\pi x}{2})$):

$$\begin{aligned} \int_0^1 x \sin\left(\frac{n\pi x}{2}\right) dx &= \left[-\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right]_0^1 + \frac{2}{n\pi} \int_0^1 \cos\left(\frac{n\pi x}{2}\right) dx \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \left[\frac{4}{n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right) \right]_0^1 \\ &= -\frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Similarly

$$\begin{aligned} \int_1^2 (2 - x) \sin\left(\frac{n\pi x}{2}\right) dx &= \left[\frac{2(2 - x)}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right]_1^2 - \frac{2}{n\pi} \int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx \\ &= \frac{2}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

Adding the two contributions cancels the cosine-terms and yields

$$b_n = \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right).$$

Thus, the Fourier-sine series of f on $[0, 2]$ is

$$f(x) = S[f](x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) = \sum_{n=1}^{\infty} \frac{8 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right),$$

where we used that f is continuous and by Corollary 1.1.4, it coincides with its Fourier-sine series.

To simplify the formula, observe that $\sin(\frac{n\pi}{2}) = 0$ for even n and $\sin(\frac{(2k+1)\pi}{2}) = (-1)^k$, an equivalent form summing only odd indices is

$$f(x) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Example 1.1.6. Consider the function $f(x) = \cos(\frac{3\pi}{2}x) \sin(\frac{\pi}{2}x)$. Find the Fourier-sine series of f .

We first use the product-to-sum identity: $\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$. With $a = \frac{\pi}{2}x$

and $b = \frac{3\pi}{2}x$ we get

$$\begin{aligned} f(x) &= \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{3\pi}{2}x\right) = \frac{1}{2} \left(\sin\left(\frac{\pi}{2}x + \frac{3\pi}{2}x\right) + \sin\left(\frac{\pi}{2}x - \frac{3\pi}{2}x\right) \right) \\ &= \frac{1}{2} (\sin(2\pi x) + \sin(-\pi x)) = \frac{1}{2} \sin(2\pi x) - \frac{1}{2} \sin(\pi x). \end{aligned}$$

Now observe that the above form is already a sum of Fourier-sine series summands, the first term corresponds to $n = 4$ and $b_4 = \frac{1}{2}$ and the second term corresponds to $n = 2$ and $b_2 = -\frac{1}{2}$. By the uniqueness of the Fourier-series, we have that

$$S[f](x) = -\frac{1}{2} \sin(\pi x) + \frac{1}{2} \sin(2\pi x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right), \quad \text{where } b_n = \begin{cases} -\frac{1}{2}, & n = 2, \\ \frac{1}{2}, & n = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we give the Fourier-sine series of some basic functions, which might appear frequently.

(I) Let $h(x) = \begin{cases} 1 & 0 < x < \pi, \\ 0 & x = 0, \pi. \end{cases}$ Then

$$h(x) = \frac{4}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \dots \right) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)} \sin((2k+1)x)$$

for $0 \leq x \leq \pi$,

(II) Let $f(x) = \begin{cases} x & 0 \leq x < \pi, \\ 0 & x = \pi. \end{cases}$ Then

$$f(x) = 2 \left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \dots \right) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$$

for $0 \leq x < \pi$,

(III) Let $g(x) = \begin{cases} x & 0 \leq x \leq \pi/2, \\ \pi - x & \pi/2 \leq x \leq \pi. \end{cases}$ Then

$$g(x) = \frac{4}{\pi} \left(\sin(x) - \frac{\sin(3x)}{3^2} + \frac{\sin(5x)}{5^2} - \frac{\sin(7x)}{7^2} + \dots \right) = \sum_{k=0}^{\infty} \frac{(-1)^k 4}{\pi(2k+1)^2} \sin((2k+1)x)$$

for $0 \leq x \leq \pi$,

(IV) Let $\varphi(x) = x(\pi - x)$ for $x \in [0, \pi]$. Then

$$\varphi(x) = \frac{8}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \frac{\sin(7x)}{7^3} + \dots \right) = \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^3} \sin((2k+1)x)$$

for $0 \leq x \leq \pi$.

The following proposition helps us to describe how the Fourier-sine series changes under scaling.

Proposition 1.1.7. *Let $p > 0$ and let $f : [0, p] \rightarrow \mathbb{R}$ be given such that f has only finitely many discontinuity points with $0 = f(0) = f(p)$. Suppose that f has Fourier-sine series*

$$S[f](x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{p} x\right).$$

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq 0$. Then the Fourier-sine series of the map $g(x) := \beta f(\alpha x)$ is

$$S[g](x) = \sum_{n=1}^{\infty} (\beta b_n) \sin\left(\frac{\pi n \alpha}{p} x\right).$$

Proof. Clearly, the map $g(x) = \beta f(\alpha x)$ is defined on the interval $[0, \frac{p}{\alpha}]$. Hence, the n th coefficient of the Fourier-sine series of g is

$$\begin{aligned} \frac{2\alpha}{p} \int_0^{p/\alpha} g(x) \sin\left(\frac{\pi n \alpha}{p} x\right) dx \\ = \beta \frac{2\alpha}{p} \int_0^{p/\alpha} f(\alpha x) \sin\left(\frac{\pi n \alpha}{p} x\right) dx \stackrel{\text{substitute } y=\alpha x}{=} \beta \frac{2}{p} \int_0^p f(y) \sin\left(\frac{\pi n}{p} y\right) dy = \beta b_n. \end{aligned}$$

□

Example 1.1.8. *Let $f : [0, 2] \rightarrow \mathbb{R}$ be the map $f(x) = x(2 - x)$. Find the Fourier-sine series of f .*

Let φ on $[0, \pi]$ by

$$\varphi(t) = t(\pi - t), \quad 0 \leq t \leq \pi,$$

which is the (IV) element of the list above. Then using $t = \frac{\pi}{2}x$, we have that

$$\varphi\left(\frac{\pi}{2}x\right) = \frac{\pi}{2}x\left(\pi - \frac{\pi}{2}x\right) = \frac{\pi}{2}x\left(\frac{\pi}{2}(2 - x)\right) = \left(\frac{\pi}{2}\right)^2 f(x), \text{ and so } f(x) = \frac{4}{\pi^2} \varphi\left(\frac{\pi}{2}x\right).$$

By the above list we have the Fourier-sine series of φ :

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{8}{\pi(2k+1)^3} \sin((2k+1)t), \quad 0 \leq t \leq \pi.$$

Hence, using Proposition 1.1.7 and substituting $t = \frac{\pi}{2}x$, we get

$$f(x) = S[f](x) = \sum_{k=0}^{\infty} \frac{4}{\pi^2} \frac{8}{\pi(2k+1)^3} \sin\left((2k+1)\frac{\pi}{2}x\right).$$

1.1.2 Exercises

1.1/1. Find the Fourier-sine series of the function $f: [0, 3] \rightarrow \mathbb{R}$, where

$$f(x) = \begin{cases} 0 & 0 \leq x < 1, \\ 1 & 1 \leq x < 2, \\ 0 & 2 \leq x \leq 3. \end{cases}$$

1.1/2. Give the coefficients of the Fourier-sine series of the function $f: [0, 3] \rightarrow \mathbb{R}$, where

$$f(x) = \cos\left(\frac{5\pi}{3}x\right) \sin\left(\frac{\pi}{3}x\right) + 2 \sin(2\pi x).$$

1.1/3. Find the Fourier-sine series of the function $f: [0, 4] \rightarrow \mathbb{R}$, where

$$f(x) = \begin{cases} 5x & 0 \leq x < 2, \\ 20 - 5x & 2 \leq x < 4. \end{cases}$$

1.2 Vibrating string

Let us consider a perfectly elastic string of length L . Such a string fixed at its ends (or otherwise constrained) and slightly displaced from its equilibrium position exhibits transverse oscillations when released. Physically the motion is governed by the balance of (vertical components of) tension forces and the inertia of small portions of the string. For small transverse displacements the motion is approximately one-dimensional: we identify the string with the interval $[0, L]$, and at each horizontal coordinate $x \in [0, L]$, the vertical displacement of the string at time t is a scalar $u(x, t)$.

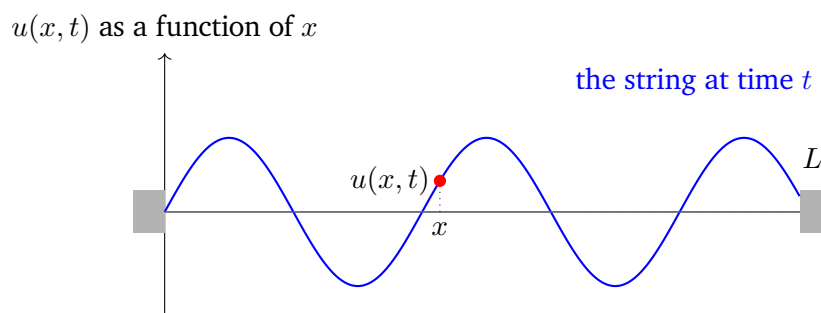


Figure 1.1: Displacement $u(x, t)$ of the vibrating string fixed at the ends.

Theorem 1.2.1. *Let us consider a perfectly elastic string of length L identified with the interval $[0, L]$. Then there exists $c > 0$ such that the vertical displacement $u(x, t)$ of the string at time $t > 0$ and at coordinate $x \in [0, L]$ satisfies the one-dimensional wave equation*

$$u''_{tt}(x, t) = c^2 u''_{xx}(x, t).$$

*This equation is called **vibrating string equation**.*

Proof. Consider a small segment of the string between x and $x + \Delta x$. The tensions at the ends, $\mathbf{T}(x, t)$ and $\mathbf{T}(x + \Delta x, t)$, act tangent to the string. Let $\theta(x, t)$ be the angle the string makes with the horizontal; by small-slope kinematics $\tan \theta \approx u'_x(x, t)$. Moreover, since the string does not move left or right, only up and down, the horizontal components must have constant horizontal magnitudes $\approx H$.

The vertical components of the end tensions are $H \tan \theta(x, t)$ at x (upward if $\theta > 0$) and $H \tan \theta(x + \Delta x, t)$ at $x + \Delta x$ (but acting on the element in the opposite direction). Thus the net vertical force F_y on the element is

$$F_y = H \tan \theta(x + \Delta x, t) - H \tan \theta(x, t) \approx H (u'_x(x + \Delta x, t) - u'_x(x, t)). \quad (1.2.1)$$

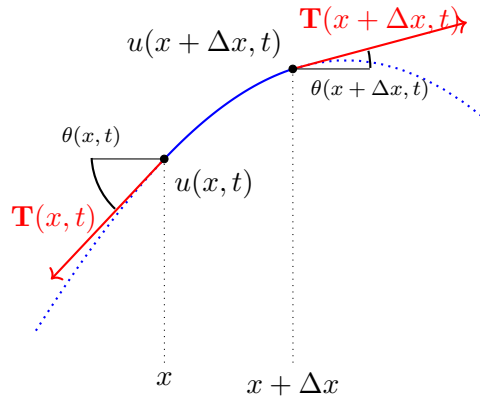


Figure 1.2: Zoomed-in view of the string segment between x and $x + \Delta x$. The tangent tensions $\mathbf{T}(x, t)$ and $\mathbf{T}(x + \Delta x, t)$ have vertical components given by $T \tan \theta(x, t)$ and $-T \tan \theta(x + \Delta x, t)$.

Suppose that our string has uniform density μ . Then the mass of the small segment is approximately mass $\mu \Delta x$, and by Newton's second law, this gives that

$$F_y \approx \mu \Delta x u''_{tt}(x, t). \quad (1.2.2)$$

Combining (1.2.1) and (1.2.2), and dividing by Δx , we get

$$\mu u''_{tt}(x, t) = H \frac{u'_x(x + \Delta x, t) - u'_x(x, t)}{\Delta x}.$$

Letting $\Delta x \rightarrow 0$ yields

$$\mu u''_{tt}(x, t) = H u''_{xx}(x, t),$$

i.e. $u''_{tt}(x, t) = c^2 u''_{xx}(x, t)$ with $c^2 = H/\mu > 0$. □

Given a perfectly elastic string of length L (identified with the interval $[0, L]$) with certain physical properties encoded in the equation $u''_{tt}(x, t) = c^2 u''_{xx}(x, t)$. We fix its endpoints ($u(0, t) = u(L, t) = 0$ for $t > 0$). One can modify its shape (like stretching a bow) and/or giving it velocity (like pluck the string on a guitar). In the following, we wish to describe the motion of $u(x, t)$ of such a string.

Definition 1.2.2. We call the boundary–initial value problem

$$\begin{cases} u''_{tt}(x, t) = c^2 u''_{xx}(x, t), & 0 < x < L, t > 0, \\ u(0, t) = u(L, t) = 0, & t \geq 0, \\ u(x, 0) = f(x) & 0 \leq x \leq L, \\ u'_t(x, 0) = g(x), & 0 \leq x \leq L, \end{cases}$$

where f is the initial shape and g the initial transverse velocity as **vibrating string problem**.

Let us begin with the following simple observation.

Proposition 1.2.3. Let $u_1(x, t)$ and $u_2(x, t)$ be function such that both satisfy the vibrating string equation $u''_{tt}(x, t) = c^2 u''_{xx}(x, t)$. Then for every constants $a, b \in \mathbb{R}$, the map $a \cdot u_1 + b \cdot u_2$ also satisfies the vibrating string equation.

This simply follows by the linearity of the differentiation.

1.2.1 Bernoulli's solution

First, we consider a solution, which takes into account the Fourier-sine representation of functions, and which is called Bernoulli's solution.

Theorem 1.2.4 (Bernoulli's solution). Let us consider the following vibrating string problem:

$$\begin{cases} u''_{tt}(x, t) = c^2 u''_{xx}(x, t), & 0 < x < L, t > 0, \\ u(0, t) = u(L, t) = 0, & t \geq 0, \\ u(x, 0) = f(x) & 0 \leq x \leq L, \\ u'_t(x, 0) = g(x), & 0 \leq x \leq L, \end{cases}$$

Then

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right) \right) \sin\left(\frac{n\pi}{L}x\right),$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \text{ and } B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx,$$

that is, A_n are the Fourier-sine coefficients of $f(x)$ and $\frac{n\pi c}{L} B_n$ are the Fourier-sine coefficients of $g(x)$.

Proof. First, assume $u(x, t) = T(t)X(x) \Rightarrow$, then by the $u''_{tt}(x, t) = c^2 u''_{xx}(x, t)$, we have

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left-hand side depends only on t and the right-hand side depends only on x , this is possible only if there exists some constant $\lambda \in \mathbb{R}$ (independent of t and x) such that

$$\frac{T''(t)}{c^2 T(t)} = \lambda \text{ and } \frac{X''(x)}{X(x)} = \lambda.$$

Hence, we get that $T(t)$ satisfies the ordinary differential equation

$$T''(t) - c^2\lambda T(t) = 0.$$

The general solution of T then is the following:

$$T(t) = \begin{cases} Ae^{-c\sqrt{\lambda}t} + Be^{c\sqrt{\lambda}t} & \text{if } \lambda > 0, \\ At + B & \text{if } \lambda = 0, \\ A \cos(\sqrt{-\lambda}ct) + B \sin(\sqrt{-\lambda}ct) & \text{if } \lambda < 0. \end{cases}$$

However, the first two case is not possible physically, since they would imply that $|T(t)| \rightarrow \infty$ as time passes $t \rightarrow \infty$ (so the string would blew away to infinity), and so, only the third option

$$T(t) = A \cos(\sqrt{-\lambda}ct) + B \sin(\sqrt{-\lambda}ct) \quad (1.2.3)$$

is physically relevant.

Thus, we see that $X(x)$ also satisfies the ordinary differential equation with initial values

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = X(L) = 0, \end{cases}$$

for some $\lambda < 0$. Therefore, and from a similar calculation for $X(x)$

$$X(x) = C \cos(\sqrt{-\lambda}x) + D \sin(\sqrt{-\lambda}x).$$

Clearly, $0 = X(0)$ can happen if and only if $C = 0$. On the other hand $X(L) = 0$ then implies that

$$0 = X(L) = D \sin(\sqrt{-\lambda}L)$$

So either $D = 0$ (but this would mean that $u(x, t) \equiv 0$, which is again a physical non-sense) or $\sqrt{-\lambda} = \frac{n\pi}{L}$ for some $n \in \mathbb{N}$. Thus, combining this with (1.2.3), we see that the maps

$$u_n(x, t) = \left(A \cos\left(\frac{n\pi}{L}ct\right) + B \sin\left(\frac{n\pi}{L}ct\right) \right) \sin\left(\frac{n\pi}{L}x\right)$$

satisfy the vibrating string equation $u_{tt}''(x, t) = c^2 u_{xx}''(x, t)$ with boundary condition $u(0, t) = u(L, t) = 0$ for every $A, B \in \mathbb{R}$ and $n \in \mathbb{N}$. Since the solutions are linear in sense of Proposition 1.2.3, we get that the general solution is of the form

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right) \right) \sin\left(\frac{n\pi}{L}x\right),$$

where $A_n, B_n \in \mathbb{R}$.

From the initial values, we will be able to get back the coefficients, A_n, B_n :

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

thus by Corollary 1.1.4, choosing

$$A_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi}{L}x\right) dx$$

we get the desired equation. Similarly,

$$g(x) = u'_x(x, 0) = \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n \sin\left(\frac{n\pi}{L}x\right)$$

gives that $\frac{n\pi}{L} B_n$ must be the Fourier-sine coefficients of $g(x)$, and so

$$B_k = \frac{2}{k\pi c} \int_0^L g(x) \sin\left(\frac{k\pi}{L}x\right) dx.$$

□

Example 1.2.5. Solve the following vibrating string problem

$$\begin{cases} u''_{tt} = 2u''_{xx}, & 0 < x < 3, t > 0, \\ u(0, t) = 0, \quad u(3, t) = 0, & t \geq 0, \\ u(x, 0) = f(x) = \sin(\pi x) \cos\left(\frac{4\pi x}{3}\right) + \sin\left(\frac{2\pi x}{3}\right), & 0 \leq x \leq 3, \\ u'_t(x, 0) = g(x) = \begin{cases} 1 & \text{if } 0 < x < 3 \\ 0 & \text{if } x = 0, 3. \end{cases} \end{cases}$$

First, we get the important constants $c^2 = 2 \Rightarrow c = \sqrt{2}$ and $L = 3$. Then the general solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{3}x\right) \left(A_n \cos\left(\frac{n\pi\sqrt{2}}{3}t\right) + B_n \sin\left(\frac{n\pi\sqrt{2}}{3}t\right) \right),$$

Decompose f using product-to-sum:

$$\sin(\pi x) \cos\left(\frac{4\pi x}{3}\right) = \frac{1}{2} [\sin\left(\frac{7\pi x}{3}\right) - \sin\left(\frac{\pi x}{3}\right)].$$

Thus

$$f(x) = \frac{1}{2} \sin\left(\frac{7\pi x}{3}\right) - \frac{1}{2} \sin\left(\frac{\pi x}{3}\right) + \sin\left(\frac{2\pi x}{3}\right).$$

Since the eigenfunctions are $\sin\left(\frac{n\pi x}{3}\right)$, the Fourier-sine coefficients $A_n = \frac{2}{3} \int_0^3 f(x) \sin\left(\frac{n\pi x}{3}\right) dx$ equal the amplitudes above:

$$A_1 = -\frac{1}{2}, \quad A_2 = 1, \quad A_7 = \frac{1}{2}, \quad A_n = 0 \text{ for } n \notin \{1, 2, 7\}.$$

For $g(x) = 1$ on $(0, 3)$:

$$\int_0^3 \sin\left(\frac{n\pi x}{3}\right) dx = \frac{3}{n\pi} (1 - (-1)^n) = \begin{cases} \frac{6}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Hence

$$B_n = \frac{2}{3 \cdot \frac{n\pi\sqrt{2}}{3}} \int_0^3 g(x) \sin\left(\frac{n\pi x}{3}\right) dx = \begin{cases} \frac{6\sqrt{2}}{n^2\pi^2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Therefore the solution is

$$\begin{aligned} u(x, t) = & -\frac{1}{2} \sin\left(\frac{\pi x}{3}\right) \cos\left(\frac{\pi\sqrt{2}t}{3}\right) + \sin\left(\frac{2\pi x}{3}\right) \cos\left(\frac{2\pi\sqrt{2}t}{3}\right) + \frac{1}{2} \sin\left(\frac{7\pi x}{3}\right) \cos\left(\frac{7\pi\sqrt{2}t}{3}\right) \\ & + \sum_{k=0}^{\infty} \frac{6\sqrt{2}}{(2k+1)^2\pi^2} \sin\left(\frac{(2k+1)\pi x}{3}\right) \sin\left(\frac{(2k+1)\pi\sqrt{2}t}{3}\right). \end{aligned}$$

Example 1.2.6. A dulcimer player plays an instrument with strings one meter long. He strikes the string exactly in the middle, from above, with a small hammer with a 2 centimetre long head, at a speed of 1 m/s. The string satisfies the vibrating string equation $u_t t''(x, t) = 2u''_{xx}(x, t)$. Describe the vibration of the string (the units are given in meter).

We model the string on the interval $0 < x < L$ with $L = 1$ m. The PDE given is

$$u_{tt}(x, t) = 2u_{xx}(x, t),$$

so $c^2 = 2$ and $c = \sqrt{2}$. The string is fixed at the ends:

$$u(0, t) = u(1, t) = 0 \quad (t \geq 0).$$

Model the initial data coming from the hammer strike. The hammer head is 0.02 m long and hits the midpoint $x = \frac{1}{2}$, therefore the contact interval is

$$\left[\frac{1}{2} - 0.01, \frac{1}{2} + 0.01\right] = [0.49, 0.51].$$

The hammer gives (approximately) a uniform downward velocity 1 m/s to the contacted portion. Choosing the upward direction positive, the initial displacement is zero and the initial velocity is

$$f(x) = u(x, 0) = 0, \quad g(x) = u_t(x, 0) = \begin{cases} -1, & 0.49 < x < 0.51, \\ 0, & \text{otherwise (on } [0, 1]). \end{cases}$$

By Theorem 1.2.4, the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} \left(A_n \cos(n\pi ct) + B_n \sin(n\pi ct) \right) \sin(n\pi x),$$

with (here $L = 1$)

$$A_n = 2 \int_0^1 f(x) \sin(n\pi x) dx, \quad B_n = \frac{2}{n\pi c} \int_0^1 g(x) \sin(n\pi x) dx.$$

Since $f \equiv 0$ we have $A_n = 0$ for every n . For B_n we compute the integral over the contact interval:

$$\int_0^1 g(x) \sin(n\pi x) dx = \int_{0.49}^{0.51} (-1) \sin(n\pi x) dx = \frac{1}{n\pi} (\cos(0.51 n\pi) - \cos(0.49 n\pi)).$$

Hence

$$B_n = \frac{2}{n^2 \pi^2 c} (\cos(0.51 n\pi) - \cos(0.49 n\pi)).$$

Therefore, using that $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$,

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi\sqrt{2}t) \sin(n\pi x), \quad B_n = -\frac{2\sqrt{2}}{n^2 \pi^2} \sin\left(\frac{n\pi}{100}\right) \sin\left(\frac{n\pi}{2}\right).$$

Figure 1.3: Animation: Vibration of a hammered string. You might need to change your PDF reader for this.

1.2.2 D'Alembert's solution

Now, we consider an alternative solution, which takes into account the movement of the peaks of the waves, called D'Alembert's solution.

Theorem 1.2.7 (D'Alembert's solution I). *Let us consider the following vibrating string problem:*

$$\begin{cases} u''_{tt}(x, t) = c^2 u''_{xx}(x, t), & 0 < x < L, t > 0, \\ u(0, t) = u(L, t) = 0, & t \geq 0, \\ u(x, 0) = f(x) & 0 \leq x \leq L, \\ u'_t(x, 0) = g(x), & 0 \leq x \leq L, \end{cases}$$

Then

$$u(x, t) = \frac{\widehat{f}(x+ct) + \widehat{f}(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \widehat{g}(y) dy,$$

where $\widehat{f}(x)$ and $\widehat{g}(x)$ are the $2L$ -periodic, odd extension of the functions $f(x)$ and $g(x)$ defined in Definition 1.1.3.

Proof. Assume that the solution has the form

$$u(x, t) = F(x + ct) + G(x - ct).$$

Then

$$u''_{xx}(x, t) = F''(x + ct) + G''(x - ct) \text{ and } u''_{tt}(x, t) = c^2 F''(x + ct) + c^2 G''(x - ct),$$

thus, $u(x, t)$ satisfies the vibrating string equation $u''_{tt}(x, t) = c^2 u''_{xx}(x, t)$.

Now, let us extend the initial conditions $2L$ -periodic and odd functions on $\mathbb{R}$. The initial conditions imply

$$\begin{aligned} \widehat{f}(x) &= u(x, 0) = F(x) + G(x), \\ \widehat{g}(x) &= u'_t(x, 0) = cF'(x) - cG'(x). \end{aligned}$$

Integrating the last equality, we get that for every $x \in \mathbb{R}$

$$\int_0^x \widehat{g}(y) dy = cF(x) - cF(0) - cG(x) + cG(0).$$

Solving the linear equation system

$$\left. \begin{aligned} cF(x) - cG(x) &= \int_0^x \widehat{g}(y) dy + cF(0) - cG(0) \\ F(x) + G(x) &= \widehat{f}(x) \end{aligned} \right\}$$

we get

$$F(x) = \frac{\widehat{f}(x)}{2} + \frac{1}{2c} \int_0^x \widehat{g}(y) dy + F(0) - G(0) \text{ and } G(x) = \frac{\widehat{f}(x)}{2} - \frac{1}{2c} \int_0^x \widehat{g}(y) dy - F(0) + G(0).$$

Hence,

$$\begin{aligned} u(x, t) &= F(x + ct) + G(x - ct) \\ &= \frac{\widehat{f}(x + ct) + \widehat{f}(x - ct)}{2} + \frac{1}{2c} \int_0^{x+ct} \widehat{g}(y) dy - \frac{1}{2c} \int_0^{x-ct} \widehat{g}(y) dy \\ &= \frac{\widehat{f}(x + ct) + \widehat{f}(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \widehat{g}(y) dy. \end{aligned}$$

Finally, let us check the boundary conditions $u(0, t) = u(L, t) = 0$. Substituting $t = 0$, we get

$$\begin{aligned} u(0, t) &= \frac{\widehat{f}(ct) + \widehat{f}(-ct)}{2} + \frac{1}{2c} \int_{-ct}^{ct} \widehat{g}(y) dy = \frac{\widehat{f}(ct) - \widehat{f}(ct)}{2} + \frac{1}{2c} \int_0^{ct} \widehat{g}(y) dy + \frac{1}{2c} \int_{-ct}^0 \widehat{g}(y) dy \\ &\stackrel{\text{substituting}}{=} \frac{1}{2c} \int_0^{ct} \widehat{g}(y) dy + \frac{1}{2c} \int_{ct}^0 \widehat{g}(-y)(-1) dy = \frac{1}{2c} \int_0^{ct} \widehat{g}(y) dy - \int_0^{ct} \widehat{g}(y) dy = 0, \end{aligned}$$

where we used that $\hat{f}$ and $\hat{g}$ are odd functions. Similarly,

$$\begin{aligned}
 u(L, t) &= \frac{\hat{f}(L + ct) + \hat{f}(L - ct)}{2} + \frac{1}{2c} \int_{L-ct}^{L+ct} \hat{g}(y) dy \\
 &\stackrel{\text{using oddity \& substituting } z=-y}{=} \frac{\hat{f}(L + ct) - \hat{f}(-L + ct)}{2} + \frac{1}{4c} \int_{L-ct}^{L+ct} \hat{g}(y) dy + \frac{1}{4c} \int_{-L+ct}^{-L-ct} \hat{g}(-y)(-1) dy \\
 &\stackrel{\text{by } 2L\text{-period.}}{=} \frac{\hat{f}(L + ct) - \hat{f}(L + ct)}{2} + \frac{1}{4c} \int_{L-ct}^{L+ct} \hat{g}(y) dy - \frac{1}{4c} \int_{L-ct}^{L+ct} \hat{g}(y) dy = 0.
 \end{aligned}$$

□

Observe that we have not used during the proof the boundary conditions to find the solutions. Contrary, we have verified it after we got the solution. This makes it possible to apply D'Alembert's solution for infinite strings, when the string is identified with the whole real line.

Theorem 1.2.8 (D'Alembert's solution II). *Let us consider the following infinite vibrating string problem:*

$$\begin{cases} u''_{tt}(x, t) = c^2 u''_{xx}(x, t), & x \in \mathbb{R}, t > 0, \\ u(x, 0) = f(x) & x \in \mathbb{R}, \\ u'_t(x, 0) = g(x), & x \in \mathbb{R}, \end{cases}$$

Then

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy.$$

Example 1.2.9. *An infinite string is linearly stretched between the "points" $(-\infty, 0)$, $(-1, 0)$, $(0, 2)$, $(1, 0)$ and $(0, \infty)$ in this order, and is released at time $t = 0$. An infinite string is fixed at the points $(-1, 0)$, $(0, 2)$, $(1, 0)$ and released at time $t = 0$. Let $u(x, t)$ be the function which describes the movement of the infinite vibrating string, which satisfies the equation $u''_{tt} = 4u''_{xx}$. Describe the movement of the string at point $x = 2$.*

The initial condition is

$$f(x) = \begin{cases} 2(1 - |x|), & |x| \leq 1, \\ 0, & |x| > 1, \end{cases} \quad \text{and} \quad g(x) \equiv 0.$$

By Theorem 1.2.8, the solution is

$$\begin{aligned}
 u(x, t) &= \frac{f(x + ct) + f(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy \\
 &= \frac{f(x + 2t) + f(x - 2t)}{2}.
 \end{aligned}$$

We are asked for the motion at $x = 2$. Thus

$$u(2, t) = \frac{f(2 + 2t) + f(2 - 2t)}{2}.$$

Observe that f is supported on $[-1, 1]$, but, for all $t \geq 0$ we have $2 + 2t \geq 2 > 1$, hence $f(2 + 2t) = 0$. Furthermore $|2 - 2t| \leq 1 \iff t \in [\frac{1}{2}, \frac{3}{2}]$. Therefore

$$u(2, t) = \frac{f(2 - 2t)}{2} = \begin{cases} (1 - |2 - 2t|), & \text{if } t \in [\frac{1}{2}, \frac{3}{2}], \\ 0, & \text{else.} \end{cases}$$

Splitting at $t = 1$ gives the piecewise linear form

$$u(2, t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ 2t - 1, & \frac{1}{2} \leq t \leq 1, \\ 3 - 2t, & 1 \leq t \leq \frac{3}{2}, \\ 0, & t > \frac{3}{2}. \end{cases}$$

Thus the motion at $x = 2$ is a triangular pulse in time: it is zero until $t = \frac{1}{2}$, then grows linearly to the maximum value $u(2, 1) = 1$ at $t = 1$, and decays linearly back to zero at $t = \frac{3}{2}$; outside $[1/2, 3/2]$ the displacement at $x = 2$ is zero.

Figure 1.4: Animation: A tent in an infinite horizon. You might need to change your PDF reader for this.

1.2.3 Exercises

1.2/1. Let us consider the following vibrating string equation:

$$\begin{cases} u''_{tt}(x, t) = 3u''_{xx}(x, t) & 0 \leq x \leq 2, t \geq 0; \\ u(0, t) = u(2, t) = 0 & t \geq 0; \\ u(x, 0) = 0 & 0 \leq x \leq 2; \\ u'_t(x, 0) = \sin(\pi x) \left(\cos(2\pi x) + \frac{1}{2} \right) & 0 \leq x \leq 2. \end{cases}$$

Determine the function $u(x, t)$!

- 1.2/2. An archer has a 2 meter-long bow. He grabs the string of the bow exactly at the middle and stretches his bow 1 meter-long and releases it. The bow satisfies the vibrating string equation $u''_{tt} = u''_{xx}$. Describe the movement of the string of the bow in time!
- 1.2/3. A meteor impacts into the ocean. (Although the ocean is a surface, we model it here with an infinite string). The meteor hits the ocean on the interval $[-1, 1]$ (the lengths are given in km) with velocity 60 km/h (note that the meteor hits the ocean from above). The vibration of the ocean is described by the equation $u''_{tt} = 4u''_{xx}$. There is a ship 15km away from the origin to the right. Describe the movement of the ship in time.

1.3 Heat transport

Finally, we will consider heat conduction in a special case. Let us consider a rod of length L identified with the $[0, L]$ interval. Let $u(x, t)$ denote temperature at position $x \in (0, L)$ and time $t > 0$. We will assume that the endpoints of the rod have constant temperature zero.

Definition 1.3.1. We call the boundary–initial value problem

$$\begin{cases} u'_t(x, t) = \kappa u''_{xx}(x, t), & 0 < x < L, \ t > 0, \\ u(0, t) = u(L, t) = 0, & t \geq 0, \\ u(x, 0) = f(x) & 0 < x < L, \end{cases}$$

where $\kappa > 0$ is called the **thermal diffusivity** constant, and $f(x)$ is the initial temperature as **heat transport problem**.

Our goal is now to describe how the temperature changes with time.

Theorem 1.3.2. Let us consider the following heat transport problem:

$$\begin{cases} u'_t(x, t) = \kappa u''_{xx}(x, t), & 0 < x < L, \ t > 0, \\ u(0, t) = u(L, t) = 0, & t \geq 0, \\ u(x, 0) = f(x) & 0 \leq x \leq L. \end{cases}$$

Then

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 \kappa t} \sin\left(\frac{n\pi}{L}x\right), \text{ where } A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx,$$

that is, A_n are the Fourier-sine coefficients of $f(x)$.

Proof. We solve the heat equation on the rod $0 < x < L$ with similar method to Bernoulli's solution in case of vibrating strings, namely, by separation of variables.

Assume that $u(x, t) = X(x)T(t)$ with $X \neq 0$, $T \neq 0$. Substitution gives

$$X(x)T'(t) = \kappa X''(x)T(t) \implies \frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left-hand side depends only on t and the right-hand side depends only on x , this is only possible if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\frac{T'(t)}{\kappa T(t)} = \lambda \text{ and } \frac{X''(x)}{X(x)} = \lambda.$$

Notice that

$$\lambda \kappa = \frac{T'(t)}{T(t)} = (\log T(t))' \implies T(t) = Ae^{\lambda \kappa t}.$$

This implies that physically the only relevant choice for the constant λ is $\lambda < 0$. Otherwise, if $\lambda > 0$ then $|T(t)| \rightarrow \infty$ as $t \rightarrow \infty$, which means that the temperature blows up (without outer source of energy), and if $\lambda = 0$ then $T(t) \equiv A$, which means that the temperature would be unchanged in time.

On the other hand, we got that the function $X(x)$ satisfies the following initial problem ordinary differential equation

$$X''(x) - \lambda X(x) = 0, \quad X(0) = X(L) = 0.$$

while similarly to the previous section, $X(t)$ has the form

$$X(x) = B \cos(\sqrt{-\lambda}x) + C \sin(\sqrt{-\lambda}x).$$

Using the boundary condition, we get $X(0) = B = 0$, and $X(L) = \sin(\sqrt{-\lambda}L) = 0$, which gives that $\sqrt{-\lambda} = \frac{n\pi}{L}$ for some $n \in \mathbb{N}$. In particular, $\lambda = -\left(\frac{n\pi}{L}\right)^2$. Hence, the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\kappa(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

where we used that the (countable) linear combination of the solutions of $u'_t = \kappa u''_{xx}$ is a solution.

Now, considering the initial condition, substituting $x = 0$ into the general solution, we get

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right),$$

which means means that

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

□

Example 1.3.3. Solve the following heat transport equation

$$\begin{cases} u'_t(x, t) = 3u''_{xx}(x, t), & 0 < x < 2, t > 0, \\ u(0, t) = u(2, t) = 0, & t \geq 0, \\ u(x, 0) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 < x < 2, \end{cases}, & 0 \leq x \leq 2, \end{cases}$$

We apply Theorem 1.3.2 with $L = 2$, $\kappa = 3$ and

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 < x < 2, \end{cases}$$

and the general solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{2}\right)^2 3t} \sin\left(\frac{n\pi x}{2}\right).$$

Let us now compute A_n . It is easy to see that f resembles to the function $g(x) = \begin{cases} x & 0 \leq x < \pi/2 \\ \pi - x & \pi/2 \leq x \leq \pi \end{cases}$ in (III) in Section 1.1.1, where we see that

$$g(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 4}{\pi(2k+1)^2} \sin((2k+1)x). \quad (1.3.1)$$

We see that $f(2) = 0$ and $g(\pi) = 0$, so we might apply the scaling $t = \frac{\pi}{2}x$ in the variable, moreover, $f(1) = 1$ and $g(\pi/2) = \pi/2$, gives the scaling $y = \frac{2}{\pi}x$. In other words,

$$f(x) = \frac{2}{\pi} g\left(\frac{\pi}{2}x\right).$$

So using Proposition 1.1.7 and (1.3.1), we get

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 8}{\pi^2(2k+1)^2} \sin\left(\frac{(2k+1)\pi}{2}x\right).$$

Therefore

$$u(x, t) = \sum_{k=0}^{\infty} \frac{(-1)^k 8}{(2k+1)^2 \pi^2} \sin\left(\frac{(2k+1)\pi}{2}x\right) e^{-\frac{3(2k+1)^2 \pi^2}{4}t}.$$

1.3.1 Exercises

- 1.3/1. A rod of length 3 meters satisfies the heat transport equation $u'_t(x, t) = 2u''_{xx}(x, t)$, where $u(x, t)$ represents the temperature of the rod at time t and at position x . We heat up the whole rod to 5 except the endpoints which are kept on 0. Determine the function $u(x, t)$!

1.4 Mixed exercises in Partial Differential Equations

1.4/1. Solve the following vibrating string problem!

$$\begin{cases} u''_{tt}(x, t) = 3u''_{xx}(x, t) & 0 \leq x \leq 2, t \geq 0 \\ u(0, t) = u(2, t) = 0 & t \geq 0 \\ u(0, x) = \begin{cases} 0 & 0 \leq x < 1/2 \\ 1 & 1/2 \leq x \leq 3/2 \\ 0 & 3/2 < x \leq 2 \end{cases} \\ u'_t(0, x) = 0 & 0 \leq x \leq 2 \end{cases}$$

1.4/2. Let us consider the following vibrating string equation:

$$\begin{cases} u''_{tt} = 3u''_{xx} & 0 \leq x \leq 2, t \geq 0; \\ u(0, t) = u(2, t) = 0 & t \geq 0; \\ u(x, 0) = 0 & 0 \leq x \leq 2; \\ u'_t(x, 0) = \sin(\pi x) \cos(2\pi x) \cos(3\pi x) & 0 \leq x \leq 2. \end{cases}$$

Determine the function $u(x, t)$!

1.4/3. Let us consider the following vibrating string equation:

$$\begin{cases} u''_{tt} = u''_{xx} & 0 \leq x \leq 3, t \geq 0; \\ u(0, t) = u(3, t) = 0 & t \geq 0; \\ u(x, 0) = \frac{x(3-x)}{2} & 0 \leq x \leq 3; \\ u'_t(x, 0) = \begin{cases} 1 & 0 < x < 3, \\ 0 & x = 0, 3. \end{cases} \end{cases}$$

Determine the function $u(x, t)$!

1.4/4. Let us consider the following infinite vibrating string equation:

$$\begin{cases} u''_{tt}(x, t) = 4u''_{xx}(x, t) & x \in \mathbb{R}, t \geq 0; \\ u(x, 0) = 0 & x \in \mathbb{R} \\ u'_t(x, 0) = \begin{cases} 3 & 0 \leq x \leq 10, \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

Determine $u(-1, 2) = ?$

1.4/5. Consider the following heat transport equation:

$$\begin{cases} u_t'(x, t) = u_{xx}''(x, t) & x \in [0, 5], t \geq 0; \\ u(0, t) = u(5, t) = 0 & t \geq 0 \\ u(x, 0) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 4 - 2x & \text{if } 1 \leq x \leq 2 \end{cases} \end{cases}$$

Determine $u(x, t) = ?$