WEAK SEPARATION PROPERTY FOR SELF-SIMILAR SETS

BACHELOR THESIS

AUTHOR

Levente Dávid

SUPERVISOR

Balázs Bárány

Department of Stochastics

BME

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Budapest University of Technology and Economics

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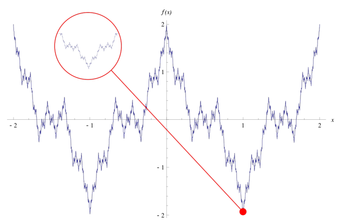
INTRODUCTION TO ITERRATED FUNCTION SYSTEMS

1.1 GENERAL THEORY OF IFS'S

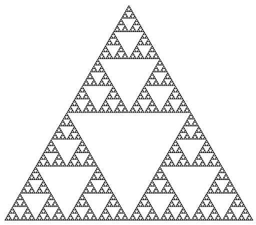
Both in nature and mathematics people observe geometric objects with deep self-reoccuring structures. The field which study these is called Fractal Geometry.

The development of the analysis of such objects started in the 19th century with mathematicians studying the diversity and the complexity of functions. One could have asked that does there exist a continuous function on $\mathbb{R}$ which is nowhere differentiable. The answer is yes, and an example is the Weierstrass function, and by that we have discovered a function with many new fractalic properties, leading us to a new area. The main breakthrough (becoming a widely know mathematical branch) came in the 20th century with Benoit Mandelbrot who founded and popularised the field. Even the name fractal, which comes from the latin fractus meaning broken, fractured was used first by him. Nowadays, fractal geometry is an actively researched field induced by the huge number of occurrences of fractals and self-similar structures not only in mathematics, but even in other scientific fields such as finance, biology and physics.

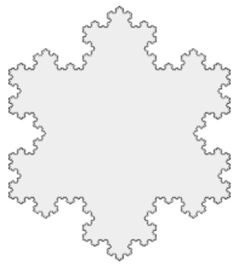


Weierstrass function

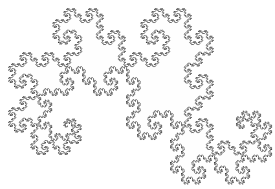
One widely used and studied way of constructing fractals are trough Iterated Function Systems (**IFS**), which in some case gave us much more simplistically



The Sierpiński gasket



Koch snowflake



Harter-Heighway dragon curve

looking fractals such as the well know ones like the Sierpiński gasket, the Koch snowflake or the Harter-Heighway dragon curve.

Definition 1.1.1 (IFS) *Let (X, dist) be a complete metric space. We say that a map $f\colon X \to X$ is a contraction if there exists $\lambda \in (0, 1)$ such that for any $x, y \in X : \mathrm{dist}(f(x), f(y)) \le \lambda \cdot \mathrm{dist}(x, y)$. The appropriate λ is called the contracting ratio of f. We call a finite collection of contractions $\Phi = \{f_1, f_2, \ldots, f_m\}$ an Iterated Function System.*

We can extend the definition of the distance to sets: let $\mathrm{dist}(A, B)$ denote $\inf\{\mathrm{dist}(a, b) \mid a \in A, b \in B\}$ throughout the document. We denote by $B(A, r) := \{x \mid \mathrm{dist}(A, x) \le r\}$ the closed r-neighbourhood of the set A. At many cases we need a distance which can distinguish between intersecting sets, the previusly defined distance fails at this. This observation leads to the definition of an other distance $\mathrm{dist}_{\mathrm{H}}(A, B) := \inf\Big\{r \ge 0 \;\Big|\; A \subseteq B(B, r) \text{ and } B \subseteq B(A, r)\Big\}$ which is called Hausdorff distance/metric. The following theorem lets us define the invariant set of the IFS, which will be the main object we will study.

Theorem 1.1.1 (Hutchinson, [9]) *For every IFS $\Phi = \{f_1, \ldots, f_m\}$ there exist a unique, non-empty, compact set $\Lambda \subset X$ such that $\Lambda = \cup_{i=1}^{m} f_i(\Lambda)$. We call Λ the attractor of the IFS.*

The draft of the construction of the attractor is the following:

- We start by just an arbitrary ball with radius R centered at x_o. To get the $\Lambda = \cup_{i=1}^m f_i(\Lambda)$ we need the R large enough so that $f_i(B(x_o, \mathrm{R})) \subseteq B(x_o, \mathrm{R})$ for any $i \in \{1, \ldots, m\}$ By the maps being contractions and the finiteness of the set of maps we can choose such $R > 0$.

- Now we iterate the ball trough the f_i-s, and then union them up getting: $\Lambda_n := \cup_{i_1=1}^m \cup_{i_2=1}^m \ldots \cup_{i_n=1}^m (f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n})(B(x_o, \mathrm{R}))$.

- Then $\Lambda_{n+1} \subseteq \Lambda_n$ by the choice of R and Λ_n is an union of finite many compact sets because the maps are contractions. These two properties let us use the Cantor intersection theorem, which gives us that there is a $\Lambda \subseteq \mathbb{R}^d$ compact, non-empty such that $\Lambda = \cap_{n=1}^\infty \Lambda_n$.

- Note that we are done because

$$\begin{aligned}\Lambda &= \cap_{n=1}^\infty \cup_{i_1=1}^m \cup_{i_2=1}^m \ldots \cup_{i_n=1}^m (f_{i_1} \circ \ldots \circ f_{i_n})(B(x_o, \mathrm{R})) \\ &= \cup_{i_1=1}^m f_{i_1}\Big(\cap_{n=2}^\infty \cup_{i_2=1}^m \ldots \cup_{i_n=1}^m (f_{i_1} \circ \ldots \circ f_{i_n})(B(x_o, \mathrm{R}))\Big) \\ &= \cup_{i_1=1}^m f_{i_1}(\Lambda).\end{aligned}$$

1.2 MEASURE AND DIMENSION

In geometry, it is usual to consider measuring length, area, volume and so on. Doing so, given nice set, we might prefer values between 0 and ∞ not containing the borders, but at many cases with fractals we can get infinite length but zero area. This phenomenon leads us to consider some sort of measure between the integer dimensions.

We first need a generalization of measure on $\mathbb{R}^d$, or even more, we would like to have a generalized $s \in [0, d]$ dimensional measure in $\mathbb{R}^d$. The following construction gives us exactly what we want, with a nice geometric intuition trough just containment in unions and limit.

Definition 1.2.1 (s-dimensional Hausdorff measure) *Let* $(\mathcal{X}, \mathrm{dist})$ *be a complete metric space,* $E \subset X$*. For* $\delta > 0$ *and* $s \geq 0$*:*

$$\mathcal{H}^s_\delta(E) := \inf\left\{ \sum_{i\in I} |U_i|^s \;\middle|\; I \text{ is countable, } E \subseteq \cup_{i\in I} U_i \right.$$
$$\left. \text{and } \forall i \in I: \; |U_i| \leq \delta \right\}.$$

Then the s-dimensional Hausdorff outer measure of E is:

$$\mathcal{H}^s(E) := \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E).$$

Another measure used in the field is the Hausdorff content, which does not always equal to the Hausdorff measure. Their relation is studied in [7].

Definition 1.2.2 (s-dimensional Hausdorff content) *Let* $(\mathcal{X}, \mathrm{dist})$ *be a complete metric space,* $E \subset X$*. For* $s \geq 0$ *the s-dimensional Hausdorff content of E is:*

$$\mathcal{H}^s_\infty(E) := \inf\left\{ \sum_{i\in I} |U_i|^s \;\middle|\; I \text{ is countable, } E \subseteq \cup_{i\in I} U_i \right\}.$$

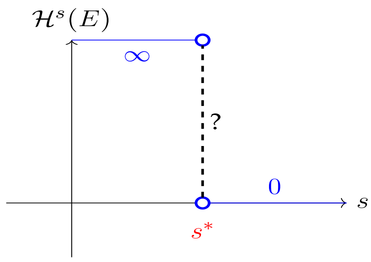
Now after we concluded a measure, we ask that which/what sets are measurable? A property that gives nice answer to this is that the Hausdorff measure is a metric-outer measure, meaning that for any two positively separated sets, the measure of their union equals to the sum of their measures. It is know that for a metric-outer measure every Borel set $B \in \mathcal{B}$ is measurable see [3].

After this we can notice that the Hausdorff Measure can be thought of as a function from $\mathbb{R}^+ \times \mathcal{B}(\mathbb{R}^d)$ to $\mathbb{R}^{+,0}$. Therefore it is natural to ask that is any continuity happening in the first variable which is the measure dimension? The following Lemma gives us an answer for exactly this.

Lemma 1.2.1 *For* $E \in \mathcal{B}(\mathbb{R}^d)$*, for any* $\alpha > 0$ *we have:*

(1) If $\mathcal{H}^\alpha(E) < \infty$, then for any $\beta > \alpha$ we have: $\mathcal{H}^\beta(E) = 0$.

(2) If $\mathcal{H}^\alpha(E) > 0$, then for any $\beta < \alpha$ we have: $\mathcal{H}^\beta(E) = \infty$.



Corollary: *This lets us to have at most one positive real γ such that $\infty > \mathcal{H}^\gamma(E) > 0$.*

Proof:

- We only give a proof for the first statement, the second one follows in a similar fashion. From these two, the Corollary comes trivially.

- Given $\mathcal{H}^\alpha(E) < \infty$, by the definition of the Hausdorff measure: for every $\varepsilon > 0$ exists $\delta > 0$, exists $U_i, i \in I$ countable cover of E such that for any i we have $|U_i| < \delta$ and $\sum_{i\in I} |U_i|^\alpha \leq \mathcal{H}^\alpha(E) + \varepsilon$.

- Then we have: $\sum_{i\in I} |U_i|^\beta \leq \delta^{(\beta-\alpha)} \sum_{i\in I} |U_i|^\alpha \leq \delta^{(\beta-\alpha)}\Big(\mathcal{H}^\alpha(E) + \varepsilon\Big) \to 0$ as δ goes to 0 proving the statement.

This phenomena let's us to define a property of the set E at the snapping point, which will be well-defined by the previous lemma.

Definition 1.2.3 (Hausdorff dimension) *Let $(\mathcal{X}, \mathrm{dist})$ be a complete metric space, for $E \subset X$ the Hausdorff dimension is:*

$$\begin{aligned}\dim_{\mathrm{H}} E &:= \inf\{\alpha > 0 \mid \mathcal{H}^\alpha(E) = 0\} \\ &:= \sup\{\alpha > 0 \mid \mathcal{H}^\alpha(E) = \infty\}.\end{aligned}$$

Lemma 1.2.2 *Let $(\mathcal{X}, \mathrm{dist})$ be a complete metric space, then for any $A \subset \mathcal{X}$: $\dim_{\mathrm{H}} A := \inf\{s \geq 0 \mid \mathcal{H}^s_\infty(A) = 0\}$.*

Some properties of the Hausdorff dimension:

1. Monotonicity: $\dim_{\mathrm{H}} A \leq \dim_{\mathrm{H}} B,\ \forall A \subseteq B$.

2. Countable stability: $\dim_{\mathrm{H}}\Big\{\cup_{i\in \mathrm{I}} E_i\Big\} = \sup_{i\in\mathrm{I}}\Big\{\dim_{\mathrm{H}} E_i\Big\}$ for all I countable set.

3. $\mathrm{Vol}(E) > 0 \implies \dim_{\mathrm{H}} E = d$, where Vol is the d-dimensional Lebesgue measure.

4. $f : X \to Y$ is α-Hölder (that is $\exists C > 0\ \forall x, y \in X : |f(x) - f(y)| \leq C \cdot |x - y|^{\alpha}) \implies \dim_{\mathrm{H}}\Big\{f(E)\Big\} \leq \frac{\dim_{\mathrm{H}} E}{\alpha}$.

5. $f : X \to Y$ is bi-Lipschitz (that is $\exists L$ such that $\forall x, y \in X : \frac{1}{L}|x - y| \leq |f(x) - f(y)| \leq L|x - y|) \implies \dim_{\mathrm{H}}\Big\{f(E)\Big\} = \dim_{\mathrm{H}} E$.

6. $f : X \to Y$ is a similarity $(\varrho(x, y) = \lambda \cdot \varrho(f(x), f(y))\ ) \implies \mathcal{H}^{\alpha}(f(E)) = \lambda^{\alpha}\mathcal{H}^{\alpha}(E)$.

For the proof, see [5].

Many more dimension concepts have been developed, we will use the following two: lower- and upper-box-counting dimension:

$$\underline{\dim}_{\mathrm{B}} E := \liminf_{\delta\to 0^+} \frac{\log\{\mathcal{N}_\delta(E)\}}{-\log\{\delta\}}, \quad \overline{\dim}_{\mathrm{B}} E := \limsup_{\delta\to 0^+} \frac{\log\{\mathcal{N}_\delta(E)\}}{-\log\{\delta\}},$$

where $\mathcal{N}_\delta(E) := \min\{m > 0 \mid \exists x_1, x_2, \dots, x_m : E \subseteq \cup_{i=1}^m B(x_i, \delta)\}$. If $\underline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{B}} E$, then we can talk about the **box-counting dimension** : $\dim_{\mathrm{B}} E := \overline{\dim}_{\mathrm{B}} E$. The box-counting dimension can also be computed with packings instead of coverings. Let $\mathcal{P}_r(E) := \max\{m > 0 \mid \exists x_1, x_2, \dots x_m \in E : B(x_i, r) \cap B(x_j, r) \neq \emptyset \implies i = j\}$ be the maximal δ packing of the set E. Then $\mathcal{P}_r(E) \leq \mathcal{N}_r(E)$ trivially and $\mathcal{P}_r(E) \geq \mathcal{N}_{2r}(E)$ because if x_i-s give $\mathcal{N}_{2r}(E) = t$ then those x_i-s give a at least t disjoint balls with radius r otherwise the balls which create intersection would also be unnecessary for the covering. Therefore:

$$\underline{\dim}_{\mathrm{B}} E := \liminf_{\delta\to 0^+} \frac{\log\{\mathcal{P}_\delta(E)\}}{-\log\{\delta\}}, \quad \overline{\dim}_{\mathrm{B}} E := \limsup_{\delta\to 0^+} \frac{\log\{\mathcal{P}_\delta(E)\}}{-\log\{\delta\}}.$$

The general relationship between the Hausdorff dimension and the box-counting dimensions is: $\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}} E$.

1.3 SYMBOLIC SPACE

Definition 1.3.1 *Let $\mathcal{A}$ be a set of finite symbols. Then we define the set of n-length words by $\Sigma_n := \mathcal{A}^n$, $\forall n \in \mathbb{N}$. The set of words with finite length is $\Sigma^* := \bigcup_{n=0}^{\infty} \mathcal{A}^n$. Finally, the **Symbolic Space** is the set of words with infinite length $\Sigma := \mathcal{A}^{\mathbb{N}}$.*

The symbolic space is particularly useful by the following:
Given an IFS $\boldsymbol{\Phi} = \{f_1, f_2, \ldots, f_m\}$, we denote $f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n}$ by just $f_{(j_1, j_2, \ldots, j_n)}$ $= f_{\mathbf{i}}$ for $\mathbf{i} = (j_1, \ldots, j_n) \in \Sigma^*$. Now it is natural ask that can we extend this to infinite words. The following map gives us just that. Introduce a natural map π from the Symbolic Space Σ given by $\mathcal{A} = \{1, 2, \ldots, m\}$ to the points of the attractor: for $\mathbf{i} = (i_1, i_2, \ldots) \in \Sigma$

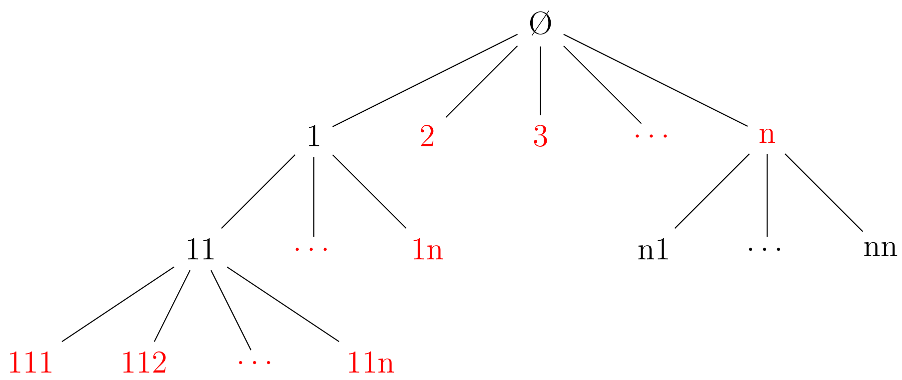
$$\begin{aligned} \pi : \mathbf{i} \longmapsto f_{\mathbf{i}}(\Lambda) &:= \bigcap_{n=1}^{\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(\Lambda) \\ &= \lim_{n \to \infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(0). \end{aligned}$$

The symbolic space allows us to analyze fractals much more. Two of the constructions for this are the measure on the symbolic space, which can be pushed to the attractor and the minimal cut-sets/partition: Let $\mathbf{p} := (p_i)_{i \in \{1, \ldots, m\}}$ be a probability vector. For $\mathbf{i} \in \Sigma^*$, let $[\mathbf{i}] = [(i_1, i_2, i_3, \ldots, i_n)] = \{\tau \in \Sigma \mid \forall l \in \{1, 2, \ldots, n\} : i_l = \tau_l\}$. Now for $\mathbf{i} = (i_1,\ i_2,\ \ldots\ , i_n)$ define $\nu([\mathbf{i}]) := p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_n}$, then the Kolmogorov extension theorem extends this to Σ giving us a measure called the Bernoulli-measure. Another useful notation is for $\mathbf{j} = (j_1, j_2, \ldots, j_{n-1}, j_n) \in \Sigma^*$ we define $\mathbf{j}^- := (j_1, j_2, \ldots, j_{n-1})$.

Definition 1.3.2 (minimal cut-sets/partition) *We call* $\Gamma \subseteq \Sigma^*$ *a minimal cut-sets/partition of* Σ *if:*

1. *For* $\forall \boldsymbol{i}, \boldsymbol{j} \in \Gamma$ *such that* $\boldsymbol{i} \neq \boldsymbol{j}$ *we have* $[\boldsymbol{i}] \cap [\boldsymbol{j}] = \emptyset$.

2. $\bigcup_{\boldsymbol{i} \in \Gamma} [\boldsymbol{i}] = \Sigma$.



A possible partition in Σ generated by $\mathcal{A} = \{1, 2, \ldots, n, \}$

The following property is the very essence of the partition:

If Γ is a partition and ν is a Bernoulli-measure on Σ, then $\sum_{\mathbf{i} \in \Gamma} \nu([\mathbf{i}]) = 1$. This also ensures us that ν is a probability measure on Σ. Furthermore if C denotes the σ-algebra generated by the cylinders then ν is C-measureble.

Definition 1.3.3 (stationary measure) *Let* μ *be a push-forward of some Bernoulli-measure on* Σ^* *to the attractor, meaning:* $\mu = \pi * \nu = \nu \circ \pi^{-1}$. *For* $A \subset X$ *Borel, define* $\mu(A) := \nu(\pi^{-1}(A))$. *We call* μ *a stationary measure on the attractor.*

1.4 PROPERTIES OF WELL SEPARATED SETS

In this section we will see some interesting results using more or less the regularity of well behaving fractals. As we move forward, the reader might ask, are these quite strict properties what we stated necessary? If not, what kind of generalization can we get? The constraint of our generality is that from now on in this paper we will be interested in only $X = \mathbb{R}^d$ with dist being the Euclidean metric.

Definition 1.4.1 (self-similar) *If the maps $f_i : \mathbb{R}^d \to \mathbb{R}^d$ of the IFS are similarities $(i = 1, \ldots, m)$:*

$$f_i(x) = \lambda_i \cdot O_i \cdot \underline{x} + \underline{t}_i \quad \text{where } \lambda_i \in (0,1) \quad t_i \in \mathbb{R}^d$$
$$O_i \in \mathcal{O}(d, \mathbb{R}) \text{ (the set of } d \times d \text{ orthonormal matrices)},$$

then we call the IFS self-similar, and the attractor self-similar set.

The following two notations will be used constantly: $\lambda_{\min} := \min_{i=1}^m(\lambda_i), \lambda_{\max} := \max_{i=1}^m(\lambda_i)$. The first use of self-similarity is a cover what we can define with it:

Definition 1.4.2 (Moran cover) *For an IFS $\Phi = \{f_1, \ldots, f_m\}$ with $f_i(x) = \lambda_i \cdot O_i \cdot \underline{x} + \underline{t}_i$ for all $i \in \{1, \ldots, m\}$, with attractor Λ. We define the Moran cut-set with parameter $r \in \mathbb{R}^+$ as follows:*

$$\widetilde{\mathcal{M}}_r := \Big\{ \mathbf{j} = (j_1, j_2, \ldots, j_{k-1}, j_k) \in \Sigma^* \;\Big|\; |f_{\mathbf{j}}(\Lambda)| \leq r < |f_{\mathbf{j}^-}(\Lambda)| \Big\}.$$

The Moran cut-set with parameter r can be interpreted as words in Σ^* such that the combined contraction of any word is at order r. The Moran cover now easily follows:

$$\mathcal{M}_r := \Big\{ \Lambda_{\mathbf{j}} = (f_{j_1} \circ \ldots \circ f_{j_k})(\Lambda) \subseteq \mathbb{R}^d \;\Big|\; (j_1, \ldots, j_k) \in \widetilde{\mathcal{M}}_r \Big\}.$$

Secondly, with self-similarity we can formalize and then decompose the defining f_i-s. Considering that rotations and transitions usually does not change the dimension, we want to try to define a formal dimension concept only using the defining functions's contracting ratios:

1. We want to give an upper bound to the Hausdorff dimension by using the attractor's level-n cylinders, that is the images of the attractor trough any n-length composition of the generating functions.
2. For a given n-length word **i** the diameter $|f_{(i_1,\ldots,i_n)}(\Lambda)| = |\Lambda| \cdot \lambda_{i_1} \ldots \lambda_{i_n}$.
3. After this, using only level-n covers, the sum in the Hausdorff measure's definition can be expressed:

$$\sum_{i\in I} |U_i|^s = |\Lambda|^s \cdot \sum_{i\in I} \lambda_{i_1}^s \ldots \lambda_{i_n}^s = |\Lambda|^s \cdot (\lambda_1^s + \ldots + \lambda_m^s)^n$$

4. As in the definition of the Hausdorff dimension, we want a number where the measure presumably snaps. In the definition of the Hausdorff measure we let δ to go to zero, which can be translated to here by letting n to go to ∞. As this happens it is clear that $(\lambda_1^s + \ldots + \lambda_m^s)^n$ can be only nontrivial if $(\lambda_1^s + \ldots + \lambda_m^s) = 1$. Hence it is natural to define a dimension at this point.

Definition 1.4.3 (similarity dimension) *For an IFS $\Phi = \{f_1, \ldots, f_m\}$ of similarities with the contracting ratios: $\lambda_1, \ldots, \lambda_m$, the similarity dimension of the IFS and the attractor is $s_o \in \mathbb{R}^+$ such that s_o is the unique solution for $\sum_{i=1}^m (\lambda_i)^{s_o} = 1$.*

The similarity dimension gives us an easy to compute and visually intuitive concept. On the other hand, it does not care about whether some $f_i(\Lambda)$-s overlap. In this case the the similarity dimension fails to give us meaningful information, it is just an upper bound of the previous mentioned Hausdorff and Box-counting dimension. At worst if we let the collisions to grow bigger, the similarity dimension can be far from the Hausdorff and the box-counting dimension.

Definition 1.4.4 (strong separation condition) *Let* $\Phi = \{f_1, \ldots, f_m\}$ *be an IFS with its attractor* Λ. *We say that* Λ *and* Φ *satisfies the strong separation condition* (**SSC**) *if for* $\forall i \neq j \in \{1, \ldots, m\} : f_i(\Lambda) \cap f_j(\Lambda) = \emptyset$.

Definition 1.4.5 (open set condition) *We say that an IFS* $\Phi = \{f_1, \ldots, f_m\}$ *and its attractor satisfies the open set condition* (**OSC**) *if* $\exists\, \mathcal{U} \in \mathbb{R}^d$ *bounded, open, non-empty set with* $f_i(\mathcal{U}) \subseteq \mathcal{U}\ \forall i \in \{1, ..., m\}$ *and* $f_i(\mathcal{U}) \cap f_j(\mathcal{U}) = \emptyset$ *for any* $i \neq j$, *both* $\in \{1, ..., m\}$.

First, it is easy to see that the SSC implies the OSC. One might like to use the SSC because it is visual clarity, and easy to use, but as mentioned in the start of the section we want a more general one which is the OSC, and as we move forward the reader will see that many theorems only use the OSC.

Theorem 1.4.1 (Hutchinson, [9],) *Let* Φ *be a self-similar IFS with attractor* Λ *on* $\mathbb{R}^d$ *such that the OSC holds. Then*

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = s_o.$$

where s_0 *is the similarity dimension. Furthermore* $0 < \mathcal{H}^{s_o}(\Lambda) < \infty$.

This proves to be very helpful with finding the Hausdorff dimension, even when computing s_o is not possible explicitly, numerical approximations help very much.

Theorem 1.4.2 (Bandt-Graf, [1]) *Let* $\Phi = \{f_i(x) = \lambda_i O_i \underline{x} + \underline{t}_i\}$ *be a self-similar IFS with attractor* Λ. *Then the following are equivalent:*

1. *OSC holds for* Φ.
2. $0 < \mathcal{H}^{s_o}(\Lambda)$.
3. Λ *is* s_o*-Ahlfors regular:* $\exists c > 0\ \forall r \leq |\Lambda|\ \forall x \in \Lambda$:

$$\frac{1}{c} \leq \frac{\mathcal{H}^{s_o}(\Lambda \cap B(x, r))}{r^{s_o}} \leq c.$$

4. $\exists c > 0\ \forall y \in \Lambda\ \forall r < |\Lambda|\ :\ \#\{\mathbf{i} \in \widetilde{\mathcal{M}}_r \mid f_{\mathbf{i}}(\Lambda) \cap B(y,r) \neq \emptyset\} \leq c$.

Theorem 1.4.3 (Falconer, [6]) *Let Λ be a self-similar set. Then:*

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda \quad \text{and for } t := \dim_{\mathrm{H}} \Lambda \text{ we have } \mathcal{H}^t(\Lambda) < \infty.$$

Proof:

- We use a packing: Let us recall $\mathcal{P}_r(E) := \max\{m > 0 \mid \exists x_1, x_2, \dots x_m \in E : B(x_i, r) \cap B(x_j, r) \neq \emptyset \implies i = j\}$, and then the upper-box-counting dimension is $\overline{\dim}_{\mathrm{B}}\Lambda := \limsup_{\delta \to 0^+} \frac{\log\{\mathcal{P}_\delta(\Lambda)\}}{-\log\{\delta\}}$.

- Fix $\delta > 0$: Let $y_1, \dots, y_N \in \mathbb{R}^d$ be the centers of balls, who-s collection attains this maximum. Let $\tau_1, \dots, \tau_N \in \Sigma$ be such that $\pi(\tau_i) = y_i\ \forall i \in \{1, \dots, N\}$. Let $\mathbf{j}_1, \dots, \mathbf{j}_N \in \widetilde{\mathcal{M}}_{\delta|\Lambda|}$, then for all i we have: $\tau_i \in [\mathbf{j}_i] := \left\{\mathbf{i} \in \Sigma \,\middle|\, \mathbf{i}_k = (\mathbf{j}_i)_k\ \forall k \in \{1, \dots, |\mathbf{j}|\}\right\}$. Denote by $\mathcal{K}_\delta := \{f_{\mathbf{j}_i}\}_{i=1}^N$ which defines a new IFS, and let us denote it's attractor by Λ_δ.

  Then $\Lambda_\delta \subseteq \Lambda$. Also we now that $\forall k \neq l$ we have $f_{\tau_k}(\Lambda) \cap f_{\tau_l}(\Lambda) = \emptyset$ by of the Moran-cover being a partition. These two property give us that Λ_δ satisfies the Strong Separation Condition.

  Hence it also satisfies the OSC and therefore Theorem 1.4.1 holds, giving us: $\dim_{\mathrm{H}} \Lambda \geq \dim_{\mathrm{H}} \Lambda_\delta = s_\delta$, where s_δ is the similarity dimension of the new fractal. Hence

$$1 = \sum_{e=1}^{N} \lambda_{\tau_e}^{s_\delta} \begin{cases} \geq \lambda_{\min}^{s_\delta} \cdot \dfrac{\delta^{s_\delta}}{|\Lambda|^{s_o}} \cdot \mathcal{P}_\delta(\Lambda) \\ \leq \dfrac{\delta^{s_\delta}}{|\Lambda|^{s_o}} \cdot \mathcal{P}_\delta(\Lambda) \end{cases} \implies \begin{cases} \delta^{-s_\delta} \cdot |\Lambda|^{s_o} \geq \lambda_{\min}^{s_\delta} \cdot \mathcal{P}_\delta(\Lambda) \\ \delta^{-s_\delta} \cdot |\Lambda|^{s_o} \leq \cdot \mathcal{P}_\delta(\Lambda). \end{cases}$$

  Therefore we have $\mathcal{P}_\delta(\Lambda) \approx \delta^{-s_\delta}$.

- Consequently along any subsequences as δ goes to 0 we have:

$$\overline{\dim}_{\mathrm{B}}\Lambda := \limsup_{\delta\to 0^+}\frac{\log\{\mathcal{P}_\delta(\Lambda)\}}{-\log\{\delta\}} = \limsup_{\delta\to 0^+}\frac{\log\{\delta^{-s_\delta}\}}{-\log\{\delta\}} \to s.$$

- Assume that $\exists\delta > 0$ such that $\mathcal{N}_{2\delta}(\Lambda) > \left(\frac{\lambda_{\min}\cdot\delta}{|\Lambda|}\right)^{-t}$ then by the inequality being strict: $\exists s > t$ such that $\mathcal{N}_{2\delta}(\Lambda) > \left(\frac{\lambda_{\min}\cdot\delta}{|\Lambda|}\right)^{-s}$ still holds. Then $\sum_{e=1}^{N}\lambda_{\tau_e}^s \geq \frac{\lambda_{\min}^s\cdot\delta^s}{|\Lambda|^s}\mathcal{N}_{2\delta}(\Lambda) > 1$. But this implies: $\dim_{\mathrm{H}}(\Lambda) \geq \dim_{\mathrm{H}}(\Lambda_\delta)$ $= s_\delta > s > t = \dim_{\mathrm{H}}(\Lambda)$ which contradicts the assumption.

- Lastly $\mathcal{H}_{2\delta}^t(\Lambda) \leq (4\delta)^t\cdot\mathcal{N}_{2\delta}(\Lambda) \leq \frac{4^t\cdot|\Lambda|^t}{\lambda_{\min}^t} < \infty$ which completes the second statement and the proof as well.

2

WEAK SEPARATION PROPERTY

2.1 DIMENSION DROP

In Section 1.4 we introduced the similarity dimension which is an easy to compute concept, but and whence it only depends on the contracting rations. On the other hand, computing the Hausdorff dimension by definition can be rather challenging. A fractal which has these two dimensions different said to exhibit dimension drop.

Hutchinson proved that the OSC is a sufficient condition for the Hausdorff dimension and the similarity dimension to be equal, but then we might ask that is it also necessary. The answer is no, there are self-similar sets without satisfying OSC but having those two dimension the same. This arrises to the next question, is there a necessary condition for it? The answer is not known, but there is a promising folklore conjecture, firstly stated by Simon in [13]:

To state the conjecture we need a new structure to analyze:

Definition 2.1.1 (Topology on Similarities) *Let us to define the space of all similarities of* $\mathbb{R}^d$ *to itself:*

$$\mathcal{G} := \Big\{ g : \mathbb{R}^d \to \mathbb{R}^d \;\Big|\; g(x) = c_g O_g x + t_g \text{ where } c_g \in \mathbb{R}^+, O_g \in \mathcal{O}(d, \mathbb{R}), t_g \in \mathbb{R}^d \Big\}.$$

Then a topology $\mathscr{G}$ *on* $\mathcal{G}$ *is generated by the following distance:*

$$d(f, g) := \max \Big\{ |c_f - c_g|, \|O_f - O_g\|, \|t_f - t_g\| \Big\}.$$

Theorem 2.1.1 (Equivalence of topologies) *Given a dimension d, let* $x_0, \ldots, x_d \subset \mathbb{R}^d$ *be in general position, then the topology of similarities* $\mathscr{G}$ *in* $\mathbb{R}^d$ *is equivalent to a second topology induced by the sets:*

$\mathscr{S}(x_0,\dots,x_d) := \Big\{U_{\varepsilon,g}(x_0,\dots,x_d)\Big|\ \varepsilon > 0, g \in \mathcal{G}\Big\}$ where $U_{\varepsilon,g}(x_0,\dots,x_d) := \Big\{f \in \mathcal{G}\ \Big|\ \|f(x_k) - g(x_k)\| < \varepsilon\ \forall k \in \{0,\dots,d\}\Big\}$ which form a neighbourhood basis of a function $g \in \mathcal{G}$.

Proof:

Without loss of generality we may fix g as the identity function in $\mathcal{G}$. Then we need that both topologies are coarser then the other:

Case 1: Given an $f \in \mathcal{G}$ with $f \in B(\mathrm{Id}, \varepsilon) \in \mathscr{G}$ we have that $\max\Big\{|c_f - 1|, \|O_f - \mathrm{Id}_{d\times d}\|, \|t_f\|\Big\} < \varepsilon$, then:

$$\begin{aligned}
|f(x_k) - x_k| &= |c_f O_f x_k + t_f - x_k| \leq \|c_f O_f x_k - x_k\| + \|t_f\| \\
&\leq \|(c_f O_f - \mathrm{Id}_{d\times d})x_k\| + \varepsilon \leq \|c_f O_f - \mathrm{Id}_{d\times d}\|\|x_k\| + \varepsilon \\
&\leq \|c_f O_f - c_f \mathrm{Id}_{d\times d} + c_f \mathrm{Id}_{d\times d} - \mathrm{Id}_{d\times d}\| \max_{k\in\{0,\dots,d\}}\{\|x_k\|\} + \varepsilon \\
&\leq \Big(\|c_f\|\|O_f - \mathrm{Id}_{d\times d}\| + \|\mathrm{Id}_{d\times d}\|\|c_f - 1\|\Big) \max_{k\in\{0,\dots,d\}}\{\|x_k\|\} + \varepsilon \\
&\leq \Big((1+\varepsilon)\varepsilon + \varepsilon\Big) \max_{k\in\{0,\dots,d\}}\{\|x_k\|\} + \varepsilon := \varepsilon'.
\end{aligned}$$

Giving us that $B(\mathrm{Id}, \varepsilon) \subseteq U_{\varepsilon',\mathrm{Id}}(x_0,\dots,x_d)$ an therefore $\mathscr{S}$ is coarser than $\mathscr{G}$.

Case 2: Given $f \in \mathcal{G}$ with $f \in U_{\varepsilon,\mathrm{Id}}(x_0,\dots,x_d) \in \mathscr{S}$ we have that $\max_{k\in\{0,\dots,d\}}\Big\{\|f(x_k) - x_k\|\Big\} < \varepsilon$. Since $x_0 \dots, x_d$ are in general position, $x_1 - x_0, \dots, x_d - x_o$ form a base of $\mathbb{R}^d$ whence there exists $c_0, \dots, c_d \in \mathbb{R}$ such that $\sum_{i=0}^d c_i x_i = 0$ and $\sum_{i=0}^d c_i = 1$. Hence

$$\begin{aligned}
\|t_f\| &= \|f(0) - 0\| = \|f(\sum_{i=0}^d c_i x_i) - \sum_{i=0}^d c_i x_i\| = \|\sum_{i=0}^d c_i\Big(f(x_i) - x_i\Big)\| \\
&\leq \sum_{i=0}^d |c_i|\|f(x_i) - x_i\| \leq \sum_{i=0}^d |c_i| \cdot \varepsilon =: \varepsilon^{(1)}
\end{aligned}$$

Let us define a norm on d by d matrices: $\|A\|' := \sum_{i=1}^d \|A(x_i - x_o)\|$. Since the vector space of matrices is finite dimensional and all norms are equivalent, there exists $D > 0$ such that: $1/D\|A\|' \leq \|A\| \leq D\|A\|'$ for any A d by d matrix. Then,

$$\begin{aligned}
|c_f - 1| &= \Big| \|c_f O_f\| - \|\mathrm{Id}_{d\times d}\| \Big| \leq \|c_f O_f - \mathrm{Id}_{d\times d}\| \\
&\leq D \cdot \sum_{i=1}^{d} \|(c_f O_f - \mathrm{Id}_{d\times d})(x_i - x_o)\| \\
&= D \cdot \sum_{i=1}^{d} \|f(x_i) - x_i - f(x_0) + x_0\| \\
&\leq D \cdot d \cdot 2 \cdot \varepsilon =: \varepsilon^{(2)},
\end{aligned}$$

$$\begin{aligned}
\|O_f - \mathrm{Id}_{d\times d}\| &= \|c_f O_f - \mathrm{Id}_{d\times d} + (1 - c_f) O_f\| \\
&\leq \|c_f O_f - \mathrm{Id}_{d\times d}\| + \|(1 - c_f) O_f\| \\
&\leq 2D \cdot d \cdot \varepsilon (\sum_{i=0}^{d} |c_i| + 1) =: \varepsilon^{(3)}.
\end{aligned}$$

Finally, letting $\varepsilon' := \min_{i=1}^{3}\{\varepsilon^{(i)}\}$ gives us that $U_{\varepsilon,\mathrm{Id}}(x_0, \ldots, x_d) \subset B(\mathrm{Id}, \varepsilon')$ and therefore, $\mathscr{G}$ is coarser than $\mathscr{S}$.

Definition 2.1.2 *For an IFS* $\Phi = \{f_1, \ldots, f_m\}$, *let* $\mathcal{E} := \Big\{ f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \ \Big| \ \mathbf{i}, \mathbf{j} \in \Sigma^*$ *such that* $\mathbf{i} \neq \mathbf{j} \Big\} \subset \mathcal{G}$ *with the inherited topology from* $\mathcal{G}$.

Lemma 2.1.2 [12] *Let* $\Phi = \{f_1, \ldots, f_m\}$ *be an IFS. Then the OSC holds for the IFS if an only if for the previously defined* $\mathcal{E}$ *we have* $\boxed{\mathrm{Id} \notin \mathrm{cl}(\mathcal{E})}$.

Proof:

Firstly the OSC has an equivalent condition by Bandt-Graf [1], which is the **SOSC** or strong open set condition: $\exists\, U \subseteq \mathbb{R}^d$ bounded, open, non-empty set with $f_i(U) \subseteq U\ \forall i \in \{1, ..., m\}$ and $f_i(U) \cap f_j(U) = \emptyset$ for any $i \neq j$, both $\in \{1, ..., m\}$ and $\Lambda \cap U \neq \emptyset$.

Given the SOSC: let $x \in \Lambda \cap U$. For any $\mathbf{i}, \mathbf{j} \in \Sigma^*, \mathbf{i} \neq \mathbf{j}$, without loss of generality we may assume that $\mathbf{i}_1 \neq \mathbf{j}_1$

$$
\begin{aligned}
|f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}}(x) - x| &= \lambda_{\mathbf{i}}^{-1} |f_{\mathbf{j}}(x) - f_{\mathbf{i}}(x)| \\
&\geq \lambda_{\mathbf{i}}^{-1} (dist(f_{\mathbf{j}}(x), f_{\mathbf{j}_1}(\partial U)) + dist(f_{\mathbf{i}}(x), f_{\mathbf{i}_1}(\partial U))) \\
&\geq \lambda_{\mathbf{i}}^{-1} (dist(f_{\mathbf{j}}(x), f_{\mathbf{j}}(\partial U)) + dist(f_{\mathbf{i}}(x), f_{\mathbf{i}}(\partial U))) \\
&= \lambda_{\mathbf{i}}^{-1} (\lambda_{\mathbf{j}}(dist(x, \partial U)) + \lambda_{\mathbf{i}}(dist(x, \partial U))) \geq dist(x, \partial U)).
\end{aligned}
$$

Given $\mathrm{Id} \notin \mathrm{cl}(\mathcal{E})$ we also have $\mathrm{Id} \notin \mathrm{cl}\Big(\mathcal{E} \backslash \{\mathrm{Id}\}\Big)$ and then by Theorem 2.3.1 *1.a.* holds: fix $p > 1$, then exists $x \in \Lambda$ and $\varepsilon > 0$ such that for any $h = f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \in \mathcal{E}$ *with* $\lambda_h = \lambda_{\mathbf{i}}^{-1} \lambda_{\mathbf{j}} \in [p^{-1}, p]$: if $h(x) \neq x$ then $|h(x) - x| > \varepsilon$, furthermore our assumption also prohibits h to be Id for $\mathbf{i} \neq \mathbf{j}$. Let

$$
U := \bigcup_{\mathbf{k} \in \Sigma^*} f_{\mathbf{k}} \Big(B^o \Big(x, \frac{\varepsilon}{2(1+p)} \Big) \Big).
$$

Where $B^o(,)$ denotes the open ball, hence by the continuity of the maps, U is open, bounded, non-empty. Also $\Lambda \cap U \ni x$, and by construction $f_\ell(\mathcal{U}) \subseteq \mathcal{U}$ for any $\ell \in \Sigma^*$. Fix $\mathbf{i}, \mathbf{j} \in \Sigma^*$ different and suppose that $f_{\mathbf{i}}(U) \cap f_{\mathbf{j}}(U) \neq \emptyset$. Now

$$
\begin{aligned}
\emptyset &\neq f_{\mathbf{i}} \Bigg(\bigcup_{\mathbf{k} \in \Sigma^*} f_{\mathbf{k}} \Big(B^o \Big(x, \frac{\varepsilon}{2(1+p)} \Big) \Big) \Bigg) \cap f_{\mathbf{j}} \Bigg(\bigcup_{\mathbf{k} \in \Sigma^*} f_{\mathbf{k}} \Big(B^o \Big(x, \frac{\varepsilon}{2(1+p)} \Big) \Big) \Bigg) \\
&= \bigcup_{\mathbf{k} \in \Sigma^*} f_{\mathbf{ik}} \Big(B^o \Big(x, \frac{\varepsilon}{2(1+p)} \Big) \Big) \cap \bigcup_{\mathbf{k} \in \Sigma^*} f_{\mathbf{jk}} \Big(B^o \Big(x, \frac{\varepsilon}{2(1+p)} \Big) \Big).
\end{aligned}
$$

Hence, there are $\mathbf{k}_1, \mathbf{k}_2 \in \Sigma^*$ such that

$$
\emptyset \neq f_{\mathbf{ik}_1} \Big(B^o \Big(x, \frac{\varepsilon}{2(1+p)} \Big) \Big) \cap f_{\mathbf{jk}_2} \Big(B^o \Big(x, \frac{\varepsilon}{2(1+p)} \Big) \Big).
$$

Then $|f_{\mathbf{ik}_1}(x) - f_{\mathbf{jk}_2}(x)| \leq \frac{(\lambda_{\mathbf{ik}_1} + \lambda_{\mathbf{jk}_2})\varepsilon}{2(1+p)}$, therefore

$$\varepsilon \le |f_{\mathbf{ik}_1}^{-1} \circ f_{\mathbf{jk}_2}(x) - x| \le (1 + \lambda_{\mathbf{jk}_2}/\lambda_{\mathbf{ik}_1})(\frac{\varepsilon}{2(1+p)})$$
$$\le (1+p)(\frac{\varepsilon}{2(1+p)}) = \frac{\varepsilon}{2}.$$

This contradiction proves that for $\mathbf{i}, \mathbf{j} \in \Sigma^*$ different $f_{\mathbf{i}}(U) \cap f_{\mathbf{j}}(U) = \emptyset$ and hence the SOSC holds.

Conjecture (Simon) In $\mathbb{R}$ dimension drop may only occur if $\mathrm{Id} \in \mathcal{E}$.

To study the conjecture, remembering Lemma 2.1.2 it is natural to separate two condition from the property $\mathrm{Id} \notin \mathrm{cl}(\mathcal{E})$. This arises to two separation condition:

For any $\mathbf{i}, \mathbf{j} \in \Sigma^*$ we define a distance $d(\mathbf{i}, \mathbf{j}) := |f_{\mathbf{i}} - f_{\mathbf{j}}|$, if $\lambda_{\mathbf{i}} = \lambda_{\mathbf{j}}$, and ∞ otherwise. Define $\triangle_n := \min\left\{d(\mathbf{i}, \mathbf{j}) \;\middle|\; |\mathbf{i}| = |\mathbf{j}| = n \text{ and } \mathbf{i} \neq \mathbf{j}\right\}$. The Exponential Separation **ES** is fulfilled if $\triangle_n \ge a^n$ for some $a > 0$ and infinitely many n. This was introduced by Hochman [8].

The other is the Weak Separation Property **WSP** which was introduced by Lau and Ngai in [10] and Zerner [14] and is the main topic of the thesis. This again allows overlapping but keeps enough structure so that the Hausdorff dimension will be computable.

2.2 A NEW SEPARATION

The main idea of the Weak Separation Property is that we restrict the self-covers in a very similar way to the OSC. Remember that the OSC holds iff $\mathrm{Id} \notin \mathrm{cl}(\mathcal{E})$. For the WSP we let the maps overlap, but if so they have to agree on some level, which translates to that the composition of functions cannot intersect, although we let for some $\mathbf{i_1}, \mathbf{i_2} \in \Sigma^*$ to have the same generated function: $f_{\mathbf{i_1}} = f_{\mathbf{i_2}}$.

Definition 2.2.1 (Weak Separation Property) *For an IFS* $\Phi = \{f_1, \ldots, f_m\}$*, denote by* $\mathcal{E} := \left\{f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \;\middle|\; \mathbf{i}, \mathbf{j} \in \Sigma^* \text{ such that } \mathbf{i} \neq \mathbf{j}\right\} \subset \mathcal{G}$ *with the inherited* *topology from $\mathcal{G}$. We say that the Weak Separation Property holds for the IFS, if* $\boxed{\mathrm{Id} \notin \mathrm{cl}\Big(\mathcal{E}\backslash\{\mathrm{Id}\}\Big)}$*. In other words the identity of $\mathcal{E}$ is not an accumulation point.*

There are many equivalent definition of the WSP, all of which reveal a different viewpoint of the same object. We choose this because it reflects in an elegant topological view, and it shows an elegant relation between the OSC and the WSP in view of Lemma 2.1.2.

Definition 2.2.2 *The following notations will be used all throughout the paper: Let $a, b, c, R > 0, \mathbf{k} \in \Sigma^*, N, M \subset \mathbb{R}^d, x \in \mathbb{R}^d$*

- $F := \Big\{ f_{\mathbf{i}} \;\Big|\; \mathbf{i} \in \Sigma^* \Big\}$,
- $F_b := \Big\{ f_{\mathbf{i}} \; \in F \;\Big|\; \lambda_{\mathbf{i}} \in \,]b\lambda_{\min}, b] \Big\}$,
- $M_b := \Big\{ f_{\mathbf{i}} \;\Big|\; \mathbf{i} \in \widetilde{\mathcal{M}}_{b/|\Lambda|} \Big\} = \Big\{ f_{\mathbf{i}} \;\Big|\; |f_{\mathbf{i}}(\Lambda)| \leq b < |f_{\mathbf{i}^-}(\Lambda)| \Big\}$,
- $\mathcal{F} := \bigcup_{b>0} \Big\{ f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \;\Big|\; f_{\mathbf{i}}, f_{\mathbf{j}} \in F_b \Big\}$,
- $\Gamma_{c,R}(\mathbf{k}) := \Big\{ f_{\mathbf{i}} \; \in M_{c\lambda_{\mathbf{k}}} \Big| \; f_{\mathbf{i}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, R)) \neq \emptyset \Big\}$
  $= \Big\{ f_{\mathbf{i}} \;\Big|\; |f_{\mathbf{i}}(\Lambda)| \leq c\lambda_{\mathbf{k}} < |f_{\mathbf{i}^-}(\Lambda)| \textit{ and } f_{\mathbf{i}}(\Lambda) \cap B(f_{\mathbf{k}}(\Lambda), R\lambda_{\mathbf{k}}) \neq \emptyset \Big\}$,
- $\gamma^{(1)}_{c,R} := \sup_{\mathbf{k} \in \Sigma^*} \Big\{ \#\Gamma_{c,R}(\mathbf{k}) \Big\}$,
- $F_{a,N,M} := \Big\{ f \in F_{a|N|} \;\Big|\; f(M) \cap N \neq \emptyset \Big\}$,
- $\gamma^{(2)}_{a,M} := \sup_{N \subseteq \mathbb{R}^d} \Big\{ \#F_{a,N,M} \Big\}$,
- $F_b(f_{\mathbf{i}}(x)) := \Big\{ g \circ f_{\mathbf{i}}(x) \;\Big|\; g \in F_b \Big\} = \Big\{ f_{\mathbf{j}} \circ f_{\mathbf{i}}(x) \;\Big|\; \lambda_{\mathbf{j}} \in]b\lambda_{min}, b] \Big\}$.

2.3 MAIN THEOREMS

The main goal of the thesis is to prove the following theorem providing all accessible equivalent condition for which the WSP is satisfied.

Theorem 2.3.1 (S.-M. Ngai, Y. Wang and M. P. W. Zerner [14, 11, 4]) *Let $\Phi = \{f_1 \dots, f_m\}$ be an IFS, Λ its attractor. If Λ is not contained in a hyperplane, then the following conditions are equivalent:*

1.a. $\forall p > 1 \quad \exists x \in \Lambda \quad \exists \varepsilon > 0 \quad \forall h \in \mathcal{E}$ *with* $\lambda_h = \lambda_{\mathbf{i}}^{-1} \lambda_{\mathbf{j}} \in [p^{-1}, p]$:

$$\text{if } h(x) \neq x \implies |h(x) - x| > \varepsilon.$$

1.b. $\exists x \in \Lambda \quad \exists \varepsilon > 0 \quad \forall h \in \mathcal{F}$: *if* $h(x) \neq x \implies |h(x) - x| > \varepsilon.$

1.c. $\exists x \in \mathbb{R}^d \quad \exists \varepsilon > 0 \quad \forall h \in \mathcal{F}$: *if* $h(x) \neq x \implies |h(x) - x| > \varepsilon.$

2.a. $\exists x_0, \dots, x_d \in \mathbb{R}^d$ *in general position* $\exists \varepsilon > 0 \quad \forall h \in \mathcal{E} \setminus Id \quad \exists j$ *with:*

$$|h(x_j) - x_j| > \varepsilon.$$

2.b. $\exists x_0, \dots, x_d \in \mathbb{R}^d$ *in general position* $\exists \varepsilon > 0 \quad \forall h \in \mathcal{F} \setminus Id \quad \exists j$ *with:*

$$|h(x_j) - x_j| > \varepsilon.$$

2.c. $\exists x_0, \dots, x_d \in \mathbb{R}^d$ *in general position* $\exists \varepsilon > 0 \quad \forall h \in \mathcal{F} \quad \forall j$

$$\text{if } h(x_j) \neq x_j \implies |h(x_j) - x_j| > \varepsilon.$$

3.a. *The identity is an isolated point of $\mathcal{E}$, that is the WSP holds for the IFS.*

3.b. *The identity is an isolated point of $\mathcal{F}$.*

4.a. $\forall c > 0 \quad \forall R > 0 : \quad \gamma_{c,R}^{(1)} < \infty.$

4.b. $\forall c > 0 \quad \forall$ *bounded* $M \subseteq \mathbb{R}^d : \quad \gamma_{c,M}^{(2)} < \infty.$

4.c. $\exists c > 0 \quad \exists R > 0 : \quad \gamma_{c,R}^{(1)} < \infty.$

4.d. $\exists c > 0 \quad \exists$ *non-empty* $M \subseteq \mathbb{R}^d : \quad \gamma_{c,M}^{(2)} < \infty.$

5.a. $\forall x \in \mathbb{R}^d \quad \exists n^{(1)} < \infty \quad \forall \mathbf{i} \in \Sigma^* \quad \forall b > 0 \quad \forall a \in \mathbb{R}^d$:

$$\#\Big\{B(a,b) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b\Big\}\Big\} \leq n^{(1)}.$$

5.b. $\forall c \geq 1 \; \forall p \in (0,1) \; \exists x \in \mathbb{R}^d \; \exists n^{(2)} < \infty \; \forall \mathbf{i} \in \Sigma^* \; \forall k \in \mathbb{N} \; \forall a \in \mathbb{R}^d$:

$$\#\Big\{B(a, cp^k) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_{cp^k}\Big\}\Big\} \leq n^{(2)}.$$

5.c. $\exists x \in \mathbb{R}^d \quad \exists n^{(3)} < \infty \quad \forall \mathbf{i} \in \Sigma^* \quad \forall b > 0 \quad \forall a \in \mathbb{R}^d :$

$$\#\Big\{B(a, b) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b\Big\}\Big\} \leq n^{(3)}.$$

5.d. $\forall x \in \mathbb{R}^d \quad \exists n^{(4)} < \infty \quad \forall \mathbf{i} \in \Sigma^* \quad \forall b > 0 \quad \forall a \in \mathbb{R}^d :$

$$\#\Big\{B(a, b) \cap F_b(f_{\mathbf{i}}(x))\Big\} \leq n^{(4)}.$$

5.e. $\exists x \in \mathbb{R}^d \quad \exists n^{(5)} < \infty \quad \forall \mathbf{i} \in \Sigma^* \quad \forall b > 0 \quad \forall a \in \mathbb{R}^d :$

$$\#\Big\{B(a, b) \cap F_b(f_{\mathbf{i}}(x))\Big\} \leq n^{(5)}.$$

6.a. $\exists n^{(6)} < \infty \quad \forall x \in \mathbb{R}^d \quad \forall b > 0$ *we have that:*

$$\#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_{b/|\Lambda|} \text{ and } f_{\mathbf{j}}(\Lambda) \cap B(x, b) \neq \emptyset\Big\} \leq n^{(6)}.$$

6.b. $\forall\, C_1 < 1 < C_2 \quad \exists n^{(7)} < \infty \quad \forall x \in \mathbb{R}^d \quad \forall b > 0$:

$$\#\Big\{f_{\mathbf{j}} \;\Big|\; C_1 \leq \tfrac{|f_{\mathbf{j}}(\Lambda)|}{|B(x,b)|} \leq C_2 \text{ and } f_{\mathbf{j}}(\Lambda) \cap B(x, b) \neq \emptyset\Big\} \leq n^{(7)}.$$

6.c. $\exists n^{(8)} < \infty \quad \exists$ *compact* $D \subset \mathbb{R}^d$ *with non-empty interior and with* $\bigcup_{i=1}^m f_i(D) \subset D$ *such that* $\forall x \in \mathbb{R}^d \quad \forall b \in (0, 1)$ *we have that:*

$$\#\Big\{f \in M_b \;\Big|\; x \in f(D)\Big\} \leq n^{(8)}.$$

We state an even stronger result from [7], but this only works assuming Λ is in the real line and not having whole Hausdorff dimension:

Theorem 2.3.2 (Á. Farkas and J. M. Fraser [7]) *Let* $\Phi = \{f_1, \ldots, f_m\}$ *be an IFS,* Λ *its attractor such that it is on the real line and* $\dim_{\mathrm{H}}(\Lambda) < 1$*. If* Λ *is not a singleton, then the following conditions are equivalent:*

I. *The WSP holds for the IFS* Φ*.*

II. $\mathcal{H}^{\dim_{\mathrm{H}}(\Lambda)}(\Lambda) > 0$*.*

III. Λ *is Ahlfors regular.*

IV. $\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{A}}(\Lambda)$ *where* $\dim_{\mathrm{A}}$*, the Assouad dimension defined:*

$$\dim_{\mathrm{A}}(E) := \inf\Big\{s \geq 0 \mid \forall x \in E\ \forall 0 < p < \delta \leq 1 :$$
$$\exists K \geq 0 : N_p(E \cap B(x,\delta)) \leq K(\delta/p)^s\Big\}.$$

V. $\forall T \in \mathrm{Tan}(\Lambda):\ \dim_{\mathrm{H}}(T) = \dim_{\mathrm{H}}(\Lambda)$ *where Tan(E) is defined by:*

$$\begin{aligned}\mathrm{Tan}(E) &:= \Big\{T \;\Big|\; T \text{ is a weak tangent set of } E\Big\}\\ &= \Big\{T \;\Big|\; \exists y_n \in \mathbb{R}^d\ \exists r_n > 0 : \mathrm{dist}_{\mathrm{H}}\Big(\frac{E - y_n}{r_n} \cap B(0,1), T\Big) \to 0\Big\}.\end{aligned}$$

VI. $\forall T \in \mathrm{Tan}(\Lambda)$ *T does not contain a line segment.*

One of the most useful corollary of the WSP is that it gives an easier way to compute the box-counting dimension and through that the Hausdorff dimension when Λ is a self-similar set remembering Theorem 1.4.3.

Theorem 2.3.3 (M. P. W. Zerner [14]) *Let $\Phi = \{f_1, \dots, f_m\}$ be an IFS satisfying the WSP, Λ its attractor. Then*

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{B}}(\Lambda) = \lim_{b \to 0^+} \frac{\log\{\#M_b\}}{-\log\{b\}} := \lim_{b \to 0^+} s_b.$$

Proof:

By the definition of the box-counting dimension it is enough to see that $\mathcal{N}_\delta(\Lambda) = \min\{m > 0 \mid \exists x_1, x_2, \dots x_m : \Lambda \subseteq \cup_{i=1}^m B(x_i,\delta)\}$ will tend to infinity in the order of $\#M_b = \#\{f_{\mathbf{i}} \;\big|\; |f_{\mathbf{i}}(\Lambda)| \leq b < |f_{\mathbf{i}-}(\Lambda)|\}$.

Firstly $\#M_b \geq N_{b/2}(\Lambda)$ because given $x \in \Lambda$ we have that $\bigcup_{f \in M_b} B(f(x), b/2)$ covers Λ. Secondly by *1.b.* we have that there exists $y \in \Lambda$ and $\varepsilon > 0$ such that for any $h \in \mathcal{F}$: if $h(y) \neq y$ we have that $|h(y) - y| > \varepsilon$. Then $\Big\{B(f(y), \varepsilon\lambda_{\min} b)\Big\}_{f \in M_b}$ are disjoint because given f, g different in M_b we have that $f^{-1} \circ g \in \mathcal{F}$, and then:

$$\varepsilon < |f^{-1}(g(y)) - y| = \lambda_f^{-1}|g(y) - f(y)| \implies |g(y) - f(y)| > \varepsilon\lambda_f \geq \varepsilon\lambda_{\min} b.$$

Therefore $\mathcal{P}_{\varepsilon\lambda_{\min}b/2}(\Lambda) \geq \# M_b$ where recall $\mathcal{P}_r(\Lambda) := \max\{m > 0 \mid \exists x_1, x_2, \ldots, x_m \in \Lambda : B(x_i, r) \cap B(x_j, r) \neq \emptyset \implies i = j\}$ is a packing of Λ. Finally using the inequality between packings and coverings:

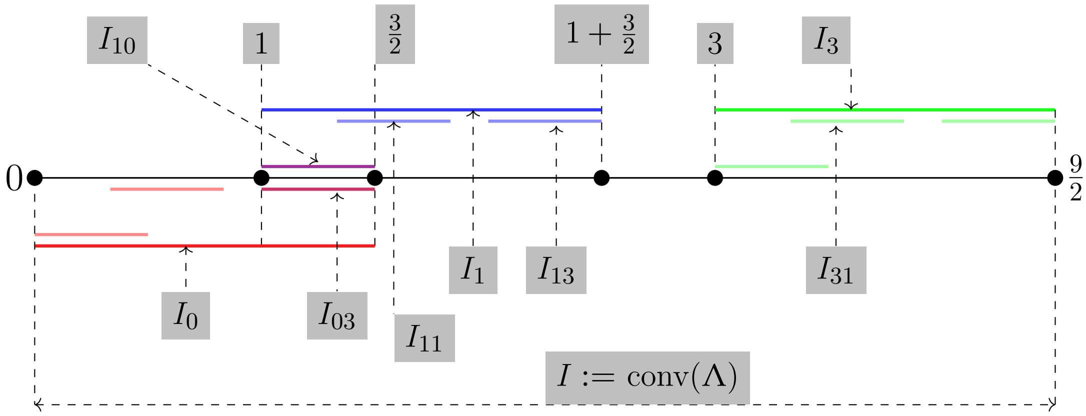
$$\# M_b \geq N_{b/2}(\Lambda) \geq P_{b/2}(\Lambda) \geq \# M_{b/(\lambda_{\min}\varepsilon)}.$$

Remark: One can think of s_b as the similarity dimension of the IFS M_b.

Before proving Theorem 2.3.1 we give an example of an IFS satisfying the WSP, but not the OSC:

Example: $\Phi_{013} = \{f_0 := \frac{x}{3}, f_1 := \frac{x}{3} + 1, f_3 := \frac{x}{3} + 3\}$,

First and second level cylinders of the convex hull of the attractor



Firstly 0 is the fixed point of f_0,$\frac{3}{2}$ for f_1 and $\frac{9}{2}$ for f_3, hence the convex hull of the attractor is $[0, \frac{9}{2}]$. Looking at the first iterations we see that $f_1 \circ f_0 = f_0 \circ f_3$ implying that the OSC falls short.

Secondly the WSP is satisfied: note that for $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ we have $f_{\mathbf{i}} = \frac{x}{3^n} + \sum_{k=1}^{n} \frac{i_k}{3^{k-1}}$. Then

$$\mathcal{E} := \left\{ f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \;\middle|\; \mathbf{i}, \mathbf{j} \in \Sigma^* \text{ such that } \mathbf{i} \neq \mathbf{j} \right\}$$
$$= \left\{ x \cdot 3^{n-m} + \sum_{k=1}^{m} \frac{j_k}{3^{k-1-n}} - \sum_{l=1}^{n} \frac{i_l}{3^{l-1-n}} \;\middle|\; \mathbf{i}, \mathbf{j} \in \Sigma^*, \mathbf{i} \neq \mathbf{j}, |\mathbf{i}| = n, |\mathbf{j}| = m \right\}.$$

Now letting $x = 0$ for any $h = f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \in \mathcal{E}$, with $\lambda_h \in [p^{-1}, p]$ we have that $|n-m| \leq \log_3 p$ where $|\mathbf{i}| = n$, $|\mathbf{j}| = m$. Then assuming $m = n$ and $h(x) \neq x$ we have

$$|h(x) - x| = \Big| \sum_{k=1}^{m} \frac{j_k}{3^{k-1-n}} - \sum_{l=1}^{n} \frac{i_l}{3^{l-1-n}} \Big| \geq \Big| \sum_{k=1}^{n} (j_{n-k+1} - i_{n-k+1})(3^k) \Big| \geq 1.$$

Hence the WSP holds by *1.a.*.

Thirdly we want to compute the Hausdorff dimension. For this we need to get the growth rate of M_b, this might be tricky but now observe that in the second level cylinders $\{f_{00}, f_{01}, f_{03}, f_{10}, f_{11}, f_{13}, f_{30}, f_{31}, f_{33}\}$ two type of intersection happends: $f_{10} = f_{03}$ and for $k = 0, 1, 3$ f_{k0} and f_{k1}.

This lets us to restrict the generating method: after 1 or 3 all $0, 1, 3$ can come, but after 0 we only let $0, 1$. Now in general given two intersecting level-n cylinders $\Lambda_{\mathbf{i}}, \Lambda_{\mathbf{j}}$ with the restriction with distinct $\mathbf{i}, \mathbf{j}$ we have that they agree on the first $n-1$ letters, denoted with $|\mathbf{i} \cap \mathbf{j}| = n-1$, and one's last letter is 0 and the other's is 1.

To prove this we use induction: observe that for the first level cylinders it holds. Now suppose that for level-(n-1) it holds, and let $\mathbf{i}, \mathbf{j}$ be two different level-n words such that $I_{\mathbf{i}} \cap I_{\mathbf{j}} \neq \emptyset$, but $k := |\mathbf{i} \cap \mathbf{j}| + 1 < n$. Denote $\omega := \mathbf{i} \cap \mathbf{j} \in \Sigma^*$. By the induction hypothesis we have $\{i_k, j_k\} = \{0, 1\}$, without loss of generality we may assume that $i_k = 0, j_k = 1$. Since $I_{\mathbf{i}} \cap I_{\mathbf{j}} \neq \emptyset$ we have $I_{\omega 0 i_{k+1}} \cap I_{\omega 1} \neq \emptyset$, therefore $I_{0 i_{k+1}} \cap I_1 \neq \emptyset$. Looking at figure we concude that $i_{k+1} = 3$, but now $(i_k, i_{k+1}) = (0, 3)$ which is the one we restricted. This contradiction proves that $|\mathbf{i} \cap \mathbf{j}| = k - 1 = n - 1$. Finally now again looking at figure we see that $\{i_n, j_n\} = \{i_k, j_k\} = \{0, 1\}$. Now $\# M_b = A^p \cdot I$, where

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad p = p(b) := \lceil \log_{1/3}\{b\} \rceil, \quad I := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The dominant eigenvalue is $a := \frac{3+\sqrt{5}}{2}$ and A is a non-negative, irreducible, primitive matrix hence by the Perron-Frobenius theorem the growth rate of $A^{p(b)} \cdot I$ is asymptotic to $a^{p(b)}$ as $b \to \infty$. Finally:

$$\begin{aligned}\dim_{\mathrm{H}}(\Lambda) &= \lim_{b\to 0^+} \frac{\log\{\#M_b\}}{-\log\{b\}} = \lim_{b\to 0^+} \frac{\log\{A^{p(b)} \cdot I\}}{-\log\{b\}} = \lim_{b\to 0^+} \frac{\log\{a^{p(b)}\}}{-\log\{b\}} \\ &= \lim_{b\to 0^+} \frac{p(b)\cdot \log\{a\}}{-\log\{b\}} = \lim_{b\to 0^+} \frac{\lceil \log_{1/3}\{b\}\rceil \cdot \log\{\frac{3+\sqrt{5}}{2}\}}{-\log\{b\}} \\ &= \frac{\log\{3+\sqrt{5}\} - \log\{2\}}{\log\{3\}}.\end{aligned}$$

For a more detailed analysis of this set we recommend [4].

3

PROOF OF THE EQUIVALENT CONDITIONS

For the proof we fix some notations: d denotes the dimension of the space where the IFS is embedded, m denotes the cardinality of the IFS. Let Σ denote the symbolic space generated by the syllables $1, \ldots, m$, Λ denotes the attractor.

For the proof we assume that $|\Lambda| = 1$ which is not too restricting because any attractor and IFS can be transformed to such and the transformation does not change the geometric properties we study. With an IFS $\Phi = \{f_1, \ldots, f_m\}$ and its attractor Λ given g invertible function, a new IFS $\Phi_g = \{g \circ f_1 \circ g^{-1}, \ldots, g \circ f_m \circ g^{-1}\}$ will have the attractor $g(\Lambda)$. Then use $g(x) := 1/|\Lambda| \cdot x$. Finally notice that the WSP's definition holds for Φ iff it holds for Φ_g.

3.1 LEMMAS

At first we start with a lemma nothing to do with fractals or self-similarity, but with the structure of $\mathbb{R}^d$. Although it's statement is trivial by any easy overestimation we have to mention that the exact values are unknown in high dimensions, for example in $d = 2$ it is called the Disc covering problem.

Lemma 3.1.1 *Given $R > r > 0$ and a dimension d. Any ball with radius R can be sufficiently covered by $\mathcal{L} = \mathcal{L}(d, R, r)$-many balls with radius r. That is there exists $\mathcal{L} > 0$ such that for any $x \in \mathbb{R}^d$ there are $x_1, \ldots, x_{\mathcal{L}} \in \mathbb{R}^d$ such that*

$$\bigcup_{i=1}^{\mathcal{L}} B(x_i, r) \supseteq B(x, R).$$

Lemma 3.1.2 *If Λ is not contained in a hyperplane, then for all $x \in \mathbb{R}^d$ which satisfy 5.c., also satisfy the following inequality:*

$$C^* := \sup_{y \in \mathbb{R}^d} \left\{ \sup_{b>0} \# \left\{ f \in M_b \;\middle|\; f(x) = y \right\} \right\} < \infty.$$

And for all $x \in \mathbb{R}^d$, which satisfy 5.e., also satisfy the following inequality:

$$C' := \sup_{y \in \mathbb{R}^d} \left\{ \sup_{b>0} \# \left\{ f \in F_b \;\middle|\; f(x) = y \right\} \right\} < \infty.$$

Proof:

Assuming 5.c., there exists an $x \in \mathbb{R}^d$ and an $n^{(3)} < \infty$ such that for any $\mathbf{i} \in \Sigma^*$ any $b > 0$ and any $a \in \mathbb{R}^d$ we have that

$$\# \left\{ B(a,b) \cap \left\{ f_{\mathbf{ji}}(x) \;\middle|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \right\} \right\} \leq n^{(3)}.$$

Now fix such pair of $x, n^{(3)}$. Since $\{f_{\mathbf{i}}(x) \mid \mathbf{i} \in \Sigma^*\}$ is dense in Λ which is not contained in any hyperplane we can choose $d+1$ many finite worlds $\mathbf{i}_0, \ldots, \mathbf{i}_d \in \Sigma^*$ such that the $d+1$ points $f_{\mathbf{i}_0}(x), \ldots, f_{\mathbf{i}_d}(x)$ are in general position.

Theorem 2.1.1 lets us to uniquely determine the similarities of $\mathbb{R}^d$ with $d+1$ general points value taken by it. Therefore we can upper bound $\#\left\{f \in M_b \;\middle|\; f(x) = y\right\}$ with the number of lenght-$(d+1)$ sequences formed from the elements of the set $\left\{f(f_{\mathbf{i}_k}(x)) \;\middle|\; k \in \{0, \ldots, d\} \text{ and } f \in M_b \text{ such that } f(x) = y\right\} =: A$. Hence:

$$\begin{aligned}
\sup_{y \in \mathbb{R}^d} \left\{ \sup_{b>0} \# \left\{ f \in M_b \;\middle|\; f(x) = y \right\} \right\} &\leq \sup_{y \in \mathbb{R}^d} \left\{ \sup_{b>0} \left\{ (\#A)^{d+1} \right\} \right\} \\
&\leq \Big(\sup_{y \in \mathbb{R}^d} \left\{ \sup_{b>0} \# \left\{ f(f_{\mathbf{i}_k}(x)) \;\middle|\; k \in \ldots,\ f \in M_b,\ f(x) = y \right\} \right\} \Big)^{d+1} \\
&\leq \Big(\sum_{k=0}^{d} \sup_{y \in \mathbb{R}^d} \left\{ \sup_{b>0} \# \left\{ f(f_{\mathbf{i}_k}(x)) \;\middle|\; f \in M_b,\ f(x) = y \right\} \right\} \Big)^{d+1}.
\end{aligned}$$

Notice that $f(x) = y$ implies $|f(f_{\mathbf{i}_k}(x)) - y| = |f(f_{\mathbf{i}_k}(x)) - f(x)| = \lambda_f |f_{\mathbf{i}_k}(x) - x| \leq b \cdot \max_k\{|f_{\mathbf{i}_k}(x) - x|\}$ which further implies $f(f_{\mathbf{i}_k}(x)) \in B(y, b \max_k\{|f_{\mathbf{i}_k}(x) - x|\})$, denote $c := \max_k\{|f_{\mathbf{i}_k}(x) - x|\}$, using this:

$$\begin{aligned}\Big\{f(f_{\mathbf{i}_k}(x)) \;\Big|\; f \in M_b,\ f(x) = y\Big\} \\ \subseteq \Big\{f(f_{\mathbf{i}_k}(x)) \;\Big|\; f \in M_b,\ f(f_{\mathbf{i}_k}(x)) \in B(y, bc)\Big\} \\ = \Big\{f(f_{\mathbf{i}_k}(x)) \;\Big|\; f \in M_b\Big\} \cap B(y, bc).\end{aligned}$$

Finally we can use *5.c.* and Lemma 3.1.1 to have that:

$$\begin{aligned}\sup_{y \in \mathbb{R}^d} \Big\{ \sup_{b>0} \#\Big\{f \in M_b \;\Big|\; f(x) = y\Big\}\Big\} \\ \leq \Big((d+1) \cdot n^{(3)} \cdot \mathcal{L}(d, c, 1)\Big)^{d+1} < \infty.\end{aligned}$$

Notice that the second equation can be proven by the same argument with only changing *5.c.* to *5.c.* and not forgetting the definition: $F_b(f_{\mathbf{i}}(x)) := \Big\{g \circ f_{\mathbf{i}}(x) \;\Big|\; g \in F_b\Big\}$.

Lemma 3.1.3 *For all $0 < c_1 < c_2$ and $R > 0$ there exists an increasing continuous function $\phi(t) > 1$ on $[1, \infty)$, depending only on the IFS, such that for every $\mathbf{k} \in \Sigma^*$:*

$$\frac{\#\Gamma_{c_2,R}(\mathbf{k})}{\phi(\frac{c_2}{c_1})} \leq \#\Gamma_{c_1,R}(\mathbf{k}) \leq \#\Gamma_{c_2,R}(\mathbf{k}) \cdot \phi(\frac{c_2}{c_1}) \quad \textit{and from that}$$

$$\frac{\gamma^{(1)}_{c_2,R}}{\phi(\frac{c_2}{c_1})} \leq \gamma^{(1)}_{c_1,R} \leq \gamma^{(1)}_{c_2,R} \cdot \phi(\frac{c_2}{c_1}).$$

Proof:

Firstly we show the left inequality:

Let $f_{\mathbf{i}} \in \Gamma_{c_2,R}(\mathbf{k})$ meaning that $\lambda_{\mathbf{i}} \leq c_2 \lambda_{\mathbf{k}} < \lambda_{\mathbf{i}-}$ and $f_{\mathbf{i}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, R)) \neq \emptyset$. Denote the set $]\mathbf{i}[:= \{\mathbf{j} \in \Sigma^* \mid \mathbf{i}$ is a prefix of $\mathbf{j}\}$, it has a non-empty intersection with $\widetilde{\mathcal{M}}_{c_1 \lambda_{\mathbf{k}}}$ and there will be at least one $\mathbf{j} = \mathbf{i}\mathbf{i}'$ in the intersection such that $f_{\mathbf{j}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, R)) \neq \emptyset$ and then $f_{\mathbf{j}} \in \Gamma_{c_1,R}(\mathbf{k})$. Fix such a word $\mathbf{j} = \mathbf{i}\mathbf{i}'$, then $\lambda_{\mathbf{i}} < c_2\lambda_{\mathbf{k}}$ implies $c_1\lambda_{\min}\lambda_{\mathbf{k}} < \lambda_{\mathbf{i}\mathbf{i}'} \leq c_2\lambda_{\max}^{|\mathbf{i}'|}\lambda_{\mathbf{k}}$ and hence:

$$|\mathbf{i}'| \leq \log_{\lambda_{\max}}\left\{\frac{c_1}{c_2}\lambda_{\min}\right\} + 1 =: \psi(\frac{c_2}{c_1})$$

Define $\phi(t) := \psi(t) \cdot m^{\psi(t)}$, recall m is the cardinality of the IFS $\{f_1, \dots, f_m\}$. Now we claim that the map $\mathcal{K} : f_{\mathbf{i}} \to f_{\mathbf{i}\mathbf{i}'}$ is at most $\phi(\frac{c_2}{c_1})$-to-1 and this follows because even if $f_{\mathbf{i}_{(1)}} \neq f_{\mathbf{i}_{(2)}}$ both in $\Gamma_{c_2,R}(\mathbf{k})$ with $f_{\mathbf{i}_{(1)}\mathbf{i}'_{(1)}} = f_{\mathbf{i}_{(2)}\mathbf{i}'_{(2)}}$ in $\Gamma_{c_1,R}(\mathbf{k})$ there are at most $\psi(t) \cdot m^{\psi(t)}$ possibilities for $\mathbf{i}'_{(2)}$ and $\mathbf{i}'_{(2)}$.

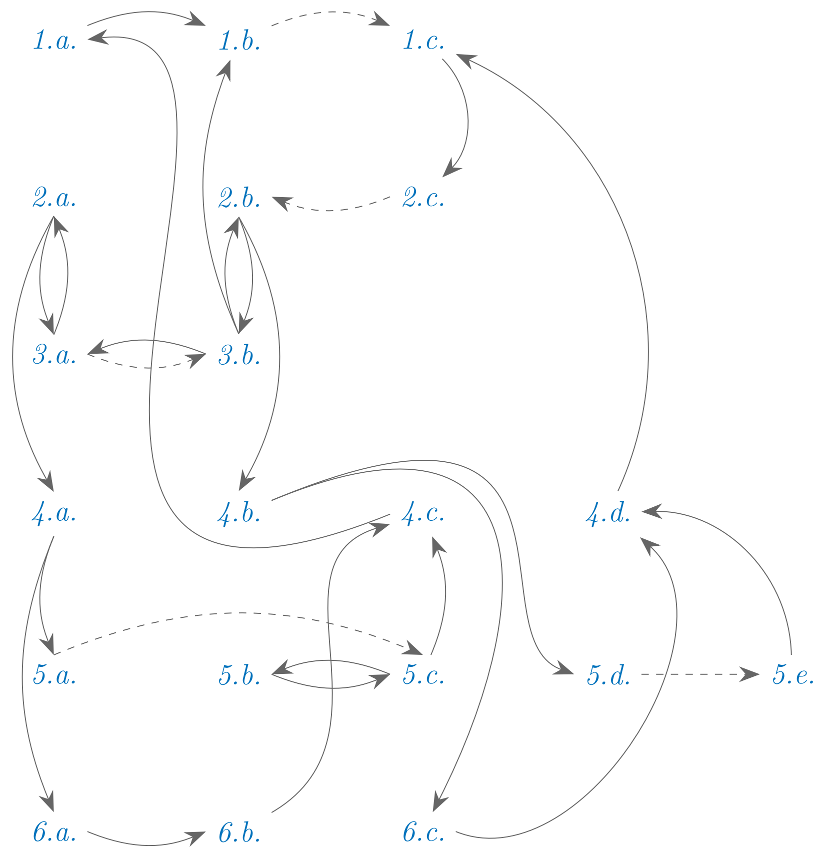
Secondly the right inequality:
Let $f_{\mathbf{j}} \in \Gamma_{c_1,R}(\mathbf{k})$ meaning that $\lambda_{\mathbf{j}} \leq c_1\lambda_{\mathbf{k}} < \lambda_{\mathbf{j}-}$ and $f_{\mathbf{j}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, R)) \neq \emptyset$. Fix such an $\mathbf{j}$, then there is an unique prefix $\mathbf{i}'$ of it such that $\mathbf{i}' \in \widetilde{\mathcal{M}}_{c_2\lambda_{\mathbf{k}}}$ and trivially $f_{\mathbf{i}'}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, R)) \neq \emptyset$ hence $f_{\mathbf{i}'} \in \Gamma_{c_1,R}(\mathbf{k})$. Now $\mathbf{j} =: \mathbf{i}'\mathbf{i}''$ and similarly we have $|\mathbf{i}''| \leq \psi(\frac{c_2}{c_1})$. Finally if $f_{\mathbf{j}_{(1)}} = f_{\mathbf{j}_{(2)}}$ both in $\Gamma_{c_2,R}(\mathbf{k})$ with $f_{\mathbf{i}'_{(1)}} \neq f_{\mathbf{i}'_{(2)}}$ in $\Gamma_{c_2,R}(\mathbf{k})$ then $\mathbf{i}''_{(1)}$ has to be different than $\mathbf{i}''_{(2)}$ and therefore $\mathcal{K}^* : f_{\mathbf{j}} \to f_{\mathbf{i}'}$ is at most $\phi(\frac{c_2}{c_1})$-to-1.

3.2 A CLEAR VIEW

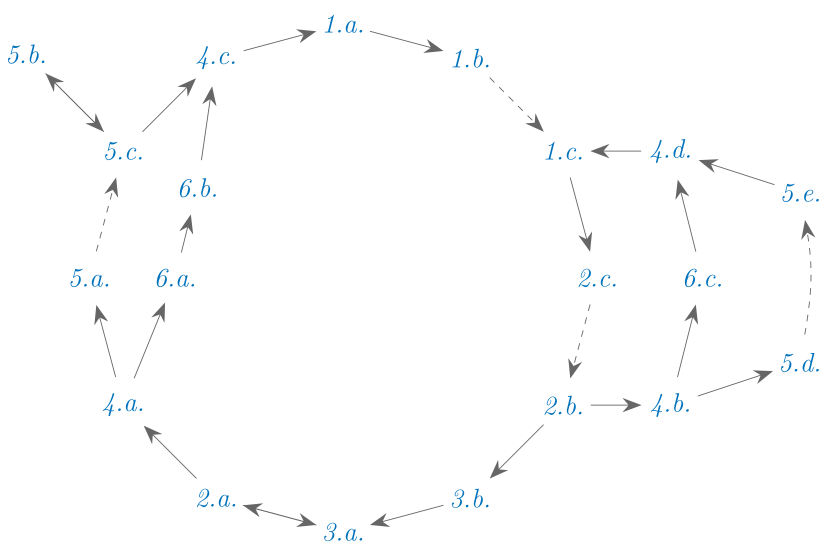
Notice that the proof of equivalence is complete if the directed graph of the statements and proofs is strongly connected, that is from any statement(vertex) we have a path of proofs to any other statement(vertex). For the easier understanding we give 2 embedding of the graph: $G1$: the one ordered by the statements and $G2$: the one by the main path of proofs.

Dashed arrows denote trivial implications, those don't need any further proof.

$G1$ showing us the overall structure of the statements



$G2$ showing us the framework of the proof



2.a. $\iff$ 3.a.

The set $\mathcal{E}$ can inherit both topology of Theorem 2.1.1 from $\mathcal{G}$ which with the statement of Theorem 2.1.1 gives us 2.a. $\implies$ 3.a.. Conversely by Λ not contained in a hyperplane we can choose $x_0, \ldots, x_d \in \Lambda$ in general position and then 3.a. $\implies$ 2.a..

2.a. $\implies$ 4.a.

Choose $\varepsilon > 0$ and $x_0, \ldots, x_d$ satisfying 2.a.. Let us fix $c, R > 0$, for any $\mathbf{k} \in \Sigma^*$ fixed, and every function $f_\mathbf{i} \in \Gamma_{c,R}(\mathbf{k})$ we have that $c\lambda_\mathbf{k}\lambda_{\min} < \lambda_\mathbf{i} \leq c\lambda_\mathbf{k}$. Furthermore for any $f_\mathbf{i}, f_\mathbf{j}$ with $f_\mathbf{i}^{-1} \circ f_\mathbf{j} \in \mathcal{E} \backslash \mathrm{Id}$, lets us to define $\ell = \ell(f_\mathbf{i}, f_\mathbf{j})$ index in $\{0, \ldots, d\}$ such that:

$$|(f_\mathbf{j}(x_\ell) - f_\mathbf{i}(x_\ell)| = \lambda_\mathbf{i}|f_\mathbf{i}^{-1}(f_\mathbf{j}(x_\ell)) - x_\ell| \geq \lambda_\mathbf{i}\varepsilon > \varepsilon c\lambda_\mathbf{k}\lambda_{\min}.$$

If there are more, then abandon all but the lexicographically smallest. From we get that: $B(f_\mathbf{i}(x_\ell), \frac{c\varepsilon\lambda_\mathbf{k}\lambda_{\min}}{3}) \cap B(f_\mathbf{j}(x_\ell), \frac{c\varepsilon\lambda_\mathbf{k}\lambda_{\min}}{3}) = \emptyset$. For all $f \in \Gamma_{c,R}(\mathbf{k})$:

$$f(\Lambda) \cap f_\mathbf{k}(B(\Lambda, R)) \neq \emptyset \implies \mathrm{dist}(f(\Lambda), B(f_\mathbf{k}(B(\Lambda, R))) = 0.$$

Now use for all $\ell \in \{0, \ldots, d\}$:

$$\mathrm{dist}(f(x_\ell), f(\Lambda)) = \lambda_f \mathrm{dist}(x_\ell, \Lambda).$$

Then,

$$\begin{aligned}\mathrm{dist}(f(x_\ell), f_\mathbf{k}(B(\Lambda, R))) &\leq \mathrm{dist}(f(x_\ell), f(\Lambda)) + \mathrm{dist}(f(\Lambda), f_\mathbf{k}(B(\Lambda, R))) \\ &+ |f(\Lambda)| \leq \lambda_f \mathrm{dist}(x_\ell, \Lambda) + 0 + \lambda_f|\Lambda| = \lambda_f \mathrm{dist}(x_\ell, \Lambda) + \lambda_f.\end{aligned}$$

Finally using the equation above and the fact that $\mathrm{dist}(a, b) \leq r$ implies that $a \in B(b, r)$ if $|a| = 0$ which is satisfied if a is a point:

$$f(x_\ell) \in B(f_\mathbf{k}(B(\Lambda, R)), \lambda_f \mathrm{dist}(x_\ell, \Lambda) + \lambda_f).$$

Using this we compute:

$$
\begin{aligned}
B(f(x_\ell), \frac{c\varepsilon\lambda_{\mathbf{k}}\lambda_{\min}}{3}) &\subset B(f_{\mathbf{k}}(B(\Lambda, R)), \lambda_f \mathrm{dist}(x_\ell, \Lambda) + \lambda_f + \frac{c\varepsilon\lambda_{\mathbf{k}}\lambda_{\min}}{3}) \\
&= B(f_{\mathbf{k}}(\Lambda), \lambda_f \mathrm{dist}(x_\ell, \Lambda) + \lambda_f + \frac{c\varepsilon\lambda_{\mathbf{k}}\lambda_{\min}}{3} + \lambda_f R) \\
&\subset B(f_{\mathbf{k}}(\Lambda), c\lambda_{\mathbf{k}} \mathrm{dist}(x_\ell, \Lambda) + c\lambda_{\mathbf{k}} + \frac{c\varepsilon\lambda_{\mathbf{k}}\lambda_{\min}}{3} + c\lambda_{\mathbf{k}} R) \\
&\subset B(f_{\mathbf{k}}(\Lambda), \lambda_{\mathbf{k}}(c \max_{\ell \in \{0,\dots,d\}} \{\mathrm{dist}(x_\ell, \Lambda)\} + c + \frac{c\varepsilon\lambda_{\min}}{3} + cR)) \\
&:= B(f_{\mathbf{k}}(\Lambda), \lambda_{\mathbf{k}} R^{(1)}).
\end{aligned}
$$

Consider the following graph: for a fixed $\Gamma_{c,R}(\mathbf{k})$ the vertices will be the functions in $\Gamma_{c,R}(\mathbf{k})$, between any two vertices $f, g \in \Gamma_{c,R}(\mathbf{k})$ there is a a suitable $\ell(f,g)$ defining a labeled edge. Now ℓ can be thought of as a coloring on a complete graph letting us to use Ramsay's theorem [2, Theorem 19.2.3], bounding the number of vertices with a constant $L = L(\#A)$ depending only on the number of vertices in the maximal connected set of vertices by the same colored edges: A. Finally consider the d-dimensional volume of the previously computed balls: $\forall \mathbf{k} \in \Sigma^*$

$$
\begin{aligned}
\#\Gamma_{c,R}(\mathbf{k}) &\leq L(\#A) \\
&\leq L\Big(\max_{p \in \{0,\dots,d\}} \Big\{ \#A \subset \Gamma_{c,R}(\mathbf{k}) \;\Big|\; \forall f \neq g \in A : \ell(f,g) = p \Big\} \Big) \\
&\leq L\Big(\max_{p \in \{0,\dots,d\}} \Big\{ \frac{\mathrm{Vol}(B(f_{\mathbf{k}}(\Lambda), \lambda_{\mathbf{k}} R^{(1)}))}{\mathrm{Vol}(B(f_{\mathbf{j}}(x_\ell), \frac{c\varepsilon\lambda_{\mathbf{k}}\lambda_{\min}}{3}))} \Big\} \Big) \\
&\leq L\Big(\max_{p \in \{0,\dots,d\}} \Big\{ \frac{\mathrm{Vol}(B(\Lambda, R^{(1)}))}{\mathrm{Vol}(B(0, \frac{c\varepsilon\lambda_{\min}}{3}))} \Big\} \Big) =: L'(c,R).
\end{aligned}
$$

The $L' < \infty$ above is independent of $\mathbf{k}$, depending only on c and R. Finally, $\gamma^{(1)}_{c,R} = \sup_{\mathbf{k} \in \Sigma^*} \big\{ \#\Gamma_{c,R}(\mathbf{k}) \big\}$ and hence by the definition of the supremum $\gamma^{(1)}_{c,R} \leq L'(c,R)$.

4.a. $\implies$ *5.a.*

Let $a \in \mathbb{R}^d, b > 0$ be arbitrary. Fix $x \in \mathbb{R}^d$, $R = R(x) > 0$ such that $x \in B(\Lambda, R)$. Now for any $\mathbf{i} \in \Sigma^*$ we have $f_{\mathbf{i}}(x) \in B(\Lambda, R)$ by the definition of the attractor Λ. Define $\mathcal{J}$ as follows:

$$\begin{aligned}
\mathcal{J} &:= \#\Big\{B(a,b) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b\Big\}\Big\} \\
&= \#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } f_{\mathbf{j}}(f_{\mathbf{i}}(x)) \in B(a,b)\Big\} \\
&\le \#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } f_{\mathbf{j}}(B(\Lambda,R)) \cap B(a,b) \neq \emptyset\Big\} \\
&= \#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } B(f_{\mathbf{j}}(\Lambda), \lambda_{\mathbf{j}} R)) \cap B(a,b) \neq \emptyset\Big\} \\
&\le \#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } f_{\mathbf{j}}(\Lambda) \cap B(a, b + \lambda_{\mathbf{j}} R) \neq \emptyset\Big\} \\
&\le \#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } f_{\mathbf{j}}(\Lambda) \cap B(a, b(1+R)) \neq \emptyset\Big\}.
\end{aligned}$$

If $\Lambda \cap B(a, b(1+R)) = \emptyset$ then $\mathcal{J} = 0$ and therefore *5.a.* holds. Otherwise $\exists \mathbf{k} \in \widetilde{\mathcal{M}}_b$ such that $f_{\mathbf{k}}(\Lambda) \cap B(a, b(1+R)) \neq \emptyset$. Choose one $\mathbf{k}$ like this, then:

$$B(a, b(1+R)) \subset B(f_{\mathbf{k}}(\Lambda), 2b(1+R)) \subset f_{\mathbf{k}}(B(\Lambda, 2\lambda_{\min}^{-1}(1+R)))$$

From this:

$$\begin{aligned}
\mathcal{J} &\le \#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } f_{\mathbf{j}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, 2\lambda_{\min}^{-1}(1+R))) \neq \emptyset\Big\} \\
&= \#\Gamma_{b\lambda_{\mathbf{k}}^{-1}, 2\lambda_{\min}^{-1}(1+R)}(\mathbf{k}) \le \gamma_{c,R}^{(1)} < \infty \quad \text{by } \textit{4.a.}
\end{aligned}$$

5.c. $\iff$ *5.b.*

Given *5.c.* : $\exists x \in \mathbb{R}^d \; \exists n^{(3)} < \infty \; \forall \mathbf{i} \in \Sigma^* \; \forall b > 0 \; \forall a \in \mathbb{R}^d$: $\#\Big\{B(a,b) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b\Big\}\Big\} \le n^{(3)}$ *5.b.* follows by Lemma 3.1.1 with $\mathcal{L}(d, c, 1)$. And then with given p^k we can easily choose any bigger b which gives us $n^{(3)} \ge n^{(2)}/\ell$, and the same x is used in both of the statements.

Conversely fix $b > 0$, $p \in (0,1)$ arbitrary. Then there exists a unique $k \in \mathbb{N}$ such that $p^{k+1} < b \le p^k$. Given $\mathbf{j} \in \widetilde{\mathcal{M}}_b$ there is a $\mathbf{j}' \in \widetilde{\mathcal{M}}_{p^k}$ prefix of $\mathbf{j}$ such that:

$$\lambda_{\mathbf{j}} \leq b < \lambda_{\mathbf{j}-} \implies \lambda_{\min} b < \lambda_{\min}\lambda_{\mathbf{j}-} \leq \lambda_{\mathbf{j}}$$

$$\lambda_{\mathbf{j}'} \leq p^k < \lambda_{\mathbf{j}'-} \implies \lambda_{\mathbf{j}'} \leq p^k$$

Now the two implies: $\lambda_{\min} p < \dfrac{\lambda_{\min} b}{p^k} \leq \dfrac{\lambda_{\mathbf{j}}}{\lambda_{\mathbf{j}'}} \leq \lambda_{\max}^{|\mathbf{j}|-|\mathbf{j}'|}$

$$\implies |\mathbf{j}| - |\mathbf{j}'| < \log_{\lambda_{\max}}\{\lambda_{\min} p\} \leq \lceil \log_{\lambda_{\max}}\{\lambda_{\min} p\}\rceil.$$

Using $c = 1$ by 5.b. we have that $\exists x \in \mathbb{R}^d \quad \exists n^{(2)} < \infty \quad \forall \mathbf{i} \in \Sigma^* \quad \forall k \in \mathbb{N} \quad \forall a \in \mathbb{R}^d$:

$$\#\Big\{B(a,b) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b\Big\}\Big\} \leq$$
$$m^{\lceil \log_{\lambda_{\max}}\{\lambda_{\min} p\}\rceil} \cdot \#\Big\{B(a,b) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_{p^k}\Big\}\Big\} \leq m^{\lceil \log_{\lambda_{\max}}\{\lambda_{\min} p\}\rceil} \cdot n^{(2)}.$$

5.c. $\implies$ 4.c.

Recall that $|\Lambda| = 1$ by assumption. With 5.c. we have that $\exists x \in \mathbb{R}^d\ \exists n^{(3)} < \infty\ \forall \mathbf{i} \in \Sigma^*\ \forall b > 0\ \forall a \in \mathbb{R}^d :\ \#\Big\{B(a,b) \cap \Big\{f_{\mathbf{ji}}(x) \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b\Big\}\Big\} \leq n^{(3)}$. Let $R > 0$ be such that $x \in B(\Lambda, R)$, then $\forall \mathbf{j} \in \Sigma^* : f_{\mathbf{j}}(x) \in B(f_{\mathbf{j}}(\Lambda), R\lambda_{\mathbf{j}})$.

Now $\forall \mathbf{k} \in \Sigma^*$:

Recall $\Gamma_{1,R}(\mathbf{k}) = \Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j} \in \widetilde{\mathcal{M}}_{\lambda_{\mathbf{k}}} \text{ and } f_{\mathbf{j}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, R)) \neq \emptyset\Big\}$, then $\lambda_{\mathbf{j}} \leq \lambda_{\mathbf{k}}$.

Let $a \in \mathbb{R}^d$ such that $f_{\mathbf{k}}(\Lambda) \subset B(a, \lambda_{\mathbf{k}})$.

$$\begin{aligned}
&f_{\mathbf{j}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda,R)) \neq \emptyset \implies f_{\mathbf{j}}(\Lambda) \subset B(f_{\mathbf{k}}(\Lambda), R\lambda_{\mathbf{k}} + \lambda_{\mathbf{j}}) \implies \\
&B(f_{\mathbf{j}}(\Lambda), R\lambda_{\mathbf{j}}) \subset B(f_{\mathbf{k}}(\Lambda), R\lambda_{\mathbf{k}} + \lambda_{\mathbf{j}} + R\lambda_{\mathbf{j}}) \subset B(f_{\mathbf{k}}(\Lambda), (2R+1)\lambda_{\mathbf{k}}) \\
&\qquad\qquad \subset B(B(a, \lambda_{\mathbf{k}}), (2R+1)\lambda_{\mathbf{k}}) \subset B(a, (2R+2)\lambda_{\mathbf{k}}) \\
&\qquad\qquad \subset \bigcup_{i=1}^{L(R)} B(a_i, \lambda_{\mathbf{k}})
\end{aligned}$$

by Lemma 3.1.1 with $L(R) = \mathcal{L}(d, (2R+1)\lambda_{\mathbf{k}}, \lambda_{\mathbf{k}}) = \mathcal{L}(d, (2R+1), 1)$.

Hence using *5.c.*: there are at most $n^{(3)}$ points $y_1, \ldots, y_{n^{(3)}} \in \mathbb{R}^d$ such that $f_{\mathbf{j}}(x) = y_p \in B(a_i, \lambda_{\mathbf{k}})$. On the other hand there are at most C^*-many $f_{\mathbf{j}}$ such that $\mathbf{j} \in \widetilde{\mathcal{M}}_{\lambda_{\mathbf{k}}}$ and $f_{\mathbf{i}}(x) = y_p$ for any p by Lemma 3.1.2, and therefore:

$$\#\Gamma_{1,R}(\mathbf{k}) \leq \sum_{i=1}^{L(R)} \#\Big\{ f_{\mathbf{j}} \,\Big|\, \mathbf{j} \in \widetilde{\mathcal{M}}_{\lambda_{\mathbf{k}}} \text{ and } f_{\mathbf{j}}(x) \in B(a_i, \lambda_{\mathbf{k}}) \Big\} \leq L(R) \cdot n^{(3)} \cdot C^*.$$

4.c. $\Longrightarrow$ *1.a.*

By *4.c.* there exist $c > 0, R > 0$ such that $\gamma^{(1)}_{c,R} < \infty$. By Lemma 3.1.3 $\gamma^{(1)}_{1,R} \leq \phi(c)\gamma^{(1)}_{c,R} < \infty$, therefore

$$\exists \ell \in \Sigma^* : \quad \#\Gamma_{1,R}(\ell) = \max_{\mathbf{k} \in \Sigma^*} \Big\{ \#\Gamma_{1,R}(\mathbf{k}) \Big\}.$$

Fix $p > 1$. Now for any $\mathbf{v} \in \Sigma^*$:

$$\begin{aligned} \Gamma_{1,R}(\mathbf{v}\ell) &= \Big\{ f_{\mathbf{i}} \,\Big|\, |f_{\mathbf{i}}(\Lambda)| \leq \lambda_{\mathbf{v}\ell} < |f_{\mathbf{i}^-}(\Lambda)| \text{ and } f_{\mathbf{i}}(\Lambda) \cap f_{\mathbf{v}\ell}(B(\Lambda, R)) \neq \emptyset \Big\} \\ &= \Big\{ f_{\mathbf{v}} \circ f_{\mathbf{j}} \,\Big|\, \mathbf{j} \in \Gamma_{1,R}(\ell) \Big\}. \end{aligned}$$

The containment $\supseteq$ holds by the definition and $\subseteq$ hold by the maximality of ℓ. For an arbitrary $y \in \Lambda$ let $x := f_\ell(y)$, then still $x \in \Lambda$, but we have $B(x, R\lambda_\ell) \subset f_\ell(B(\Lambda, R))$. Let q be an integer such that $\lambda^q_{\max} < 1/p$. We may assume that $|\ell| > q$, this is possible because by $\Gamma_{1,R}(\mathbf{v}\ell) = \Big\{ f_{\mathbf{v}} \circ f_{\mathbf{j}} \,\Big|\, \mathbf{j} \in \Gamma_{1,R}(\ell) \Big\}$ we can choose a new $\ell' := \mathbf{v}\ell$ with $|\ell'| > q$. Let us define

$$\varepsilon_1 := \min \Big\{ |x - f(f_{\mathbf{k}}(y))| \,\Big|\, f \in \Gamma_{1,R}(\ell),\ 0 \leq |\mathbf{k}| \leq q \text{ and } f(f_{\mathbf{k}}(y)) \neq x \Big\},$$

$$\text{and } \varepsilon := \min \Big\{ \varepsilon_1, R\lambda_\ell \Big\}.$$

Let us use the convention: $f_{\mathbf{k}} := \mathrm{Id}$ when $|\mathbf{k}| = 0$. Let $h = f_{\mathbf{v}}^{-1} \circ f_{\mathbf{w}} \in \mathcal{E} \backslash \mathrm{Id}$ with $\lambda_h \in [p^{-1}, p]$. We may assume that $\lambda_{\mathbf{v}} \geq \lambda_{\mathbf{w}}$, since otherwise we might just work with h^{-1}. Then $1/p \leq \lambda_h = \lambda_{\mathbf{w}}/\lambda_{\mathbf{v}} \leq 1$. Then $\lambda_{\mathbf{w}\ell} \leq \lambda_{\mathbf{v}\ell}$ and exists ℓ' a prefix of ℓ such that $\mathbf{w}\ell' \in \widetilde{\mathcal{M}}_{\lambda_{\mathbf{v}\ell}}$. Defining ℓ'' as $\ell = \ell'\ell''$ we have that $0 \leq |\ell''| \leq q$. Now there are two cases:

If $f_{\mathbf{w}\ell'} \in \Gamma_{1,R}(\mathbf{v}\ell) = \left\{ f_{\mathbf{v}} \circ f_{\mathbf{j}} \;\middle|\; \mathbf{j} \in \Gamma_{1,R}(\ell) \right\}$ then $f_{\mathbf{w}\ell'} = f_{\mathbf{v}} \circ f_{\mathbf{i}}$ for some $\mathbf{i} \in \Gamma_{1,R}(\ell)$. Then:

$$\begin{aligned}
|h(x) - x| &= |f_{\mathbf{v}}^{-1}(f_{\mathbf{w}}(x)) - x| = |f_{\mathbf{v}}^{-1}(f_{\mathbf{w}}(f_{\ell}(y))) - x| \\
&= |f_{\mathbf{v}}^{-1}(f_{\mathbf{w}}(f_{\ell'}(f_{\ell''}(y)))) - x| \\
&= |f_{\mathbf{v}}^{-1}(f_{\mathbf{v}}(f_{\mathbf{i}}(f_{\ell''}(y)))) - x| \\
&= |f_{\mathbf{i}}(f_{\ell''}(y)) - x| \geq \varepsilon_1 \geq \varepsilon \quad \text{by definition.}
\end{aligned}$$

If $f_{\mathbf{w}\ell'} \notin \Gamma_{1,R}(\mathbf{v}\ell) = \left\{ f_{\mathbf{v}} \circ f_{\mathbf{j}} \;\middle|\; \mathbf{j} \in \Gamma_{1,R}(\ell) \right\}$ then by $f_{\mathbf{w}\ell}(\Lambda) \subset f_{\mathbf{w}\ell'}(\Lambda)$ we have that for $x \in f_{\ell}(\Lambda) : |f_{\mathbf{w}}(x) - f_{\mathbf{v}}(x)| \geq R\lambda_{\mathbf{v}}\lambda_{\ell}$. Finally:

$$\begin{aligned}
|h(x) - x| &= |f_{\mathbf{v}}^{-1}(f_{\mathbf{w}}(x)) - x| = \lambda_{\mathbf{v}}^{-1}|f_{\mathbf{w}}(x) - f_{\mathbf{v}}(x)| \text{ by linearity} \\
&\geq \lambda_{\mathbf{v}}^{-1} R\lambda_{\mathbf{v}}\lambda_{\ell} = R\lambda_{\ell} \geq \varepsilon.
\end{aligned}$$

In both cases for $h \in \mathcal{E} \backslash \mathrm{Id}$ we have that $|h(x) - x| \geq \varepsilon$.

1.a. $\implies$ *1.b.*

Let $\mathcal{W} := \left\{ h \in \mathcal{G} \;\middle|\; \lambda_h \in]\lambda_{\min}, \lambda_{\min}^{-1}[\right\}$, clearly $\mathcal{F} \subseteq \mathcal{E} \cap \mathcal{W}$, now we show the other containment. Let $h \in \mathcal{E} \cap \mathcal{W}$ meaning that $h = f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}}$ for some $\mathbf{i}, \mathbf{j} \in \Sigma^*$ and $\lambda_h = \lambda_{\mathbf{i}}^{-1}\lambda_{\mathbf{j}} \in]\lambda_{\min}, \lambda_{\min}^{-1}[$. Suppose that $h \notin \mathcal{F}$, so $\nexists b > 0 : f_{\mathbf{i}}, f_{\mathbf{j}} \in F_b$ which implies $\forall b > 0$ such that $f_{\mathbf{j}} \in F_b$ we have $f_{\mathbf{i}} \notin F_b$ and hence

$$\begin{aligned}
f_{\mathbf{i}} \notin \bigcup_{b: f_{\mathbf{j}} \in F_b} F_b \implies \lambda_{\mathbf{i}} \notin \bigcup_{b: f_{\mathbf{j}} \in F_b}]b\lambda_{\min}, b] = \bigcup_{b \in [\lambda_{\mathbf{j}}, \lambda_{\mathbf{j}}\lambda_{\min}^{-1}[}]b\lambda_{\min}, b] = \\
=]\lambda_{\mathbf{j}}\lambda_{\min}, \lambda_{\mathbf{j}}\lambda_{\min}^{-1}[\implies \lambda_{\mathbf{i}}\lambda_{\mathbf{j}}^{-1} \notin]\lambda_{\min}, \lambda_{\min}^{-1}[,
\end{aligned}$$

which contradicts that $h \in \mathcal{W}$ proving that $\mathcal{F} = \mathcal{E} \cap \mathcal{W}$. Finally we can choose $p := \lambda_{\min}^{-1} > 1$ to provide the implication.

1.c. $\implies$ *2.c.*

By *1.c.*: there exists $x \in \mathbb{R}^d$ $\exists \varepsilon > 0$ $\forall h \in \mathcal{F}$: if $h(x) \neq x \implies |h(x) - x| > \varepsilon$, fix this x. Let $F_b(x) = \{f(x) \mid f \in F_b\} = \left\{ f_{\mathbf{i}}(x) \;\middle|\; \lambda_{\mathbf{i}} \in]b\lambda_{\min}, b] \right\}$, as b

decreases towards 0, $F_b(x)$ converge to the attractor Λ in the Hausdorff metric. By assumption Λ is not contained in any hyperplane, therefore there will be $f_{\mathbf{i}_1}, \dots, f_{\mathbf{i}_d} \in F_b$ such that $x_j := f_{\mathbf{i}_j}(x), j = 0, \dots, d$ will be in general position. Now $\forall j \in \{0, \dots, d\}, \forall h \in \mathcal{F}$: either $h(x_j) = x_j$, or

$$\begin{aligned} \left|h(x_j) - x_j\right| = \left|h(f_{\mathbf{i}_j}(x)) - f_{\mathbf{i}_j}(x)\right| &= \lambda_{\mathbf{i}_j}\left|f_{\mathbf{i}_j}^{-1}(h(f_{\mathbf{i}_j}(x))) - (x)\right| \\ &\geq \min_{j \in \{0,\dots,d\}} \left\{\lambda_{\mathbf{i}_j}\right\} \cdot \varepsilon. \end{aligned}$$

We remark that with a similar argument one can prove *1.a.* $\Longrightarrow$ *2.a.*.

2.b. $\Longleftrightarrow$ *3.b.*

Trivially $\mathcal{F} := \bigcup_{b>0} \left\{f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \;\middle|\; f_{\mathbf{i}}, f_{\mathbf{j}} \in F_b\right\}$ is a subset of $\mathcal{E} := \left\{f_{\mathbf{i}}^{-1} \circ f_{\mathbf{j}} \mid \mathbf{i}, \mathbf{j} \in \Sigma^* \text{ such that } \mathbf{i} \neq \mathbf{j}\right\}$, so we can use the topology defined in the proof of *2.a.* $\Longleftrightarrow$ *3.a.* gives us ther result.

3.b. $\Longrightarrow$ *3.a.*

As we have seen before $\mathcal{F} = \mathcal{E} \cap \mathcal{W}$ where $\mathcal{W} := \left\{h \in \mathcal{G} \;\middle|\; \lambda_h \in]\lambda_{\min}, \lambda_{\min}^{-1}[\right\}$ which is an open subset in $\mathcal{G}$, because for any $g \in \mathcal{G} : g(x) = c_g U_g x + t_g$ where $c_g \in \mathbb{R}^+, U_g \in O(d, \mathbb{R}), t_g \in \mathbb{R}^d$, if $c_g = \lambda_g \in]\lambda_{\min}, \lambda_{\min}^{-1}[$, then $B(g, \min\{\frac{|\lambda_{\min} - \lambda_g|}{2}, \frac{|\lambda_{\min}^{-1} - \lambda_g|}{2}\}) \subset \mathcal{W}$.

3.4 THE SECONDARY PATHS

2.b. $\Longrightarrow$ *4.b.*

The proof is similar to *2.a.* $\Longrightarrow$ *4.a.*. By *2.b.* there exists $x_0, \dots, x_d \in \mathbb{R}^d$ in general position $\exists \varepsilon > 0\ \forall h \in \mathcal{F} \setminus \mathrm{Id}\ \exists j$ with: $|h(x_j) - x_j| > \varepsilon$. Fix $M \subset \mathbb{R}^d$ bounded, fix $c > 0$. We want to see that

$$\gamma_{c,M}^{(2)} := \sup_{N \subseteq \mathbb{R}^d} \left\{\# F_{c,N,M}\right\} = \sup_{N \subseteq \mathbb{R}^d} \left\{\#\left\{f \in F_{c|N|} \;\middle|\; f(M) \cap N \neq \emptyset\right\}\right\} =$$

$$= \sup_{N\subseteq\mathbb{R}^d} \Big\{\#\Big\{f \in \Big\{f_{\mathbf{i}} \;\Big|\; \lambda_{\mathbf{i}} \in]c|N|\lambda_{\min}, c|N|]\Big\} \;\Big|\; f(M)\cap N \neq \emptyset\Big\}\Big\}$$

$$= \sup_{N\subseteq\mathbb{R}^d} \Big\{\#\Big\{f_{\mathbf{i}} \;\Big|\; f_{\mathbf{i}}(M)\cap N \neq \emptyset \text{ and } \lambda_{\mathbf{i}} \in]c|N|\lambda_{\min}, c|N|]\Big\}\Big\} < \infty.$$

We may assume that $x_0, \ldots, x_d \in M$. For any $f \neq g \in F_{c,N,M}$ we have $\Big|f(x_j) - g(x_j)\Big| \geq c|N|\lambda_{\min}\varepsilon$ for a $j = j(f,g)$ given for $h := g^{-1} \circ f \in \mathcal{F} \setminus \mathrm{Id}$. Now define for all j $= 0, \ldots, d : F_j$ subfamily of $F_{c,N,M}$ such that $j(f_1, f_2) = j$ is a constant for any $f_1, f_2 \in F_j$.

Fix $j \in \{0, \ldots, d\}$, then for all $f, g \in F_j$:

$$B(f(x_j), c|N|\lambda_{\min}\varepsilon/2) \cap B(g(x_j), c|N|\lambda_{\min}\varepsilon/2) = \emptyset.$$

On the other hand the their centers are contained in a ball of radius $|N|(1+2c|M|)$ therefore $\#F_j \leq (1 + 2c|M| + 2c\lambda\text{min}\varepsilon/2)^d/(2c\lambda\text{min}\varepsilon/2)^d$ independent of N. And again Ramsey's theorem 19.2.3 [2] yields that there exist $L(c, M)$ independent of N such that $\#F_{c,N,M} \leq L(c,M) < \infty$ giving us a bound to $\gamma^{(2)}_{c,M}$.

4.b. $\implies$ *5.d.*

Fix $x \in \mathbb{R}^d$, for any $a \in \mathbb{R}^d, y > 0$:

$$\sup_{b>0} \sup_{a\in\mathbb{R}^d} \sup_{f\in F} \#\Big\{F_b(f(x)) \cap B(a,b)\Big\}$$

$$\leq \sup_{b,a,f} \#\Big\{g \in F_b \;\Big|\; g(f(x)) \in B(a,b)\Big\}$$

$$\leq \sup_{b,a} \#\Big\{g \in F_b \;\Big|\; g(F(x)) \cap B(a,b) \neq \emptyset\Big\}$$

$$\leq \sup_{b,U} \#\Big\{g \in F_{\frac{1}{2}\cdot|U|} \;\Big|\; g(F(x)) \cap U \neq \emptyset \text{ and } |U| = 2b\Big\}$$

$$\leq \sup_{U\subset\mathbb{R}^d} \#\Big\{g \in F_{\frac{1}{2}\cdot|U|} \;\Big|\; g(F(x)) \cap U \neq \emptyset\Big\} \leq \gamma^{(2)}_{1/2,F(x)}.$$

And $F(x) = \{f_{\mathbf{i}}(x) \mid \mathbf{i} \in \Sigma^*\}$ is bounded hence $\gamma^{(2)}_{1/2,F(x)} < \infty$ by *4.b.*.

5.e. $\implies$ 4.d.

$$\begin{aligned}\gamma^{(2)}_{1/2,\{x\}} &= \sup_{N\subseteq\mathbb{R}^d} \#\Big\{f\in F_{1/2|N|}\ \Big|\ f(x)\cap N\neq\emptyset\Big\}\\ &\le \sup_{N\subseteq\mathbb{R}^d} \mathcal{L}\sup_{z\in\mathbb{R}^d} \#\Big\{f\in F_{|N|/2}\ \Big|\ f(x)\in B(z,|N|/2)\Big\}\\ &\le \mathcal{L}\cdot\sup_{b>0}\sup_{z\in\mathbb{R}^d}\sum_{y\in F_b(x)\cap B(z,b)} \#\Big\{f\in F_b\ \Big|\ f(x)=y\Big\},\end{aligned}$$

where $\mathcal{L}=\mathcal{L}(d,2,1)$ is the number of unit balls needed to cover a radius two ball by Lemma 3.1.1. Now using Lemma 3.1.2 and 5.e.:

$$\gamma^{(2)}_{1/2,\{x\}} \le \mathcal{L}\cdot C'\cdot\sup_{b>0}\sup_{z\in\mathbb{R}^d}\#\Big\{F_b(x)\cap B(z,b)\Big\}\le\mathcal{L}\cdot C'\cdot n^{(5)}.$$

4.d. $\implies$ 1.c.

The proof will be very similar to 4.c. $\implies$ 1.a.. By 4.d. $\exists c>0$ $\exists$ non-empty $M\subseteq\mathbb{R}^d$: $\gamma^{(2)}_{c,M}<\infty$. Without loss of generality we may assume that $M=\{y\}, y\in\mathbb{R}^d$. Define for any $f_{\mathbf{i}}$:

$$\mathcal{I}_{B(y,1/2c)}(f_{\mathbf{i}}):=\Big\{f_{\mathbf{j}}\in F_{\lambda_{\mathbf{i}}}\ \Big|\ f_{\mathbf{j}}(y)\cap f_{\mathbf{i}}(B(y,1/2c))\neq\emptyset\Big\}=F_{c,f_{\mathbf{i}}(B(y,1/2c)),\{y\}}.$$

By 4.d. $\#F_{c,f_{\mathbf{i}}(B(y,1/2c)),\{y\}}$ is bounded, also it can only admit whole numbers, so there exists a $f_{\mathbf{k}}\in F$ such that $\#F_{c,f_{\mathbf{k}}(B(y,1/2c)),\{y\}}$ is maximal. Now for any $f_{\mathbf{i}}\in F$:

$$\mathcal{I}_{B(y,1/2c)}(f_{\mathbf{i}}\circ f_{\mathbf{k}})=f_{\mathbf{i}}\circ\mathcal{I}_{B(y,1/2c)}(f_{\mathbf{k}}).$$

Indeed the maximality property of $f_{\mathbf{k}}$ gives $\subseteq$ and for any $h=f_{\mathbf{i}}\circ f_{\mathbf{j}}, f_{\mathbf{j}}\in\mathcal{I}_{B(y,1/2c)}(f_{\mathbf{k}})$ we have that $h\in F_{\lambda_{\mathbf{i}}\cdot\lambda_{\mathbf{k}}}=F_{\lambda_{\mathbf{ik}}}$ and $f_{\mathbf{k}}(y)\cap f_{\mathbf{j}}(B(y,1/2c))\neq\emptyset$ implies $f_{\mathbf{i}}(f_{\mathbf{k}}(y))\cap f_{\mathbf{i}}(f_{\mathbf{j}}(B(y,1/2c)))\neq\emptyset$.

Set $x:=f_{\mathbf{k}}(y)$. Let $h=f_{\mathbf{i}}^{-1}\circ f_{\mathbf{j}}\in\mathcal{F}$ be arbitrary, for some $f_{\mathbf{i}},f_{\mathbf{j}}\in F_b$. We may assume that $\lambda_{\mathbf{i}}\le\lambda_{\mathbf{j}}$ otherwise h^{-1} works. Case 1: $f_{\mathbf{j}}\circ f_{\mathbf{k}}\in\mathcal{I}_{B(y,1/2c)}(f_{\mathbf{i}}\circ f_{\mathbf{k}})$. Then

$$|h(x)-x| = |f_{\mathbf{i}}^{-1}(f_{\mathbf{j}}(f_{\mathbf{k}}(y))) - f_{\mathbf{k}}(y)| = |\tilde{g}(y) - f(y)|.$$

If $\tilde{g}(y) \neq f(y)$, then $|\tilde{g}(y) - f(y)| \geq \min\left\{|g(y)-f(y)| \;\middle|\; g \in \mathcal{I}_{B(y,1/2c)}(f) \text{ and } g(y) \neq f(y)\right\}$

Case 2: $f_{\mathbf{j}} \circ f_{\mathbf{k}} \notin \mathcal{I}_{B(y,1/2c)}(f_{\mathbf{i}} \circ f_{\mathbf{k}})$. Then $f_{\mathbf{i}}^{-1}(f_{\mathbf{j}}(f_{\mathbf{k}}(y))) \notin f_{\mathbf{k}}(B(y,1/2c)) = B(x, \lambda_{\mathbf{k}}/2c)$, so $|h(x)-x| \geq \lambda_{\mathbf{k}}/2c$. Finally now *1.b.* holds with

$$\varepsilon := \min\left\{\lambda_{\mathbf{k}}/2c, \min\left\{|g(y)-f(y)| \;\middle|\; g \in \mathcal{I}_{B(y,1/2c)}(f) \text{ and } g(y) \neq f(y)\right\}\right\}.$$

3.5 REMAINDER IMPLICATIONS

4.a. $\Longrightarrow$ *6.a.*

We need to show that exists $n^{(6)} < \infty \quad \forall x \in \mathbb{R}^d \quad \forall b > 0$ such that:

$$\#\left\{f_{\mathbf{j}} \;\middle|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } f_{\mathbf{j}}(\Lambda) \cap B(x,b) \neq \emptyset\right\} \leq n^{(6)}.$$

Fix $B(x,b)$ we may assume that $b \leq 1$ and $\Lambda \cap B(x,b) \neq \emptyset$ otherwise $\widetilde{\mathcal{M}}_b$ would be empty or the intersection would be empty. Then there exist $\mathbf{k} \in \Sigma^*$ such that $f_{\mathbf{k}}(\Lambda) \cap B(x,b) \neq \emptyset$ and $b\lambda_{\min} < \lambda_{\mathbf{k}} \leq b$. By this again using $|\Lambda| = 1$:

$$B(x,b) \subset B(f_{\mathbf{k}}(\Lambda), 2b) \subset B(f_{\mathbf{k}}(\Lambda), 2\lambda_{\mathbf{k}}\lambda_{\min}^{-1}) = f_{\mathbf{k}}(B(\Lambda, 2\lambda_{\min}^{-1})).$$

Therefore for any $\mathbf{j} \in \widetilde{\mathcal{M}}_b$ if $f_{\mathbf{j}}(\Lambda) \cap B(x,b) \neq \emptyset$ then we have that $f_{\mathbf{j}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, 2\lambda_{\min}^{-1})) \neq \emptyset$. Now fix $\mathbf{k} \in \Sigma^*$:

$$\begin{aligned}
&\left\{f_{\mathbf{j}} \;\middle|\; \mathbf{j} \in \widetilde{\mathcal{M}}_b \text{ and } f_{\mathbf{j}}(\Lambda) \cap B(x,b) \neq \emptyset\right\} \\
&\subset \left\{f_{\mathbf{j}} \;\middle|\; \mathbf{j} \in \widetilde{\mathcal{M}}_{\lambda_{\mathbf{k}}\lambda_{\mathbf{k}}^{-1}b} \text{ and } f_{\mathbf{j}}(\Lambda) \cap f_{\mathbf{k}}(B(\Lambda, 2\lambda_{\min}^{-1})) \neq \emptyset\right\}. \\
&= \Gamma_{\lambda_{\mathbf{k}}^{-1}b, 2\lambda_{\min}^{-1}}(\mathbf{k}) \leq \gamma_{\lambda_{\mathbf{k}}^{-1}b, 2\lambda_{\min}^{-1}}.
\end{aligned}$$

Finally, using Lemma 3.1.3: for any $\mathbf{k}$ such that $b\lambda_{\min} < \lambda_{\mathbf{k}} \leq b$

$$\gamma_{\lambda_{\mathbf{k}}^{-1}b,2\lambda_{\min}^{-1}} \leq \phi(\frac{\lambda_{\min}^{-1}}{\lambda_{\mathbf{k}}^{-1}b})\gamma_{\lambda_{\min}^{-1},2\lambda_{\min}^{-1}} \leq \phi(\lambda_{\min}^{-1})\gamma_{\lambda_{\min}^{-1},2\lambda_{\min}^{-1}} < \infty.$$

Because ϕ is increasing and continuous and $\lambda_{\mathbf{k}} \in]b\lambda_{\min}, b]$.

6.a. $\implies$ *6.b.*

Fix $0 < C_1 < 1 < C_2$. Notice that the assumption on $|\Lambda| = 1$ converts $C_1 \leq \frac{|f_{\mathbf{j}}(\Lambda)|}{|B(x,b)|} \leq C_2$ into $C_1 \leq \frac{\lambda_{\mathbf{j}}}{2b} \leq C_2$, or in other words: $\lambda_{\mathbf{j}} \in [2bC_1, 2bC_2]$. Now consider the following covering union of sets of that interval:

$$\begin{aligned}\widetilde{\mathcal{M}}_{2bC_2} &= \left\{\mathbf{i} \in \Sigma^* \;\middle|\; \lambda_{\mathbf{i}} = |f_{\mathbf{i}}(\Lambda)| \leq 2bC_2 < |f_{\mathbf{i}^-}(\Lambda)| = \lambda_{\mathbf{i}^-}\right\} \\ &\supset \left\{\mathbf{i} \in \Sigma^* \;\middle|\; 2bC_2\lambda_{\max} \leq \lambda_{\mathbf{i}} \leq 2bC_2\right\},\end{aligned}$$

$$\begin{aligned}\widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}} &= \left\{\mathbf{i} \in \Sigma^* \;\middle|\; \lambda_{\mathbf{i}} \leq 2bC_2\lambda_{\max} < \lambda_{\mathbf{i}^-}\right\} \\ &\supset \left\{\mathbf{i} \in \Sigma^* \;\middle|\; 2bC_2\lambda_{\max}^2 \leq \lambda_{\mathbf{i}} \leq 2bC_2\lambda_{\max}\right\},\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}^k} &= \left\{\mathbf{i} \in \Sigma^* \;\middle|\; \lambda_{\mathbf{i}} \leq 2bC_2\lambda_{\max}^k < \lambda_{\mathbf{i}^-}\right\} \\ &\supset \left\{\mathbf{i} \in \Sigma^* \;\middle|\; 2bC_2\lambda_{\max}^{k+1} \leq \lambda_{\mathbf{i}} \leq 2bC_2\lambda_{\max}^k\right\},\end{aligned}$$

$$\vdots$$

Then there will be an h such that from that index $2bC_2\lambda_{\max}^h$ becomes smaller than $2bC_1$ because $\lambda_{\max} < 1$: $2bC_2\lambda_{\max}^k = 2bC_1 \implies k = \log_{\lambda_{\max}}\left\{\frac{C_1}{C_2}\right\}$, so $h := \left\lfloor \log_{\lambda_{\max}}\left\{\frac{C_1}{C_2}\right\}\right\rfloor + 1$. Finally we have that:

$$\begin{aligned}&\left\{f_{\mathbf{j}} \;\middle|\; C_1 \leq \frac{|f_{\mathbf{j}}(\Lambda)|}{|B(x,b)|} \leq C_2\right\} = \left\{f_{\mathbf{j}} \;\middle|\; \lambda_{\mathbf{j}} \in [2bC_1, 2bC_2]\right\} \\ &\subset \bigcup_{k=0}^{h} \left\{f_{\mathbf{j}} \;\middle|\; \lambda_{\mathbf{j}} \in [2bC_2\lambda_{\max}^{k+1}, 2bC_2\lambda_{\max}^k]\right\} \subset \bigcup_{k=0}^{h} \widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}^k}.\end{aligned}$$

For a given pair $k, B(x,b)$, we define $x_i, i \in \{0,\dots,L(k)\}$ as the centers of balls with radius $2bC_2\lambda_{\max}^k$ such that they cover $B(x,b)$ by Lemma 3.1.1 with $L(k) = \mathcal{L}(d,b,2bC_2\lambda_{\max}^k)$. Hence:

$$
\begin{aligned}
&\#\Big\{f_{\mathbf{j}} \;\Big|\; C_1 \le \frac{|f_{\mathbf{j}}(\Lambda)|}{|B(x,b)|} \le C_2 \text{ and } f_{\mathbf{j}}(\Lambda)\cap B(x,b)\neq\emptyset\Big\}\\
&\le \#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j}\in\bigcup_{k=1}^{h}\widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}^k} \text{ and } f_{\mathbf{j}}(\Lambda)\cap B(x,b)\neq\emptyset\Big\}\\
&\le \sum_{k=1}^{h}\#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j}\in\widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}^k} \text{ and } f_{\mathbf{j}}(\Lambda)\cap B(x,b)\neq\emptyset\Big\}\\
&\le \sum_{k=1}^{h}\#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j}\in\widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}^k} \text{ and } f_{\mathbf{j}}(\Lambda)\cap \bigcup_{i=1}^{L(k)} B(x_i,2bC_2\lambda_{\max}^k)\neq\emptyset\Big\}\\
&\le \sum_{k=1}^{h}\sum_{i=1}^{L(k)}\#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j}\in\widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}^k} \text{ and } f_{\mathbf{j}}(\Lambda)\cap B(x_i,2bC_2\lambda_{\max}^k)\neq\emptyset\Big\}\\
&\le \sum_{k=1}^{h}\sum_{i=1}^{L(h)}\#\Big\{f_{\mathbf{j}} \;\Big|\; \mathbf{j}\in\widetilde{\mathcal{M}}_{2bC_2\lambda_{\max}^k} \text{ and } f_{\mathbf{j}}(\Lambda)\cap B(x_i,2bC_2\lambda_{\max}^k)\neq\emptyset\Big\}\\
&\le h(C_1,C_2)\cdot n^{(6)}\cdot L(h).
\end{aligned}
$$

6.b. $\implies$ *4.c.*

Fix $c > 0$ and $R > 0$.

$$
\begin{aligned}
\gamma_{c,R}^{(1)} &= \sup_{\mathbf{k}\in\Sigma^*}\#\Big\{f_{\mathbf{i}} \;\Big|\; \lambda_{\mathbf{i}} \le c\lambda_{\mathbf{k}} < \lambda_{\mathbf{i}-} \text{ and } f_{\mathbf{i}}(\Lambda)\cap B(f_{\mathbf{k}}(\Lambda),R\lambda_{\mathbf{k}})\neq\emptyset\Big\}\\
&\le \sup_{\mathbf{k}\in\Sigma^*}\#\Big\{f_{\mathbf{i}} \;\Big|\; \lambda_{\mathbf{i}} \le c\lambda_{\mathbf{k}} < \lambda_{\mathbf{i}-} \text{ and } f_{\mathbf{i}}(\Lambda)\cap \bigcup_{i=1}^{L(R)} B(f_{\mathbf{k}}(x_i),R\lambda_{\mathbf{k}})\neq\emptyset\Big\}\\
&\le \sum_{i=1}^{L(R)}\sup_{\mathbf{k}\in\Sigma^*}\#\Big\{f_{\mathbf{i}} \;\Big|\; \lambda_{\mathbf{i}} \le c\lambda_{\mathbf{k}} < \lambda_{\mathbf{i}-} \text{ and } f_{\mathbf{i}}(\Lambda)\cap B(f_{\mathbf{k}}(x_i),R\lambda_{\mathbf{k}})\neq\emptyset\Big\}\\
&\le \sum_{i=1}^{L(R)}\sup_{\mathbf{k}\in\Sigma^*}\#\Big\{f_{\mathbf{i}} \;\Big|\; \lambda_{\min}c\lambda_{\mathbf{k}} < \lambda_{\mathbf{i}} \le c\lambda_{\mathbf{k}} \text{ and } f_{\mathbf{i}}(\Lambda)\cap B(f_{\mathbf{k}}(x_i),R\lambda_{\mathbf{k}})\neq\emptyset\Big\}.
\end{aligned}
$$

By Lemma 3.1.1 for some $L(R) = \mathcal{L}(d,R|\Lambda|,R) = \mathcal{L}(d,|\Lambda|,1)$. Now $\lambda_{\min}c\lambda_{\mathbf{k}} < \lambda_{\mathbf{i}} \le c\lambda_{\mathbf{k}}$ implies: $\frac{\lambda_{\min}c}{2R} = \frac{\lambda_{\min}c\lambda_{\mathbf{k}}}{2\lambda_{\mathbf{k}}R} < \frac{\lambda_{\mathbf{i}}}{2\lambda_{\mathbf{k}}R} \le \frac{c\lambda_{\mathbf{k}}}{2\lambda_{\mathbf{k}}R} = \frac{c}{2R}$. Finally:

$$\leq \sum_{i=1}^{L(R)} \sup_{k\in\Sigma^*} \#\Big\{f_{\mathbf{i}} \;\Big|\; \frac{\lambda_{\min} c}{2R} < \frac{\lambda_{\mathbf{i}}}{2\lambda_{\mathbf{k}} R} \leq \frac{c}{2R} \text{ and } f_{\mathbf{i}}(\Lambda) \cap B(f_{\mathbf{k}}(x_i), R\lambda_{\mathbf{k}}) \neq \varnothing\Big\}$$
$$\leq L(R)\cdot n^{(7)} \quad \text{where } n^{(7)} \text{ is given by 6.b. for } C_1 := \frac{\lambda_{\min} c}{2R}, C_2 := \frac{c}{2R}.$$

4.b. $\implies$ *6.c.*

Let D be fixed compact in $\mathbb{R}^d$, for any $x \in \mathbb{R}^d, b > 0$, then using $M_b \subseteq F_b$ we have that:

$$\begin{aligned}\#\Big\{f \in M_b \;\Big|\; x \in f(D)\Big\} &\leq \#\Big\{f \in M_b \;\Big|\; x \in f(D) \cap B(x, b/2) \neq \varnothing\Big\} \\ &\leq \#\Big\{f \in F_b \;\Big|\; x \in f(D) \cap B(x, b/2) \neq \varnothing\Big\} \\ &\leq \#F_{1,B(x,b/2),D} \leq \gamma^{(2)}_{1,D} < \infty.\end{aligned}$$

6.c. $\implies$ *4.d.*

We have that $\#\Big\{f \in M_b \;\Big|\; x \in f(D)\Big\} \leq n^{(8)}$. Note that $f(D) \subset B(x, (1 + |D|)b)$. Now it follows that

$$\sum_f \Big\{\mathrm{Vol}(f(D)) \;\Big|\; f \in M_b \text{ and } f(D) \cap B(x, b) \neq \varnothing\Big\}$$
$$\leq n^{(8)} \cdot \mathrm{Vol}(B(x, (1 + |D|)b)) = n^{(8)} \cdot ((1 + |D|)b)^d \cdot c,$$

where Vol denotes the d-dimensional Lebesgue measure, and for a constant $c = \mathrm{Vol}(B(0, 1))$. On the other hand, $(\lambda_{\min} b)^d \cdot \mathrm{Vol}(D) \leq \lambda_f^d \cdot \mathrm{Vol}(D) = \mathrm{Vol}(f(D))$ because $f \in M_b$. Hence

$$\#\Big\{f \in M_b \;\Big|\; f(D) \cap B(x, b) \neq \varnothing\Big\} \leq \frac{n^{(8)} \cdot (1 + |D|)^d \cdot c}{\lambda^d_{\min} \mathrm{Vol}(D)}.$$

Finally,

$$\begin{aligned}
\gamma_{1,D}^{(2)} &= \sup_{N\subseteq\mathbb{R}^d}\Big\{\#F_{1,N,D}\Big\} = \sup_{N\subseteq\mathbb{R}^d}\#\Big\{f\in F_{|N|}\;\Big|\; f(D)\cap N\neq\emptyset\Big\}\\
&\leq \sup_{x\in\mathbb{R}^d,b>0}\#\Big\{f\in F_b\;\Big|\; f(D)\cap B(y,b/2)\neq\emptyset\Big\}\\
&\leq \sup_{x\in\mathbb{R}^d,b>0}\#\Big\{f\in F_b\;\Big|\; f(D)\cap B(y,b)\neq\emptyset\Big\}\\
&\leq \sup_{x\in\mathbb{R}^d,b>0}\#\Big\{f\in \bigcup_{i=0}^{h(\lambda_{\min})} M_{b\lambda_{\max}^i}\;\Big|\; f(D)\cap B(y,b)\neq\emptyset\Big\}\\
&\leq h(\lambda_{\min})\cdot \sup_{x\in\mathbb{R}^d,b>0,i}\#\Big\{f\in M_{b\lambda_{\max}^i}\;\Big|\; f(D)\cap B(y,b)\neq\emptyset\Big\}\\
&\leq h(\lambda_{\min})\cdot\frac{n^{(8)}\cdot(1+|D|)^d\cdot c}{\lambda_{\min}^d\mathrm{Vol}(D)},
\end{aligned}$$

where we used the construction from *6.a.* $\Longrightarrow$ *6.b.* to get an $h = h(\lambda_{\min}) = \Big\lfloor \log_{\lambda_{\max}}\Big\{\lambda_{\min}\Big\}\Big\rfloor + 1$ independent of b such that $F_b \subseteq \bigcup_{i=0}^{h(\lambda_{\min})} M_{b\lambda_{\max}^i}$.

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