**Budapest University of Technology and Economics**

Institute of Mathematics

Department of Stochastics

# Self-similar iterated function systems with non-distinct fixed points

Thesis

*Author*

Edina Szvák

MSc student

*Supervisor*

Dr. Balázs Bárány

Docent,

BME, Department of Stochastics

May 31, 2019

# Contents

# Acknowledgements

I would like to express my gratitude to my supervisor Balázs Bárány for his patience, motivation and enthusiasm. I am thankful for his invaluable support and encouraging. It brings me great pleasure for the opportunity to study from him. I am much obliged to him for his continuous assistance for a long time.

Besides my advisor, I would like to acknowledge the teaching of Károly Simon. I wish thank him for the beginning my studies regarding to iterated function systems.

I duly acknowledge the help, direct or indirect of my teachers. I grateful to them for sharing the knowlede with us.

# Chapter 1

# Preliminary

Hutchinson showed that if the cylinder sets of a self-similar iterated function system (IFS) are disjoint, then the Hausdorff dimension of its attractor is equals with the similarity dimension. Also, he showed similar result for self-similar measures which belongs to such self-similar IFS for which some strong separation condition holds.

When the cylinder sets of an IFS has significant overlap, the dimension is difficult to understand, because we have to consider complicated overlapping system of cylinder sets.

Using transversality condition for a self-similar IFS family, then K. Simon, B. Solomyak and M. Urbanski calculated this dimensions for almost every paramaters of the IFS family. B. Bárány also proved almost everywhere results, when the self-similar IFS's have fix points that coincide.

Kamalutdinov and Tetenov studied twofold Cantor sets, which are very similar to the forward separated systems (Definition 3.3). In a system of a twofold Cantor set there are total overlaps. They have results for the properties of the attractor. They calculated the exact value of the Hausdorff dimension of twofold Cantor sets. They do not mentioned about the self-similar measures of those systems.

## Results of this dissertation

In this work we study self-similar IFS's on the interval $[0, 1]$ for which the so-called forward separated condition holds (Definition 3.3). In the considered IFS's there is also total overlap between the cylinder sets.

Using the argument of Kamalutdinov and Tetenov we proved that forward separated systems exist. The main result of this dissertation is Theorem 7.1, which states everywhere result for the Hausdorff dimension of a self-similar measure with respect to a forward separated system.

**Theorem 7.1** Let $\alpha, \beta, \gamma \in (0, \frac{1}{9})$. Let $\mathcal{S} = \{S_1, S_2, S_3\}$ be a self-similar IFS on $[0,1]$ such that

$$\begin{gathered} \mathcal{S} = \{S_1, S_2, S_3\} \\ S_1(x) = \alpha x, \quad S_2(x) = \beta x, \quad S_3(x) = \gamma x + 1 - \gamma. \end{gathered} \tag{1.1}$$

Let $K$ denote the attractor of $\mathcal{S}$. Moreover, we suppose that

$$\text{for every } m, n \in \mathbb{N}^+, \quad S_1^m S_3(K) \cap S_2^n S_3(K) = \emptyset. \tag{1.2}$$

The natural projection of $\mathcal{S}_{\alpha,\beta,\gamma}$ is $\Pi_{\alpha,\beta,\gamma}$. Let $\mu = (p_1, p_2, p_3)^{\mathbb{N}^+}$ be a Bernoulli measure on $\Sigma$ for the probability vector $\mathbf{p} = (p_1, p_2, p_3)$. Let $\nu = \Pi_{\alpha,\beta,\gamma *}\mu = \mu \circ \Pi_{\alpha,\beta,\gamma}^{-1}$ be the self-similar measure on the attractor. Then the Hausdorff dimension of $\nu$ can be exactly determined.

The exact value of the dimension is in Chapter 7. To achieve this statement we use ergodic CP-shift system.

# Chapter 2

# Introduction of self-similar iterated function systems

In this chapter we would like to define the most fundamental notions and we collect the most important theorems concerning self-similar iterated function systems (IFS).

## 2.1 Definitions of self-similar IFS

**Definition 2.1** *Let $m \geq 2$, $m \in \mathbb{Z}$ and $d \geq 1$, $d \in \mathbb{Z}$. We say that $\mathcal{S}$ is a self-similar iterated function system (IFS) on $\mathbb{R}^d$, if*

$$\mathcal{S} = \{S_1, \dots, S_m\}, \tag{2.1}$$

*where $S_i : \mathbb{R}^d \to \mathbb{R}^d$ is contracting similarity transformation with contraction ratio $0 < r_i < 1$ for all $i$. This means, that*

$$\forall i \in \{1, \dots, m\} \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \quad \|S_i(\mathbf{x}) - S_i(\mathbf{y})\| = r_i \|\mathbf{x} - \mathbf{y}\|. \tag{2.2}$$

Frequently we use the notation $S_{i_1} \circ \cdots \circ S_{i_n} = S_{i_1,\dots,i_n}$.

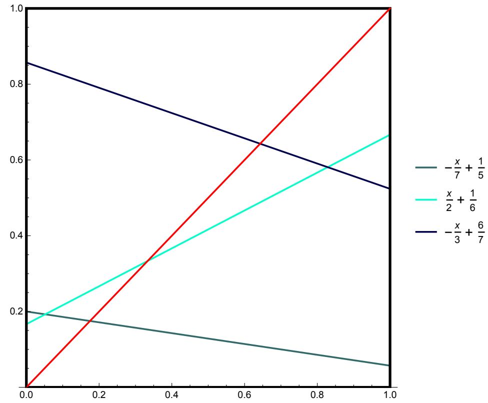


Figure 2.1: Example for a self-similar IFS on the line

**Definition 2.2** *Let* $B = \overline{B}(0, R)$, *where* $R = \max_{1 \le i \le m} \left\{ \frac{\|S_i(\mathbf{0})\|}{1 - r_i} \right\}$. *The set* $\Lambda$ *is the attractor of the self-similar IFS* $\mathcal{S}$, *if*

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in \{1, \dots, m\}^n} S_{i_1, \dots, i_n}(B). \tag{2.3}$$

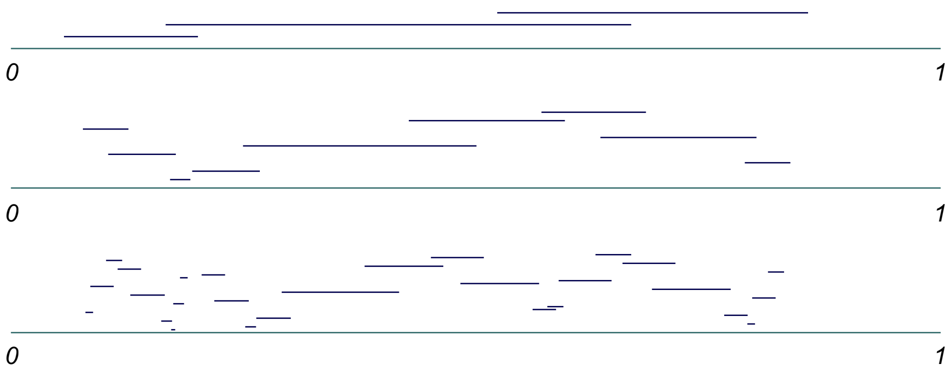


Figure 2.2: The first, second and third level cylinder sets of the IFS $\mathcal{S} = \{-\frac{x}{7} + \frac{1}{5}, \frac{x}{2} + \frac{1}{6}, -\frac{x}{3} + \frac{6}{7}\}$

**Definition 2.3** *We call* $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$ *the symbolic space of the IFS* $\mathcal{S}$ *defined in equation* (2.1).

On the symbolic space we use the following notation. If $\underline{i} = (i_1, \dots, i_k) \in \{1, \dots, m\}^k$ and $\underline{j} \in \{1, \dots, m\}^l$, then let $\underline{i} * \underline{j} = (i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_l)$. Denote $\underline{i}^2 = \underline{i} * \underline{i}$ and $\underline{i}^k = \underline{i}^{k-1} * \underline{i}$. This definition is also proper for $l = \infty$.

Let us denote the set of all finite length word by $\Sigma^* = \bigcup_{k=1}^{\infty} \{1, \dots, m\}^k$.

We denote the left shift on the symbolic space with $\sigma : \Sigma \to \Sigma$ for all $\underline{j} = (j_1, j_2, \dots) \in \Sigma$ $\sigma(\underline{j}) = (j_2, j_3, \dots)$.

**Definition 2.4** *The map* $\Pi$ *is the natural projection of the IFS* $\mathcal{S}$, *if*

$$\Pi : \Sigma \to \Lambda \quad \Pi(\underline{i}) = \lim_{n\to\infty} S_{i_1,\dots,i_n}(\mathbf{0}), \tag{2.4}$$

*where* $\underline{i} = (i_1, i_2, \dots) \in \Sigma$.

It is easy to see that

$$\Lambda = \Pi(\Sigma). \tag{2.5}$$

**Theorem 2.5 (Hutchinson)** *The* $\Lambda$ *attractor of the IFS* $\mathcal{S}$ (2.1) *is the only non-empty compact set solution of the following equation on sets*

$$X = \bigcup_{i=1}^{m} S_i(X), \tag{2.6}$$

*where* $X$ *is the variable.*

The proof can be found in [2].

**Definition 2.6** *Let* $\Sigma = \{1, \dots, m\}^{\mathbb{N}^+}$ *and* $\underline{i} = (i_1, \dots i_k) \in \{1, \dots, m\}^k$, *then the set*

$$[i_1, \dots, i_k] = \{\underline{j} \in \Sigma : j_1 = i_1, \dots j_k = i_k\} \tag{2.7}$$

*is called a cylinder set.*

Let $\mathbf{p} = (p_1, \dots, p_m)$ be a probability vector. Then, let $\mu = \mathbf{p}^{\mathbb{N}}$ be the infinite product measure or Bernolli measure on $\Sigma$. That is

$$\mu([i_1, \dots, i_k]) = p_{i_1} \dots p_{i_k}, \tag{2.8}$$

where $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$. Using Kolmogorov's extension theorem, we can see that there exists a unique $\mu$ Borel measure on $\Sigma$ defined on the $\sigma$-algebra generated by the cylinder sets and for which the equation (2.8) holds.

**Definition 2.7** *Let* $\mathbf{p} = (p_1, \dots, p_m)$ *be a probability vector. We say that* $\nu$ *is a self-similar measure or invariant measure of the self-similar IFS* $\mathcal{S}$ *with the probabilty vector* $\mathbf{p}$, *if* $\nu$ *is the following push-down measure*

$$\nu(E) = \Pi_* \mathbf{p}^{\mathbb{N}}(E) = \mathbf{p}^{\mathbb{N}} \circ \Pi^{-1}(E). \tag{2.9}$$

**Theorem 2.8** *Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *a probabilty vector and* $\mathcal{S}$ *is a self-similar IFS in the form* (2.1)*. Then* $\nu$ *self-similar measure of* $\mathcal{S}$ *with the probabilty vector* $\mathbf{p}$ *if it is the only* $\nu$ *Borel probabilty measure on* $\mathbb{R}^d$ *for which*

$$\nu = \sum_{k=1}^{m} p_k(\nu \circ S_k^{-1}) \tag{2.10}$$

*holds.*

The proof can be found in [2].

## 2.2 The size of the attractor

Most of the time the attractor has zero Lebesgue measure, thus we need some definition to be able to compare the size of sets with zero Lebesgue measure.

**Definition 2.9** *Let* $t \geq 0$*. The measure* $\mathcal{H}^t$ *is called the t-dimensional Hausdorff measure on* $\mathbb{R}^d$*, if it is the restriction of the following outer measure for the* $\sigma$*-algebra of the measurable sets. Let*

$$\mathcal{H}^t(E) = \lim_{\delta \to 0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t \, : \, E \subseteq \bigcup_{i=1}^{\infty} A_i, \, |A_i| \leq \delta \right\} \right\} = \lim_{\delta \to 0} \mathcal{H}^t_\delta(E), \tag{2.11}$$

*where* $A \subseteq \mathbb{R}^d$ $|A|$ *is the diameter of the set* $A$*.*

**Remark 2.10** *The limit in the equation* (2.11) *is exists, because the function*

$$\delta \mapsto \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t \, : \, E \subseteq \bigcup_{i=1}^{\infty} A_i, \, |A_i| \leq \delta \right\} \tag{2.12}$$

*is monoton decreasing.*

Now, let us introduce some basic facts regarding to Hausdorff measure.

**Theorem 2.11** *For every* $t > 0$*, all Borel set in* $\mathbb{R}^d$ *is measurable with respect to the t-dimensional Hausdorff measure.*

**Theorem 2.12** *For every* $n \in \mathbb{N}^+$*, there exists* $c \in \mathbb{R}^+$ *such that for all Borel set* $B \subseteq \mathbb{R}^n$ $\mathcal{H}^n(B) = c\mathcal{L}^n(B)$ *hold.*

**Lemma 2.13** *For every Borel set* $B \subseteq \mathbb{R}^d$ *and every* $0 \leq \alpha < \beta$*, we have the following implications:*

*(i)* $\mathcal{H}^\alpha(B) < \infty \Longrightarrow \mathcal{H}^\beta(B) = 0$

*(ii)* $\mathcal{H}^\beta(B) > 0 \Longrightarrow \mathcal{H}^\alpha(B) = \infty$

**Definition 2.14** *By Lemma 2.13 we can define the Hausdorff dimension of a $B \subseteq \mathbb{R}^d$ Borel set by*

$$\dim_H(B) = \inf_{t\geq 0}\{\mathcal{H}^t(B) = 0\} = \sup_{t\geq 0}\{\mathcal{H}^t(B) = \infty\}. \tag{2.13}$$

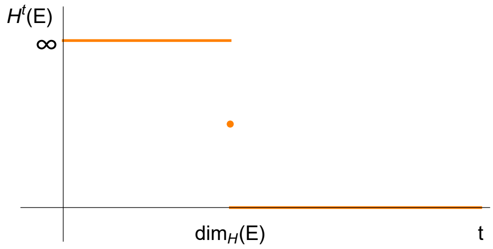


Figure 2.3: The definition of the Hausdorff dimension.

**Definition 2.15** *If $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a $C^1$ IFS, then the value of upper and lower Lyapunov exponents in $\underline{i} = (i_1, i_2, \ldots) \in \Sigma$ is defined respectively by*

$$\begin{aligned}\overline{\lambda}(\underline{i}) &= \limsup_{n\to\infty}\left(-\frac{1}{n}\log\left\|S'_{i_1 i_2 \ldots i_n}(\Pi(\sigma^n \underline{i}))\right\|\right),\\ \underline{\lambda}(\underline{i}) &= \liminf_{n\to\infty}\left(-\frac{1}{n}\log\left\|S'_{i_1 i_2 \ldots i_n}(\Pi(\sigma^n \underline{i}))\right\|\right).\end{aligned} \tag{2.14}$$

*When $\overline{\lambda}(\underline{i}) = \underline{\lambda}(\underline{i})$, then the common value is denoted by $\lambda(\underline{i})$ and we call it the Lyapunov exponent of the system $\mathcal{S}$ at the point $\underline{i} \in \Sigma$.*

**Definition 2.16** *If $\mathcal{S}$ is a $C^1$ IFS and $\mu$ is a Bernoulli measure on $\Sigma$, then we call the system $\mathcal{S}$ is $\mu$-conformal, if $\lambda(\underline{i})$ exists for $\mu$-almost every $\underline{i} \in \Sigma$.*

**Definition 2.17** *Suppose that $\nu$ is a Borel probability measure on $\mathbb{R}^d$, then the definition of upper and lower local dimension of $\nu$ at $x \in \mathbb{R}^d$ is respectively*

$$\begin{aligned}\overline{\dim}_\nu(x) &= \limsup_{r\to 0}\frac{\log \nu(B(x,r))}{\log r},\\ \underline{\dim}_\nu(x) &= \liminf_{r\to 0}\frac{\log \nu(B(x,r))}{\log r},\end{aligned} \tag{2.15}$$

*where $B(x,r)$ denotes the open ball of radius $r$ centered at $x$. If $\overline{\dim}_\nu(x) = \underline{\dim}_\nu(x)$, then the common value is denoted by $\dim_\nu(x)$ and we call it the local dimension of $\nu$ at $x$.*

**Definition 2.18** *We can also define the Hausdorff dimension of a Borel probability measure $\nu$ on $\mathbb{R}^d$ with*

$$\dim_H(\nu) = \inf\{\dim_H(E) \,:\, \nu(E) = 1\}. \tag{2.16}$$

**Theorem 2.19** *If $\nu$ is a Borel probability measure on $\mathbb{R}^d$ with compact support, then* $\dim_H(\nu) = \operatorname{ess\,sup}\{\underline{\dim}_\nu(x) : x \in \mathbb{R}^d\} = \inf\{\alpha : \nu(\{x : \underline{\dim}_\nu(x) \leq \alpha\}) = 1\}$

**Lemma 2.20** *If $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a self-similar IFS and $\mu$ is a $\sigma$ invariant, ergodic Borel probability measure on $\Sigma$, then $\mathcal{S}$ is $\mu$-conformal.*

**Proof:** Let $\phi_n : \Sigma \to \mathbb{R}$ such that for $\underline{i} = (i_1, i_2, \ldots) \in \Sigma \quad \phi_n(\underline{i}) = -\frac{1}{n}\log\left\|S'_{i_1 i_2 \ldots i_n}(\Pi(\sigma^n \underline{i}))\right\|$. Using $\mathcal{S}$ is self-similar and the chain rule, we get $\left\|S'_{i_1 i_2 \ldots i_n}(\Pi(\sigma^n \underline{i}))\right\| = \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}$. Thus

$$\phi_n(\underline{i}) = -\frac{1}{n}\sum_{k=1}^{n}\log(\lambda_{i_k}) = \frac{1}{n}\sum_{k=1}^{n}\psi(\sigma^{k-1}\underline{i}), \tag{2.17}$$

where $\psi(\underline{i}) = -\log(\lambda_{i_1})$. Using Birkhoff ergodic theorem, we get

$$\lim_{n\to\infty}\phi_n(\underline{i}) = \int_\Sigma \psi(\underline{i})\mathrm{d}\mu(\underline{i}) \text{ for } \mu\text{-almost every } \underline{i} \in \Sigma. \tag{2.18}$$

Thus $\lambda$ is a constant $\mu$-almost everywhere. So $\mathcal{S}$ is $\mu$-conformal. □

**Lemma 2.21** *If $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a self-similar IFS. The Lipschitz constant of $S_i$ is $\lambda_i$. Assume $\mu$ is a Bernoulli measure on $\Sigma$ for the probability vector $\mathbf{p} = (p_1, \ldots, p_m)$. Then $\mathcal{S}$ is $\mu$-conformal and*

$$\int_\Sigma \lambda(\underline{i})\mathrm{d}\mu(\underline{i}) = -\sum_{k=1}^{m} p_k \log(\lambda_k). \tag{2.19}$$

**Proof:** It is a well-known fact that if $\mu$ is a Bernoulli measure on $\Sigma$, then it is $\sigma$ invariant and ergodic, thus due to the previous lemma $\mathcal{S}$ is $\mu$-conformal. Using the argument in the previous proof, we can see that

$$\int_\Sigma \lambda(\underline{i})\mathrm{d}\mu(\underline{i}) = \int_\Sigma \psi(\underline{i})\mathrm{d}\mu(\underline{i}) = -\sum_{k=1}^{m} p_k \log(\lambda_k). \tag{2.20}$$

□

## 2.3 Dimension theorems without separation condition

**Definition 2.22** *We call $s$ the similarity dimension of the self-similar IFS defined in* (2.1)*, if $s$ is the solution of*

$$\sum_{i=1}^{m} r_i^s = 1. \tag{2.21}$$

**Theorem 2.23** *Let $\mathcal{S}$ be a self-similar IFS on $\mathbb{R}^d$, defined in* (2.1)*. Let $\Lambda$ be the attractor of $\mathcal{S}$ and $s$ is the similarity dimension of $\mathcal{S}$. Then*

$$\dim_H(\Lambda) \le s. \tag{2.22}$$

The proof can be found in [2].

**Theorem 2.24** *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a self-similar IFS on $\mathbb{R}^d$. The vector $\mathbf{r} = (r_1, \ldots, r_m)$ contains the contraction ratios of $\mathcal{S}$.The $\nu$ is the invariant measure of $\mathcal{S}$ with the probabilty vector $\mathbf{p} = (p_1, \ldots, p_m)$. Then we have*

$$\dim_H(\nu) \le \frac{-\sum_{i=1}^{m} p_i \log p_i}{-\sum_{i=1}^{m} p_i \log r_i} = \frac{h_{\mathbf{p}}}{\chi_{\mathbf{r}}^{\mathbf{p}}}. \tag{2.23}$$

The proof can be found in [2].

## 2.4 Dimension theorems with separation condition

In the special case, when the cylinder sets satify certain separation condition we are able to estimate the Hausdorff dimension of the attractor of such IFS. Moreover, in this case we can study the self-similar measure of the IFS.

**Definition 2.25** *The Strong Separation Property (SSP) holds for the self-similar IFS $\mathcal{S}$ defined in* (2.1)*, if*

$$\forall i \ne j \quad S_i(\Lambda) \cap S_j(\Lambda) = \emptyset. \tag{2.24}$$

**Definition 2.26** *The Open Set Condition (OSC) holds for the self-similar IFS $\mathcal{S}$ defined in* (2.1)*, if*

$$\exists V \subseteq \mathbb{R}^d \text{ open set } V \ne \emptyset \quad \forall i\, S_i(V) \subseteq V \text{ and } \forall i \ne j\, S_i(V) \cap S_j(V) = \emptyset. \tag{2.25}$$

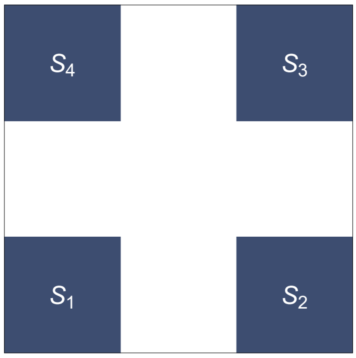


Figure 2.4: The IFS $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$ satisfies the OSC.

**Theorem 2.27 (Moran, Hutchinson)** *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a self-similar IFS on $\mathbb{R}^d$ for which the OSC holds. We denote the attractor of $\mathcal{S}$ with $\Lambda$ and the similarity dimension of $\mathcal{S}$ with $s$. Then,*

$$\dim_H(\Lambda) = s. \tag{2.26}$$

The proof can be found in [2].

**Theorem 2.28** *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a self-similar IFS on $\mathbb{R}^d$ for which the OSC holds. The vector $\mathbf{r} = (r_1, \ldots, r_m)$ contains the contraction ratios of $\mathcal{S}$.The $\nu$ is the invariant measure of $\mathcal{S}$ with the probabilty vector $\mathbf{p} = (p_1, \ldots, p_m)$. Then we have*

$$\dim_H(\nu) = \frac{-\sum_{i=1}^m p_i \log p_i}{-\sum_{i=1}^m p_i \log r_i} = \frac{h_{\mathbf{p}}}{\chi_{\mathbf{r}}^{\mathbf{p}}}. \tag{2.27}$$

The proof can be found in [2].

**Remark 2.29** *In the case, when we do not know any separation condition holds for the self-similar IFS $\mathcal{S}$ the values $s$ and $\frac{h_{\mathbf{p}}}{\chi_{\mathbf{r}}^{\mathbf{p}}}$ in Theorem 2.27 and 2.28 is only an upper bound on the Hausdorff dimension.*

# Chapter 3

# The systems $\mathcal{S}_{\alpha,\beta,\gamma}$

We study a family of self-similar iterated function systems (IFS) on the interval $[0, 1]$ such that there is total overlap and for which some separation condition holds.

Kamalutdinov and Tetenov in [3] studied similar iterated function systems, which called twofold Cantor set.

We follow their argument with similar statements in this chapter.

**Definition 3.1** *Let* $\alpha, \beta, \gamma \in (0, 1)$ *arbitrary. Then* $\mathcal{S}_{\alpha,\beta,\gamma}$ *is a system of contractive similarities such that*

$$\begin{gathered} \mathcal{S}_{\alpha,\beta,\gamma} = \{S_1, S_2, S_3\} \\ S_1(x) = \alpha x, \quad S_2(x) = \beta x, \quad S_3(x) = \gamma x + 1 - \gamma \end{gathered} \tag{3.1}$$


Figure 3.1: The first level cylinder sets of the IFS $\mathcal{S}_{\alpha,\beta,\gamma} = \{S_1, S_2, S_3\}$.

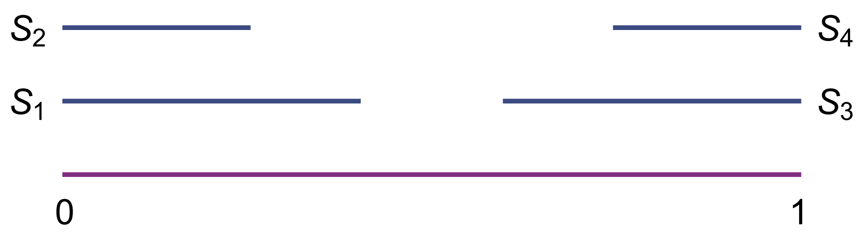


Figure 3.2: The first level cylinder sets of a system which belongs to a twofold Cantor set.

B. Bárány has already considered the Hausdorff dimension of the attractor of the system introduced in Definition 3.1. He showed this result for Lebesgue almost every $\alpha, \beta, \gamma(0, \frac{1}{2})$.

Let $K_{\alpha,\beta,\gamma}$ be the attractor of the system $\mathcal{S}_{\alpha,\beta,\gamma}$. Let $L_{\alpha,\beta,\gamma} = S_1(K_{\alpha,\beta,\gamma}) \cup S_2(K_{\alpha,\beta,\gamma})$ and $R_{\alpha,\beta,\gamma} = S_3(K_{\alpha,\beta,\gamma})$.
It is easy to see that $K_{\alpha,\beta,\gamma} = L_{\alpha,\beta,\gamma} \cup R_{\alpha,\beta,\gamma}$.

We denote the symbolic space of $\mathcal{S}_{\alpha,\beta,\gamma}$ with $\Sigma = \{1,2,3\}^{\mathbb{N}^+}$.
Let $\Pi_{\alpha,\beta,\gamma} : \Sigma \to K_{\alpha,\beta,\gamma}$ be the natural projection of the system $\mathcal{S}_{\alpha,\beta,\gamma}$.

First, we consider some obvious properties of the systems $\mathcal{S}_{\alpha,\beta,\gamma}$:

**Lemma 3.2** *If $\alpha, \beta, \gamma \in (0, \frac{1}{2})$, then:*

*(i)* $S_1 \circ S_2 = S_2 \circ S_1$,

*(ii) for all $i \in \{1,2\}$ and every $m, n \in \mathbb{N}$ with $m \neq n$, $S_i^m(R_{\alpha,\beta,\gamma}) \cap S_i^n(R_{\alpha,\beta,\gamma}) = \emptyset$,*

*(iii) for all $m, n \in \mathbb{N}$, $S_1^m S_2^n(K_{\alpha,\beta,\gamma}) \subseteq S_1^m(K_{\alpha,\beta,\gamma}) \cap S_2^n(K_{\alpha,\beta,\gamma})$,*

*(iv)* $K_{\alpha,\beta,\gamma} \backslash \{0\} = \bigcup_{n,m=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma})$ .

**Proof:**

(i) For every $x \in [0,1]$ $\quad S_1(S_2(x)) = \alpha(\beta x) = \beta(\alpha x) = S_2(S_1(x))$.

(ii) We prove only for $i = 1$, the case $i = 2$ is similar. Let $m, n \in \mathbb{N}$ $\quad m > n$. $R_{\alpha,\beta,\gamma} \subseteq (\frac{1}{2}, 1)$, thus $S_1^m(R_{\alpha,\beta,\gamma}) \subseteq (\frac{1}{2}\alpha^m, \alpha^m)$ and $S_1^n(R_{\alpha,\beta,\gamma}) \subseteq (\frac{1}{2}\alpha^n, \alpha^n)$. Since we can see that the right endpoint of one interval is smaller than the left endpoint of the other interval that is $\alpha^m = \alpha \cdot \alpha^{m-1} < \frac{1}{2}\alpha^{m-1} \leq \frac{1}{2}\alpha^n$.

(iii) Let $m, n \in \mathbb{N}$, then $S_1^m(K_{\alpha,\beta,\gamma}) \subseteq K_{\alpha,\beta,\gamma}$ and $S_2^n(K_{\alpha,\beta,\gamma}) \subseteq K_{\alpha,\beta,\gamma}$. So, we conclude that $S_2^n S_1^m(K_{\alpha,\beta,\gamma}) \subseteq S_2^n(K_{\alpha,\beta,\gamma})$ and $S_1^m S_2^n(K_{\alpha,\beta,\gamma}) \subseteq S_1^m(K_{\alpha,\beta,\gamma})$. Using commutativity, which is property (i) we get the statements.

(iv) Consider the natural projection $\Pi_{\alpha,\beta,\gamma}$ of $\mathcal{S}_{\alpha,\beta,\gamma}$. The map $\Pi_{\alpha,\beta,\gamma}$ is surjective. It is easy to see that

$$\Pi_{\alpha,\beta,\gamma}^{-1}\Big(\bigcup_{m,n=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma})\Big) = \{\underline{i} \in \Sigma : \exists k \quad i_k = 3\}. \tag{3.2}$$

For those $\underline{i} \in \Sigma$ such that there is no $k$ for which $i_k = 3$, then the image of $\underline{i}$ is 0.

$\square$

Using Theroem 2.23, we can conclude that the dimension of $K_{\alpha,\beta,\gamma}$ is less than $\frac{1}{2}$ if $\alpha, \beta, \gamma \in (0, \frac{1}{9})$.

**Definition 3.3** *We call the system $\mathcal{S}_{\alpha,\beta,\gamma}$ forward separated, if $\alpha, \beta, \gamma \in (0, \frac{1}{9})$ and*

$$\forall m, n \in \mathbb{N} \quad m, n > 0 \quad S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) = \emptyset. \tag{3.3}$$

We denote the disjoint union with $\sqcup$.

**Lemma 3.4** *The system $\mathcal{S}_{\alpha,\beta,\gamma}$ is forward separated if and only if*

$$K_{\alpha,\beta,\gamma} \backslash \{0\} = \bigsqcup_{n,m=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma}), \tag{3.4}$$

*where $\sqcup$ denotes the disjoint union.*

**Proof:** ($\Rightarrow$)First, we assume that $\mathcal{S}_{\alpha,\beta,\gamma}$ is forward separated. Let $(m_1, n_1) \neq (m_2, n_2)$, then

$$\begin{aligned} S_1^{m_1} S_2^{n_1}(R_{\alpha,\beta,\gamma}) &= S_1^{min\{m_1,m_2\}} S_2^{min\{n_1,n_2\}}(S_1^{k_1} S_2^{l_1}(R_{\alpha,\beta,\gamma})) \\ S_1^{m_2} S_2^{n_2}(R_{\alpha,\beta,\gamma}) &= S_1^{min\{m_1,m_2\}} S_2^{min\{n_1,n_2\}}(S_1^{k_2} S_2^{l_2}(R_{\alpha,\beta,\gamma})) \end{aligned} \tag{3.5}$$

hold. At least one of $k_1, k_2$ is zero and one of $l_1, l_2$ is zero. So if we use the forward separated property we get the statement.
($\Leftarrow$) Now, assume that $K_{\alpha,\beta,\gamma} \backslash \{0\} = \bigsqcup_{n,m=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma})$ holds. Then we can get the statement by using the conditon for the indeces $(m, 0)$ and $(0, n)$.

$\square$

**Lemma 3.5** *If the system $\mathcal{S}_{\alpha,\beta,\gamma}$ is forward separated, then for every*

$$m, n \in \mathbb{N} \quad S_1^m(K_{\alpha,\beta,\gamma}) \cap S_2^n(K_{\alpha,\beta,\gamma}) = S_1^m S_2^n(K_{\alpha,\beta,\gamma}). \tag{3.6}$$

**Proof:** Using the above results, we get

$$\begin{aligned} S_1^m(K_{\alpha,\beta,\gamma}) \cap S_2^n(K_{\alpha,\beta,\gamma}) \backslash \{0\} &= S_1^m(\bigcup_{k,l=0}^{\infty} S_1^k S_2^l(R_{\alpha,\beta,\gamma})) \cap S_2^n(\bigcup_{k,l=0}^{\infty} S_1^k S_2^l(R_{\alpha,\beta,\gamma})) = \\ = \bigcup_{k,l=0}^{\infty} S_1^{k+m} S_2^{l+n}(R_{\alpha,\beta,\gamma}) &= S_1^m S_2^n(\bigcup_{k,l=0}^{\infty} S_1^k S_2^l(R_{\alpha,\beta,\gamma})) = S_1^m S_2^n(K_{\alpha,\beta,\gamma}) \backslash \{0\}. \end{aligned}$$

In the first and last equation we use Lemma 3.2 (iv) point and in the second equation we use Lemma 3.4. □

# Chapter 4

# Existence of forward separated systems

Kamalutdinov and Tetenov proved that twofold Cantor sets exist in [3]. In this whole section we follow their arguement with similar statements.

Due to requirement of completeness we take over the same proof of this Theorem from [3].

**Theorem 4.1 (General Position Theorem [3])** *Let $(D, d_D), (L_1, d_{L_1}), (L_2, d_{L_2})$ be compact metric spaces and let $\varphi_i(\xi, x) : D \times L_i \to \mathbb{R}^n$ for $i \in \{1, 2\}$ be continuous functions. If these functions satisfies:*

*(i) The functions $\varphi_i$ are $\alpha$-Hölder with respect to $x$ which is*

$$\text{there exists } \alpha > 0 \text{ for all } i \in \{1,2\} \text{ there exists } C_i > 0 \text{ for all } \xi \in D \text{ for all } x, y \in L_i$$
$$\|\varphi_i(\xi, x) - \varphi_i(\xi, y))\| \le C_i d_{L_i}(x, y)^\alpha,$$

*where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^n$.*

*(ii) Let $\Phi : D \times L_1 \times L_2 \to \mathbb{R}^n \quad \Phi(\xi, x_1, x_2) = \varphi_1(\xi, x_1) - \varphi_2(\xi, x_2)$ such that*

$$\begin{gathered}\text{there exist } M > 0 \text{ for all } \xi, \xi' \in D \text{ for all } x_1 \in L_1 \text{ for all } x_2 \in L_2 \\ \|\Phi(\xi, x_1, x_2) - \Phi(\xi', x_1, x_2)\| \ge M d_D(\xi, \xi').\end{gathered} \tag{4.1}$$

*Then the set $\Delta = \{\xi \in D : \varphi_1(\xi, L_1) \cap \varphi_2(\xi, L_2) \neq \emptyset\}$ is a compact in $D$ and*

$$\dim_H(\Delta) \le \frac{\dim_H(L_1 \times L_2)}{\alpha}. \tag{4.2}$$

**Proof:** Let $\tilde{\Delta} = \{(\xi, x_1, x_2) \in D \times L_1 \times L_2 : \varphi_1(\xi, x_1) = \varphi_2(\xi, x_2)\} = \{(\xi, x_1, x_2) \in D \times L_1 \times L_2 : \Phi(\xi, x_1, x_2) = 0\}$ be the set of those parameters where $\varphi_1(\xi, L_1)$ and $\varphi_2(\xi, L_2)$ intersects. Then $\Delta = \operatorname{proj}_D\left(\tilde{\Delta}\right)$. Let $L = L_1 \times L_2$ and $\Delta_L = \operatorname{proj}_L\left(\tilde{\Delta}\right)$.

The map $\Phi$ is a continuous map and $\tilde{\Delta} = \Phi^{-1}(\{0\})$, thus $\tilde{\Delta}$ is closed. Then $\tilde{\Delta}$ is closed in a compact metric space, so it is compact. The projection is continuous, thus $\Delta$ is also compact.

The functions $\text{proj}_D : \tilde{\Delta} \to \Delta$ and $\text{proj}_L : \tilde{\Delta} \to \Delta_L$ are surjective. Moreover, $\text{proj}_L$ is also injective, because if exist $(\xi, x_1, x_2) \neq (\xi', x'_1, x'_2) \in \tilde{\Delta}$ such that $\text{proj}_L(\xi, x_1, x_2) = \text{proj}_L(\xi', x'_1, x'_2)$, then $x'_1 = x_1$, $x'_2 = x_2$ and $\xi' \neq \xi$. By the definition of $\tilde{\Delta}$ $\Phi(\xi, x_1, x_2) = \Phi(\xi', x_1, x_2) = 0$ and this is contradicts with the second assumption. So $\text{proj}_L$ is injective, thus it is invertible.

Let $g = \text{proj}_D \circ {\text{proj}_L}^{-1} : \Delta_L \to \Delta$. This is surjective. Let $g(x_1, x_2) = \xi$ and $g(x'_1, x'_2) = \xi'$. Then $\Phi(\xi, x_1, x_2) = 0$ and $\Phi(\xi', x'_1, x'_2) = 0$.

$$M \cdot d_D(\xi, \xi') \leq \|\Phi(\xi, x_1, x_2) - \Phi(\xi', x_1, x_2)\| = \|\Phi(\xi', x'_1, x'_2) - \Phi(\xi', x_1, x_2)\| \leq$$
$$\|\varphi_1(\xi', x'_1) - \varphi_1(\xi', x_1)\| + \|\varphi_2(\xi', x'_2) - \varphi_2(\xi', x_2)\| \leq C\left(d_{L_1}(x_1, x'_1)^\alpha + d_{L_2}(x_2, x'_2)^\alpha\right),$$

where $C = \max\{C_1, C_2\}$. In the first inequation we use the (ii) assumption, the next inequation is triangle inequality and the last inequation is the Hölder continuity in (i). □

Also for the completeness we take over the same proof of thefollwing theorem from [3].

**Lemma 4.2 (Displacement theorem)** *Let $\mathcal{S} = \{S_1, \dots, S_m\}$ and $\tilde{\mathcal{S}} = \{\tilde{S}_1, \dots, \tilde{S}_m\}$ be two iterated function systems on $\mathbb{R}^n$. We denote the natural projection of $\mathcal{S}$ with $\Pi : \Sigma \to \mathbb{R}^n$ and the natural projection of $\tilde{\mathcal{S}}$ with $\tilde{\Pi} : \Sigma \to \mathbb{R}^n$, where $\Sigma = \{1, \dots, m\}^{\mathbb{N}^+}$ is the symbolic space. Let $V \subseteq \mathbb{R}^n$ be a compact set such that for every $i \in \{1, \dots, m\}$, $S_i(V) \subseteq V$ and $\tilde{S}_i(V) \subseteq V$. Then*

$$\forall \underline{i} = (i_1, i_2, \dots) \in \Sigma \quad \left\|\Pi(\underline{i}) - \tilde{\Pi}(\underline{i})\right\| \leq \frac{\delta}{1-p}, \tag{4.3}$$

*where*

$$\begin{gathered} \delta = \max\{\left\|S_i(x) - \tilde{S}_i(x)\right\| : i \in \{1, \dots, m\}, \quad x \in V\} \text{ and} \\ p = \max_{1 \leq i \leq m}\{\max\{\text{Lip}(S_i), \text{Lip}(\tilde{S}_i)\}\}. \end{gathered} \tag{4.4}$$

**Proof:** Let $\underline{i} \in \Sigma$ arbitrary. We can conclude

$$\begin{aligned} \left\|\Pi(\underline{i}) - \tilde{\Pi}(\underline{i})\right\| &= \left\|S_{i_1}(\Pi(\sigma\underline{i})) - \tilde{S}_{i_1}(\tilde{\Pi}(\sigma\underline{i}))\right\| \leq \\ \leq \left\|S_{i_1}(\Pi(\sigma\underline{i})) - S_{i_1}(\tilde{\Pi}(\sigma\underline{i}))\right\| &+ \left\|S_{i_1}(\tilde{\Pi}(\sigma\underline{i})) - \tilde{S}_{i_1}(\tilde{\Pi}(\sigma\underline{i}))\right\| \leq \\ &\leq p\left\|\Pi(\sigma\underline{i}) - \tilde{\Pi}(\sigma\underline{i})\right\| + \delta. \end{aligned} \tag{4.5}$$

Using the above inequation $n$ times, then we get

$$\left\|\Pi(\underline{i}) - \tilde{\Pi}(\underline{i})\right\| \leq p^n \left\|\Pi(\sigma^n\underline{i}) - \tilde{\Pi}(\sigma^n\underline{i})\right\| + \delta \cdot \sum_{i=0}^{n-1} p^i. \tag{4.6}$$

If $n \to \infty$, then we get $\left\|\Pi(\underline{i}) - \tilde{\Pi}(\underline{i})\right\| \leq \frac{\delta}{1-p}$, because $V$ is compact. □

**Notation 4.3** *Let $\Sigma = \{1, \dots, m\}^{\mathbb{N}^+}$ and $a \in (0,1)$. We can construct a metric space $(\Sigma, \rho_a)$ with metric $\rho_a$. We define $\forall \underline{i}, \underline{j} \in \Sigma \quad s(\underline{i}, \underline{j}) = \min\{k-1 : i_k \neq j_k\}$, then let $\rho_a(\underline{i}, \underline{j}) = a^{s(\underline{i},\underline{j})}$.*

It is a well known fact that the metric space $(\Sigma, \rho_a)$ is compact.

**Lemma 4.4** *Let $(\Sigma, \rho_a)$ be a metric space as above. Then*

$$\dim_H(\Sigma) = -\frac{\log(m)}{\log(a)}. \tag{4.7}$$

**Proof:** Let $\mathcal{G} = \{G_1, \dots, G_m\}$ be a set of $\Sigma \to \Sigma$ functions.For all $k \in \{1, \dots, m\}$ and for all $\underline{i} \in \Sigma$

$$G_k(\underline{i}) = (k, i_1, i_2, \dots). \tag{4.8}$$

Then every $G_k$ is a contractive function with Lipschitz constant $a$, so $\mathcal{G}$ is an IFS. The attractor of $\mathcal{G}$ is $\Sigma$. $\mathcal{G}$ satisfies the strong separation property, so the Hausdorff-dimension of its attractor is equal to the similarity dimension of the IFS. It is also a well known fact. The proof can be found in [2]. Thus $\dim_H(\Sigma) = -\frac{\log(m)}{\log(a)}$. □

So using this fact for the symbolic space of the system $\mathcal{S}_{\alpha,\beta,\gamma}$. Remind that $\Sigma = \{1,2,3\}^{\mathbb{N}^+}$, then

$$\dim_H(\Sigma) < \frac{1}{2} \text{ in the metric } \rho_a \iff a \in \left(0, \frac{1}{9}\right). \tag{4.9}$$

**Lemma 4.5** *Let $\alpha, \beta, \gamma < a$ and $a \in \left(0, \frac{1}{9}\right)$. Then $\Pi_{\alpha,\beta,\gamma}$ natural projection of the system $\mathcal{S}_{\alpha,\beta,\gamma}$ is 1-Lipschitz with respect to the metric space $(\Sigma, \rho_a)$.*

**Proof:** Let $\underline{i}, \underline{j} \in \Sigma$ with $s(\underline{i}, \underline{j}) = k$. Then $\rho_a(\underline{i}, \underline{j}) = a^k$ and $i_1 = j_1, \dots, i_k = j_k$, thus $\Pi_{\alpha,\beta,\gamma}(\underline{i}), \Pi_{\alpha,\beta,\gamma}(\underline{j}) \in S_{i_1 \dots i_k}(K_{\alpha,\beta,\gamma})$. The diameter of $S_{i_1 \dots i_k}(K_{\alpha,\beta,\gamma})$ is $\mathrm{Lip}(S_{i_1}) \cdot \dots \cdot \mathrm{Lip}(S_{i_k})$, which is strictly smaller than $a^k$. So

$$\left|\Pi_{\alpha,\beta,\gamma}(\underline{i}) - \Pi_{\alpha,\beta,\gamma}(\underline{j})\right| < a^k = \rho_a(\underline{i}, \underline{j}). \tag{4.10}$$

□

**Lemma 4.6** *Let $m, n \in \mathbb{N}^+$. $\alpha, \beta, \gamma \in \left(0, \frac{1}{9}\right)$ and consider the system $\mathcal{S}_{\alpha,\beta,\gamma}$. If $S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \neq \emptyset$, then $\frac{8}{9} \leq \frac{\alpha^m}{\beta^n} \leq \frac{9}{8}$.*

**Proof:** If $\alpha, \beta, \gamma \in \left(0, \frac{1}{9}\right)$, then $R_{\alpha,\beta,\gamma} \subseteq [\frac{8}{9}, 1]$. Thus $S_1^m(R_{\alpha,\beta,\gamma}) \subseteq [\frac{8}{9}\alpha^m, \alpha^m]$ and $S_2^n(R_{\alpha,\beta,\gamma}) \subseteq [\frac{8}{9}\beta^n, \beta^n]$. The intersection can not happen if $\alpha^m < \frac{8}{9}\beta^n$ or $\beta^n < \frac{8}{9}\alpha^m$. Thus we do not have intersection if $\frac{\alpha^m}{\beta^n} < \frac{8}{9}$ or $\frac{\alpha^m}{\beta^n} > \frac{9}{8}$. So $\frac{8}{9} \leq \frac{\alpha^m}{\beta^n} \leq \frac{9}{8}$. □

**Lemma 4.7** *Let $m, n \in \mathbb{N}^+$ and $\beta, \gamma \in (0, \frac{1}{9})$ be fixed. We denote*

$$D_{m,n}(\beta, \gamma) = \left\{\alpha \in \left(0, \frac{1}{9}\right) : \frac{8}{9} \leq \frac{\alpha^m}{\beta^n} \leq \frac{9}{8}\right\}. \tag{4.11}$$

*Let $\varphi_i : D_{m,n}(\beta, \gamma) \times \Sigma \to \mathbb{R}$ for $i = 1, 2$. We define*

$$\begin{aligned}
\forall \alpha \in D_{m,n}(\beta, \gamma) \quad \forall \underline{i} \in \Sigma \quad \varphi_1(\alpha, \underline{i}) = \Pi_{\alpha,\beta,\gamma}((1)^m * (3) * \underline{i}) = S_1^m S_3(\Pi_{\alpha,\beta,\gamma}(\underline{i})), \\
\forall \alpha \in D_{m,n}(\beta, \gamma) \quad \forall \underline{i} \in \Sigma \quad \varphi_2(\alpha, \underline{i}) = \Pi_{\alpha,\beta,\gamma}((2)^n * (3) * \underline{i}) = S_2^n S_3(\Pi_{\alpha,\beta,\gamma}(\underline{i})),
\end{aligned}$$

*where $\Pi_{\alpha,\beta,\gamma}$ is the natural projection of $\mathcal{S}_{\alpha,\beta,\gamma}$. Then for every $\alpha, \alpha' \in D_{m,n}(\beta, \gamma)$ and for every $\underline{i}, \underline{j} \in \Sigma$*

$$\left|\varphi_1(\alpha, \underline{i}) - \varphi_2(\alpha, \underline{j}) - \varphi_1(\alpha', \underline{i}) + \varphi_2(\alpha', \underline{j})\right| \geq M \left|\alpha - \alpha'\right|, \tag{4.12}$$

*where $M(m, n, \beta, \gamma) > 0$ constant.*

**Proof:** Let $\alpha, \alpha' \in D_{m,n}(\beta, \gamma)$ and $\underline{i}, \underline{j} \in \Sigma$ be arbitrary. We introduce the notation $\mathcal{S} = \mathcal{S}_{\alpha,\beta,\gamma} = \{S_1, S_2, S_3\}$, $\mathcal{S}' = \mathcal{S}_{\alpha',\beta,\gamma} = \{S_1', S_2', S_3'\}$, let $\Pi = \Pi_{\alpha,\beta,\gamma}$ and $\Pi' = \Pi_{\alpha',\beta,\gamma}$. Then $S_2' = S_2$ and $S_3' = S_3$.

Let $\alpha < \alpha'$ and $\delta = |\alpha' - \alpha|$, then using Lagrange mean value theorem

$$m\alpha^{m-1} \leq \frac{\alpha'^m - \alpha^m}{\alpha' - \alpha} = \frac{|\alpha'^m - \alpha^m|}{\delta} \leq m\alpha'^{m-1}. \tag{4.13}$$

We defined $\delta = |\alpha' - \alpha|$ and using displacement Theorem 4.2 for $\mathcal{S}$ and $\mathcal{S}'$, then we get

$$\text{for every } \underline{i} \in \Sigma \quad |\Pi(\underline{i}) - \Pi'(\underline{i})| \leq \frac{9}{8}\delta. \tag{4.14}$$

Consider the difference that we have to estimate

$$\begin{aligned}
&\varphi_1(\alpha, \underline{i}) - \varphi_1(\alpha', \underline{i}) + \varphi_2(\alpha', \underline{j}) - \varphi_2(\alpha, \underline{j}) = \\
&= S_1^m S_3(\Pi(\underline{i})) - S_1'^m S_3'(\Pi'(\underline{i})) + S_2'^n S_3'(\Pi'(\underline{j})) - S_2^n S_3(\Pi(\underline{j})) = \\
&= S_1^m S_3(\Pi(\underline{i})) - S_1'^m S_3(\Pi'(\underline{i})) + S_2^n S_3(\Pi'(\underline{j})) - S_2^n S_3(\Pi(\underline{j})) = \\
&= \underbrace{S_1^m S_3(\Pi(\underline{i})) - S_1^m S_3(\Pi'(\underline{i}))}_{A} + \underbrace{S_1^m S_3(\Pi'(\underline{i})) - S_1'^m S_3(\Pi'(\underline{i}))}_{B} + \underbrace{S_2^n S_3(\Pi'(\underline{j})) - S_2^n S_3(\Pi(\underline{j}))}_{C}.
\end{aligned}$$

We will use the estimate

$$|A + B + C| \geq |B| - |A + C| \geq |B| - |A| - |C|, \tag{4.15}$$

where first we use the reversed triangle inewquality and second the triangle inequality.

Consider $|A|$ part of the above calculation

$$|A| = |S_1^m S_3(\Pi(\underline{i})) - S_1^m S_3(\Pi'(\underline{i}))| = \alpha^m \gamma \, |\Pi(\underline{i}) - \Pi'(\underline{i})| \leq \frac{9}{8}\alpha^m \gamma \delta,$$

where in the inequation we use (4.14).

The next part is

$$|B| = |S_1^m S_3(\Pi'(\underline{i})) - S_1'^m S_3(\Pi'(\underline{i}))| = |\alpha^m - \alpha'^m| \, |S_3(\Pi'(\underline{i}))| \geq \frac{8}{9} m\alpha^{m-1}\delta,$$

where in the inequation we use (4.13).

The last part is

$$|C| = \left|S_2^n S_3(\Pi'(\underline{j})) - S_2^n S_3(\Pi(\underline{j}))\right| = \beta^n \gamma \left|\Pi(\underline{j}) - \Pi'(\underline{j})\right| \leq \frac{9}{8}\beta^n \gamma \delta,$$

where in the inequation we use (4.14).

Now estimate

$$|B| - |A| \geq \left(\frac{8m}{9\alpha} - \frac{9}{8}\gamma\right)\alpha^m\delta \geq \left(8 - \frac{9}{8}\right)\alpha^m\delta \geq \left(8 - \frac{9}{8}\right)\frac{8}{9}\beta^n\delta > 6\beta^n\delta, \tag{4.16}$$

where in the second inequation we use $\gamma < 1$, $m \geq 1$, $\alpha < \frac{1}{9}$.

The following

$$|C| \leq \frac{9}{8}\gamma\beta^n\delta < \beta^n\delta \tag{4.17}$$

is true, beacuse $\gamma < \frac{1}{9}$. Thus

$$\left|\varphi_1(\alpha, \underline{i}) - \varphi_2(\alpha, \underline{j}) - \varphi_1(\alpha', \underline{i}) + \varphi_2(\alpha', \underline{j})\right| \geq 5\beta^n \left|\alpha' - \alpha\right|, \tag{4.18}$$

so $M = 5\beta^n$. $\square$

**Lemma 4.8** *Let $m, n \in \mathbb{N}^+$ and $\beta, \gamma \in (0, \frac{1}{9})$. Then the set*

$$\Delta_{m,n}(\beta, \gamma) = \left\{\alpha \in \left(0, \frac{1}{9}\right) : S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \neq \emptyset\right\} \tag{4.19}$$

*is closed in $(0, \frac{1}{9})$ and $\mathcal{L}(\Delta_{m,n}(\beta, \gamma)) = 0$, where $\mathcal{L}$ means the Lebesgue measure on $\mathbb{R}$.*

**Proof:** Let $\varepsilon > 0$ be such that $\frac{1}{9} - \varepsilon > \beta, \gamma$. Then $E_{m,n}(\beta, \gamma) = D_{m,n}(\beta, \gamma) \cap [\varepsilon, \frac{1}{9} - \varepsilon]$ is a closed interval in $\mathbb{R}$, so it is compact. We consider the compact metric space $(\Sigma, \rho_a)$, where $\Sigma = \{1, 2, 3\}^{\mathbb{N}^+}$ and $a = \frac{1}{9} - \varepsilon$.

Let $\varphi_i : E_{m,n}(\beta, \gamma) \times \Sigma \to \mathbb{R}$ for $i = 1, 2$. We define

$$\begin{aligned} \varphi_1(\alpha, \underline{i}) &= \Pi_{\alpha,\beta,\gamma}((1)^m * (3) * \underline{i}) = S_1^m S_3(\Pi_{\alpha,\beta,\gamma}(\underline{i})), \\ \varphi_2(\alpha, \underline{i}) &= \Pi_{\alpha,\beta,\gamma}((2)^n * (3) * \underline{i}) = S_2^n S_3(\Pi_{\alpha,\beta,\gamma}(\underline{i})). \end{aligned} \tag{4.20}$$

Let

$$\Xi^{\varepsilon}_{m,n}(\beta, \gamma) = \Delta_{m,n}(\beta, \gamma) \cap \left[\varepsilon, \frac{1}{9} - \varepsilon\right].$$

For an $\alpha$ the $S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \neq \emptyset$ holds if and only if there exist $\underline{i}, \underline{j} \in \Sigma$ such that $\varphi_1(\alpha, \underline{i}) = \varphi_2(\alpha, \underline{j})$, thus

$$\Xi^{\varepsilon}_{m,n}(\beta, \gamma) = \{\alpha \in E_{m,n}(\beta, \gamma) : \varphi_1(\alpha, \Sigma) \cap \varphi_2(\alpha, \Sigma) \neq \emptyset\}. \tag{4.21}$$

Using Lemma 4.5 one can see that $\varphi_i$ is Hölder continuous with respect to the second variable for $i = 1, 2$. Applying Lemma 4.7, we get that the conditions of the General Position Theorem 4.1 holds. Using General Position Theorem 4.1, then get

$$\dim_H(\Xi^{\varepsilon}_{m,n}(\beta,\gamma)) \le \dim_H(\Sigma \times \Sigma) \le 2\dim_H(\Sigma) < 1, \tag{4.22}$$

the last inequation is true because of the equation (4.9). So $\mathcal{L}(\Xi^{\varepsilon}_{m,n}(\beta,\gamma)) = 0$. Moreover,

$$\Delta_{m,n}(\beta,\gamma) = \bigcup_{k=1}^{\infty} \Xi^{1/k}_{m,n}(\beta,\gamma), \tag{4.23}$$

thus the continuity of measure yields that $\mathcal{L}(\Delta_{m,n}(\beta,\gamma)) = 0$.

General Position Theorem 4.1 implies that $\Xi^{\varepsilon}_{m,n}(\beta,\gamma)$ is closed for every $\varepsilon > 0$, so $\Delta_{m,n}(\beta,\gamma)$ is also closed. □

**Lemma 4.9** *Let $m, n \in \mathbb{N}^+$ be arbitrary. Then the set*

$$\tilde{\Delta}_{m,n} = \left\{ (\alpha,\beta,\gamma) \in \left(0, \frac{1}{9}\right)^3 : S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \ne \emptyset \right\} \tag{4.24}$$

*is closed in $(0,\frac{1}{9})^3$ and $\mathcal{L}^3(\tilde{\Delta}_{m,n}) = 0$, that is its Lebesgue measure is zero in $\mathbb{R}^3$.*

**Proof:** Let $I = (0,\frac{1}{9})^3$ and $\Psi : I \times \Sigma^2 \to \mathbb{R}$ be such that

$$\begin{aligned} \Psi(\alpha,\beta,\gamma,\underline{i},\underline{j}) &= \Pi_{\alpha,\beta,\gamma}((1)^m * (3) * \underline{i}) - \Pi_{\alpha,\beta,\gamma}((2)^n * (3) * \underline{j}) = \\ &= S_1^m S_3(\Pi_{\alpha,\beta,\gamma}(\underline{i})) - S_2^n S_3(\Pi_{\alpha,\beta,\gamma}(\underline{j})). \end{aligned} \tag{4.25}$$

Then $\Psi$ is a continuous function and $\tilde{\Delta}_{m,n} = \mathrm{proj}_I(\Psi^{-1}(\{0\}))$. Because $\Psi$ is continuous $\Psi^{-1}(\{0\})$ is closed, and because $\Sigma$ is compact $\tilde{\Delta}_{m,n}$ is closed. This is because if $x \in I$ is an accumulation point of $\mathrm{proj}_I(\Psi^{-1}(\{0\}))$, then there exists an $(x,y) \in I \times \Sigma^2$ such that is an accumulation point of $\Psi^{-1}(\{0\})$. Consider the integral

$$\begin{aligned}\mathcal{L}^3(\tilde{\Delta}_{m,n}) &= \iiint\limits_{(0,\frac{1}{9})^3} \mathbb{1}_{\tilde{\Delta}_{m,n}}(\alpha,\beta,\gamma)\,\mathrm{d}\mathcal{L}^3(\alpha,\beta,\gamma) = \\ &= \iint\limits_{(0,\frac{1}{9})^2} \int\limits_{(0,\frac{1}{9})} \mathbb{1}_{\tilde{\Delta}_{m,n}}(\alpha,\beta,\gamma)\,\mathrm{d}\mathcal{L}(\alpha)\mathrm{d}\mathcal{L}^2(\beta,\gamma) = \\ &= \iint\limits_{(0,\frac{1}{9})^2} \int\limits_{(0,\frac{1}{9})} \mathbb{1}_{\Delta_{m,n}(\beta,\gamma)}(\alpha)\,\mathrm{d}\mathcal{L}(\alpha)\mathrm{d}\mathcal{L}^2(\beta,\gamma) = \\ = \iint\limits_{(0,\frac{1}{9})^2} \mathcal{L}(\Delta_{m,n}(\beta,\gamma))\mathrm{d}\mathcal{L}^2(\beta,\gamma) &= \iint\limits_{(0,\frac{1}{9})^2} 0\mathrm{d}\mathcal{L}^2(\beta,\gamma) = 0,\end{aligned} \tag{4.26}$$

where we use Fubini's theorem in the second equality and the fourth equality we use Lemma 4.8. $\square$

**Theorem 4.10** *Let $J = \left(0, \frac{1}{9}\right)^3$. We define*

$$\Omega = \{(\alpha,\beta,\gamma) \in J : \mathcal{S}_{\alpha,\beta,\gamma} \textit{ is a forward separated system}\}. \tag{4.27}$$

*Then $\mathcal{L}^3(J/\Omega) = 0$ and $J/\Omega$ is uncountable and dense in $J$.*

**Proof:** The set

$$J/\Omega = \bigcup_{m,n=1}^{\infty} \tilde{\Delta}_{m,n}\ , \text{ thus } \mathcal{L}^3(J/\Omega) \leq \sum_{m,n=1}^{\infty} \mathcal{L}^3(\tilde{\Delta}_{m,n}) = 0, \tag{4.28}$$

where we use Lemma 4.9 in the last equality. Thus $\mathcal{L}^3(J/\Omega) = 0$.

If $\alpha^m = \beta^n$ for $m, n \in \mathbb{N}^+$, then $\mathcal{S}_{\alpha,\beta,\gamma}$ is not a forward separated system, so

$$\tilde{\Delta} = \left\{(\alpha,\beta,\gamma) \in J : \frac{\log\alpha}{\log\beta} \in \mathbb{Q}\right\} \subseteq J/\Omega. \tag{4.29}$$

Then for every $z \in (0, \frac{1}{9})$ the set

$$\{(\alpha,\beta,\gamma) \in \tilde{\Delta} : \gamma = z\} = \bigcup_{q\in\mathbb{Q}^+} \left\{(\alpha, f_q(\alpha), z) : \alpha \in \left(0, \frac{1}{9}\right), \quad f_q(x) = x^q\right\} \tag{4.30}$$

is union of smooth curves. From this one can easily see that $\tilde{\Delta}$ is dense and uncountable. This implies $J/\Omega$ is also dense and uncountable. $\square$

# Chapter 5

# The tools of calculating Hausdorff-dimension of the self-similar measure

We would like to calculate the Hausdorff dimension of the self-similar measure of the forward separated system $\mathcal{S}_{\alpha,\beta,\gamma}$ and for this we require the following statements.

## 5.1 Conditional expectation

**Definition 5.1** *Let $\mathcal{G} \subseteq \mathcal{B}$ be an arbitrary $\sigma$-algebra. Let $\varphi \in L^1(Z, \mathcal{B}, \mu)$, then the function $\psi \in L^1(Z, \mathcal{B}, \mu)$ is the conditional expectation of $\varphi$ with respect to the $\sigma$-algebra $\mathcal{G}$, if*

*(i) $\psi$ is $\mathcal{G}$-measurable,*

*(ii) for every $G \in \mathcal{G}$*

$$\int\limits_Z \varphi(x)\mathbb{1}_G(x)\,\mathrm{d}\mu(x) = \int\limits_Z \psi(x)\mathbb{1}_G(x)\,\mathrm{d}\mu(x). \tag{5.1}$$

**Theorem 5.2** *Let $\mathcal{G} \subseteq \mathcal{B}$ be an arbitrary $\sigma$-algebra. If $\psi$ and $\tilde{\psi}$ are conditional expectations of the function $\varphi \in L^1(Z, \mathcal{B}, \mu)$ with respect to $\mathcal{G}$, then $\psi(z) = \tilde{\psi}(z)$ for $\mu$-almost every $z \in Z$.*

We denote the conditional expectation of $\varphi \in L^1(Z, \mathcal{B}, \mu)$ with respect to $\mathcal{G}$ with $\mathbb{E}_\mu(\varphi|\mathcal{G})$.

## 5.2 Conditional measure

The proof of the statements that contained in this section can be found in [4].

Let $Z$ be a compact metric space. We consider the probability space $(Z, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of $Z$ and $\mu$ is a probability measure on $Z$.

Moreover, let $\mathcal{F}$ be a $\sigma$-algebra such that there exists some $E_1, E_2, \dots \in \mathcal{B}$ for which

$$\mathcal{F} = \bigvee_{i=1}^{\infty} \{E_i, Z/E_i\}, \tag{5.2}$$

where $\vee$ denotes the generated $\sigma$-algebra. Indeed, if $\mathcal{A}_i \subseteq \mathcal{B}$ is a $\sigma$-algebra for all $i = 1, 2, \dots$, then $\bigvee\limits_{i=1}^{\infty} \mathcal{A}_i$ is the generated $\sigma$-algebra by $\bigcup\limits_{i=1}^{\infty} \mathcal{A}_i$.

**Definition 5.3** *The $\mathcal{P} \subseteq \mathcal{B}$ is a partition of $Z$, if for every $P_1 \neq P_2 \in \mathcal{P}$ $\quad P_1 \cap P_2 = \emptyset$ and $\bigcup\limits_{P \in \mathcal{P}} P = Z$.*

Let $\mathcal{P}$ be a partition of $Z$. Then for $z \in Z$ the set $\mathcal{P}(z)$ denotes those $\mathcal{P}(z) \in \mathcal{P}$ such that $z \in \mathcal{P}(z)$.

For every $n = 1, 2, \dots$ let $\mathcal{P}_n$ be a partition of $Z$ such that

$$\sigma(\mathcal{P}_n) = \bigvee_{i=1}^{n} \{E_i, Z/E_i\}, \tag{5.3}$$

where $\sigma(\mathcal{A})$ denotes the generated sigma algebra by $\mathcal{A}$.

**Definition 5.4** *The set $\{\mu_z\}_{z \in Z}$ of Borel probability measures on $Z$ is a system of conditional measures of $\mu$ with respect to the $\sigma$-algebra $\mathcal{F}$, if*

*(i) for every $E \in \mathcal{F}$, $z \in E$ $\quad \mu_z(E) = 1$ holds for $\mu$-almost every $z \in Z$ ,*

*(ii) for every bounded measurable function $\varphi : Z \to \mathbb{R}$ the function $z \mapsto \int\limits_Z \varphi \, \mathrm{d}\mu_z$ is $\mathcal{F}$-measurable and*

$$\int\limits_Z \varphi(x) \, \mathrm{d}\mu(x) = \int\limits_Z \int\limits_Z \varphi(x) \, \mathrm{d}\mu_z(x) \, \mathrm{d}\mu(z). \tag{5.4}$$

**Theorem 5.5** *If $\{\mu_z\}_{z \in Z}$ and $\{\nu_z\}_{z \in Z}$ are two systems of condtional measures of $\mu$ with respect to $\mathcal{F}$, then $\mu_z = \nu_z$ for $\mu$-almost every $z \in Z$.*

The proof is in [4].

**Theorem 5.6** *The limit of the measures*

$$\mu_z^{\mathcal{F}} = \lim_{n\to\infty} \frac{\mu|_{\mathcal{P}_n(z)}}{\mu(\mathcal{P}_n(z))} \textit{ exists for } \mu\textit{-almost every } z \in Z, \tag{5.5}$$

*where the limit is meant in the weak-star topology.*

*Moreover, the set* $\{\mu_z^{\mathcal{F}}\}_{z\in Z}$ *is a system of conditional measures of* $\mu$ *with respect to the* $\sigma$*-algebra* $\mathcal{F}$*.*

The proof can be found in [4].

**Theorem 5.7** *Let* $\varphi : Z \to \mathbb{R}$ *is bounded and measurable, then the function*

$$\Phi : Z \to \mathbb{R} \textit{ for which}$$
$$\Phi(z) = \int\limits_Z \varphi(x)\mathrm{d}\mu_z(x) \textit{ for } \mu\textit{-almost every } z \in Z \tag{5.6}$$

*is the conditional expectation of* $\varphi$ *with respect to* $\mathcal{F}$*, thus* $\mathbb{E}_\mu(\varphi|\mathcal{F}) = \Phi$*.*

The proof is in [4].

# Chapter 6

# Dimension conservation

In this chapter we follow the paper [1], but we reach different formulas with similar argument.

In this chapter we study the relation between the dimension of a measure and the dimension of its projected measure.

If $k \in \mathbb{Z}^+$, then we will use the notation $Q^k = [0,1]^k \subset \mathbb{R}^k$. In this chapter we fix the dimension, so let $Q = Q^k = Q^{k_1+k_2} = Q' \times Q''$ for $k_1, k_2 \in \mathbb{Z}^+$, where $Q' = Q^{k_1}$ and $Q'' = Q^{k_2}$. Let $P' : Q \to Q'$ be the projection from $Q$ to $Q'$ and let $P'' : Q \to Q''$ be the projection from $Q$ to $Q''$. For $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ the images are $P'(\mathbf{x}, \mathbf{y}) = \mathbf{x} \in \mathbb{R}^{k_1}$ and $P''(\mathbf{x}, \mathbf{y}) = \mathbf{y} \in \mathbb{R}^{k_2}$.

Further, let $m \in \mathbb{Z}^+$ and $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a self-similar IFS on $\mathbb{R}^k$ such that for every $i$ the function is $F_i(\mathbf{x}) = \alpha_i \mathbf{x} + \mathbf{t}_i$, where $\alpha_i \in (0,1)$ and $\mathbf{t}_i \in \mathbb{R}^k$. For every $i$ the image $F_i(Q) \subseteq Q$. We suppose that

$$\text{for every } i \neq j \quad F_i(Q) \cap F_j(Q) = \emptyset. \tag{6.1}$$

Let $\Sigma = \{1, \ldots, m\}^{\mathbb{Z}^+}$ be the symbolic space and $A = \{1, \ldots, m\}$ be the set of the characters. Let $\Lambda$ be the attractor of $\mathcal{F}$, then $\Lambda \subseteq Q$. Moreover, we denote the natural projection of $\mathcal{F}$ with $\Pi$. Let $\mu = (p_1, \ldots, p_m)^{\mathbb{Z}^+}$ be a product measure on $\Sigma$ and $\nu = \Pi_* \mu$.

Let $\mathcal{S} = \{S_1, \ldots, S_m\}$, where for every $i$ $S_i : Q' \to Q'$ such that $S_i(\mathbf{x}) = P'(F_i(P'^{-1}(\mathbf{x})))$. The $P'^{-1}$ is not a function, it is the inverse image of the set $\{\mathbf{x}\}$. Then $\mathcal{S}$ is a self-similar IFS on $\mathbb{R}^{k_1}$ with the same contraction ratios as $\mathcal{F}$.

## 6.1 Natural partition of the attractor

We introduce a natural partition of the attractor $\Lambda$ which is defined by the IFS $\mathcal{F}$.

Let $\underline{i} = (i_1, \ldots, i_k) \in A^k$ for some $k \in \mathbb{Z}^+$, then

$$Q_{\underline{i}} = Q_{i_1,\ldots,i_k} = F_{i_1,\ldots,i_k}(Q) = (F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_k})(Q). \tag{6.2}$$

First we consider some elementary properties of the above notation:

- for all $\underline{i} = (i_1, \ldots, i_{k+1}) \in A^{k+1} \quad Q_{i_1,\ldots,i_{k+1}} \subset Q_{i_1,\ldots,i_k}$,
- for every $\underline{i} = (i_1, \ldots, i_k) \in A^k$ the side length of $Q_{\underline{i}}$ is $\alpha_{i_1,\ldots,i_k} = \alpha_{i_1} \cdot \alpha_{i_2} \cdot \ldots \cdot \alpha_{i_k}$,
- for every $\underline{i} = (i_1, i_2, \ldots) \in \Sigma$, if $\mathbf{x} \in \bigcap_{k=1}^{\infty} Q_{i_1,\ldots,i_k}$, then $\mathbf{x} = \Pi(\underline{i})$.

Due to equation (6.1), we can easily see that for every $k \in \mathbb{Z}^+$ and for all $\underline{i} \neq \underline{j} \in A^k \quad Q_{\underline{i}} \cap Q_{\underline{j}} = \emptyset$.

Using equation (2.6) k-times, then we get

$$\Lambda = \bigcup_{(i_1,\ldots,i_k)\in A^k} F_{i_1,\ldots,i_k}(\Lambda) \subseteq \bigcup_{(i_1,\ldots,i_k)\in A^k} F_{i_1,\ldots,i_k}(Q) = \bigcup_{(i_1,\ldots,i_k)\in A^k} Q_{i_1,\ldots,i_k}. \tag{6.3}$$

Thus we get a disjoint cover of $\Lambda$

$$\Lambda \subseteq \bigsqcup_{\underline{i}\in A^k} Q_{\underline{i}}, \tag{6.4}$$

which we call a natural partition defined by $\mathcal{F}$.

**Notation 6.1** *Let $\mathbf{x} \in \Lambda$, then for some $\underline{i} \in \Sigma \quad \Pi(\underline{i}) = \mathbf{x}$. We will use the notation $Q_n(\mathbf{x}) = Q_{i_1,\ldots,i_n}$.*

## 6.2 Dimensions regarded to the natural partitions

**Notation 6.2** *Let $A^* = \bigcup_{k=1}^{\infty} A^k$ be the set of finite length words which is formed from the character set $A$.*

**Notation 6.3** *For $\underline{i} \in A^*$ the length of $\underline{i}$ is $k$, if $\underline{i} \in A^k$. We denote the lenght of $\underline{i}$ with $l(\underline{i})$.*

**Definition 6.4** *For a Borel set $E \subseteq \Lambda$ and $t \geq 0$, let*

$$\beta^t_{\mathcal{F},n}(E) = \inf \left\{ \sum_{k=1}^{K} (\alpha_{\underline{i}_k})^t : K \in \mathbb{Z}^+,\ E \subseteq \bigcup_{k=1}^{K} Q_{\underline{i}_k},\ \forall k \in \{1, \ldots, K\} \quad l(\underline{i}_k) \geq n \right\}. \tag{6.5}$$

*The t-dimensional $\mathcal{F}$-measure of the set E is*

$$\beta^t_{\mathcal{F}}(E) = \lim_{n\to\infty} \beta^t_{\mathcal{F},n}(E). \tag{6.6}$$

**Lemma 6.5** *For $t \geq 0$ the function $\beta^t_{\mathcal{F}}$ is a Borel measure on $\Lambda$.*

**Proof:** It can be proved by using Carathéodory's extension theorem for the outer measure. $\square$

**Lemma 6.6** *For a Borel set $E \subseteq \Lambda$ and $0 < s < t$ the following are true:*

*(i) if $\beta^t_{\mathcal{F}}(E) = \infty$, then $\beta^s_{\mathcal{F}}(E) = \infty$,*

*(ii) if $\beta^s_{\mathcal{F}}(E) = 0$, then $\beta^t_{\mathcal{F}}(E) = 0$.*

**Proof:** (i) Suppose that $\beta^t_{\mathcal{F}}(E) = \infty$. Let $\varepsilon > 0$ be arbitrary and fix. For $n \in \mathbb{Z}^+$ let $\{Q_{\underline{i}^n_k}\}_{k=1}^{K_n}$ be a cover such that $E \subseteq \bigcup_{k=1}^{K_n} Q_{\underline{i}^n_k}$, for every $k$ $l(\underline{i}^n_k) \geq n$ and $\sum_{k=1}^{K_n}(\alpha_{\underline{i}^n_k})^s \leq \beta^s_{\mathcal{F},n}(E) + \varepsilon$. For every $\underline{i} \in A^*$ $0 < \alpha_{\underline{i}} < 1$, thus $(\alpha_{\underline{i}})^t \leq (\alpha_{\underline{i}})^s$. Using this we get

$$\beta^t_{\mathcal{F},n}(E) \leq \sum_{k=1}^{K_n}(\alpha_{\underline{i}^n_k})^t \leq \sum_{k=1}^{K_n}(\alpha_{\underline{i}^n_k})^s \leq \beta^s_{\mathcal{F},n}(E) + \varepsilon. \tag{6.7}$$

If we take the limit in $n$ , then we get $\beta^s_{\mathcal{F}}(E) = \infty$.

(ii) Suppose that $\beta^s_{\mathcal{F}}(E) = 0$. Let $\varepsilon > 0$ be arbitrary and fix. Again, for $n \in \mathbb{Z}^+$ let $\{Q_{\underline{i}^n_k}\}_{k=1}^{K_n}$ be a cover such that $E \subseteq \bigcup_{k=1}^{K_n} Q_{\underline{i}^n_k}$, for every $k$ $l(\underline{i}^n_k) \geq n$ and $\sum_{k=1}^{K_n}(\alpha_{\underline{i}^n_k})^s \leq \beta^s_{\mathcal{F},n}(E) + \varepsilon$. Then as above, we get

$$0 \leq \beta^t_{\mathcal{F},n}(E) \leq \sum_{k=1}^{K_n}(\alpha_{\underline{i}^n_k})^t \leq \sum_{k=1}^{K_n}(\alpha_{\underline{i}^n_k})^s \leq \beta^s_{\mathcal{F},n}(E) + \varepsilon. \tag{6.8}$$

If we take the limit in $n$, then we get $0 \leq \beta^t_{\mathcal{F}}(E) \leq \varepsilon$. This holds for every $\varepsilon > 0$, thus $\beta^t_{\mathcal{F}}(E) = 0$. $\square$

**Definition 6.7** *Let $E \subseteq \Lambda$ be a Borel set, then*

$$\dim_{\mathcal{F}}(E) = \sup\{t : \beta^t_{\mathcal{F}}(E) = \infty\} = \inf\{t : \beta^t_{\mathcal{F}}(E) = 0\} \tag{6.9}$$

*is the $\mathcal{F}$-dimension of E.*

**Lemma 6.8** *If $E \subseteq \Lambda$ is a compact set, then*

$$\dim_{\mathcal{F}}(E) = \dim_H(E). \tag{6.10}$$

**Proof:** If we use the definition of Hausdorff measure (is defined in equation (2.11)) for $E$, then it is enough to see the finite cover of $E$ with balls, beacuse $E$ is a compac set.

Let $t > 0$ arbitrary fixed. Suppose that $\mathcal{H}^t(E) < \infty$.

Let $\varepsilon > 0$ arbitrary. For a $\delta > 0$, let $\{A_k^\delta\}_{k=1}^{K_\delta}$ be such that $E \subseteq \bigcup_{k=1}^{K_\delta} A_k^\delta$, for every $k$ $A_k^\delta$ is a ball with $\left|A_k^\delta\right| \leq \delta$ and $\sum_{k=1}^{K_\delta} \left|A_k^\delta\right|^t \leq \mathcal{H}_\delta^t(E) + \varepsilon$.

We would like to get a cover with $Q_{\underline{i}}$ cubes. So let $\varphi_\delta : \{1, \ldots, K_\delta\} \to A^*$ be the map which corresponds each $A_k^\delta$ to a cube $Q_{i_1,\ldots,i_m}$. Consider the ball $A_k^\delta$. Let $\mathbf{x} \in A_k^\delta \cap E$, if there is no such $\mathbf{x}$, then we did not need the set $A_k^\delta$ in the cover. Because $E \subseteq \Lambda$, there is an $\underline{i} = (i_1, i_2, i_3, \ldots) \in \Sigma$ such that $\Pi(\underline{i}) = \mathbf{x}$. Let $\Delta = \min\{d(F_i(Q), F_j(Q)) : i \neq j \in \{1, \ldots, m\}\}$, where we use the following notation. If $A, B \subseteq Q$, then $d(A, B) = \inf\{\|\mathbf{a} - \mathbf{b}\| : \mathbf{a} \in A, \mathbf{b} \in B\}$, where $\|\cdot\|$ is the Euclidean norm on $Q$.

$$\text{For every } (i_1, \ldots, i_n) \neq \underline{j} \in A^n \quad d(Q_{i_1,i_2,\ldots,i_n}, Q_{\underline{j}}) \geq \Delta\alpha_{i_1,\ldots i_{n-1}}. \tag{6.11}$$

There is a unique $m \in \mathbb{Z}^+$ such that

$$\Delta\alpha_{i_1,\ldots,i_m} \leq \left|A_k^\delta\right| < \Delta\alpha_{i_1,\ldots,i_{m-1}}. \tag{6.12}$$

The only $m$-level cylinder set, which intersects with $A_k^\delta$ is $Q_{i_1,\ldots,i_m}$. Thus we can conclude that

$$A_k^\delta \cap E \subseteq A_k^\delta \cap \Lambda \subseteq Q_{i_1,\ldots,i_m}. \tag{6.13}$$

Using the above observations we can define $\varphi_\delta(k) = (i_1, \ldots, i_m) \in A^*$.

The set $\{Q_{\varphi_\delta(k)}\}_{k=1}^{K_\delta}$ satisfies $E \subseteq \bigcup_{k=1}^{K_\delta} Q_{\varphi_\delta(k)}$. Moreover, for every $k$ the $\Delta\alpha_{\varphi_\delta(k)} \leq \left|A_k^\delta\right| \leq \delta$ holds. Let $\tilde{\alpha} = \min\{\alpha_k : k \in A\}$. Then using this, we get $\Delta\tilde{\alpha}^{m(\delta,k)} \leq \Delta\alpha_{\varphi_\delta(k)}$. If we order the equation $\Delta\tilde{\alpha}^{m(\delta,k)} \leq \delta$, then we get

$$m(\delta, k) \geq \frac{\ln\left\{\frac{\delta}{\Delta}\right\}}{\ln(\tilde{\alpha})} = \frac{\ln(\delta) - \ln(\Delta)}{\ln(\tilde{\alpha})}. \tag{6.14}$$

Using equation (6.12), we can see

$$\Delta^t \beta^t_{\mathcal{F},\left[\frac{\ln(\delta)-\ln(\Delta)}{\ln(\bar{\alpha})}\right]}(E) \leq \Delta^t \sum_{k=1}^{K_\delta} (\alpha_{\varphi_\delta(k)})^t \leq \sum_{k=1}^{K_\delta} \left|A_k^\delta\right|^t \leq \mathcal{H}^t_\delta(E) + \varepsilon. \tag{6.15}$$

If we let $\delta \to 0$, then we get $\beta^t_{\mathcal{F}}(E) < \infty$, because we supposed that $\mathcal{H}^t(E) < \infty$.

Now, we fixed an arbitrary $t > 0$ number. We suppose that $\beta^t_{\mathcal{F}}(E) < \infty$.

Let $\varepsilon > 0$ be arbitrary. For every $n$ let $\{Q_{\underline{i}^n_k}\}_{k=1}^{K_n}$ be such that $E \subseteq \bigcup_{k=1}^{K_n} Q_{\underline{i}^n_k}$, for every $k$ the length $l(\underline{i}^n_k) \geq n$ and $\sum_{k=1}^{K_n} (\alpha_{\underline{i}^n_k})^t \leq \beta^t_{\mathcal{F},n}(E) + \varepsilon$.

The diameter of $Q_{\underline{i}^n_k}$ is $\left|Q_{\underline{i}^n_k}\right| = \sqrt{k}\alpha_{\underline{i}^n_k}$, where $k = \dim(Q)$. Moreover, for a fix $n$ for every $k \in \{1, \ldots, K_n\}$ the diameter $\left|Q_{\underline{i}^n_k}\right| \leq \sqrt{k}(\max_{i \in A} \alpha_i)^n$. Thus

$$\mathcal{H}^t_{\sqrt{k}(\max_{i\in A}\alpha_i)^n}(E) \leq \sum_{k=1}^{K_n} (\sqrt{k}\alpha_{\underline{i}^n_k})^t = \left(\sqrt{k}\right)^t \sum_{k=1}^{K_n} (\alpha_{\underline{i}^n_k})^t \leq \left(\sqrt{k}\right)^t (\beta^t_{\mathcal{F},n}(E) + \varepsilon). \tag{6.16}$$

If $n \to \infty$, then we get $\mathcal{H}^t(E) < \infty$, because we supposed that $\beta^t_{\mathcal{F}}(E) < \infty$. $\square$

**Notation 6.9** *Let $X$ be a metric space. We denote the set of Borel probability measures on $X$ with $\mathcal{P}(X)$.*

**Definition 6.10** *Let $\theta \in \mathcal{P}(Q)$ and $\mathbf{x} \in \Lambda$. The $\underline{i} = (i_1, i_2 \ldots) \in \Sigma$ satisfies $\Pi(\underline{i}) = \mathbf{x}$. Then the $\mathcal{F}$-local dimension of $\theta$ at the point $\mathbf{x}$ is*

$$\dim_{\mathcal{F}}(\theta, \mathbf{x}) = \lim_{n\to\infty} \frac{\log(\theta(Q_n(\mathbf{x})))}{\log(\alpha_{i_1,\ldots,i_n})}, \tag{6.17}$$

*if the limit exists.*

**Definition 6.11** *We say that the measure $\theta \in \mathcal{P}(Q)$ is $\mathcal{F}$-regular, if there is a constant $C$ such that*

$$\dim_{\mathcal{F}}(\theta, \mathbf{x}) = C \quad \textit{for } \theta\textit{-a.e. } \mathbf{x} \in Q. \tag{6.18}$$

*We denote this $C$ constant with $\dim_{\mathcal{F}}(\theta)$.*

**Lemma 6.12** *Let $\theta \in \mathcal{P}(Q)$ be a measure such that $\operatorname{spt}(\theta) \subseteq \Lambda$. Let $\mathbf{x} \in \Lambda$ and $\underline{i} \in \Sigma$ for which $\Pi(\underline{i}) = \mathbf{x}$. Then we can conclude that*

$$\textit{if there exists } \dim_{\mathcal{F}}(\theta, \mathbf{x}), \textit{ then there exists } \dim_\theta(\mathbf{x}), \tag{6.19}$$

*where* $\dim_\theta(\mathbf{x})$ *is the local dimension of the measure* $\theta$ *at* $\mathbf{x}$*, which can be found in Definition 2.17. Moreover, if* $\dim_{\mathcal{F}}(\theta,\mathbf{x})$ *exists, then*

$$\dim_{\mathcal{F}}(\theta,\mathbf{x}) = \dim_\theta(\mathbf{x}). \tag{6.20}$$

**Proof:** We suppose that $\dim_{\mathcal{F}}(\theta,\mathbf{x})$ exists. Then for an arbitrary $r > 0$ consider the ball $B_r(\mathbf{x})$ centered at $\mathbf{x}$ with radius $r$.

Let $\Delta = \min\{d(F_i(Q), F_j(Q)) : i \neq j \in \{1,\ldots,m\}\}$, where we use $d(A,B) = \inf\{\|\mathbf{a}-\mathbf{b}\| : \mathbf{a} \in A, \mathbf{b} \in B\}$ to denote the distance of the sets $A, B \subseteq Q$. The $\|\cdot\|$ is the Euclidean norm on $Q$.

For every $n \in \mathbb{Z}^+$

$$\text{for every } \underline{j} \in A^n \quad d(Q_{i_1,\ldots,i_n}, Q_{\underline{j}}) \geq \Delta\alpha_{i_1,\ldots i_{n-1}}. \tag{6.21}$$

There is a unique $m(r)$ such that

$$\Delta\alpha_{i_1,\ldots,i_{m(r)}} \leq r < \Delta\alpha_{i_1,\ldots,i_{m(r)-1}}. \tag{6.22}$$

Thus, $B_r(\mathbf{x})$ intersects with only one $m(r)$ cylinder set. The support of the measure $\theta$ is a subset in $\Lambda$, so we can see that $\theta(B_r(\mathbf{x})) \leq \theta(Q_{i_1,\ldots,i_{m(r)}})$.

We remind that the dimension of $Q$ is $k$. Let us introduce $\alpha_{\max} = \max_{i\in A}\alpha_i$. Let $L$ be a fixed integer such that

$$L > \frac{\ln(\Delta) - \ln(\sqrt{k})}{\ln(\alpha_{\max})}. \tag{6.23}$$

If $\tilde{Q} = [a_1, b_1] \times \ldots \times [a_k, b_k] \subseteq Q$ is a cube with side length $c = b_1 - a_1 = \ldots = b_k - a_k$, then the diameter $\left|\tilde{Q}\right| = \sqrt{k}c$. Now, we can get the estimate

$$\begin{aligned} \left|Q_{i_1,\ldots,i_{m(r)+L}}\right| &= \sqrt{k}\alpha_{i_1,\ldots i_{m(r)+L}} = \sqrt{k}\cdot\frac{1}{\Delta}\cdot(\Delta\alpha_{i_1,\ldots,i_{m(r)}})\cdot\alpha_{i_{m(r)+1}}\cdot\ldots\cdot\alpha_{i_{m(r)+L}} \leq \\ &\leq \sqrt{k}\cdot\frac{1}{\Delta}\cdot r\cdot\alpha_{i_{m(r)+1}}\cdot\ldots\cdot\alpha_{i_{m(r)+L}} \leq \sqrt{k}\cdot\frac{1}{\Delta}\cdot r\cdot\alpha_{\max}^L < r, \end{aligned} \tag{6.24}$$

where we use equation (6.22) in the first inequality and the definition of L in the last step, which is in equation (6.23).

The definition of $L$ does not depend on $r$ and $Q_{i_1,\ldots,i_{m(r)+L}} \subseteq B_r(\mathbf{x})$. This implies that

$\theta(Q_{i_1,\dots,i_{m(r)+L}}) \leq \theta(B_r(\mathbf{x})) \leq \theta(Q_{i_1,\dots,i_{m(r)}})$. Using equation (6.22) we get that

$$r < \Delta \cdot \alpha_{i_1,\dots,i_{m(r)-1}} \leq \Delta \cdot \alpha_{i_1,\dots,i_{m(r)-1}} \cdot \frac{\alpha_{i_{m(r)}}}{\alpha_{\min}} \cdot \dots \cdot \frac{\alpha_{i_{m(r)+L}}}{\alpha_{\min}} = \frac{\Delta}{\alpha_{\min}^{L+1}} \cdot \alpha_{i_1,\dots,i_{m(r)+L}}, \quad (6.25)$$

where we use the notation $\alpha_{\min} = \min_{i \in A} \alpha_i$. Summarize the above observations

$$\frac{\ln(\theta(Q_{i_1,\dots,i_{m(r)+L}}))}{\ln(\frac{\Delta}{\alpha_{\min}^{L+1}}) + \ln(\alpha_{i_1,\dots,i_{m(r)+L}})} \leq \frac{\ln(\theta(B_r(\mathbf{x})))}{\ln(r)} \leq \frac{\ln(\theta(Q_{i_1,\dots,i_{m(r)}}))}{\ln(\Delta) + \ln(\alpha_{i_1,\dots,i_{m(r)}})}. \quad (6.26)$$

For $m(r)$ the $m(r) \geq \frac{\ln(r)-\ln(\Delta)}{\ln(\alpha_{\min})}$ holds, thus if we let $r \to 0$, then $\lim_{r\to 0} m(r) = \infty$. Take $r \to 0$ in equality (6.25), then we can see that $\dim_\theta(\mathbf{x})$ exists and $\dim_\theta(\mathbf{x}) = \dim_{\mathcal{F}}(\theta, \mathbf{x})$. $\square$

**Lemma 6.13** *Let $\theta \in \mathcal{P}(Q)$ such that the support of $\theta$ $\mathrm{spt}(\theta) \subseteq \Lambda$. Assume that for every $\Pi(i_1, i_2, \dots) = \mathbf{x} \in \Lambda$ outside of $\theta$-meaure 0*

$$\liminf_{n\to\infty} \frac{\log \theta(Q_n(\mathbf{x}))}{\log(\alpha_{i_1}, \dots, \alpha_{i_n})} \geq \beta. \quad (6.27)$$

*Then $\dim_H(\mathrm{spt}(\theta)) \geq \beta$.*

**Proof:** Indirectly suppose that $\dim_H(\mathrm{spt}(\theta)) < \beta$, then there is $\beta'$ such that $\dim_H(\mathrm{spt}(\theta)) < \beta' < \beta$. Using Egorov's theorem there exist a set $A$ and $N \geq 1$ such that $\theta(A) > 1/2$ and

$$\text{for every } \mathbf{x} \in A \text{ and for every } n \geq N \quad \frac{\log \theta(Q_n(\mathbf{x}))}{\log(\alpha_{i_1}, \dots, \alpha_{i_n})} \geq \beta'. \quad (6.28)$$

This implies that

$$\text{for every } \mathbf{x} \in A \text{ and for every } n \geq N \quad \theta(Q_n(\mathbf{x})) \leq \alpha_{i_1,\dots,i_n}^{\beta'}. \quad (6.29)$$

Using Lemma 6.8 for any $\varepsilon > 0$ there exists $\tilde{N} \geq N$ and $\{Q_{\underline{i}_k}\}_{k=1}^K$ such that

$$A \subseteq \bigcup_{k=1}^K Q_{\underline{i}_k}, \text{ for every } k \ l(\underline{i}_k) \geq \tilde{N} \text{ and } \sum_{k=1}^K \alpha_{\underline{i}_k}^{\beta'} < \varepsilon. \quad (6.30)$$

We can assume that for every $k$ $\quad Q_{\underline{i}_k} \cap A \neq \emptyset$. Thus for every $k$ $\quad \theta(Q_{\underline{i}_k}) \leq \alpha_{\underline{i}_k}^{\beta'}$. We get the following

$$\theta(A) \leq \sum_{k=1}^K \theta(Q_{\underline{i}_k}) \leq \sum_{k=1}^K \alpha_{\underline{i}_k}^{\beta'} < \varepsilon. \quad (6.31)$$

This is a contradiction, if $\varepsilon < 1/2$. $\square$

## 6.3 Ergodic CP-shift systems

In this section we would like to introduce dinamical systems which we will call ergodic CP-shift systems.

**Definition 6.14** *For every $n \in \mathbb{Z}^+$ let $\theta_n \in \mathcal{P}(Q)$ and $\theta \in \mathcal{P}(Q)$. We say that the sequence of $\theta_n$ converges to $\theta$ in the weak-* topology, if*

$$\text{for every } f : Q \to \mathbb{R} \text{ continuous function} \quad \lim_{n\to\infty} \int_Q f(\mathbf{x})\mathrm{d}\theta_n(\mathbf{x}) = \int_Q f(\mathbf{x})\mathrm{d}\theta(\mathbf{x}). \qquad (6.32)$$

*We denote this convergence with $\lim_{n\to\infty} \theta_n = \theta$.*

**Lemma 6.15** *For every $n \in \mathbb{Z}^+$ let $\theta_n \in \mathcal{P}(Q)$ and $\theta \in \mathcal{P}(Q)$. Then the following are equivalent:*

*(i)* $\lim_{n\to\infty} \theta_n = \theta$,

*(ii) for every closed set $C \subseteq Q$* $\quad \limsup_{n\to\infty} \theta_n(C) \le \theta(C)$,

*(iii) for every open set $O \subseteq Q$* $\quad \liminf_{n\to\infty} \theta_n(O) \ge \theta(O)$.

**Lemma 6.16** *The weak-* topology on $\mathcal{P}(Q)$ is induced by a metric on $\mathcal{P}(Q)$.*

The symbolic space $\Sigma$ is also a metric space with the metric, which is introduced in Notation 4.3.

Let $\mathcal{M} = \{\theta \in \mathcal{P}(Q) : \text{ for every } L \text{ (k-1)-dimensional subspace } \quad \theta(L \cap Q) = 0 \ \}$. That is $\mathcal{M}$ is the set of those $\theta$ measures on $Q$ such that the $\theta$ measure of all (k-1)-dimensional subspace intersected with $Q$ is zero.

Let

$$\Phi = \{(\theta, \underline{i}) \in \mathcal{M} \times \Sigma \, : \text{ for every } n \in \mathbb{Z}^+ \quad \theta(Q_n(\Pi(\underline{i}))) > 0\} \subset \mathcal{P}(Q) \times \Sigma. \qquad (6.33)$$

**Lemma 6.17** *The set $\Phi$ is closed in $\mathcal{P}(Q) \times \Sigma$, thus it is a Borel set.*

**Proof:** For every $n \in \mathbb{Z}^+$ let $(\theta^n, \underline{i}^n) \in \Phi$ such that $\lim_{n\to\infty} (\theta^n, \underline{i}^n) = (\theta, \underline{i}) \in \mathcal{P}(Q) \times \Sigma$. For every $n$ let $\underline{i}^n = (i_1^n, i_2^n, \ldots)$ and $\underline{i} = (i_1, i_2, \ldots)$.

First we want to prove $(\theta, \underline{i}) \in \Phi$. For this, it is enough to prove that for every $k \geq 1 \quad \theta(Q_k(\Pi(\underline{i}))) > 0$. Let $k \geq 1$ be arbitrary. There is an $M$ such that for every $m \geq M \quad i_1 = i_1^m, \ldots, i_k = i_k^m$. Using this, we get

$$\theta(Q_k(\Pi(\underline{i}))) \geq \limsup_{m\to\infty} \theta_m(Q_k(\Pi(\underline{i}))) = \limsup_{m\to\infty} \theta_m(Q_k(\Pi(\underline{i}^m))) > 0, \tag{6.34}$$

where we use Lemma 6.15.

Now, we would like to see that $\theta \in \mathcal{M}$. Let $L \subset \mathbb{R}^k$ be a (k-1)-dimensional subspace in $\mathbb{R}^k$ and $A = L \cap Q$. Then by continuity of measure implies $\theta(A) = \lim_{\varepsilon\to 0} \theta(N_\varepsilon(A))$. Let $\tilde{\varepsilon} > 0$ be arbitrary. Then for every $m$ there exists $E_m > 0$ such that for every $0 < \varepsilon < E_m$ the inequality $\theta_m(N_\varepsilon(A)) \leq \theta_m(A) + \tilde{\varepsilon}$, because the continuity of the measures. Thus

$$\theta(A) = \lim_{\varepsilon\to 0} \theta(N_\varepsilon(A)) \leq \limsup_{\varepsilon\to 0} \liminf_{m\to\infty} \theta_m(N_\varepsilon(A)) \leq \liminf_{m\to\infty} (\theta_m(A) + \tilde{\varepsilon}) = \tilde{\varepsilon}, \tag{6.35}$$

where we use Lemma 6.15. If we let $\tilde{\varepsilon} \to 0$, then we get $\theta \in \mathcal{M}$. $\square$

We restrict $F_i$ for every $i$, that is we consider $F_i$ as $F_i : Q \to Q_i$ function and $F_i^{-1} : Q_i \to Q$ function. For $(\theta, \underline{i}) \in \Phi$, $\underline{i} = (i_1, i_2, \ldots)$ let

$$T(\theta, \underline{i}) = \left( \frac{(F_{i_1}^{-1})_* \theta}{\theta(Q_{i_1})}, \sigma \underline{i} \right) \in \mathcal{P}(Q) \times \Sigma, \tag{6.36}$$

where $\sigma : \Sigma \to \Sigma$ is the left-shift on $\Sigma$.

**Lemma 6.18** *The map $T$ is a continuous function.*

The proof of this Lemma can be easily see. As a corollary, $T$ is a Borel measurable map.

**Lemma 6.19** *For $(\theta, \underline{i}) \in \Phi$ the image $T(\theta, \underline{i}) \in \Phi$.*

**Proof:** Let $\underline{i} = (i_1, i_2, \ldots) \in \Sigma$.

We denote the image

$$(\tilde{\theta}, \tilde{\underline{i}}) = T(\theta, \underline{i}) = \left( \frac{(F_{i_1}^{-1})_* \theta}{\theta(Q_{i_1})}, \sigma \underline{i} \right). \tag{6.37}$$

The meaure $\tilde{\theta}$ is in $\mathcal{M}$, because if $L$ is a (k-1)-dimensional subspace in $\mathbb{R}^k$, then

$$\frac{(F_{i_1}^{-1})_* \theta}{\theta(Q_{i_1})} (L \cap Q) = \frac{\theta(F_{i_1}(L \cap Q))}{\theta(Q_{i_1})} \leq \frac{\theta(F_{i_1}(L) \cap Q)}{\theta(Q_{i_1})} = 0, \tag{6.38}$$

because $F_{i_1}$ preserves the (k-1)-dimensional subspaces.

Let $n$ be an arbitrary positive integer. Then we would like to prove that $\tilde{\theta}(Q_n(\Pi(\tilde{\underline{i}}))) > 0$. We use the definition of $T$ to calculate

$$\tilde{\theta}(Q_n(\Pi(\tilde{\underline{i}}))) = \frac{\theta(F_{i_1}(Q_n(\Pi(\sigma\underline{i}))))}{\theta(Q_{i_1})} = \frac{\theta(Q_{n+1}(\Pi(\underline{i})))}{\theta(Q_{i_1})} > 0. \tag{6.39}$$

□

**Definition 6.20** *A probability measure $\eta$ on $\mathcal{P}(Q)\times\Sigma$ is said to be adapted, if there exists a measure $\rho$ on $\mathcal{P}(Q)$ such that for every bounded, measurable map $f : \mathcal{P}(Q) \times \Sigma \to \mathbb{R}$*

$$\int\limits_{\mathcal{P}(Q)\times\Sigma} f(\theta,\underline{i})\mathrm{d}\eta(\theta,\underline{i}) = \int\limits_{\mathcal{P}(Q)}\int\limits_{\Sigma} f(\theta,\underline{i})\mathrm{d}(\Pi^{-1})_*\theta(\underline{i})\mathrm{d}\rho(\theta) = \int\limits_{\mathcal{P}(Q)}\int\limits_{\Lambda} f(\theta,\Pi^{-1}(\mathbf{x}))\mathrm{d}\theta(\mathbf{x})\mathrm{d}\rho(\theta).$$

**Definition 6.21** *The $(\Phi, T, \eta)$ is an ergodic CP-shift system (ECPS system), if the measure $\eta$ on $\Phi$ is adapted, $T$-invariant and the corresponding system is ergodic.*

The CP refers to conditional probability measure, which appears in the image of $T(\theta,\underline{i})$.

**Lemma 6.22** *Suppose that $(\Phi, T, \eta)$ is an ECPS system. Then $\int_\Phi -\log(\alpha_{i_1})\mathrm{d}\eta(\theta,\underline{i}) < \infty$ and*

$$\lim_{n\to\infty} -\frac{1}{n}\log(\alpha_{i_1\dots,i_n}) = \int_\Phi -\log(\alpha_{i_1})\mathrm{d}\eta(\theta,\underline{i}) \quad \textit{for } \eta\textit{-a.e. } (\theta,\underline{i}), \tag{6.40}$$

*where $\underline{i} = (i_1, i_2, \dots) \in \Sigma$.*

**Proof:** Let $L : \Phi \to \mathbb{R}$ be such that $L(\theta,\underline{i}) = -\log(\alpha_{i_1})$, where $\underline{i} = (i_1, i_2, \dots) \in \Sigma$. The function $L$ is integrable, because $L(\theta,\underline{i}) \leq -\log(\alpha_{\min})$, where $\alpha_{\min} = \min_{i\in A}\alpha_i$. We can easily calculate $L(T^k(\theta,\underline{i})) = -\log(\alpha_{i_k})$. Then the Birkhoff's sum is

$$\frac{1}{n}\sum_{k=0}^{n-1} L(T^k(\theta,\underline{i})) = -\frac{1}{n}\log(\alpha_{i_1,\dots,i_n}). \tag{6.41}$$

We get the statement, if we use Birkhoff's ergodic theorem. □

**Lemma 6.23** *If $(\Phi, T, \eta)$ is an ECPS system. Then $\int_\Phi -\log\theta(Q_1(\Pi(\underline{i})))\mathrm{d}\eta(\theta,\underline{i}) < \infty$ and*

$$\lim_{n\to\infty} -\frac{1}{n}\ln(\theta(Q_{i_1,\dots,i_n})) = \int_\Phi -\log\theta(Q_1(\Pi(\underline{i})))\mathrm{d}\eta(\theta,\underline{i}) \quad \textit{for } \eta\textit{-a.e. } (\theta,\underline{i}), \tag{6.42}$$

*where $\underline{i} = (i_1, i_2, \dots)$.*

**Proof:** Let $I : \Phi \to \mathbb{R}$ such that $I(\theta, \underline{i}) = -\ln(\theta(Q_{i_1}))$ for $(\theta, \underline{i}) \in \Phi$, where $\underline{i} = (i_1, i_2, \ldots)$. The map $I$ is a positive measurable map.

We will use Birkhoff's ergodic theorem for the function $I$. First, we calulate the Birhoff's sum.

We iterate the map $T$ $k$-times, then we get

$$T^k(\theta, \underline{i}) = \left( \frac{(F_{i_k}^{-1} \circ \ldots \circ F_{i_1}^{-1})_* \theta}{\theta(Q_{i_1, \ldots, i_k})}, \sigma^k \underline{i} \right). \tag{6.43}$$

We replace this into $I$

$$I(T^k(\theta, \underline{i})) = -\ln \left( \frac{\theta((F_{i_1} \circ \ldots \circ F_{i_k})(Q_{i_{k+1}}))}{\theta(Q_{i_1, \ldots, i_k})} \right) = -\ln \left( \frac{\theta(Q_{i_1, \ldots, i_{k+1}})}{\theta(Q_{i_1, \ldots, i_k})} \right). \tag{6.44}$$

We can calculate the sum

$$\frac{1}{n} \sum_{k=0}^{n-1} I(T^k(\theta, \underline{i})) = -\frac{1}{n} \ln(\theta(Q_{i_1, \ldots, i_n})). \tag{6.45}$$

Using Birkhoff's ergodic theorem if $\int_\Phi I(\theta, \underline{i}) \mathrm{d}\eta(\theta, \underline{i})$ exists, then

$$\lim_{n \to \infty} -\frac{1}{n} \ln(\theta(Q_{i_1, \ldots, i_n})) = \int\limits_{\Phi} I(\theta, \underline{i}) \mathrm{d}\eta(\theta, \underline{i}) \quad \text{for } \eta\text{-a.e. } (\theta, \underline{i}). \tag{6.46}$$

If $I$ is not integrable, then because $I \geq 0$ then in this case the limit is $\infty$ for $\eta$ almost every $(\theta, \underline{i})$. Thus there exists a constant $0 \leq C \leq \infty$ such that

$$\lim_{n \to \infty} -\frac{1}{n} \ln(\theta(Q_{i_1, \ldots, i_n})) = C \quad \text{for } \eta\text{-a.e. } (\theta, \underline{i}), \tag{6.47}$$

where $\underline{i} = (i_1, i_2, \ldots)$. If $C = \infty$, then using Lemma 6.22, we get that the $\mathcal{F}$-local dimension of $\theta$ is infinity almost everywhere. This contradicts with the assertion in Lemma 6.13, because $\dim_H(\mathrm{spt}(\theta)) \leq k = \dim(Q)$. Thus $C$ is finite, so $\int_\Phi -\log \theta(Q_1(\Pi(\underline{i}))) \mathrm{d}\eta(\theta, \underline{i}) < \infty$. □

## 6.4 Fubini decomposition

Let $\theta$ be a Borel probability measure on $Q = Q' \times Q''$, where $Q' = [0,1]^{k_1}$ and $Q'' = [0,1]^{k_2}$. We denoted the projections from$Q$ to $Q'$ with $P'$ and from $Q$ to $Q''$ with $P''$.

Let $\mathcal{A} = \sigma(\{B \times Q'' : B \subseteq Q' \text{ is a Borel set}\})$, where $\sigma$ denotes the generated $\sigma$-algebra by sets. Let

$$\mathcal{P}_n = \left\{ \left[\frac{j_1 - 1}{2^n}, \frac{j_1}{2^n}\right) \times \ldots \times \left[\frac{j_{k_1} - 1}{2^n}, \frac{j_{k_1}}{2^n}\right) \ : \ j_1, j_2, \ldots, j_{k_1} \in \{1, \ldots, 2^n\} \right\} \tag{6.48}$$

Then $\mathcal{P}_n$'s are finite measurable partitions of $Q'$ such that $\bigvee_{n=1}^{\infty} \sigma(\mathcal{P}_n)$ is the Borel $\sigma$-algebra on $Q'$, where $\bigvee$ denotes the generated $\sigma$-algebra by the elements of $\sigma$-algebras. Thus, $\mathcal{A}$ is a $\sigma$-algebra with the same property as $\mathcal{F}$ in equation (5.2). Using Theorem 5.6 for $\theta$ almost every $\mathbf{x}$ there exists $\theta_{\mathbf{x}}^{\mathcal{A}}$ and the set $\{\theta_{\mathbf{x}}^{\mathcal{A}}\}_{\mathbf{x}\in Q}$ is a system of conditional measures of $\theta$ with respect to the $\sigma$-algebra $\mathcal{A}$. The conditional measures are the same on the sets $\mathbf{x}_1 \times Q''$, where $\mathbf{x}_1 \in Q'$. That is, if $\mathbf{x}, \mathbf{y} \in \mathbf{x}_1 \times Q''$, then $\theta_{\mathbf{x}}^{\mathcal{A}} = \theta_{\mathbf{y}}^{\mathcal{A}}$. We know that for every bounded, measurable $f : Q \to \mathbb{R}$

$$\int_Q f(\mathbf{x}_1, \mathbf{x}_2) \mathrm{d}\theta(\mathbf{x}_1, \mathbf{x}_2) = \int_Q \int_Q f(\mathbf{x}_1, \mathbf{x}_2) \mathrm{d}\theta_{(\mathbf{y}_1,\mathbf{y}_2)}(\mathbf{x}_1, \mathbf{x}_2) \mathrm{d}\theta(\mathbf{y}_1, \mathbf{y}_2), \tag{6.49}$$

where $\mathbf{x}_1, \mathbf{y}_1 \in Q'$ and $\mathbf{x}_2, \mathbf{y}_2 \in Q''$. Let $\hat{\theta} = P'_* \theta$ be a measure on $Q'$. We can think of the measure $\theta_{(\mathbf{x}_1,\mathbf{x}_2)}$ as a measure on $\mathbf{x}_1 \times Q''$. So $\theta_{(\mathbf{x}_1,\mathbf{x}_2)} = \delta_{\mathbf{x}_1} \times \theta_{\mathbf{x}_1}$, where $\theta_{\mathbf{x}_1} \in \mathcal{P}(Q'')$.

Thus, we can write equation (6.49) in the form that for every bounded, measurable $f : Q \to \mathbb{R}$

$$\int_Q f(\mathbf{x}_1, \mathbf{x}_2) \mathrm{d}\theta(\mathbf{x}_1, \mathbf{x}_2) = \int_{Q'} \int_Q f(\mathbf{x}_1, \mathbf{x}_2) \mathrm{d}(\delta_{\mathbf{y}_1} \times \theta_{\mathbf{y}_1})(\mathbf{x}_1, \mathbf{x}_2) \mathrm{d}\hat{\theta}(\mathbf{y}_1), \tag{6.50}$$

where $\mathbf{x}_1, \mathbf{y}_1 \in Q'$ and $\mathbf{x}_2 \in Q''$. We call this measure decomposition as Fubini decomposition.

**Lemma 6.24** *If $\theta \in \mathcal{M}$ is a measure on $Q$, which has Fubini decomposition $\hat{\theta} \in \mathcal{P}(Q')$ and for $\hat{\theta}$-a.e. $\mathbf{x} \in Q'$ $\quad \theta_{\mathbf{x}}$ as in equation* (6.50). *Then*

$$\theta_{\mathbf{x}}(E) = \lim_{n\to\infty} \frac{\theta(\mathcal{P}_n(\mathbf{x}) \times E)}{\theta(\mathcal{P}_n(\mathbf{x}) \times Q'')}, \tag{6.51}$$

*where $\mathbf{x} \in Q'$ and $E = I_1 \times \ldots \times I_{k_2} \subseteq Q''$ is a cube such that $I_k$'s are intervals. The $\mathcal{P}_n(\mathbf{x}) \in \mathcal{P}_n$ is the unique set in $\mathcal{P}_n$ such that $\mathbf{x} \in \mathcal{P}_n(\mathbf{x})$.*

**Proof:** We can see if we use Theorem 5.6, Lemma 6.15 and that $\theta \in \mathcal{M}$. $\square$

## 6.5 Dimension conservation regarded to ECPS systems

**Definition 6.25** *Let $\theta \in \mathcal{P}(Q)$ with Fubini decomposition $\theta = \int_{Q'} \delta_{\mathbf{x}} \times \theta_{\mathbf{x}} \mathrm{d}\hat{\theta}(\mathbf{x})$, where $\hat{\theta} \in \mathcal{P}(Q')$ and for $\hat{\theta}$-a.e. $\mathbf{x} \in Q'$ $\theta_{\mathbf{x}} \in \mathcal{P}(Q'')$. Then we say that $\theta$ satisfies dimension conservation, if $\theta$, $\hat{\theta}$ are $\mathcal{F}$-regular and for $\hat{\theta}$-a.e. $\mathbf{x} \in Q'$ the measure $\theta_{\mathbf{x}}$ is $\mathcal{F}$-regular, moreover*

$$\dim_{\mathcal{F}}(\theta) = \dim_{\mathcal{F}}(\hat{\theta}) + \dim_{\mathcal{F}}(\theta_{\mathbf{x}}) \text{ holds for } \hat{\theta}\text{-a.e. } \mathbf{x} \in Q'. \tag{6.52}$$

**Theorem 6.26 (Maker)** *Let $(X, \mathcal{B}, \eta, T)$ be a measure-preserving system, where $(X, \mathcal{B}, \eta)$ be a probability space with the Borel measure $\eta$. For every $n \in \mathbb{Z}^+$ let $f_n : X \to \mathbb{R}$ be a measurable function such that for $\eta$ almost every $x \in X$ $\lim_{n\to\infty} f_n(x) = f_\infty(x)$. We suppose that $\sup_n |f_n(x)| = g(x)$ is an integrable function.*

*Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f_{n-k}(T^k(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f_\infty(T^k(x)) \quad \text{for } \eta\text{-a.e } x \in X. \tag{6.53}$$

**Lemma 6.27** *Let $(X, \mathcal{B}, \eta)$ be a Borel probability space. The set $A \in \mathcal{B}$ is a measurable set. For every $n \in \mathbb{Z}^+$ let $\mathcal{F}_n$ be a finite $\sigma$-algebra such that for every $n$ $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$. We use the notation $\eta(A|\mathcal{F}_n) = \mathbb{E}_\eta(\mathbb{1}_A|\mathcal{F}_n)$ for the conditional expectation of $\mathbb{1}_A$ with respect to $\mathcal{F}_n$. Let*

$$f(x) = \mathbb{1}_A(x) \sup_n \left(-\ln \eta(A|\mathcal{F}_n)(x)\right), \tag{6.54}$$

*then $f$ is an integrale function and $\int_X f(x)\mathrm{d}\eta(x) < \infty$.*

**Proof:** It is enough to prove that $\sum_{N=0}^{\infty} \eta(\{x : f(x) \geq N\}) < \infty$. For every $n$ let $\mathcal{P}_n = \{P_1^n, \ldots, P_{K_n}^n\}$ be a measurable partition of $X$ such that $\sigma(\mathcal{P}_n) = \mathcal{F}_n$, where $\sigma$ denotes the generated $\sigma$-algebra by sets. For every $n$ the map $\eta(A|\mathcal{F}_n)$ is constant on each $P_i^n$ and for $x \in P_i^n$ $\eta(A|\mathcal{F}_n)(x) = \frac{\eta(A\cap P_i^n)}{\eta(P_i^n)}$.

We define inductively the following sets

$$\begin{gathered}
\mathcal{B}_1 = \left\{E \in \mathcal{P}_1 : -\ln\left(\frac{\eta(A\cap E)}{\eta(E)}\right) \geq N\right\},\ \mathcal{B}_1^c = \mathcal{P}_1 \setminus \mathcal{B}_1, \\
\mathcal{B}_k = \left\{E \in \mathcal{P}_k : -\ln\left(\frac{\eta(A\cap E)}{\eta(E)}\right) \geq N\right\} \cap \mathcal{B}_{k-1}^c \cap \ldots \cap \mathcal{B}_1^c,\ \mathcal{B}_k^c = \mathcal{P}_k \setminus \mathcal{B}_k.
\end{gathered} \tag{6.55}$$

We can easily see that if $E, \tilde{E} \in \bigcup_{n=1}^{\infty} \mathcal{B}_n$, then $E \cap \tilde{E} = \emptyset$ and

$$\{x : f(x) \geq N\} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{E \in \mathcal{B}_n} A \cap E. \tag{6.56}$$

For every $n$ and arbitrary $E \in \mathcal{B}_n$ $\eta(A \cap E) \leq e^{-N}\eta(E)$. Thus

$$\eta(\{x : f(x) \geq N\}) \leq \sum_{n=1}^{\infty} \sum_{E \in \mathcal{B}_n} \eta(A \cap E) \leq e^{-N} \sum_{n=1}^{\infty} \sum_{E \in \mathcal{B}_n} \eta(E) \leq e^{-N}. \tag{6.57}$$

If we take the sum over $N$, then we get $\sum_{N=0}^{\infty} \eta(\{x : f(x) \geq N\}) < \infty$. □

**Theorem 6.28** *Let $(\Phi, T, \eta)$ be an ECPS system such that the corresponding measure on $\mathcal{P}(Q)$ is $\rho$, then $\rho$-a.e. $\theta \in \mathcal{P}(Q)$ satisfies dimension conservation.*

**Proof:** For $n \in \mathbb{Z}^+$ define the function $J_n : \Phi \to \mathbb{R}$ such that

$$J_n(\theta, \underline{i}) = \frac{\theta(P'Q_n(\Pi(\underline{i})) \times P''Q_1(\Pi(\underline{i})))}{\theta(P'Q_n(\Pi(\underline{i})) \times Q'')}. \tag{6.58}$$

Using Lemma 6.24, we can easily see that

$$J_\infty(\theta, \underline{i}) = \theta_{P'(\Pi(\underline{i}))}(P''Q_1(\Pi(\underline{i}))). \tag{6.59}$$

So, we can see that $\lim_{n\to\infty} J_n(\theta, \underline{i}) = J_\infty(\theta, \underline{i})$ holds for every $(\theta, \underline{i}) \in \Phi$. We repeat that

$$T^k(\theta, \underline{i}) = \left( \frac{(F_{i_k}^{-1} \circ \ldots \circ F_{i_1}^{-1})_* \theta}{\theta(Q_{i_1,\ldots,i_k})}, \sigma^k \underline{i} \right). \tag{6.60}$$

We calculate

$$\begin{aligned} J_n(T^k(\theta, \underline{i})) =& \frac{\theta(F_{i_1} \ldots F_{i_k}(P'Q_n(\Pi(\sigma^k \underline{i})) \times P''Q_1(\Pi(\sigma^k \underline{i}))))}{\theta(F_{i_1} \ldots F_{i_k}(P'Q_n(\Pi(\sigma^k \underline{i})) \times Q''))} = \\ =& \frac{\theta(P'Q_{n+k}(\Pi(\underline{i})) \times P''Q_{k+1}(\Pi(\mathbf{i})))}{\theta(P'Q_{n+k}(\Pi(\underline{i})) \times P''Q_k(\Pi(\mathbf{i})))}. \end{aligned} \tag{6.61}$$

Replace $n$ by $n - k$, then

$$J_{n-k}(T^k(\theta, \underline{i})) = \frac{\theta(P'Q_n(\Pi(\underline{i})) \times P''Q_{k+1}(\Pi(\mathbf{i})))}{\theta(P'Q_n(\Pi(\underline{i})) \times P''Q_k(\Pi(\mathbf{i})))}. \tag{6.62}$$

Let the functions $K_n : \Phi \to \mathbb{R}$ and $K_\infty : \Phi \to \mathbb{R}$ such that $K_n(\theta, \underline{i}) = -\ln(J_n(\theta, \underline{i}))$ and $K_\infty(\theta, \underline{i}) = -\ln(J_\infty(\theta, \underline{i}))$.

Then

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} K_{n-k}(T^k(\theta, \underline{i})) &= \\ = -\frac{1}{n}\ln\left(\prod_{k=0}^{n-1}\frac{\theta(P'Q_n(\Pi(\underline{i}))\times P''Q_{k+1}(\Pi(\mathbf{i})))}{\theta(P'Q_n(\Pi(\underline{i}))\times P''Q_k(\Pi(\mathbf{i})))}\right) &= \\ = -\frac{1}{n}\ln\left(\frac{\theta(P'Q_n(\Pi(\underline{i}))\times P''Q_n(\Pi(\mathbf{i})))}{\theta(P'Q_n(\Pi(\underline{i}))\times P''Q_0(\Pi(\mathbf{i})))}\right) &= \\ = -\frac{1}{n}\ln\theta(Q_n(\Pi(\underline{i}))) + \frac{1}{n}\ln\hat{\theta}(P'Q_n(\Pi(\underline{i}))) &= \\ = W_n(\theta, \underline{i}) - R_n(\theta, \underline{i}). \end{aligned} \tag{6.63}$$

Fix $k$ and let $n$ tends to infinity

$$K_\infty(T^k(\theta, \underline{i})) = -\ln\theta_{\Pi(\underline{i})}(P''Q_{k+1}(\Pi(\underline{i}))) + \ln\theta_{\Pi(\underline{i})}(P''Q_k(\Pi(\underline{i}))). \tag{6.64}$$

We can easily calculate the Birkhoff's sum

$$\frac{1}{n}\sum_{k=0}^{n-1} K_\infty(T^k(\theta, \underline{i})) = -\frac{1}{n}\ln\theta_{P'\Pi(\underline{i})}(P''Q_n(\Pi(\underline{i}))) = S_n(\theta, \underline{i}). \tag{6.65}$$

If we use Birkhoff's ergodic theorem for the function $K_\infty(\theta, \underline{i}) = -\ln(\theta_{P'(\Pi(\underline{i}))}(P''Q_1(\Pi(\underline{i})))) \geq 0$, then there exists a constant $0 \leq \tilde{C} \leq \infty$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} K_\infty(T^k(\theta, \underline{i})) = \tilde{C} \quad \text{for } \eta\text{-a.e. } (\theta, \underline{i}) \in \Phi. \tag{6.66}$$

We want to use Maker's theorem 6.26. Thus, we have to verify that $\sup_n K_n(\theta, \underline{i})$ is an integrable function.

We already know, that

$$K_n(\theta, \underline{i}) = -\ln\left(\frac{\theta(P'Q_n(\Pi(\underline{i}))\times P''Q_1(\Pi(\underline{i})))}{\theta(P'Q_n(\Pi(\underline{i}))\times Q'')}\right). \tag{6.67}$$

We want to use Lemma 6.27, so we try to write $K_n(\theta, \underline{i})$ as a function of conditional measures. Let $\mathcal{F}_n = \sigma(\{P'Q_{\underline{i}} \times Q'' : \underline{i} \in A^n\})$ be a finite $\sigma$-algebra for every $n$. The $\sigma$ denotes the generated $\sigma$-algebra by sets. We can express the following with conditional measure for $\underline{i} = (i_1, i_2, \ldots) \in \Sigma$ and $j \in A$

$$\theta(Q' \times P''Q_j|\mathcal{F}_n)(\Pi(\underline{i})) = \begin{cases} \frac{\theta(P'Q_n(\Pi(\underline{i})) \times P''Q_1(\Pi(\underline{i})))}{\theta(P'Q_n(\Pi(\underline{i})) \times Q'')}, & \text{if } i_1 = j \\ 0, & \text{if } i_1 \neq j \end{cases}. \tag{6.68}$$

We can use this to get

$$K_n(\theta, \underline{i}) = \sum_{j=1}^{m} \mathbb{1}_{Q' \times P''Q_j}(\Pi(\underline{i}))\left(-\ln \theta(Q' \times P''Q_j|\mathcal{F}_n)(\Pi(\underline{i}))\right) = \sum_{j=1}^{m} M_n^j(\theta, \Pi(\underline{i})),$$

where $M_n^j(\theta, \cdot) : Q \to \mathbb{R}$.

If we use Lemma 6.27 for the functions $M_n^j(\theta, \cdot)$, then we get that the supremum in $n$ is an integrable function with respect to the measure $\theta$, thus the supremum in $n$ of its finite sum is also an integrable function.

Let

$$M(\theta, \Pi(\underline{i})) = \sup_n K_n(\theta, \underline{i}), \tag{6.69}$$

we have seen that $M(\theta, \cdot)$ is integrable on $Q$ with respect to $\theta$. That is $\int_Q M(\theta, \mathbf{x}) \mathrm{d}\theta(\mathbf{x}) < \infty$. This implies

$$L(\theta) = \int_{\Sigma} M(\theta, \Pi(\underline{i})) \mathrm{d}(\Pi)_*^{-1}\theta(\underline{i}) = \int_{\Lambda} M(\theta, \mathbf{x}) \mathrm{d}\theta(\mathbf{x}) < \infty. \tag{6.70}$$

The function $L : \mathcal{P}(Q) \to \mathbb{R}$ is continuous and finite on a compact set, thus $L$ is an integrable function with respect to the measure $\rho$ and $\int_{\mathcal{P}(Q)} L(\theta) \mathrm{d}\rho(\theta) < \infty$. Now, using that $\eta$ is adapted we can easily see that $\sup_n K_n(\theta, \underline{i})$ is an integrable function on $\Phi$ with respect to $\eta$.

We have seen that Maker's theorem 6.26 is applicable. Using this theorem for the sequence $K_n$, then we get that

$$\tilde{C} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} K_\infty(T^k(\theta, \underline{i})) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} K_{n-k}(T^k(\theta, \underline{i})) \quad \text{for } \eta\text{-a.e. } (\theta, \underline{i}) \in \Phi. \tag{6.71}$$

With the above notation, this is

$$\lim_{n\to\infty} S_n(\theta, \underline{i}) = \lim_{n\to\infty} W_n(\theta, \underline{i}) - \lim_{n\to\infty} R_n(\theta, \underline{i}) \quad \text{for } \eta\text{-a.e. } (\theta, \underline{i}) \in \Phi, \tag{6.72}$$

where the limits exist and are finite $\eta$ almost everywhere, due to the arguement, which is in the proof of Lemma 6.23.

Thus the corresponding integrals exist

$$\begin{aligned}
&\int_{\Phi} -\ln \theta_{P'\Pi(\underline{i})}(P''Q_1(\Pi(\underline{i}))\mathrm{d}\eta(\theta, \underline{i}) = \\
&= \int_{\Phi} -\ln \theta(Q_1(\Pi(\underline{i}))\mathrm{d}\eta(\theta, \underline{i}) - \int_{\Phi} -\ln \hat{\theta}(P''Q_1(\Pi(\underline{i}))\mathrm{d}\eta(\theta, \underline{i}).
\end{aligned} \tag{6.73}$$

If we divide equation (6.72) with $\lim_{n\to\infty} -\frac{1}{n}\ln(\alpha_{i_1,\dots,i_n})$, then we get

$$\dim_{\mathcal{F}}(\theta_{P'\Pi(\underline{i})}, P''\Pi(\underline{i})) = \dim_{\mathcal{F}}(\theta, \Pi(\underline{i})) - \dim_{\mathcal{F}}(\hat{\theta}, P'\Pi(\underline{i})) \quad \text{for } \eta\text{-a.e. } (\theta, \underline{i}) \in \Phi \tag{6.74}$$

and each $\mathcal{F}$-local dimension is constant for $\eta$-a.e. $(\theta, \underline{i})$. Thus, if we use that the measure $\eta$ is adapted, then we get the statement for $\rho$ almost every measure. □

**Corollary 6.29** *Let $(\Phi, T, \eta)$ be an ECPS system with the corresponding measure $\rho$ on $\mathcal{P}(Q)$. Then $\rho$ almost every $\theta \in \mathcal{P}(Q)$ is such that $\theta, \hat{\theta}$ is $\mathcal{F}$-regular and*

$$\dim_{\mathcal{F}}(\hat{\theta}) = \frac{\int_{\Phi} -\ln \theta(Q_1(\Pi(\underline{i}))\mathrm{d}\eta(\theta, \underline{i}) - \int_{\Phi} -\ln \theta_{P'\Pi(\underline{i})}(P''Q_1(\Pi(\underline{i}))\mathrm{d}\eta(\theta, \underline{i})}{\int_{\Phi} -\log(\alpha_{i_1})\mathrm{d}\eta(\theta, \underline{i})}. \tag{6.75}$$

## 6.6 Dimension conservation in a special case

In this section we construct a measure $\eta$ such that $(\Phi, T, \eta)$ is an ECPS system.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. Let $X_1, X_2, \ldots : \Omega \to A$ be independent, identically distributed random variables such that for every $k$ $\mathbb{P}(X_k = i) = p_i$. Let $\underline{X} : \Omega \to \Sigma$ such that for $\omega \in \Omega$ $\underline{X}(\omega) = (X_1(\omega), X_2(\omega)\ldots)$. Then the distribution of $\underline{X}$ is $\underline{X}_*\mathbb{P} = \mu = (p_1, \ldots, p_m)^{\mathbb{Z}^+}$. Further, let $Z_n : \Omega \to \Lambda$ random variables such that $Z_n(\omega) = \Pi(\sigma^n \underline{X}(\omega))$.

**Lemma 6.30** *Let $\eta$ be the distribution of $(\delta_{Z_0}, \underline{X})$, that is $\eta = (\delta_{Z_0}, \underline{X})_*\mathcal{P}$. Then $(\Phi, T, \eta)$ forms an ECPS system with the corresponding measure $\rho = \delta_\nu$ on $\mathcal{P}(Q)$.*

**Proof:** First, we would like to see that $\eta$ is a $T$-invariant measure. Let $A \subseteq \Phi$ be an arbitrary measurable set. Because $\eta$ is the distribution function of $(\delta_{Z_0}, \underline{X})$, thus it is enough to see the elements of $A$ which is in the form $(\delta_{Z_0(\omega)}, \underline{X}(\omega))$. So,

$$A = \{(\delta_{Z_0(\omega)}, \underline{X}(\omega)) : \omega \in V\}. \tag{6.76}$$

The map $T$ acts on the image by $(\delta_{Z_0}, \underline{X})$ the way

$$T(\delta_{Z_0(\omega)}, \underline{X}(\omega)) = (\delta_{Z_1(\omega)}, \sigma\underline{X}(\omega)). \tag{6.77}$$

Then

$$T^{-1}(A) = \{(\delta_{Z_0(\omega)}, \underline{X}(\omega)) : (\delta_{Z_1(\omega)}, \sigma\underline{X}(\omega)) \in A\}. \tag{6.78}$$

The $\eta$ measure of $T^{-1}(A)$ is

$$\begin{aligned} \eta(T^{-1}A) = \mathbb{P}(\omega : (\delta_{\Pi(\sigma\underline{X}(\omega))}, \sigma\underline{X}(\omega)) \in A\}) = \\ = \mathbb{P}(\{\omega : (\delta_{Z_0(\omega)}, \underline{X}(\omega)) \in A\}) = \eta(A), \end{aligned} \tag{6.79}$$

because the distribution of $\underline{X}$ is invariant under $\sigma$.

We prove that $\eta$ defines an ergodic system. For this let $A$ be a $T$-invariant set, that is $T^{-1}A = A$ with respect to $\eta$. Then

$$(\delta_{Z_0(\omega)}, \underline{X}(\omega)) \in A \iff (\delta_{Z_0(\omega)}, \underline{X}(\omega)) \in T^{-1}A \iff (\delta_{Z_1(\omega)}, \sigma\underline{X}(\omega)) \in A \tag{6.80}$$

holds for $\eta$ almost every $(\delta_{Z_0(\omega)}, \underline{X}(\omega))$. We can write

$$A = \{(\delta_{Z_0(\omega)}, \underline{X}(\omega)) : \underline{X}(\omega) \in W\}, \tag{6.81}$$

because the set $W$ determines $A$. Then we can easily see that $\sigma^{-1}W = W$. Thus

$$\begin{aligned} \eta(A) = \mathbb{P}(\{\omega : (\delta_{Z_0(\omega)}, \underline{X}(\omega)) \in A\}) = \\ \mathbb{P}(\{\omega : \underline{X}(\omega) \in W\}) = \underline{X}_*\mathbb{P}(W) = \mu(W) = 0 \text{ or } 1, \end{aligned} \tag{6.82}$$

because $(\Sigma, \sigma, \mu)$ is an ergodic, $\sigma$-invariant dynamical system.

Last, we have to prove that the measure $\eta$ is adapted. Take an $f$ integrable function, then

$$\int_\Phi f(\theta, \underline{i}) \mathrm{d}\eta(\theta, \underline{i}) = \int_\Omega f(\delta_{Z_0(\omega)}, \underline{X}(\omega)) \mathrm{d}\mathbb{P}(\omega) = \int_\Omega f(\delta_{\Pi(\underline{X}(\omega))}, \underline{X}(\omega)) \mathrm{d}\mathbb{P}(\omega). \tag{6.83}$$

We can easily see that

$$\int_\Lambda f(\delta_{\Pi(\underline{X}(\omega))}, \Pi^{-1}y) \mathrm{d}\delta_{\Pi(\underline{X}(\omega))}(y). \tag{6.84}$$

Replace this into the above calculation

$$\begin{aligned}\int\limits_{\Phi} f(\theta,\underline{i})\mathrm{d}\eta(\theta,\underline{i}) &= \int\limits_{\Omega}\int\limits_{\Lambda} f(\delta_{\Pi(\underline{X}(\omega))},\Pi^{-1}y)\mathrm{d}\delta_{\Pi(\underline{X}(\omega))}(y) = \\ &= \int\limits_{\mathcal{P}(Q)}\int\limits_{\Lambda} f(\theta,\Pi^{-1}y)\mathrm{d}\theta(y)\mathrm{d}(\delta_{\Pi\underline{X}})_*\mathbb{P}(\theta) = \\ &= \int\limits_{\mathcal{P}(Q)}\int\limits_{\Lambda} f(\theta,y)\mathrm{d}(\Pi^{-1})_*\theta(y)\mathrm{d}\delta_{\nu}(\theta),\end{aligned} \tag{6.85}$$

where we use in the last step that $(\delta_{\Pi\underline{X}})_*\mathbb{P} = \delta_{\nu}$. This is because

$$(\delta_{\Pi\underline{X}})_*\mathbb{P}(\nu) = 1. \tag{6.86}$$

□

**Theorem 6.31** *The self-similar measure $\nu$ on $\Lambda$ satisfies dimension conservation and*

$$\dim_{\mathcal{F}}(\hat{\nu}) = \frac{\int_{\Lambda} -\ln\nu(Q_1(\mathbf{x}))\mathrm{d}\nu(\mathbf{x}) - \int_{\Lambda} -\ln\nu_{P'\mathbf{x}}(P''Q_1(\mathbf{x}))\mathrm{d}\nu(\mathbf{x})}{\int_{\Sigma} -\log(\alpha_{i_1})\mathrm{d}\mu(\underline{i})}. \tag{6.87}$$

**Proof:** The proof is the use of Lemma 6.30, Theorem 6.28 and Corollary 6.29. □

# Chapter 7

# Hausdorff-dimension of a self-similar measure of a forward separated system $\mathcal{S}_{\alpha,\beta,\gamma}$

**Theorem 7.1** *Let $\mathcal{S}_{\alpha,\beta,\gamma} = \{S_1, S_2, S_3\}$ be a forward separated system. (See Definition 3.1 and 3.3). The symbolic space is $\Sigma = \{1,2,3\}^{\mathbb{N}^+}$ and the natural projection of $\mathcal{S}_{\alpha,\beta,\gamma}$ is $\Pi_{\alpha,\beta,\gamma}$. Let $\mu = (p_1, p_2, p_3)^{\mathbb{N}^+}$ be a Bernoulli measure on $\Sigma$ for the probability vector $\mathbf{p} = (p_1, p_2, p_3)$. And let $\hat{\nu} = \Pi_{\alpha,\beta,\gamma *}\mu = \mu \circ \Pi^{-1}_{\alpha,\beta,\gamma}$ is the self-similar measure on the attractor with respect to $\mathbf{p}$. Then the Hausdorff dimension of $\hat{\nu}$ can be exactly determined as*

$$\dim_H(\hat{\nu}) = \frac{-(p_1\log(p_1) + p_2\log(p_2) + p_3\log(p_3)) + \phi(p_1,p_2,p_3)}{-(p_1\log(\alpha) + p_2\log(\beta) + p_3\log(\gamma))}, \tag{7.1}$$

*where*

$$\Phi(p_1,p_2,p_3) = \sum_{k=1}^{\infty}\left(\sum_{m=1}^{k}\binom{k-1}{m-1}\log\left(\frac{m}{k}\right)p_1^m p_2^{k-m}p_3 + \sum_{m=0}^{k-1}\binom{k-1}{m}\log\left(\frac{k-m}{k}\right)p_1^m p_2^{k-m}p_3\right).$$

**Proof:** We introduce easier notation for further use. We denote $\mathcal{S}_{\alpha,\beta,\gamma}$ with $\mathcal{S}$ and $\Pi_{\alpha,\beta,\gamma}$ is $\Pi$. The $K_{\alpha,\beta,\gamma}$ attractor of $\mathcal{S}$ is $K$. Let $R = S_3(K)$ and $L = S_1(K) \cup S_2(K)$.

Let $\mathcal{F} = \{F_1, F_2, F_3\}$ be a self-similar IFS on $[0,1]^2$ such that $F_1(\mathbf{x}) = \alpha\mathbf{x}$, $F_2(\mathbf{x}) = \beta\mathbf{x} + (0, 1-\beta)$, $F_3(\mathbf{x}) = \gamma\mathbf{x} + (1-\gamma, 0)$. Let $\tilde{\Pi}$ the natural projection of $\mathcal{F}$ and $\nu = \tilde{\Pi}_*\mu$.

The invariant measure $\hat{\nu}$ is the projection of the measure $\nu$. Thus using Theorem 6.31, it is enough to calculate

$$\int_{\Lambda} -\ln \nu_{P'\mathbf{x}}(P''Q_1(\mathbf{x}))\mathrm{d}\nu(\mathbf{x}). \tag{7.2}$$

Let $\gamma$ be the Borel $\sigma$-algebra on $[0,1] \subseteq \mathbb{R}$.

The $\sigma$-algebra $\gamma$ can be generated by countable many finite partition. Let

$$\mathcal{P}_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) : 0 \leq k \leq 2^n - 1 \right\}, \tag{7.3}$$

this is a finite partition of $[0,1]$ and $\bigvee_{i=1}^{\infty} \mathcal{P}_i$ is the Borel $\sigma$-algebra on $[0,1]$. Using this fact and Theorem 5.6, then we get that $\{\mu_{\underline{i}}\}_{\underline{i}\in\Sigma}$ system of conditional measures with respect to the $\sigma$-algebra $\Pi^{-1}\gamma$ exists. We can see that

$$-\int_{\Lambda} \ln \nu_{P'\mathbf{x}}(P''Q_1(\mathbf{x}))\mathrm{d}\nu(\mathbf{x}) = -\int_{\Sigma} \log(\mu_{\underline{i}}([i_1]))\mathrm{d}\mu(\underline{i}). \tag{7.4}$$

Using Theorem 5.6, we conclude that

$$\mu_{\underline{i}} = \lim_{n\to\infty} \frac{\mu|_{\Pi^{-1}(\mathcal{P}_n(\Pi(\underline{i})))}}{\mu(\Pi^{-1}(\mathcal{P}_n(\Pi(\underline{i}))))}, \tag{7.5}$$

where limit is meant in the weak-star topology. We know the property of weak-star convergence, that if $\nu_n, \nu$ are Borel probability measures on the compact metric space $X$ and $\lim_{n\to\infty} \nu_n = \nu$ in weak-star sense, then for all $U \subseteq X$ open and $Z \subseteq X$ closed $\liminf_{n\to\infty} \nu_n(U) \geq \nu(U)$ and $\limsup_{n\to\infty} \nu_n(Z) \leq \nu(Z)$ hold. Because $[k] \subseteq \Sigma$ is open and closed, then

$$\mu_{\underline{i}}([i_1]) = \lim_{n\to\infty} \frac{\mu(\Pi^{-1}(\mathcal{P}_n(\Pi(\underline{i}))) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_n(\Pi(\underline{i}))))}. \tag{7.6}$$

For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ enough large such that $\mathcal{P}_n(\Pi(\underline{i})) \subseteq B(\Pi(\underline{i}), \varepsilon)$.

We define for all $m, n = 0, 1, \ldots$ the set

$$\begin{aligned} H(m,n) = \{\underline{i} = (i_1, i_2, \ldots) \in \Sigma \, : \, i_l \neq 3 \quad \forall l = 1, \ldots m+n, \\ i_{m+n+1} = 3, \ |\{1 \leq k \leq m+n : i_k = 1\}| = m\}. \end{aligned} \tag{7.7}$$

We can see that $\Pi(H(m,n)) = S_1^m S_2^n S_3(K) = S_1^m S_2^n(R)$.

Let $\underline{i} \in H(m,n) \subseteq \Sigma$. Then $\Pi(\underline{i}) \neq 0$, because $i_{m+n+1} = 3$, thus $\Pi(\underline{i}) \in S_1^m S_2^n S_3(K) \subseteq [a(m,n), 1]$, where $a(m,n) = \frac{1}{2}(min\{\alpha, \beta\})^{m+n}$

If $(\max\{\alpha,\beta\})^{k+l} < \frac{a(m,n)}{2}$, then the sets $S_1^k S_2^l(R) \subseteq (0, \frac{a(m,n)}{2})$. Thus there exist $M(m,n) \in \mathbb{N}$ such that if $k+l > M(m,n)$, then $S_1^k S_2^l(R) \subseteq (0, \frac{a(m,n)}{2})$.

We introduce

$$\mathcal{H} = \left\{ S_1^k S_2^l(R) \,:\, k+l \leq M(m,n), \quad k,l = 0,1,\dots \right\}. \tag{7.8}$$

We can notice that $S_1^m S_2^n(R) \in \mathbb{H}$. The set $\mathcal{H}$ is finite and the elements are disjoint compact sets, thus there exists $\varepsilon_1 > 0$ such that

$$\forall H_1 \neq H_2 \in \mathcal{H} \quad N_{\varepsilon_1}(H_1) \cap N_{\varepsilon_1}(H_2) = \emptyset, \tag{7.9}$$

where $N_\varepsilon(H)$ means the $\varepsilon$ neighbourhood of the set $H$.

Using $K/\{0\} = \bigcup_{k,l=0}^{\infty} S_1^k S_2^l(R)$, then there exists $\varepsilon_2$ such that

$$\forall k,l = 0,1\dots \quad N_{\varepsilon_2}(S_1^m S_2^n(R)) \cap N_{\varepsilon_2}(S_1^k S_2^l(R)) = \emptyset, \tag{7.10}$$

Thus we can conclude that $B(\Pi(\underline{i}),\varepsilon_2) \cap K \subseteq S_1^m S_2^n(R)$. For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ enough large such that $\mathcal{P}_n(\Pi(\underline{i})) \subseteq B(\Pi(\underline{i}),\varepsilon)$. So there exist $N(n,m) \in \mathbb{N}$ such that for all $N(m,n) < z \quad \mathcal{P}_z(\Pi(\underline{i})) \subseteq B(\Pi(\underline{i}),\varepsilon_2)$. Moreover, for all $N(m,n) < z \quad \Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i})) \subseteq H(m,n)$.

Let $z > N(m,n)$ be fix, then

$$\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i}))) = E \times T_z, \text{ where}$$
$$E = \{(i_1,\dots,i_{m+n+1}) \,:\, i_{m+n+1} = 3, \ |\{k : i_k = 1\}| = m, \ |\{k : i_k = 2\}| = n\}$$
$$T_z = \{(j_1,j_2,\dots) \in \{1,2,3\}^{\mathbb{N}^+} \,:\, \exists \underline{k} \in \Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i}))) \quad j_{n+m+1} = k_{n+m+1},\, j_{n+m+2} = k_{n+m+2},\dots\}.$$

The above equation is true, because if $\underline{k} \in \Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i})))$, then if permute the first $n+m$ coordinates and in this way we get $\underline{k}'$, then $\Pi(\underline{k}) = \Pi(\underline{k}')$. Then

$$\frac{\mu(E \times T_z \cap [i_1])}{\mu(E \times T_z)} = \frac{\mu((E \cap [i_1]) \times T_z)}{\mu(E \times T_z)} = \frac{\mu([E \cap [i_1]\,])\mu(T_z)}{\mu([E])\mu(T_z)} = \frac{\mu([E \cap [i_1]\,])}{\mu([E])}, \tag{7.11}$$

where $[E] = \{\underline{i} \in \Sigma : \exists \underline{j} \in E \quad \exists \underline{k} \in \Sigma \quad \underline{i} = \underline{j} * \underline{k}\}$.

Because $\mu$ is a Bernoulli measure, then

$$\mu(E) = \frac{(m+n)!}{m!n!} p_1^m p_2^n p_3. \tag{7.12}$$

Suppose that $i_1 = 1$, then

$$\mu(E \cap [i_1]) = \frac{(m+n-1)!}{(m-1)!n!} p_1^m p_2^n p_3. \tag{7.13}$$

In this case for large $z$

$$\frac{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i})) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i}))))} = \frac{\mu(E \times T_z \cap [i_1])}{\mu(E \times T_z)} = \frac{m}{m+n}. \tag{7.14}$$

Suppose that $i_1 = 2$, then

$$\mu(E \cap [i_1]) = \frac{(m+n-1)!}{m!(n-1)!} p_1^m p_2^n p_3. \tag{7.15}$$

In this case for large $z$

$$\frac{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i})) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i}))))} = \frac{\mu(E \times T_z \cap [i_1])}{\mu(E \times T_z)} = \frac{n}{m+n}. \tag{7.16}$$

Suppose that $i_1 = 3$, then for enough large $z$ we get $\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i})) \cap [3] = \Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i})))$, thus for large $z$

$$\frac{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i})) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\underline{i}))))} = 1. \tag{7.17}$$

From the above observations, if $\underline{i} \in H(m,n)$ then

$$\mu_{\underline{i}}([i_1]) = \begin{cases} \frac{m}{m+n}, & \text{if } i_1 = 1, \\ \frac{n}{m+n}, & \text{if } i_1 = 2, \\ 1, & \text{if } i_1 = 3. \end{cases} \tag{7.18}$$

Using Kolmogorov 0-1 law we get

$$\mu(\bigcup_{m,n=0}^{\infty} H(m,n)) = 1. \tag{7.19}$$

The integral that we want to calculate is

$$\int_\Sigma \log(\mu_{\underline{i}}([i_1])) \mathrm{d}\mu(\underline{i}) =$$

$$= \sum_{k=1}^{\infty} \sum_{m=1}^{k} \log\left(\frac{m}{k}\right) \frac{(k-1)!}{(m-1)!(k-m)!} p_1^m p_2^{k-m} p_3 + \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \log\left(\frac{k-m}{k}\right) \frac{(k-1)!}{m!(k-m-1)!} p_1^m p_2^{k-m} p_3,$$

where we use combinatorics calculation. $\square$

# List of Figures

# Bibliography

[1] Furstenberg, H. Ergodic fractal measures and dimension conservation. *Ergodic Theory and Dynamical Systems 28*, 2 (2008), 405–422.

[2] Hutchinson, J. E. Fractals and self-similarity. *Indiana Univ. Math. J. 30*, 5 (1981), 713–747.

[3] Kamalutdinov, K., and Tetenov, A. Twofold cantor sets in r. *arXiv preprint arXiv:1802.03872* (2018).

[4] Rohlin, V. A. On the fundamental ideas of measure theory. *Amer. Math. Soc. Translation 1952*, 71 (1952), 55.