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DIMENSION THEORY OF SELF-AFFINE SYSTEMS WITH SINGULAR MATRICES

MASTER THESIS

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1 INTRODUCTION

The foundation of fractal theory belongs to Benoit Mandelbrot, who called these irregular and fragmented objects as fractals in his book "The Fractal Geometry of Nature" [Man83]. As Mandelbrot said "*It describes many of the irregular and fragmented patterns around us, and leads to full-fledged theories, by identifying a family of shapes I call fractals. The most useful fractals involve chance and both their regularities and their irregularities are statistical. Also, the shapes described here tend to be scaling, implying that the degree of their irregularity and/or fragmentation is identical at all scales*". There is no proper definition for fractals, but the mathematical society agreed about fractals have unique features. In nature we can found many examples for fractals, like coastlines, pine cone, cauliflower, etc.. Some fractals can be described via Iterated Function System (IFS), which is a finite collection of contracting maps. There exists a unique non-empty compact set, which is called the Attractor or if the IFS is formed by similarity transformations often called the self-similar set [HUT81]. The determination of the dimension of the attractor of a general iterated function systems and a general graph-directed iterated function systems (GDIFS) is an open problem, but under some conditions, we can determine it. To measure the dimension, we usually use the Hausdorff-dimension, the Box-dimension but there are another dimensions. The self-similarity means that a set or object is exacly similar to a part of itself. One of the most famous example for self-similar set is the Sierpiński triangle.

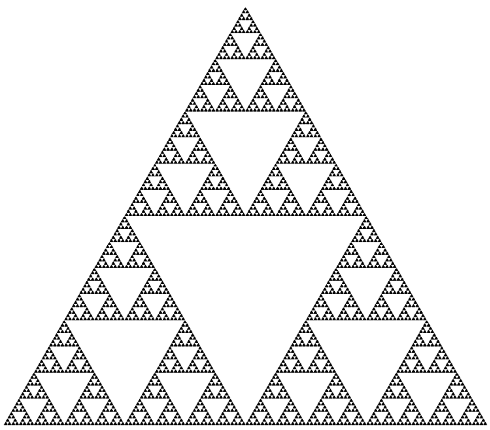


Figure 1: The Sierpiński triangle

A Self-affine set is affine image to a part of itself, scaled by different amount in different directions. A famous illustration for a self-affine set is the Barnsley fern.

Figure 2: The Barnsley fern

For self-similar separated regular IFS, Hutchinson ([HUT81]) showed that if the cylinder sets are disjoint, then the Hausdorff dimension of the attractor is equal to the similarity dimension. Later, Mauldin and Williams ([MW88]) determined the dimension of regular Graph-directed self-similar IFS (GDSSIFS), which is a natural generalization of an ordinary IFS. Recently Bárány, Hochman and Rapaport ([BHR19]) determined the dimension of the attractor of self-affine IFS on the plane, when the matricies of the contracting affine transformations are strongly irreducible and regular. Our question is, what can we say about the dimension, when the matricies of the affine functions are singular?

If we have a self-similar IFS or a self-affine IFS with regular matrices then the attractor of these systems is a perfect set. But if the matrices of the affine mappings are singular, it can happen, that the attractor of the IFS is not a perfect set. We explain this via the following example.

Example 1.1. Let $\mathcal{F} = \{f_i(x) = A_i x + t_i\}_{i=1}^{2}$ be an IFS, such that

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \; t_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \; t_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix},$$

and $x \in [0,1]^2$. Denote $I_1 := [0,1] \times \{0\}$ and $I_2 := \{0\} \times [0,1]$. Then

$$f_1(I_1) = [\frac{1}{2}, 1] \times \{0\}, \qquad f_1(I_2) = \left\{ \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right\},$$
$$f_2(I_1) = \left\{ \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right\}, \qquad f_2(I_2) = \{0\} \times [\frac{1}{2}, 1].$$

To determine the attractor, we need to continue applying f_1 and f_2 on the intervals. By symmetry we can split the attractor into two parts, and we can investigate how it behaves on each axis. Figure 3 shows this behaviour on the **x** axis by each layer means an iteration.

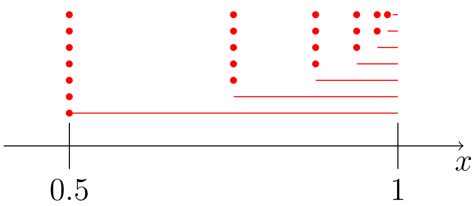


Figure 3: Iterates on x axis.

Let Λ_x and Λ_y be non-empty compact sets, such that

$$\Lambda_x = \bigcup_{n=1}^{\infty} \left\{ \begin{pmatrix} 1-2^{-n} \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \; \Lambda_y = \bigcup_{n=1}^{\infty} \left\{ \begin{pmatrix} 0 \\ 1-2^{-n} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \tag{1}$$

Then for the compact set $\Lambda = \Lambda_x \bigcup \Lambda_y$,

$$f_1(\Lambda) \cup f_2(\Lambda) = \Lambda. \tag{2}$$

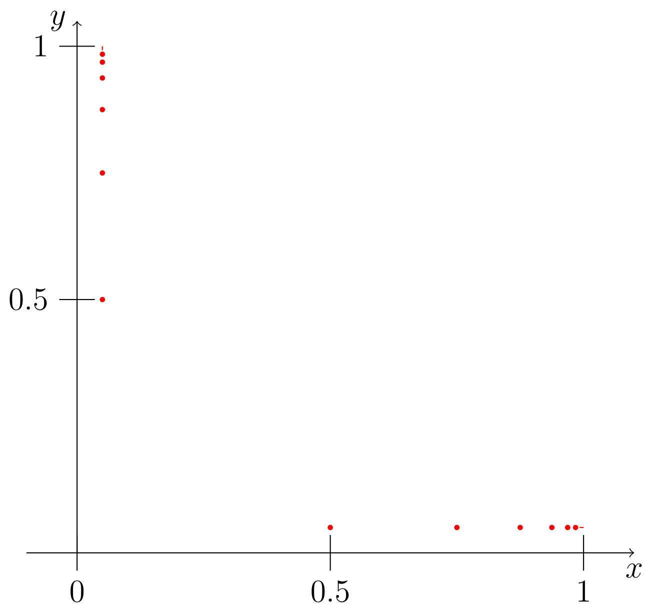


Figure 4: Attractor of IFS defined in **Example** 1.1

In this Thesis, we investigate the connection between self-affine IFS on the plane and GDSSIFS on the line. Using this connection we would like to state separation conditions, for which the dimension of the attractor can be defined with the sub-additive pressure.

2 PRELIMINARY

2.1 THE HAUSDORFF MEASURE AND DIMENSION

In this Thesis, we use the Hausdorff dimension to determine the dimension of the attractor of an IFS. Some theorems state the Box dimension too, which is

$$\underline{\dim}_B(\Lambda) = \liminf_{\delta\to 0+} \frac{N_\delta(\Lambda)}{-\log\delta}, \quad \overline{\dim}_B(\Lambda) = \limsup_{\delta\to 0+} \frac{N_\delta(\Lambda)}{-\log\delta},$$

where $N_\delta(\Lambda)$ denotes the measures the minimal number of balls with radius δ that are covering the attractor Λ. If the limit above exists then we denote it by $\dim_B(\Lambda)$. By the properties below it is more sophisticated to use the Hausdorff dimension instead of Box dimension in theoretical point of view. In this section, we discuss the basic definitions and properties of the Hausdorff measure and the Hausdorff dimension as in [Fal88].

Definition 2.1. Let (X,ϱ) be a complete metric space and E,F be subsets of X. Then we define the Hausdorff premetric of E and F as

$$d_H(E,F) = \inf\left\{\delta > 0 : F \subseteq [E]_\delta \text{ and } E \subseteq [F]_\delta\right\},$$

where $[E]_\delta = \{y \in X : \text{ there exists } x \in E,\ \varrho(x,y) < \delta\}$. Let $\mathcal{C} = \{E \subseteq X : E \text{ is compact}\}$ be a collection of compact sets, then d_H is a metric on $\mathcal{C}$.

We denote the diameter of a set A by $|A|$.

Definition 2.2. Let $E \subset \mathbb{R}^d$ and $t \geq 0$. Then the collecton of set $\{A_i\}_{i=1}^\infty$ is a δ-cover of E for $\delta > 0$, if $E \subset \bigcup_{i=1}^\infty A_i$ and $|A_i| < \delta$. We call $\mathcal{H}^t(E)$ the t-dimensional Hausdorff pre-measure of E, where

$$\mathcal{H}^t(E) = \lim_{\delta\to 0}\left\{\inf\left\{\sum_{i=1}^\infty |A_i|^t \,:\, E \subset \bigcup_{i=1}^\infty A_i,\ |A_i| \leq \delta\right\}\right\}.$$

Lemma 2.3. *For any Borel set $E \subset \mathbb{R}^d$ and $0 \leq \alpha < \beta$ we obtain the following implications:*

$$\mathcal{H}^\alpha(E) < \infty \Rightarrow \mathcal{H}^\beta(E) = 0,$$
$$0 < \mathcal{H}^\beta(E) \Rightarrow \mathcal{H}^\alpha(E) = \infty.$$

The proof can be found in the book called "The Geometry of Fractal Sets" by Falconer ([Fal85]).

Definition 2.4. For any set $E \subset \mathbb{R}^d$, the Hausdorff-dimension of E is the following,

$$\dim_H(E) = \inf\{t > 0 : \mathcal{H}^t(E) = 0\} = \sup\{t : \mathcal{H}^t(E) = \infty\}.$$

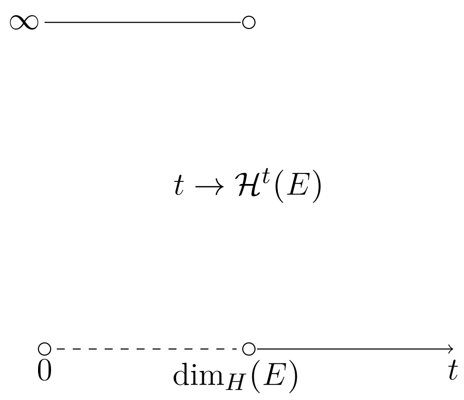


Figure 5: Hausdorff measure of a set E.

The following lemma shows some properties of the Hausdorff-dimension.

Lemma 2.5. *1. Every countable set has Hausdorff-dimension zero.*

2. *For every $F \subset \mathbb{R}^d$, we have $\dim_H(F) \leq d$.*

3. *If $\mathcal{L}^d(E) > 0$ then $\dim_H(E) = d$.*

4. *For any $k < d$ the $k-$dimensional smooth surface in $\mathbb{R}^d$ has a Hausdorff dimension k.*

5. *For a Lipschitz map $f\colon \mathbb{R}^d \to \mathbb{R}^n$ and a Borel set $E \subset \mathbb{R}^d$ we have $\dim_H(f(E)) \leq \dim_H(E)$.*

6. *Let E be a Borel set and let $f\colon \mathbb{R}^d \to \mathbb{R}^n$ be a bi-Lipschitz map. Then $\dim_H(E) = \dim_H(f(E))$.*

7. *Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of Borel sets in $\mathbb{R}^d$. Then*

$$\dim_H\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_i \dim_H(E_i).$$

2.2 SELF-SIMILAR IFS

This section is a brief introduction to Self-similar Iterated Functions Systems. First we define the basics of an IFS, then we show how to measure the dimension of the attractor of an IFS via the Hausdorff-dimension.

Definition 2.6. Let (X, ϱ) be a complete metric space, then we call the map $f: X \to X$ contraction, if there exists a $\lambda \in (0,1)$ such that

$$\varrho(f(x), f(y)) \leq \lambda \cdot \varrho(x, y)$$

for every $x, y \in X$.

Definition 2.7. We say that a finite collection of contractions $\mathcal{F} = \{f_1, \ldots, f_n\}$ is an iterated function system (IFS).

Now we show that, the previously defined Hausdorff pre-metric is a metric on a compact set $\mathcal{C}$. Let us check the properties of a metric. First, $d_H(E,F) = d_H(F,E)$ is trivial. Let $\delta_1 > d_H(E,G)$ and $\delta_2 > d_H(G,F)$ then $E \subseteq [G]_{\delta_1}$ and $G \subseteq [F]_{\delta_2}$. Which means $E \subseteq [F]_{\delta_1+\delta_2}$, then

$$d_H(E,F) \leq d_H(E,G) + d_H(G,F).$$

$d_H(E,F) \geq 0$ can be seen by the definition. If $d_H(E,F) = 0$, then $[F]_\delta \supseteq E$ for every $\delta > 0$. So

$$\bigcap_{n=1}^{\infty} [F]_{\frac{1}{n}} \supseteq E,$$

where $\bigcap_{n=1}^{\infty} [F]_{\frac{1}{n}} = \overline{F}$. Since F is compact $\overline{F} = F$. That is $d_H(E,F) = 0$ if and only if $E = F$.

The following lemmas are essential to proof the existence and uniqueness of the attractor of an IFS.

Lemma 2.8. *If $f: X \to X$ is a contraction with ratio $\lambda \in (0,1)$ and $E, F \subseteq X$, then*

$$d_H(f(E), f(F)) \leq \lambda \cdot d_H(E,F). \tag{3}$$

Proof. Let $\delta > d_H(E,F)$. Since $[E]_\delta \supseteq F$ then

$$f(F) \subseteq f([E]_\delta).$$

By definition

$$[E]_\delta = \{y\colon \text{ there exists } x, \quad \varrho(x,y) < \delta\}.$$
$$f([E]_\delta) = \{f(y)\colon \text{ there exists } x \in E, \quad \varrho(x,y) < \delta\}$$
$$\subseteq \{f(y)\colon \text{ there exists } x \in E, \quad \varrho(f(x), f(y)) < \lambda \cdot \delta\}.$$

By previous calculations, $f(F) \subseteq [f(E)]_{\lambda \cdot \delta}$, which implies

$$d_H(f(E), f(F)) \leq \lambda \cdot d_H(E, F).$$

□

Lemma 2.9. *For every subsets $A, B, C, D \subseteq X$ we have*

$$d_H(A \cup B, C \cup D) \leq \max\{d_H(A, C),\ d_H(B, D)\}. \tag{4}$$

Proof. Let $\delta > d_H(A, C), d_H(B, D)$, then $A \subseteq [C]_\delta$ and $B \subseteq [D]_\delta$. By definition

$$A \cup B \subseteq [C]_\delta \cup [D]_\delta.$$

Since $[C]_\delta \cup [D]_\delta = \{y :$ there exists $x \in C$ or there exists $x \in D,\ \varrho(x, y) < \delta\}$. Then

$$A \cup B \subseteq [C \cup D]_\delta.$$

□

Theorem 2.10 (Hutchinson [HUT81])**.** *Assume that the IFS $\mathcal{F} = \{f_i\}_{i=1}^m$ consits of functions $f_i : X \to X$ with Lipschitz constants $\lambda_i < 1$. Then for the closed ball*

$$B := \overline{B}(x_0, R) \text{ where } R := \max_i \left\{ \frac{\varrho(f_i(x_0), x_0)}{1 - \lambda_i} \right\} \tag{5}$$

we have $f_i(B) \subseteq B$ for all $i = 1, \ldots, m$. Furthermore, we call the non-empty compact set

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \ldots, i_n) \in \{1, \ldots, m\}^n} f_{i_1} \circ \cdots \circ f_{i_n}(B)$$

the attractor or invariant set of the IFS $\mathcal{F}$. Then Λ is the unique non-empty compact set which satisfies

$$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda).$$

Proof. First, we show the existence. Let (X, ϱ) be a compact metric space, let $x_0 \in X$ be arbitrary and choose $R > 0$ sufficiently large, such that for every $i = \{1, \ldots, n\}$

$$f_i(\overline{B(x_0, R)}) \subseteq \overline{B(x_0, R)}, \tag{6}$$

where $B(x_0, R)$ is an open ball around x_0 with radius R. Then for every $y \in f_i(\overline{B(x_0,R)})$, $\varrho(x_0, y) \leq R$. There exists $z \in \overline{B(x_0,R)}$ such that $y = f_i(z)$.

$$\begin{aligned}\varrho(f_i(z), x_0) &\leq \varrho(f_i(z), f_i(x_0)) + \varrho(x_0, f_i(x_0)) \leq \lambda_i \cdot \varrho(z, x_0) + \varrho(f_i(x_0), x_0) \\ &\leq \lambda_i \cdot R + \varrho(f_i(x_0), x_0) < R.\end{aligned}$$

Which means that

$$\max_i \left\{ \frac{\varrho(f_i(x_0), x_0)}{1 - \lambda_i} \right\} < R.$$

Let

$$\Lambda_n = \bigcup_{i_1=1}^m \bigcup_{i_2=1}^m \cdots \bigcup_{i_n=1}^m f_{i_1} \circ \cdots \circ f_{i_n} \left(\overline{B(x_0, R)} \right).$$

By the property (6), $\Lambda_{n+1} \subseteq \Lambda_n$ that is Λ_n is a sequence of shrinking compact sets. We use Cantor's intersection theorem on Λ_n and let Λ be the non-empty compact set

$$\Lambda := \bigcap_{n=0}^{\infty} \Lambda_n.$$

Clearly, $\Lambda := \bigcup_{i=0}^m f_i(\Lambda)$. Now, we show the uniqueness of the attractor. Let Λ' be another non-empty compact set such that,

$$\Lambda' = \bigcup_{i=1}^m f_i(\Lambda').$$

Then investigate the distance of Λ and Λ' by the Hausdorf metric,

$$\begin{aligned}&0 < d_H(\Lambda, \Lambda') = d_H \left(\bigcup_{i=1}^m f_i(\Lambda), \bigcup_{i=1}^m f_i(\Lambda') \right) \overset{(4)}{\leq} \max_{i=1,\ldots,m} d_H \left(f_i(\Lambda), f_i(\Lambda') \right) \\ &\overset{(3)}{\leq} \max_{i=1,\ldots,m} \lambda_i \cdot d_H \left(\Lambda, \Lambda' \right) < d_H \left(\Lambda, \Lambda' \right).\end{aligned}$$

Which is a contradiction, so that means the distance of Λ and Λ' has to be zero. □

Definition 2.11. Let (X, ϱ) be a compact metric space, then the IFS of maps $f_i \colon X \to X$ is called self-similar if it contains only similarities, i.e

$$\varrho(f_i(x), f_i(y)) = \lambda_i \cdot \varrho(x, y),$$

or some $\lambda_i \in (0, 1)$ and for every $x, y \in X$.

Now we show a construction for a Graph-directed self-similar IFS by Mauldin and Williams [MW88]. Furthermore, for each Graph-directed self-similar IFS there exists an attractor.

Definition 2.12. Let$\{I_i\}_{i=1}^N$ be a set of closed intervals of $\mathbb{R}$ and let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph, where $\mathcal{E}_{i,j}$ denotes the set of edges from i to j, and $\mathcal{V} = (I_j)_{j=1}^N$. Furthermore, a contracting similarity mapping $f_e \colon I_j \to I_i$ on the following metric spaces (I_j, d), $j = 1, \ldots, N$ with contraction ratio $r_e \in (0,1)$. Then

$$d(f_e(x), f_e(y)) = r_e \cdot d(x,y) \text{ for every } x, y \in I_j.$$

We call the system $(\mathcal{G}, \{f_e, e \in \mathcal{E}\})$ as graph directed self-similar IFS (GDSSIFS).

Theorem 2.13 (Mauldin and Williams [MW88])**.** *For each geometric construction, there exists a unique collection of compact sets,* $(\Lambda_1, \ldots, \Lambda_N)$ *such that for* $N \in \mathbb{N}$

$$\Lambda_i = \bigcup_{j=1}^{N} \bigcup_{e \in \mathcal{E}_{i,j}} f_e(\Lambda_j) \text{ for every } i = 1, \ldots, N.$$

The construction object is defined as

$$\Lambda := \bigcup_{i=1}^{N} \Lambda_i$$

called the attractor.

The proof can be found in [MW88, Theorem 1]. Note that, in **Theorem** 2.13 the authors reefer to the previously define Graph-directed system as a geometric consruction.

Definition 2.14. Let $\mathcal{F}$ be an IFS and let $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ be the symbolic space. Then $\Pi \colon \Sigma \to \Lambda$ is the natural projection, that is

$$\Pi(\bar{\imath}) = \lim_{n \to \infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(0) \text{ for every } \bar{\imath} = (i_1, i_2, \ldots, i_n, \ldots).$$

Denote σ the left shift on the symbolic space Σ.

2.3 DIMENSION THEOREMS FOR SELF-SIMILAR IFS

This section is an introduction to the dimension theory of IFS. We give conditions and cases, when we can determine the Hausdorff dimension of the attractor of an IFS.

2.3.1 *Non-overlapping case*

This case is the simplest one. The images of the contracting similarity transformations does not intersect, then we can give nice separation conditions as follows.

Definition 2.15. Let $\mathcal{F} = \{f_1, \dots, f_m\}$ be a contracting IFS and Λ the attractor, then the Strong Separation Property holds for $\mathcal{F}$ if,

$$f_i(\Lambda) \cap f_j(\Lambda) = \emptyset \text{ for all } i \neq j.$$

Definition 2.16. The Open Set Condition (OSC) hold for $\mathcal{F}$ if there exists a non-empty open set $V \in \mathbb{R}^d$ such that

$$f_i(V) \subset V \text{ holds for every } i = 1, \dots, m;$$
$$f_i(V) \cap f_j(V) = \emptyset \text{ for every } i \neq j.$$

Definition 2.17. The Strong Open Set Condition holds for $\mathcal{F}$ if the set V in teh definition of OSC can be chosen such that

$$V \cap \Lambda \neq \emptyset.$$

Using the previously defined separations conditions, we can state two different theorems for the dimension. The first one by Moran and Hutchinson for Self-similar IFS, the second by Mauldin and Williams for Graph-directed IFS.

Theorem 2.18 (Moran [Mor46], Hutchinson [HUT81])**.** *Let $\mathcal{F}$ be a self-similar IFS which satisfies the OSC. Let $0 < \lambda_i < 1$ be the contraction ratio of f_i and let s be the similarity dimension, that is $\lambda_1^s + \cdots + \lambda_m^s = 1$, then for the attractor Λ of $\mathcal{F}$ we have*

$$0 < \mathcal{H}^s(\Lambda) < \infty.$$

Moreover,

$$\dim_H(\Lambda) = \dim_B(\Lambda) = s.$$

Now we define a matrix called $Mauldin - Williams$ matrix by the construction of a Graph Directed Self-Similar IFS (see in **Definition 2.12**) as the following.

Definition 2.19. Let $(B_{MW}^{(s)})i, j = \left(b^{(s)}(i,j)\right)$ be an $n \times n$ matrix, where

$$b^{(s)}(i,j) = \begin{cases} 0, & \text{if } \mathcal{E}_{i,j} = \emptyset, \\ \sum\limits_{e \in \mathcal{E}_{i,j}} r_e^s, & \text{otherwise,} \end{cases} \qquad \text{for every } \ s > 0.$$

Lemma 2.20. *Let $\rho(B^{(s)}_{MW})$ be the spectral radius of $B^{(s)}_{MW}$. The mapping $s \mapsto \rho(B^{(s)}_{MW})$ is continuous, strictly decreasing, $\rho(s) \geq 1$ if $s = 0$ and $\rho(B^{(s)}_{MW}) \xrightarrow[s\to\infty]{} 0$. Then, there exists a unique $s_0 \geq 0$ for which*

$$\rho(B^{(s_0)}_{MW}) = 1.$$

The proof can be found in [MW88, Theorem 2].

Theorem 2.21. *(Mauldin and Williams [MW88]) Consider a GDSSIFS such that, $\mathcal{G}$ is strongly connected and $\{f_{i,j}(\Lambda_j) : \quad (i,j) \in \mathcal{E}\}$ is a disjoint family of sets, then for every $i = 1, \dots, N$ we have*

$$\dim_H \Lambda_i = \dim_B \Lambda_i = s_0 \text{ and } 0 < \mathcal{H}^{s_0}(\Lambda_i) < \infty.$$

2.3.2 *IFS with overlaps*

This section it is clear that we investigate those IFS, which image spaces can intersect.

Theorem 2.22 (Simon, Solomyak [SS02])**.** *Let $\mathcal{F} = \{\lambda_i x + t_i\}_{i=1}^m$ be a self-similar IFS on $\mathbb{R}$ such that $0 < |\lambda_i| < 1$ and $t_i \in \mathbb{R}$ for every $i = 1, \dots, m$. Denote Λ the attractor of $\mathcal{F}$ and let s be the similarity dimension, that is $\sum_{i=1}^m |\lambda_i|^s = 1$. Then*

$$\dim_H(\Lambda) = \dim_B(\Lambda) = \min\{1, s\} \text{ for Lebesgue almost every } \underline{t} = (t_1, \dots, t_m) \in \mathbb{R} \times \dots \times \mathbb{R}.$$

Moreover if $s > 1$, then

$$\mathcal{L}^1(\Lambda) > 0 \text{ for almost every } \underline{t} = (t_1, \dots, t_m) \in \mathbb{R} \times \dots \times \mathbb{R},$$

where $\mathcal{L}^1$ denotes the Lebesgue measure.

Let $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$ be a set of every infinite words and let Σ^n be a set of every finite words with lenght at most n. For a finite word $\bar{\imath} \in \Sigma$, denote $|\bar{\imath}|$ the length of $\bar{\imath}$. If $\bar{\imath} = i_1 \cdot \ldots i_n$ we denote by $A_{\bar{\imath}}$ the finite product $A_{i_1} \cdot \ldots \cdot A_{i_n}$. Let $f_i = \lambda_i \underline{x} + \underline{t}_i$ be a contracting similarity transformations with contraction ratio λ_i and $f_{\bar{\imath}} = f_{i_1} \circ \dots \circ f_{i_n}$ for every $\bar{\imath} \in \Sigma^n$. Let I be the set of parameters, then $\lambda_i \colon I \to (-1,1) \setminus \{0\}$ and $a_i \colon I \to \mathbb{R}$ for every $i \in \{1, \dots, m\}$. For each $t \in I \subseteq \mathbb{R}$ define $f_{i,t} \colon \mathbb{R} \to \mathbb{R}$ such that $f_{i,t}(x) = \lambda_i(t)(x - a_i(t))$. For any $\bar{\imath} \in \Sigma^n$ let $f_{\bar{\imath},t} = f_{i_1,t} \circ \dots \circ f_{i_n,t}$ and $\Delta_{\bar{\imath},\bar{\jmath}} = f_{\bar{\imath},t}(0) - f_{\bar{\jmath},t}(0)$, and for every $\bar{\imath}, \bar{\jmath} \in \Sigma$ let $\Delta_{\bar{\imath},\bar{\jmath}} = \Pi_t(\bar{\imath}) - \Pi_t(\bar{\jmath})$.

Theorem 2.23 (Hochman [Hoc14]). *Let $\mathcal{F}_t = \{f_{i,t}\}_{i=1}^m$ be a parametrized IFS with attractor Λ_t. For every $\varepsilon > 0$ let*

$$E_\varepsilon = \bigcup_{N=1}^{\infty} \bigcap_{n>N} \left(\bigcup_{\bar{\imath},\bar{\jmath}\in\Sigma^n} (\Delta_{\bar{\imath},\bar{\jmath}})^{-1}(-\varepsilon^n, \varepsilon^n) \right),$$

and

$$E = \bigcap_{\varepsilon>0} E_\varepsilon.$$

Then for every $t \in I \setminus E$ and for the attractor Λ_t of $\mathcal{F}_t$ satisfies $\dim_H \Lambda_t = \dim_B(\Lambda_t) = \min\{1, s(t)\}$, where $\sum_{i=1}^m |\lambda_i(t)|^{s(t)} \equiv 1$.

Theorem 2.24 (Hochman [Hoc14]). *Let $I \subset \mathbb{R}$ be a compact interval, let $\lambda_i\colon I \to (-1,1) \setminus \{0\}$ and $a_i\colon I \to \mathbb{R}$ be real analytic, and let $\mathcal{F}_t = \{f_{i,t}\}_{i=1}^m$ be associated parametric family of IFS-s, as above. Suppose that*

$$\Delta_{\bar{\imath},\bar{\jmath}} \equiv 0 \text{ on } I \text{ if and only if } \bar{\imath} = \bar{\jmath} \in \Sigma.$$

Then the set E of "exceptional" parameters in Theorem 2.23 has Hausdorff dimension 0.

2.4 SELF-AFFINE IFS

This section is a brief introduction to self-affine IFS. Every affine transformation can be represented by a matrix. In this section we assume that, these matrices are regular. We show the earlier results on the dimension of self-affine IFS.

Definition 2.25. An IFS $\mathcal{F} = \{f_1, \ldots, f_m\}$ called self-affine if it is only contains affinities i.e.

$$f_i(x) = A_i\underline{x} + \underline{t}_i,$$

where $A_i \in \mathbb{R}^{d\times d}$, $\underline{t}_i \in \mathbb{R}^d$ and $\|A_i\| < 1$.

Definition 2.26. Let T be a $d \times d$ real valued matrix, then the singular value function $\varphi^t(T)$ of T can be defined,

$$\varphi^t(T) := \begin{cases} \alpha_1 \cdots \alpha_{k-1} \cdot \alpha_k^{t-(k-1)}, & \text{if } k-1 < t \le k \le d; \\ (\alpha_1 \cdots \alpha_d)^{\frac{t}{d}}, & \text{if } t \ge d, \end{cases}$$

where $\alpha_1 \ge \cdots \ge \alpha_d$ are the singular values of T.

If $\bar{\imath} = i_1 \cdot \ldots i_n$ we denote by $A_{\bar{\imath}}$ the finite product $A_{i_1} \cdot \ldots \cdot A_{i_n}$ for every $i = 1, \ldots, n$ and for every $n \in \mathbb{N}$.

Definition 2.27. Denote $s(A_1, \ldots, A_m)$ the affinity dimension of the self-affine IFS $\mathcal{F} = \{A_i x + t_i\}_{i=1}^m$, that is,

$$s(A_1, \ldots, A_m) = \inf \left\{ t > 0 : \sum_{m=0}^{\infty} \sum_{|\bar{\imath}|=m} \varphi^t(A_{\bar{\imath}}) < \infty \right\}.$$

Falconer ([Fal94]) defined the sub-additive pressure function $P : \mathbb{R}^+ \to \mathbb{R}$,

$$P(s) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{|\bar{\imath}|=n} \varphi^s(A_{\bar{\imath}}) \right). \tag{7}$$

Because of the submultiplicativity of the singular value function (see in [Fal88, Lemma 2]) the following holds,

$$\sum_{|\bar{\imath}|=n+m} \varphi^t(A_{\bar{\imath}}) = \sum_{|\bar{\jmath}|=n} \sum_{|\bar{k}|=m} \varphi^t(A_{\bar{\jmath}} \cdot A_{\bar{k}}) \leq \sum_{|\bar{\jmath}|=n} \sum_{|\bar{k}|=m} \varphi^t(A_{\bar{\jmath}}) \cdot \varphi^t(A_{\bar{k}}).$$

Since the logarithm function is strictly monotone increasing and continuous, then the sequence $\{\log \sum_{|\bar{\imath}|=n} \varphi^t(A_{\bar{\imath}})\}_n$ is subadditive. Then by Fekete's lemma [Fek23], there exists a limit of the equation (7).

Theorem 2.28 (Falconer [Fal88])**.** *Let $\mathcal{F} = \{A_i \underline{x} + \underline{t}_i\}_{i=1}^m$ be a self-affine IFS in $\mathbb{R}^d$ with attractor Λ, such that the matrices A_i are regular. Then*

$$\overline{\dim}_B(\Lambda) \leq \min\{d, s(A_1, \ldots, A_m)\}.$$

Theorem 2.29 (Falconer [Fal88], Solomyak [SOL98])**.** *For $m \geq 2$ let $\{A_1, \ldots, A_m\}$ be non-singular $d \times d$ matrices, such that their Euklidean norm satisfies,*

$$\|A_i\| < \frac{1}{2}, \qquad i = 1, \ldots, m.$$

For $\underline{t} := (t_1, \ldots, t_m) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ define the m-parameter family of self-affine IFS on $\mathbb{R}^d$,

$$\mathcal{F}^{\underline{t}} := \{A_i x + t_i\}_{i=1}^m.$$

Let $\Lambda^{\underline{t}}$ be the attractor of $\mathcal{F}^t$. Then for $\mathcal{L}^{md}$-almost all $\underline{t}$ we have

$$\dim_H(\Lambda) = \dim_B(\Lambda) = \min\{d, s(A_1, \ldots, A_m)\}.$$

The original version of the previous theorem was weaker than we stated here. It was proved by Falconer for regular matrices, such that $\|A_i\| \leq \frac{1}{3}$. Later in 1998, Boris Solomyak proved for $\|A_i\| \leq \frac{1}{2}$.

Definition 2.30. Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a collection of regular $d \times d$ matrices. We say that $\mathcal{A}$ is strongly irreducible if there is no finite collection $V_1, \dots, V_k$ of proper subspaces of $\mathbb{R}^d$ such that

$$A_i \left(\bigcup_{j=1}^{k} V_j \right) = \bigcup_{j=1}^{k} V_j \quad \text{for every } i = 1, \dots, n.$$

Theorem 2.31 (Bárány, Hochman and Rapaport [BHR19])**.** *Let $\mathcal{F} = \{f_i(x) = A_i x + t_i\}_{i=1}^{m}$ be a planar self-affine IFS, such that $\mathcal{F}$ satisfies the Strong Open-set Condition and the collection of regular matrices $\mathcal{A} = \{A_1, \dots, A_m\}$ is strongly irreducible. Then,*

$$\dim_H(\Lambda) = \dim_B(\Lambda) = s(A_1, \dots, A_m).$$

3 MAIN RESULTS

In the previous Chapter we showed dimension theorems for regular IFS. Now we move on to the main question of this Thesis: What happens to the dimension if we allow singular matrices in the IFS? In the following, we show our results about dimension and the connection between self-affine sets and graph directed iterated function systems with singular matrices. Let $\Sigma = \{1, \ldots, n\}^{\mathbb{N}}$ be a symbolic space, furthermore we can define $\Sigma_{reg} = \{i \colon A_i \text{ are regular matrices}\}$ and $\Sigma_{sing} = \{i \colon A_i \text{ are singular matrices}\}$. Let Σ^n be the set of words with lenght of at most n and let Σ^* be the set of all finite words. Then define Σ^n_{reg} and Σ^*_{reg} as

$$\Sigma^n_{reg} = \{(i_1, \ldots, i_k) : k \leq n \text{ and } i_l \in \Sigma_{reg}\} \text{ for every } l = (1, \ldots, k),$$
$$\Sigma^*_{reg} = \bigcup_{k=1}^{\infty} \Sigma^k_{reg}.$$

Now we construct a matrix formed by the contracting ratios and motivated by the Mauldin-Williams matrix.

Definition 3.1. Let A be a 2×2 matrix and let V be a subspace of $\mathbb{R}^2$, then for every $x \in V$ the norm of A conditioned on V defined as

$$\|A|V\| = \sup_{x \in V} \frac{\|Ax\|}{\|x\|}$$

Let $B^{(s)}$ be an $n \times n$ matrix and $n = |\Sigma_{sing}|$, where

$$(B^{(s)})_{i,j} = \left(\sum_{\bar{\imath} \in \Sigma^*_{reg}} \|A_i A_{\bar{\imath}} | Im(A_j)\|^s \right)_{i,j \in \Sigma_{sing}}. \tag{8}$$

Observe that, the previously defined $B^{(s)}$ matrix will be not well defined for every s. In the following we will show some cases, when we can determine the dimension of a singular IFS via the $B^{(s)}$ matrix. We use the notation of

$$P_{sing}(s) = \log\left(\rho(B^{(s)})\right),$$
$$P_{reg}(s) = \lim_{k \to \infty} \frac{1}{k} \log \left(\sum_{\bar{\jmath} \in \Sigma^k_{reg} \setminus \Sigma^{k-1}_{reg}} \varphi^s(A_{\bar{\jmath}}) \right).$$

Furthermore, let us define s_{sing} as a unique solution for the equation

$$\rho\left(B^{(s_{sing})}\right) = 1.$$

And let s_{reg} be the unique root of P_{reg}.

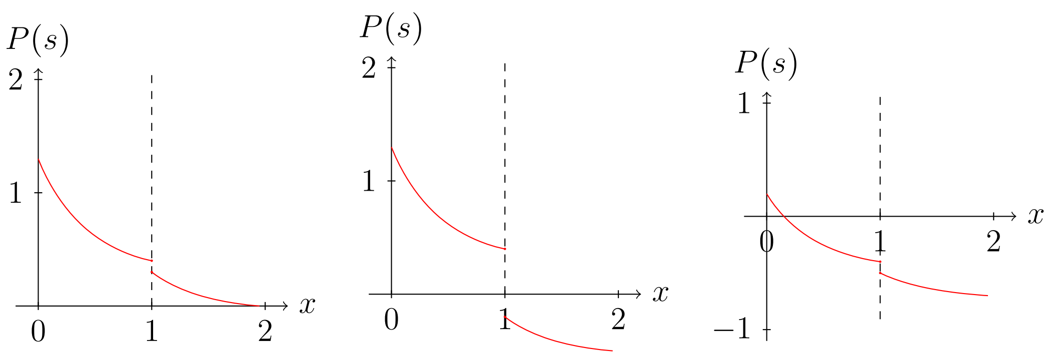
3.1 UPPER BOUND

This section is a brief introduction to the theorems we gave for the upper bound of the dimension of the attractor of a self-affine IFS.

Theorem 3.2. *For every* $\mathcal{F} = \{A_i \underline{x} + \underline{t}_i\}_{i=1}^n$ *let* $s_0 = \inf\{s\colon \sum_{\bar{\imath}\in\Sigma^*} \varphi^s(A_{\bar{\imath}}) < \infty\}$ *be the affinity dimension, then for the attractor* Λ *of* $\mathcal{F}$*,*

$$\dim_H(\Lambda) \le s_0.$$

We will proof the previous **Theorem 3.2** in the following Chapter 4. By definition of s_{reg} and s_{sing} we can investigate the root of the pressure function $P(s)$. Since the self-affine IFS contains singular matrix, then the pressure function will shift at $s = 1$. The question is, what can we say about the dimension of the attractor of the self-affine IFS? The following figure represents that, how does the changes in s_{reg} and s_{sing} affect the shift of the pressure function.



(a) $s_{reg} > 1$ (b) $s_{sing} > 1 > s_{reg}$ (c) $1 > s_{sing}$

In the following we will answer the previous question. First let us investigate the affinity dimension in different cases of s_{reg} and s_{sing}.

Proposition 3.3. *Consider* $s_0 = \inf\{s > 0 : \sum_{\bar{\jmath}\in\Sigma^*} \varphi^s(A_{\bar{\jmath}}) < \infty\}$, s_{reg} *and* s_{sing} *as defined previously, then*

$$s_0 = \begin{cases} s_{reg} & \text{if } s_{reg} > 1, \\ \min\{1, \max\{s_{sing}, s_{reg}\}\} & \text{if } s_{reg} \le 1. \end{cases}$$

Theorem 3.4. *Assume that* $s_{reg} > 1$*, consider* $\mathcal{F}$ *a self-affine IFS with singular and regular matrices, then let* $\mathcal{F}_{reg} \subset \mathcal{F}$ *be a self-affine IFS with regular matrices only, such that* $\mathcal{F}_{reg}$ *satisfies the Strong Separation Property and the collection of regular matrices* $\mathcal{A}_{reg} = \{A_1, \dots, A_m\}$ *is strongly irreducible, then*

$$\dim_H(\Lambda) = s_{reg}.$$

Proof. By **Theorem 3.2** s_0 will be an upper for $\dim_H(\Lambda)$, but by **Proposition 3.3** $s_0 = s_{reg}$. On the other hand, by **Theorem 2.31** and by the definition of s_{reg} we have $\dim_H(\Lambda) \geq s_{reg}$. □

3.2 SELF-AFFINE SETS WITH SINGULAR MATRICES ONLY

This section is an introduction to the case when the matrices of the contracting similarity functions of the IFS $\mathcal{F} = \{\varphi_i = A_i\underline{x} + \underline{t}_i\}_{i=1}^n$ are singular.

Remark 3.5. If we only consider singular matrices, then the matrix $B^{(s)}$ defined in (8) simplifies to the following form.

$$b^{(s)}(i,j) = \|A_i|Im(A_j)\|^s \quad \text{for every } i,j = 1,\dots,n.$$

By this construction we eliminated the problem that, the dimension of $B^{(s)}$ will not equal to the number of transformations. Now, we will show the dimension of the attractor of an IFS by the $B^{(s)}$ matrix.

Theorem 3.6. *Let* $\mathcal{F} = \{\varphi_i = A_i x + t_i\}_{i=1}^n$ *be an IFS for every matrix* A_i *is singular. If* $\varphi_i \circ \varphi_j(\Lambda) \cap \varphi_i \circ \varphi_k(\Lambda) = \emptyset$ *for every* i *and* $j \neq k$ *then*

$$\dim_H(\Lambda) = s_0,$$

where s_0 *is the unique solution of* $\rho(B^{(s_0)}) = 1$.

3.3 ONE SINGULAR BETWEEN REGULARS

In this section we investigate those self-affine IFS, which contains regular matrices and only one singular matrix. Then by our new assumption we will determine the dimension of the attractor such self-affine IFS.

Definition 3.7. Let B be a ball of $\mathbb{R}^d$ and an IFS $\mathcal{F} = \{\varphi_1, \dots, \varphi_n\}$ such that $\varphi_i(B) \subseteq B$ for every i. Then we say that $\mathcal{F}$ is Elliptically Strong Separated or satisfies the Elliptic Strong Separation Property, if

$$\varphi_i(B) \cap \varphi_j(B) = \emptyset, \text{ if } i \neq j.$$

Proposition 3.8. *We assume* $\Sigma_{sing} = \{1\}$ *and* $\Sigma_{reg} = \{2, \dots, n\}$, *furthermore let* $A_1(\beta)$ *be the singular matrix as follows*

$$A_1(\beta) = \varrho \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\beta) & \sin(\beta) \end{pmatrix}, \quad \textit{for every } \beta \in [0, 2\pi] \textit{ and } \varrho < 1.$$

Let $\mathcal{F}_\beta = \{A_1(\beta)\underline{x} + \underline{t}_1, \dots, A_n\underline{x} + \underline{t}_n\}$ *be the modified IFS and define the set* $\mathcal{I} = \{\beta \in [0, 2\pi] \colon$ *Elliptic separation property holds*$\}$. *Then there exists a set* $E \subset \mathcal{I}$ *such that* $\dim_H(E) = 0$ *and for every* $\beta \in \mathcal{I} \setminus E$

$$\dim_H(\Lambda_\beta) = s_0(\beta),$$

where Λ_β *is the attractor of the IFS* $\mathcal{F}_\beta$.

Remark 3.9. For every singular matrix A with $rank(A) = 1$ there is a transformation, B such that

$$BAB^{-1} = \varrho \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\beta) & \sin(\beta) \end{pmatrix}, \quad \text{for some } \beta \in [0, 2\pi] \text{ and } \varrho > 0.$$

The transformation $g(\underline{x}) = B\underline{x}$ is a bi-Lipschitz, so it does not affect the dimension of the attractor $g(\Lambda)$, which is the attractor of the IFS $\{g \circ F_i \circ g^{-1}\}_{i=1}^n$.

4 PROOFS

In this chapter we give a detailed proofs of the results in the previous chapter. First, it is practical to prove a general upper bound, then we show the lower bound for the dimension in specific cases. In $\mathbb{R}^2$ with regular matrices by **Theorem 2.29**, 2 will be an upper bound for the dimension of the attractor at any time.

4.1 UPPER BOUND

Lemma 4.1. *For every singular matrices A and B and for every subspace V we have*

$$\|AB|V\| = \|A|Im(B)\| \cdot \|B|V\|.$$

Proof. For every $v \in V$ and $\|v\| = 1$,

$$\|AB|V\| = \|AB\underline{v}\| =$$

$$= \begin{cases} 0, & \text{if } Ker(B) = V \\ \frac{\|AB\underline{v}\|}{\|B\underline{v}\|} \cdot \|B\underline{v}\| = \|A|Im(B)\| \cdot \|B|V\| & \text{if } Ker(B) \neq V \end{cases}$$

If $Ker(B) = V$ then $\|B|V\| = 0$ and $\|A|Im(B)\| \cdot \|B|V\| = 0$. □

Lemma 4.2. *Let $\mathcal{F} = \{A_i\underline{x} + \underline{t}_i\}_{i=1}^n$ then for every $s \leq 1$*

$$\sum_{\bar{\imath}\in\Sigma^*} \varphi^s(A_{\bar{\imath}}) < \infty \textit{ if and only if } \sum_{k=0}^{\infty} \left\| \left(B^{(s)}\right)^k \right\|_1 < \infty \textit{ and } \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) < \infty.$$

Remark 4.3. In **Lemma 4.2** for a matrix A we use

$$\|A\|_1 = \sum_{i,j} |a_{ij}|.$$

Proof. For the reverse direction of the implication we have,

$$\sum_{\bar{\imath}\in\Sigma_*} \varphi^s(A_{\bar{\imath}}) \geq \sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}\in\Sigma_*} \varphi^s(A_i A_{\bar{\imath}} A_j) + \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}). \tag{9}$$

By **Lemma 4.1**,

$$(9) = \sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}\in\Sigma_*} \|A_i A_{\bar{\imath}}|Im(A_j)\|^s \cdot \|A_j\|^s + \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}). \tag{10}$$

Then by the definition of $B^{(s)}$, see in equation (8)

$$
\begin{aligned}
(10) &= \sum_{k=0}^{\infty} \sum_{i,j\in\Sigma_{sing}} \left((B^{(s)})^k\right)_{i,j} \cdot \|A_j\|^s + \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) \\
&\geq \min_{j\in\Sigma_{sing}} \|A_j\|_1^s \cdot \sum_{k=0}^{\infty} \left\| \left(B^{(s)}\right)^k \right\|_1 + \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}).
\end{aligned}
$$

On the other hand similarly to previous calculations we have,

$$
\begin{aligned}
&\sum_{\bar{\imath}\in\Sigma_*} \varphi^s(A_{\bar{\imath}}) = \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) + \sum_{i\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}_1}A_iA_{\bar{\imath}_2}) + \\
&\sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \sum_{\bar{\jmath}\in\Sigma_*} \varphi^s(A_{\bar{\imath}_1}A_iA_{\bar{\jmath}}A_jA_{\bar{\imath}_2}) \leq \\
&\leq \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) + \sum_{i\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}_1})\|A_i\|^s\varphi^s(A_{\bar{\imath}_2}) + \\
&+ \sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \sum_{\bar{\jmath}\in\Sigma_*} \varphi^s(A_{\bar{\imath}_1})\|A_iA_{\bar{\jmath}}|Im(A_j)\|^s \cdot \|A_j\|^s\varphi^s(A_{\bar{\imath}_2}) \leq \\
&\leq \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) + \left(\sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}})\right)^2 \left(\sum_{j\in\Sigma_{sing}} \|A_j\|^s\right) + \\
&+ \left(\sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}})\right)^2 \cdot \sum_{k=0}^{\infty} \sum_{i,j\in\Sigma_{sing}} \left((B^{(s)})^k\right)_{i,j} \cdot \|A_j\|^s \\
&\leq \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) \cdot \left(\sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}})\right) \cdot \left(\sum_{j\in\Sigma_{sing}} \|A_j\|^s\right) + \left(\sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}})\right)^2 \\
&\cdot \max_j \|A_j\|^s \cdot \left(\sum_{k=0}^{\infty} \left\|(B^{(s)})^k\right\|_1\right).
\end{aligned}
$$

□

Proof of Proposition 3.3. For $s_{reg} > 1$,

$$
\sum_{\bar{\imath}\in\Sigma^*} \varphi^s(A_{\bar{\imath}}) = \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}).
$$

Hence, it is clear that $s_{reg} = s_0$.

On the other hand, if $s_{reg} \leq 1$, then we have two cases. In first case, for every $s \in (0, 1]$ the $\sum_{\bar{\imath}\in\Sigma^*} \varphi^s(A_{\bar{\imath}}) = \infty$. Then it is clear, $s_0 = 1$ and by **Lemma 4.2**

$$
\sum_{k=0}^{\infty} \left\| \left(B^{(s)}\right)^k \right\|_1 = \infty.
$$

Then by **Lemma 2.20** $\rho(B^{(s)}) \geq 1$, which implies $s_{sing} \geq 1$. In the second case, there exists $0 \leq s \leq 1$, such that $\sum_{\bar{\imath}\in\Sigma^*} \varphi^s(A_{\bar{\imath}}) < \infty$. Then $s_0 \leq 1$ and by **Lemma** 4.2

$$\sum_{k=0}^{\infty} \left\|(B^{(s)})^k\right\|_1 < \infty \text{ and } \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) < \infty.$$

By **Lemma 2.20** $\rho(B^{(s)}) \leq 1$, which implies that $s_{sing} \leq s$. Since, $\sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) < \infty$ then by **Lemma** 4.2 $s_{reg} \leq s$. The above implies that,

$$\max\{s_{reg}, s_{sing}\} \leq s.$$

□

***Proof of Theorem** 3.2.* If $s_0 > 1$ then the affinity dimension $s_0 = s_{reg}$, by **Proposition 3.3** and Falconer **Theorem 2.29** On the other hand, let $x \in \Lambda$ be a point. Then we can determine x as

$$x = \lim_{n\to\infty} \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(\underline{0}) \text{ for some } \bar{\imath} = (i_1, \ldots i_n, \ldots) \in \Sigma.$$

Then we have those x which can be represented by all regular matrices Λ_{reg} and those, which contains at least one singular Λ_{sing}. So we can devide the attractor,

$$\Lambda = \Lambda_{reg} \bigcup \Lambda_{sing}, \text{ where}$$
$$\Lambda_{sing} = \bigcup_{i\in\Sigma_{sing}} \bigcup_{\bar{\jmath}\in\Sigma^*_{reg}} \varphi_{\bar{\jmath}} \circ \varphi_i(\Lambda)$$
$$\Lambda_{reg} = \bigcup_{j\in\Sigma_{reg}} \varphi_j\left(\Lambda_{reg}\right).$$

We know that, for every $i \in \Sigma_{sing}$, $\varphi_i(\Lambda)$ is contained in a line segment. Let B be a ball as defined in **Theorem 2.10**, then $\varphi_i(B) \subset B$. So $\varphi_i(\Lambda) \subset \varphi_i(B)$. Since $\dim \varphi_i(B) = 1$ if $i \in \Sigma_{sing}$, then

$$\dim_H(\Lambda_{sing}) = \sup_i \sup_{\bar{\jmath}\in\Sigma^*_{reg}} \dim_H \varphi_{\bar{\jmath}} \circ \varphi_i(\Lambda) \leq 1.$$

By the 7. property in **Lemma 2.5**

$$\dim_H(\Lambda) = \max\{\dim_H(\Lambda_{reg}), \dim_H(\Lambda_{sing})\} \leq \max\{s_{reg}, 1\} = s_{reg}.$$

Now we turn to the case when $s_0 \leq 1$. By definition of a ball B, $\varphi_i(B) \subset B$. So $\Lambda \subset \bigcup_{\bar{\imath}\in\Sigma^*} \varphi_{\bar{\imath}}(B)$. For the diameter of $\varphi_{\bar{\imath}}$ it follows,

$$|\varphi_{\bar{\imath}}(B)| = \|A_{\bar{\imath}}\| \cdot |B|. \tag{11}$$

Then (11) implies $\varphi^s(A_{\bar{\imath}}) = \|A_{\bar{\imath}}\|^s$. If $\sum_{\bar{\imath}\in\Sigma^*} \varphi^s(A_{\bar{\imath}}) < \infty$, then

$$\mathcal{H}^s(\Lambda) = 0.$$

This implies that, $\dim_H(\Lambda) \leq s_0$. □

4.2 STUDY OF PRESSURE FUNCTION

In this section we investigate the previously defined $B^{(s)}$ matrix with the pressure. We will show under which condition the matrix will be well defined. First we consider the case, when we have singular and regular matrices in the IFS.

Proposition 4.4. *Let A_j be a regular matrix and $A_{\bar{j}} = A_{j_1} \cdot \ldots \cdot A_{j_n}$. Then for every $\varepsilon > 0$ there exists $c > 0$, such that for every $n \in \mathbb{N}$*

$$c^{-1} \cdot e^{n(P_{reg}(s)+\varepsilon)} \leq \sum_{\substack{\bar{\imath} \in \Sigma^*_{reg} \\ |\bar{\imath}|=n}} \varphi^s(A_{\bar{\imath}}) \leq c \cdot e^{n(P_{reg}(s)+\varepsilon)}.$$

Proof. By definition of $P_{reg}(s)$, there exists $N, n > 0$, for every $\varepsilon > 0$ if $n > N$ then,

$$\left| \frac{1}{n} \log \left(\sum_{\substack{\bar{\imath} \in \Sigma^*_{reg} \\ |\bar{\imath}|=n}} \varphi^s(A_{\bar{\imath}}) \right) - P_{reg}(s) \right| < \varepsilon.$$

□

Lemma 4.5. *If $s_{reg} < 1$ then the matrix $B^{(s)}$ is well defined for every $s > s_{reg}$.*

Proof.

$$\sum_{\bar{j} \in \Sigma^*_{reg}} \|A_i A_{\bar{j}} | Im(A_j)\|^s \leq \sum_{\bar{j} \in \Sigma^*_{reg}} \|A_i A_{\bar{j}}\|^s. \tag{12}$$

By the sub-additivity of the norm,

$$(12) \leq \sum_{\bar{j} \in \Sigma^*_{reg}} \|A_i\| \cdot \|A_{\bar{j}}\|^s. \tag{13}$$

By **Proposition 4.4**,

$$(13) \leq \|A_i\|^s \sum_{n=0}^{\infty} c \cdot e^{n(P_{reg}(s)+\varepsilon)} < \infty, \text{ if } P_{reg}(s) + \varepsilon < 0.$$

But there exists such ε if $s > s_{reg}$.

□

In the following we investigate the case, when the IFS contains only singular matrices.

Lemma 4.6. *Let $A_{\bar{\imath}}$ be an arbitrary product of the singular matrices $A_{i_1}, \dots, A_{i_n}$ for every $n \in \mathbb{N}$. Then,*

$$Im(A_{\bar{\imath}}) = Im(A_{i_1}) \text{ or } Im(A_{\bar{\imath}}) = \{0\}.$$

Proof. Let A_i be a matrix with $rank(A_i) = 1$. Assume that,

$$A_i = \begin{pmatrix} c_i \cdot a_i & c_i \cdot b_i \\ d_i \cdot a_i & d_i \cdot b_i \end{pmatrix} = \begin{pmatrix} c_i \\ d_i \end{pmatrix} \cdot \begin{pmatrix} a_i & b_i \end{pmatrix},$$

for every $i = 1, \dots, n$ and for every $a, b, c, d \in \mathbb{R}$. Obviously,

$$Im(A_i) = span \left\langle \begin{pmatrix} c_i \\ d_i \end{pmatrix} \right\rangle \text{ and } Ker(A_i) = span \left\langle \begin{pmatrix} a_i \\ b_i \end{pmatrix} \right\rangle.$$

If $n = 1$ then

$$A_{\bar{\imath}} = A_{i_1}, \text{ so } Im(A_{\bar{\imath}}) = Im(A_{i_1}).$$

If $n = 2$ we have,

$$A_{\bar{\imath}} = A_{i_1} \cdot A_{i_2} =$$
$$= \begin{pmatrix} c_{i_1} \cdot a_{i_2} (c_{i_2} \cdot a_{i_1} + b_{i_1} \cdot d_{i_2}) & c_{i_1} \cdot b_{i_2} (c_{i_2} \cdot a_{i_1} + b_{i_1} \cdot d_{i_2}) \\ d_{i_1} \cdot a_{i_2} (c_{i_2} \cdot a_{i_1} + b_{i_1} \cdot d_{i_2}) & d_{i_1} \cdot b_{i_2} (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2} \cdot c_{i_2}) \end{pmatrix}.$$

The image of $A_{\bar{\imath}}$

$$A_{\bar{\imath}} \cdot \underline{x} = \begin{pmatrix} c_{i_1} \cdot (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2}) \cdot (a_{i_2} \cdot x + b_{i_2} \cdot y) \\ d_{i_1} \cdot (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2}) \cdot (a_{i_2} \cdot x + b_{i_2} \cdot y) \end{pmatrix} =$$
$$= (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2}) \cdot (a_{i_2} \cdot x + b_{i_2} \cdot y) \begin{pmatrix} c_{i_1} \\ d_{i_1} \end{pmatrix}.$$

Then unless $Ker(A_{i_1}) = Im(A_{i_2})$,

$$Im(A_{\bar{\imath}}) = span \left\langle \begin{pmatrix} c_{i_1} \\ d_{i_1} \end{pmatrix} \right\rangle = Im(A_{i_1}).$$

If $|\bar{\imath}| = n$ we can split the product $A_{\bar{\imath}}$ into two parts

$$A_{\bar{\imath}} = A_{i_1} \cdot A_{|\bar{\imath}|-1}.$$

Then using the case $n = 2$ and induction we get the result.

□

Lemma 4.7. *A generalization of **Lemma** 4.1, for every $\bar{\imath} = (i_1, \ldots, i_n)$ we have*

$$\|A_{i_1}A_{i_2}\ldots A_{i_{n-1}}|Im(A_{i_n})\| = \|A_{i_1}|Im(A_{i_2})\|\ldots\|A_{i_{n-1}}|Im(A_{i_n})\|.$$

Proof. For every $\underline{v} \in Im(A_{i_n})$ and $\|\underline{v}\|$ if $A_{i_2}\cdots A_{i_{n-1}}\underline{v} \neq 0$,

$$\|A_{i_1}\cdots A_{i_{n-1}}|Im(A_{i_n})\| = \|A_{i_1}\cdots A_{i_n}\underline{v}\| =$$

$$= \|A_{i_1}\frac{A_{i_2}\ldots A_{i_{n-1}}\underline{v}}{\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\|}\|\cdot\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\|$$

We know that $A_{i_2}\ldots A_{i_{n-1}}\underline{v} \in Im(A_{i_2}\ldots A_{i_{n-1}})$ by **Lemma** 4.6 $Im(A_{i_2}\ldots A_{i_{n-1}}) = Im(A_{i_2})$ then

$$\|A_{i_1}\frac{A_{i_2}\ldots A_{i_{n-1}}\underline{v}}{\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\|}\| = \|A_{i_1}|Im(A_{i_2})\|.$$

By induction we use the previous calculation for $\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\|$. □

Proposition 4.8. *Let $A_1, \ldots, A_n$ be singular matrices of an IFS $\mathcal{F}$ and $\alpha_1(A_i)$ be maximum of the singular values of A_i. Then there exists constants $m, M > 0$ such that for every $k \geq 1$ and $s \in (0, 1]$,*

$$m \cdot \|(B^{(s)})^k\|_1 \leq \sum_{|\bar{\imath}|=k+1} \alpha_1(A_{\bar{\imath}})^s \leq M \cdot \|(B^{(s)})^k\|_1. \tag{14}$$

Proof. We know that, $\alpha_1(A_i) = \|A_i\|$. Observe that

$$\sum_{|\bar{\imath}|=k+1} \alpha_1(A_{\bar{\imath}}) = \sum_{i,j=1,\ldots,n}\sum_{|\bar{\jmath}|=k-1} \|A_iA_{\bar{\jmath}}|Im(A_j)\|\cdot\|A_j\|. \tag{15}$$

Then denote $M = \max_j\{\|A_j\|\}$ and $m = \min_j\{\|A_j\|\}$. By using m and M in equation (15)

$$m \cdot \|(B^{(s)})^k\|_1 \leq \sum_{|\bar{\imath}|=k+1} \alpha_1(A_{\bar{\imath}})^s \leq M \cdot \|(B^{(s)})^k\|_1.$$

□

Proposition 4.9. *Let $A_1, \ldots, A_n$ be singular matrices. Then,*

$$P(s) = \log\left(\rho(B^{(s)})\right) \text{ for } 0 \leq s \leq 1.$$

Proof. By using inequality (14) and taking logarithm and dividing by n we have

$$\frac{1}{n}\log(\rho(B^{(s)})^n) \leq \frac{1}{n}\log\left(\sum_{|\bar{\imath}|=n+1}\alpha_1(A_{\bar{\imath}})^s\right) \leq \frac{1}{n}\log(\rho(B^{(s)})^n).$$

Then by Gelfand's formula $\|B^n\|^{1/n} \to \rho(B)$ as $n \to \infty$. Thus, we have

$$P(s) = \log(\rho(B^{(s)})).$$

□

4.3 LOWER BOUND

To give a proper lower bound for the dimension of the attractor is not as simple as the upper bound.

4.3.1 *Singular case*

Let us recall the definition of $B^{(s)}_{MW}$ from equation (2.19) and our constructed matrix $B^{(s)}$ from equation (8). To begin the investigation of these matrices, one can see that if there are more than one mapping to the same affine space, the sum of their contraction ratio will appear in the Mauldin-Williams matrix. On the other hand, in matrix $B^{(s)}$, we record every contraction ratio one by one. Hence the dimension of the matrices satisfies $\dim(B^{(s)}_{MW}) \leq \dim(B^{(s)})$. We construct an eigenvector for the same positive eigenvalue from $B^{(s)}$ to $B^{(s)}_{MW}$. By Perron-Frobenius theorem this eigenvector is unique and positive, so the spectral radius of the matrices should equal. This is necessary step which allows us to use the Mauldin-Williams theorem 2.21 for the matrix $B^{(s)}$.

Now we show a Grap-directed IFS corresponds to the self-affine IFS $\mathcal{F} = \{\varphi_i(\underline{x}) = A_i\underline{x} + t_i\}_{i=1}^m$. By definition $Im(\varphi_i) = Im(A_i) + t_i$ for every $i = 1, \ldots, m$. Let I be the set of all image spaces of φ_i which are distinct. In other words

$$I = \{Im(\varphi_i) : i = 1, \ldots, m\} := \{V_1, \ldots, V_M\},$$

where $V_i \neq V_j$ for $i \neq j$. Furthermore, by definition $\mathcal{E}_{i,j} := \{k : Im(\varphi_k) = V_i \text{ and } Ker(A_k) \neq V_j\}$ and define $\mathcal{E}^i = \{k\colon Im(\varphi_k) = V_i\}$. Then we can define $f_e\colon Im(\varphi_i) \to Im(\varphi_j)$ for every i, j and $e\colon i \to j$, $e \in \mathcal{E}$. In this case

$$f_e(\underline{x}) = \varphi_j(\underline{x}) \text{ for every } \underline{x} \in Im(\varphi_i).$$

Since for every i, V_i is a hyperspace, we need to construct a set with subspaces to determine the norm of the matrix $B^{(s)}_{MW}$. For every V_i there is a unique W_i subspace in $\mathbb{R}^2$ such that for every $\underline{x}, \underline{y} \in V_i$, $\underline{x} - \underline{y} \in W_i$. Then the elements of the matrix $B^{(s)}_{MW}$ will be

$$\left(B^{(s)}_{MW}\right)_{i,j} = \sum_{k \in \mathcal{E}_{i,j}} \|A_i|W_j\|^s \quad \text{for every } i, j = 1, \ldots, M.$$

Lemma 4.10. *Let* $\{\varphi_i(\underline{x}) = A_i\underline{x} + t_i\}_{i=1}^m$ *be an IFS for every* $\underline{x} \in \mathbb{R}^2$*and for every* $m > 1$. *Consider the following matrices* $B^{(s)}_{MW}$ *and* $B^{(s)}$ *for the IFS. Then the spectral radius of the matrices are equal,*

$$\rho = \rho(B^{(s)}_{MW}) = \rho(B^{(s)}).$$

Proof. Previously we saw the elements of the $B^{(s)}_{MW}$ matrix. On the other hand, the elements of the matrix $B^{(s)}$ will be

$$\left(B^{(s)}\right)_{i,j} = \|A_i|Im(A_j)\|^s \quad \text{for every } i, j = 1, \ldots, m.$$

There exists a unique vector $\underline{v} \in \mathbb{R}^m$ such that $\|\underline{v}\| = 1$ and for every $i = 1, \ldots, m$, $v_i > 0$. The spectral radius of $\rho(B^{(s)}) = \rho$ and by the Perron Frobenius theorem,

$$B^{(s)}\underline{v} = \rho\underline{v}.$$

Now we construct a vector, by $\underline{v}$ then we will see this constructed vector is an eigenvector of $B^{(s)}_{MW}$.

Let $\underline{z} \in \mathbb{R}^M$ be a vector such that $z_j = \sum_{k\in\mathcal{E}^j} v_k$. Then

$$\left(B^{(s)}_{MW}\underline{z}\right)_i = \sum_{j=1}^{M}\sum_{k\in\mathcal{E}^i} \|A_k|W_j\|^s z_j = \sum_{j=1}^{M}\sum_{k\in\mathcal{E}^i}\sum_{l\in\mathcal{E}^j} \|A_k|W_j\|^s v_l =$$
$$\sum_{j=1}^{M}\sum_{k\in\mathcal{E}^i}\sum_{l\in\mathcal{E}^j} \|A_k|Im(A_l)\|^s v_l = \sum_{k\in\mathcal{E}^i}\sum_{j=1}^{M}\sum_{l\in\mathcal{E}^j} \|A_k|Im(A_l)\|^s v_l =$$
$$\sum_{k\in\mathcal{E}^i}\sum_{l=1}^{m}\|A_k|Im(A_l)\|^s v_l = \sum_{k\in\mathcal{E}^i} \rho v_k = \rho \underline{z}_i.$$

Then for every $z_i > 0$, $\underline{z}$ is an eigenvector of $B^{(s)}_{MW}$ with $\rho > 0$ eigenvalue. So, by Perron-Frobenius theorem

$$\rho = \rho(B^{(s)}_{MW}).$$

□

***Proof of Theorem* 3.6.** If $\varphi_i \circ \varphi_j(\Lambda) \cap \varphi_i \circ \varphi_l(\Lambda) = \emptyset$ if $j \neq l$ for every $i, j, l = 1, \ldots, n$. By definiton of Λ_j and φ_j, we have $\Lambda_j = \varphi_j(\Lambda)$. Since $f_e(\underline{x}) = \varphi_j(\underline{x})$ for every $\underline{x} \in Im(\varphi_i)$ then $f_e(\Lambda_j) = \varphi_i \circ \varphi_j(\Lambda)$. So $\varphi_i \circ \varphi_j(\Lambda) \cap \varphi_i \circ \varphi_l(\Lambda) = \emptyset$ satisfies the eparation condition in Mauldin-Williams theorem. Then by applying **Theorem 2.21**, s_0 is a unique solution for the equation

$$\rho(B^{(s_0)}_{MW}) = 1.$$

By **Lemma 4.10**, $\rho = \rho(B^s_{MW}) = \rho(B^{(s)})$ and by Mauldin-Williams theorem, s_0 is the unique solution for the equation $\rho(B^{(s_0)}) = 1$. Then

$$\dim_H(\Lambda) = s_0.$$

□

4.3.2 *One singular between regulars*

In this subsection we prove **Proposition 3.8**, but first we show some necessary conditions when the statement will be true. Let us begin with the basic definitions. Assume, in this section for a singular matrix $A_1(\beta)$ we use the notation of

$$A_1 = \varrho \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos(\beta) & \sin(\beta) \end{pmatrix} \quad \text{for every } \beta \in [0, 2\pi] \text{ and } \varrho < 1.$$

We would like to associate $Im(A_1(\beta))$ to the real line $\mathbb{R}$. Let $\underline{v}$ be a fixed vector, such that $\|\underline{v}\| = 1$ and $\underline{v} \in Im(A_1)$. Let $p\colon Im(A_1) \to \mathbb{R}$ be a function, such that

$$p(\underline{w}) = \langle \underline{v}, \underline{w} \rangle = \tau. \tag{16}$$

By rearrange (16), we have $\underline{w} = p(\underline{w}) \cdot \underline{v}$. Now we will investigate the function p on the similarity transformations φ_1 and $\varphi_{\bar{\imath}}$.

$$p(\varphi_1(\varphi_{\bar{\imath}}(\underline{w}))) = \langle \underline{v}, A_1 A_{\bar{\imath}} \underline{w} + A_1 \underline{t}_{\bar{\imath}} \rangle = p(\underline{w}) \cdot \langle \underline{v}, A_1 A_{\bar{\imath}} \underline{v} \rangle + \langle A_1 \underline{t}_{\bar{\imath}}, \underline{v} \rangle .$$

Definition 4.11. Let $A_{\bar{\imath}}$ be a regular matrix where $\bar{\imath} \in \Sigma_{reg}^n$ and A_1 be singular matrix. Then let $g_{1\bar{\imath}} : \mathbb{R} \to \mathbb{R}$ be a function such that, for an arbitrary $\tau \in \mathbb{R}$

$$g_{1\bar{\imath}}^{(\beta)}(\tau) = \langle \underline{v}, A_1 A_{\bar{\imath}} \underline{v} \rangle \cdot \tau + \langle \underline{v}, A_1 \underline{t}_{\bar{\imath}} \rangle . \tag{17}$$

Observe that, $|\langle \underline{v}, A_1 A_{\bar{\imath}} \underline{v} \rangle| = \|A_1 A_{\bar{\imath}} | Im(A_1)\|$.

Definition 4.12. Let $g_{1\bar{\imath}}^{(\beta)} : \mathbb{R} \to \mathbb{R}$ be a function as defined in equation (17), then let $\Pi_\beta(\hat{\imath}) : \Omega_n \to \Gamma_{n,\beta}$ be a function such that,

$$\Pi_\beta(\hat{\imath}) = \lim_{n \to \infty} g_{1\bar{\imath}_1}^{(\beta)} \circ g_{1\bar{\imath}_2}^{(\beta)} \circ \ldots g_{1\bar{\imath}_n}^{(\beta)}(0),$$

where $\Omega_n = \left(\Sigma_{sing} \times \Sigma_{reg}^n\right)^{\mathbb{N}}$ and Γ_n is the attractor of the IFS $\{g_{1,\bar{\imath}}\}$.

Lemma 4.13. *Let $A_1(\beta)$ be a singular matrix and let A_i be regular matrices for every $i = 2, \ldots, n$. Let $\varphi_1(\underline{x}) = A_1(\beta)\underline{x}$ and $\varphi_i(\underline{x}) = A_i \underline{x} + \underline{t}_i$ be contracting similarity functions. If the Elliptic Strong Separation Condition holds,*

$$\Pi_\beta(\hat{\imath}) \equiv \Pi_\beta(\hat{\jmath}) \textit{ for every } \beta \textit{ if and only if } \hat{\imath} = \hat{\jmath} \in \Omega_n.$$

Proof. First, we note that the contraction rate $\beta \mapsto \langle \underline{v}, A_1(\beta) A_{\bar{\imath}} \underline{v} \rangle$ and the translation $\beta \mapsto \langle \underline{v}, A_1(\beta) \underline{t}_{\bar{\imath}} \rangle$ are real analytic for every $\bar{\imath} \in \Sigma_{reg}^*$. Let us argue by contradiction, so assume

$$\Pi_\beta(\hat{\imath}) \equiv \Pi_\beta(\hat{\jmath}) \text{ for } \hat{\imath} \neq \hat{\jmath}. \tag{18}$$

Then without loss of generality, assume $\bar{\imath}_1 \neq \bar{\jmath}_1$, there are two possibe position to the iterates of $\varphi_{\bar{\imath}}$ because of the Elliptic Strong Separation. First, when they are disjoint, and the second, when one of them is contained in the other one.

$$\varphi_{\bar{\imath}_1}(S^1) \cap \varphi_{\bar{\jmath}_1}(S^1) = \emptyset \tag{19}$$
$$\varphi_{\bar{\imath}_1}(S^1) \subset \varphi_{\bar{\jmath}_1}(S^1) \tag{20}$$

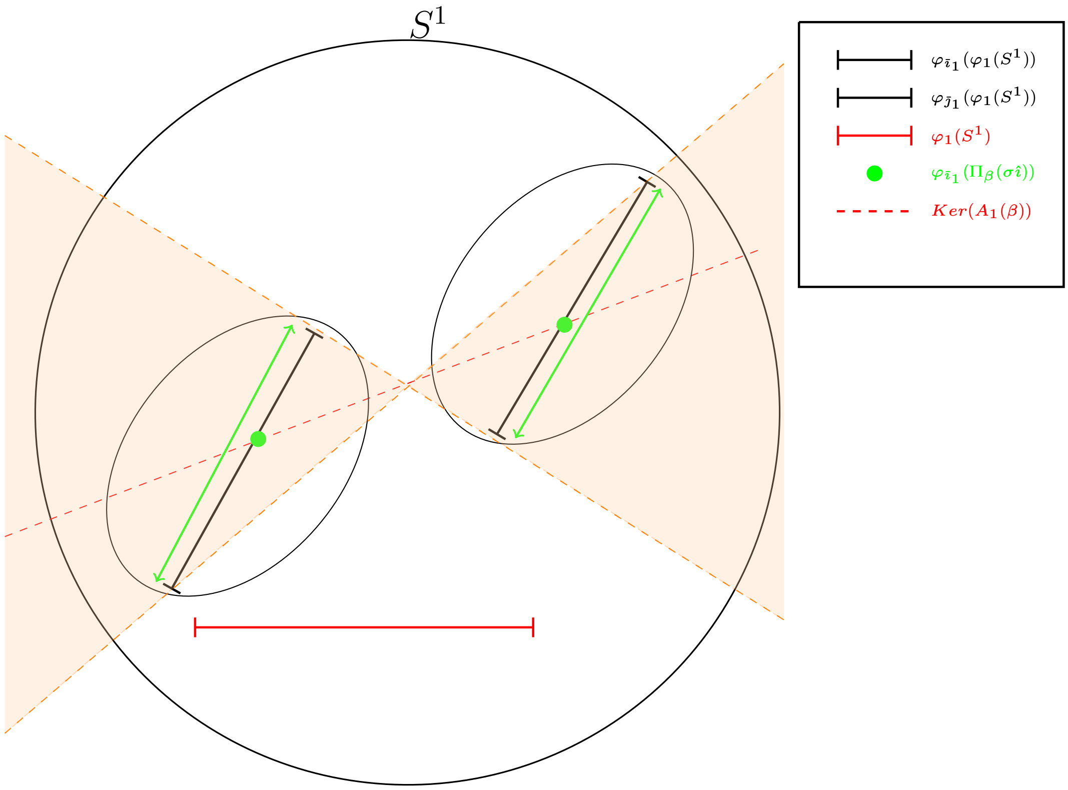


Figure 6: Cylinders in case (19)

By our assumption on $\hat{\imath}$ and $\hat{\jmath}$, we have

$$A_1(\beta) \cdot \left(\varphi_{\bar{\imath}_1} \begin{pmatrix} \Pi_\beta(\sigma\hat{\imath}) \\ 0 \end{pmatrix} - \varphi_{\bar{\jmath}_1} \begin{pmatrix} \Pi_\beta(\sigma\hat{\jmath}) \\ 0 \end{pmatrix} \right) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent with

$$\varphi_{i_1} \begin{pmatrix} \Pi_\beta(\sigma\hat{\imath}) \\ 0 \end{pmatrix} - \varphi_{\bar{\jmath}_1} \begin{pmatrix} \Pi_\beta(\sigma\hat{\jmath}) \\ 0 \end{pmatrix} \in \left\langle \begin{pmatrix} -\sin(\beta) \\ \cos(\beta) \end{pmatrix} \right\rangle = Ker(A_1(\beta)). \tag{21}$$

The image of φ_1 is independent of β, so we can define a cone in S^1 such that

$$\mathcal{C}_1 = \left\{ \frac{\varphi_{\bar{\imath}_1}(\underline{x}) - \varphi_{\bar{\jmath}_1}(\underline{y})}{\left\|\varphi_{\bar{\imath}_1}(\underline{x}) - \varphi_{\bar{\jmath}_1}(\underline{y})\right\|} \; : \; \text{for every } \underline{x}, \underline{y} \in Im(\varphi_1) \right\}.$$

Since the range of $Ker(A_1(\beta))$ is the whole S^1, and $\frac{\varphi_{\bar{\imath}_1}(\underline{x}) - \varphi_{\bar{\jmath}_1}(\underline{y})}{\|\varphi_{\bar{\imath}_1}(\underline{x}) - \varphi_{\bar{\jmath}_1}(\underline{y})\|} \in \mathcal{C}_1$, for all $\underline{x}, \underline{y} \in Im(A_1(\beta))$. Hence $\mathcal{C}_1 \subsetneq S^1$, there is a $\beta \in [0, 2\pi]$ for which our assumption (18) does not holds, i.e there is a vector $\underline{w}_1$, such that

$$\underline{w}_1 = \begin{pmatrix} -\sin(\beta) \\ \cos(\beta) \end{pmatrix} \notin \mathcal{C}_1,$$

which is a contradiction.

On the other hand, if we have $\varphi_{\bar{\imath}_1}(S^1) \subset \varphi_{\bar{\jmath}_1}(S^1)$, then we can lead back the solution to the previous case. By the Elliptic Strong Separation Condition $\varphi_{\bar{\imath}_1} = \varphi_{\bar{\jmath}_1} \circ \varphi_{\overline{k}_1}$, then we have

$$\varphi_1\left(\varphi_{\overline{k}_1}\left(\begin{pmatrix} \Pi_\beta(\sigma\hat{\imath}) \\ 0 \end{pmatrix}\right)\right) \equiv \varphi_1\left(\varphi_{\bar{\jmath}_1}\left(\begin{pmatrix} \Pi_\beta(\sigma\hat{\jmath}) \\ 0 \end{pmatrix}\right)\right),$$

which is equivalent to

$$A_{\bar{\jmath}_1}\left(\begin{pmatrix} \Pi_\beta(\sigma\hat{\imath}) \\ 0 \end{pmatrix} - \varphi_{\overline{k}_1}\left(\begin{pmatrix} \Pi_\beta(\sigma\hat{\jmath}) \\ 0 \end{pmatrix}\right)\right) \in Ker(A_1(\beta)) \text{ for every } \beta,$$

and so

$$\begin{pmatrix} \Pi_\beta(\sigma\hat{\imath}) \\ 0 \end{pmatrix} - \varphi_{\overline{k}_1}\left(\begin{pmatrix} \Pi_\beta(\sigma\hat{\jmath}) \\ 0 \end{pmatrix}\right) \in A_{\bar{\jmath}_1}^{-1} Ker(A_1(\beta)) \text{ for every } \beta.$$

Since $\varphi_{\overline{k}_1}(\Pi_\beta(\sigma\hat{\imath})) \in \varphi_{\overline{k}_1}(\varphi_1(S^1))$ and $\varphi_{\bar{\jmath}_1}(\Pi_\beta(\sigma\hat{\imath})) \in \varphi_{\bar{\jmath}_1}(\varphi_1(S^1))$. Again, we might define a cone

$$\mathcal{C}_2 = \left\{ \frac{\underline{x} - \varphi_{\overline{k}_1}(\underline{y})}{\|\underline{x} - \varphi_{\overline{k}_1}(\underline{y})\|} \; : \; \text{for every } \underline{x}, \underline{y} \in Im(\varphi_1) \right\}.$$

Similarily to the previous case, $A_{\bar{\jmath}_1}^{-1} Ker(A_1(\beta))$ ranges over the whole S^1, and $\frac{\underline{x} - \varphi_{\overline{k}_1}(\underline{y})}{\|\underline{x} - \varphi_{\overline{k}_1}(\underline{y})\|} \in \mathcal{C}_2$, where $\underline{x}, \underline{y} \in \mathbb{R}$. Hence $\mathcal{C}_2 \subsetneq S^1$, so there is a $\beta \in [0, 2\pi]$ for which our assumption does not holds i.e there is a vector $\underline{w}_2$, such that

$$\underline{w}_2 = A_{\bar{\jmath}_1}^{-1} \begin{pmatrix} -\sin(\beta) \\ \cos(\beta) \end{pmatrix} \notin \mathcal{C}_2,$$

which is again a contradiction.

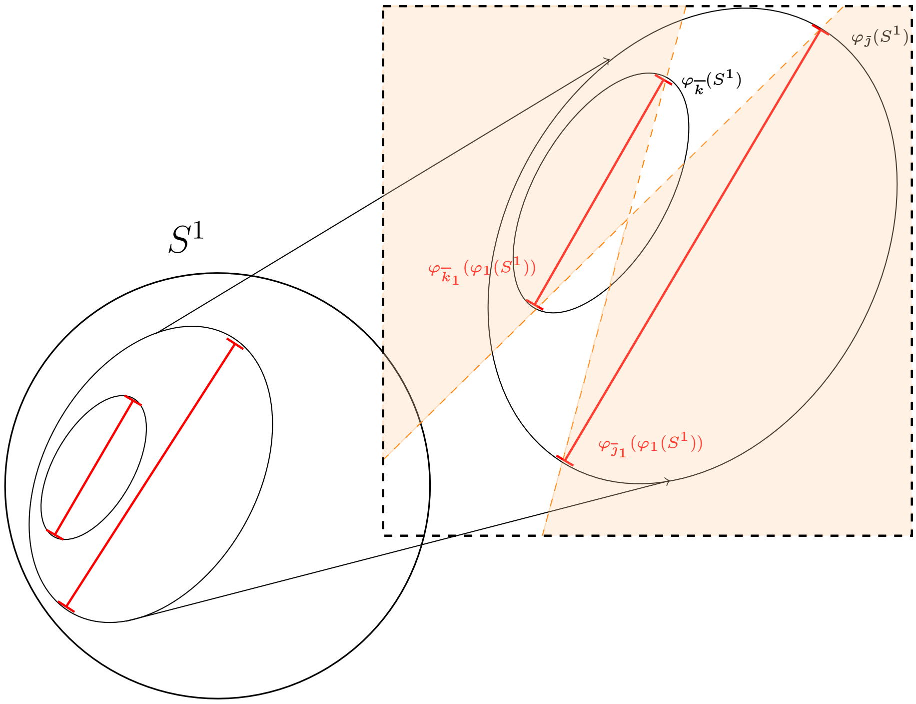


Figure 7: Cylinders in case (20)

□

Proof of Proposition 3.8. If $s_0 > 1$ we proved this case before, so without loss of generality, we can assume $s_0 \leq 1$. We know that, by **Proposition 3.3** $s_0 = \max\{s_{reg}, s_{sing}\}$. Let $\mathbf{\Phi}_n = \{g_{1,\bar{\imath}}^{(\beta)}(\tau)\}$ be an IFS formed by $g_{1,\bar{\imath}}^{(\beta)}(\tau)$ functions and let $\Pi_\beta(\hat{\imath}) = \lim_{n\to\infty} g_{1,\bar{\imath}_1}^{(\beta)} \circ \cdots \circ g_{1,\bar{\imath}_n}^{(\beta)}(\tau)$. Furthermore, let $\Gamma_{n,\beta}$ be the attractor of $\mathbf{\Phi}_n$. Then by **Lemma 4.13**, $\mathbf{\Phi}_n$ satisfies the condition of Hochman's **Theorem 2.24**. Hence, there is a set $E \subseteq [0, 2\pi]$ such that $\dim_H(E) = 0$ for every $\beta \in [0, 2\pi] \setminus E$. Then by Hochman's **Theorem 2.23**

$$\dim_H(\Gamma_n) = \min\{s_n(\beta), 1\},$$

where $s_n(\beta)$ is the unique solution of the equation

$$\sum_{\bar{\imath} \in \Sigma_{reg}^n} \|A_1 A_{\bar{\imath}} | Im(A_1)\|^{s_n} = 1.$$

By definition $s_{sing}(\beta)$ is the unique solution of the following equation

$$\sum_{\bar{\imath}\in\Sigma^*_{reg}} \|A_1 A_{\bar{\imath}}|Im(A_1)\|^{s_{sing}} = 1,$$

hence $s_n(\beta)$ converges to $s_{sing}(\beta)$. This implies that, $\dim_H(\Lambda_\beta) \geq \min\{s_n(\beta), 1\}$ for every $\beta \in [0, 2\pi] \setminus E$. Then by Bárány, Hochman and Rapaport **Theorem 2.31**,

$$\dim_H(\Lambda_\beta) \geq s_{reg}.$$

All of the above implies

$$\dim_H(\Lambda_\beta) \geq s_0(\beta) \text{ for every } \beta \in [0, 2\pi] \setminus E.$$

Then, by **Theorem 3.6**

$$\dim_H(\Lambda_\beta) = s_0(\beta).$$

□

5 CONCLUSIONS

In contrast to regular self-affine IFS, the attractor of a singular self-affine IFS is not always a perfect set, but these systems behaviour does not change too much. In general, we do not have any answer for the general question of this Thesis, but we have results in more specific areas.

For singular self-affine IFS we can construct a graph-directed IFS and a matrix $B^{(s)}$ from equation (8), then by Perron-Frobenius theorem and Mauldin-Williams theorem, the dimension of the attractor of the self-affine IFS, will equal to the affinity dimension s_0.

If the matrices of the contracting similarity transformations are both singular and regular, we need more assumptions to determine the dimension of the attractor. First we need to define $P_{sing}(s)$ the pressure of the system and we have a subsystem, which contains only regular matrices with its pressure function $P_{reg}(s)$. Furthermore we defined s_{sing} as the unique solution to $\rho(B^{(s_{sing})}) = 1$ and s_{reg} as the unique root of $P_{reg}(s)$. By the singular matrices in the self-affine IFS, the sub-additive pressure function has a shiftat $s = 1$. We showed that, the affinity dimension $s_0 = s_{reg}$ in the case if $s_{reg} > 1$ and $s_0 = \min\{1, \max\{s_{sing}, s_{reg}\}\}$ in the case if $s_{reg} \leq 1$. The main result of this section is a theorem, which determines the dimension of the attractor of such a self-affine IFS. If $s_{reg} > 1$ and the regular sub-system satisfies the Strong Separation Property, and the collection of the regular matrices is strongly irreducible, then the Hausdorff-dimension of the attractor equals to s_{reg}.

In the last part of the Thesis, we investigated self-affine IFS with only one singualr matrix. In this case, with a new assumption called the Elliptic Separation Property, we can determine the dimension of the attractor, if it depends on some $\beta \in [0, 2\pi]$. Then, the dimension of $\Lambda(\beta)$ will equal to s_0 in both cases when $s_{reg} > 1$ or $s_{reg} \leq 1$.

There are a lot more open question related to this topic. For example: if $s_{reg} \leq 1$ will the dimension of the attractor always equals to s_0 ? Maybe we could answer it later.

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