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DIMENSION OF SELF-AFFINE SETS WITH SINGULAR MATRICES

TDK DISSERTATION

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CONTENTS

1 PRELIMINARY

To determine the dimension of the attractor of a general iterated function systems (IFS) and a general graph-directed iterated function systems (GDIFS) is an open problem, but with some conditions, we can determine it. For self-similar separated IFS Hutchinson (showed that, when the cylinder sets are disjoint, then the Hausdorff dimension of the attractor is equal to the similarity dimension.) solved the problem, for self-similar separeted GDIFS Mauldin and Williams determined the Hausdorff dimension. Nowadays Bárány, Hochman and Rapaport determined the dimension of the attractor of self-affine IFS on the plane, when the matricies of the contracting similarity transformation are strongly irreducible. Our question is, what can we say about the dimension, when those matricies are singular?

In this TDK dissertation, we investigate the connection between self-affine IFS on the plane and GDIFS on the line. Using this connection we would like to state separation conditions, for which the dimension of the attractor can be defined with the sub-additive pressure.

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INTRODUCTION

2.1 HAUSDORFF-DIMENSION

In this section we collect some basic propertes of the Hausdorff dimension from the book [BBep] under preparation by Balázs Bárány, Károly Simon and Boris Solomyak. These properties are crutial tools to study the size of fractal sets.

Definition 2.1. Let $E \subset \mathbb{R}^d$ and $t \geq 0$. Then the collecton of set $\{A_i\}_{i=1}^{\infty}$ is a δ-cover of E for $\delta > 0$, if $E \subset \bigcup_{i=1}^{\infty} A_i$ and $|A_i| < \delta$. We call $\mathcal{H}^t(E)$ the t-dimensional Hausdorff measure of E if,

$$\mathcal{H}^t(E) = \lim_{\delta \to 0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t \,:\, E \subset \bigcup_{i=1}^{\infty} A_i,\, |A_i| \leq \delta \right\} \right\} \tag{1}$$

The measure $\mathcal{H}^t$ is a metric outer measure. But if we restrict $\mathcal{H}^t$ to the $\mathcal{H}^t$-measurable sets on the σ-algebra, we get a Borel measure, called the t-dimensional Hausdorff measure.

Lemma 2.2. *For any Borel set $E \subset \mathbb{R}^d$ and $0 \leq \alpha < \beta$ we obtain the following implications:*

$$\mathcal{H}^\alpha(E) < \infty \Rightarrow \mathcal{H}^\beta(E) = 0 \tag{2}$$

$$0 < \mathcal{H}^\beta(E) \Rightarrow \mathcal{H}^\alpha(E) = \infty \tag{3}$$

The previous lemma heuristically means, that for a given E if we choose t "too small", then $\mathcal{H}^t(E) = \infty$. Or if we choose t "too large", then $\mathcal{H}^t(E) = 0$. So, there exists a unique t_0, when the t-dimensional Hausdorff measure "drops down" from infinity to zero. The value of this unique t_0 is the Hausdorff dimension of E.

Definition 2.3. For any $E \subset \mathbb{R}^d$ Hausdorff-measurable set, the Hausdorff-dimension of E is the following,

$$\dim_H(E) = \inf\{t \,:\, \mathcal{H}^t(E) = 0\} = \sup\{t \,:\, \mathcal{H}^t(E) = \infty\}. \tag{4}$$

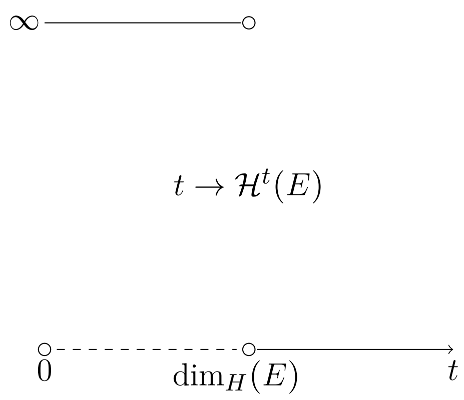

Figure 1: Hausdorff measure of the set E.

In the following we will see some properties of the Hausdorff dimension.

Lemma 2.4. *Some properties of the Hausdorff-dimension.*

1. *Every countable set has Hausdorff-dimension zero.*
2. *For every* $F \subset \mathbb{R}^d$ *we have* $\dim_H(F) \leq d$.
3. *If* $\mathcal{L}^d(E) > 0$ *then* $\dim_H(E) = d$.
4. *For any* $k < d$ *the* $k-$*dimensional smooth surface in* $\mathbb{R}^d$ *has a Hausdorff dimension* k.
5. *For a Lipschitz map* $f : \mathbb{R}^d \to \mathbb{R}^d$ *and a Borel set* $E \subset \mathbb{R}^d$ *we have* $\dim_H(f(E)) \leq \dim_H(E)$.
6. *Let* E *be a Borel set and let* $f : \mathbb{R}^d \to \mathbb{R}^d$ *be a bi-Lipschitz map. Then* $\dim_H(E) = \dim_H(f(E))$.
7. *Let* $\{E_i\}_{i=1}^{\infty}$ *be a sequence of Borel sets in* $\mathbb{R}^d$. *Then*

$$\dim_H\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_i \dim_H(E_i).$$

2.2 GRAPH DIRECTED SELF-SIMILAR IFS

Definition 2.5. Let $m \geq 2$ and $d \geq 1$ be integers, we say $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a self-similar iterated function system (IFS) a collection of contracting similarity transformations on $\mathbb{R}^d$ with contraction ratios $0 < r_i < 1$, $i = 1, \ldots, m$, if

$$\|S_i(x) - S_i(y)\| = r_i\|x - y\| \text{ for every } i \leq m \text{ and for every } x, y \in \mathbb{R}^d. \quad (5)$$

Definition 2.6. Let$\{I_i\}_{i=1}^N$ be the set of closed intervals of $\mathbb{R}$ and let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph, where $\mathcal{E}_{i,j}$ denotes the set of edges from i to j, and $\mathcal{V} = (I_j)_{j=1}^N$. Furthermore, a contracting similarity mapping $f_e : I_j \to I_i$ on the following metric spaces (I_j, d), $j = 1, \ldots, N$ with contraction ratio $r_e \in (0, 1)$. Then ,

$$d(f_e(x), f_e(y)) = r_e \cdot d(x, y) \text{ for every } x, y \in I_j. \tag{6}$$

We call the system $(\mathcal{G}, \{f_e, e \in \mathcal{E}\})$ as graph directed self-similar IFS (GDSSIFS).

Remark 2.7. The article by Mauldin and Williams [MW88] call the definition above a *geometric* graph directed construction.

Theorem 2.8. *Assume that the IFS $\mathcal{F} = \{f_i\}_{i=1}^m$ consits of functions $f_i : \mathbb{R} \to \mathbb{R}$ with Lipschitz constants $r_i < 1$. Let $\boldsymbol{b}_i$ be the fix point of f_i. Then for the closed ball*

$$B := \overline{B}(0, R) \text{ where } R := \max_i \left\{ \|b_i\| \cdot \frac{1 + r_i}{1 - r_i} \right\} \tag{7}$$

we have $f_i(B) \subset B$ for all $i = 1, \ldots, m$. Furthermore, we call the non-empty compact set

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\ldots,i_n)\in\{1,\ldots,m\}^n} f_{i_1\ldots i_n}(B) \tag{8}$$

the attractor or invariant set of the IFS $\mathcal{F}$. Then Λ is the unique non-empty compact set which satisfies

$$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda). \tag{9}$$

Theorem 2.9. *[MW88, Theorem 1] For each geometric construction, there exists a unique collection of compact sets, $(\Lambda_1, \ldots, \Lambda_N)$ such that for $N \in \mathbb{N}$*

$$\Lambda_i = \bigcup_{j=1}^{N} \bigcup_{e\in\mathcal{E}_{i,j}} f_e(\Lambda_j) \text{ for every } i = 1, \ldots, N. \tag{10}$$

The construction object is defined as

$$\Lambda := \bigcup_{i=1}^{N} \Lambda_i \tag{11}$$

called the attractor.

Definition 2.10. Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a graph with the set of verticies $\mathcal{V}$ and the set of edges $\mathcal{E}$. We say that, $\mathcal{G}$ is strongly connected if for every $i, j \in \mathcal{V}$, there is a direcetd path in the graph from i to j.

Definition 2.11. Let $(B_{MW}^{(s)})i, j = \left(b^{(s)}(i,j)\right)$ be an $n \times n$ matrix, where

$$b^{(s)}(i,j) = \begin{cases} 0, & \text{if } \mathcal{E}_{i,j} = \emptyset \\ \sum\limits_{e \in \mathcal{E}_{i,j}} r_e^s, & \text{otherwise} \end{cases}, \qquad \text{for every } \ s > 0.$$

Lemma 2.12. *Let $\rho(B_{MW}^{(s)})$ be the spectral radius of $B_{MW}^{(s)}$. The mapping $s \mapsto \rho(B_{MW}^{(s)})$ is continuous, strictly decreasing, $\rho(s) \geq 1$ if $s = 0$ and $\rho(B_{MW}^{(s)}) \xrightarrow[s \to \infty]{} 0$. Then, there exists a unique $s_0 \geq 0$ for which*

$$\rho(B_{MW}^{(s_0)}) = 1. \tag{12}$$

2.3 SIZE OF THE ATTRACTOR

Definition 2.13. For every $i = 1, \ldots, n$ the family of sets $\{f_{i,j}(\Lambda_j) : \quad (i,j) \in \varepsilon\}$ is non-overlapping if the interiors of the sets $f_{i,j}(\Lambda_j)$ are pairwise disjoint.

Theorem 2.14. *(Mauldin and Williams [MW88]) Consider a GDSSIFS such that, $\mathcal{G}$ is strongly connected and $\{f_{i,j}(\Lambda_j) : \quad (i,j) \in \mathcal{E}\}$ is a non-overlapping family of sets, then for every $i = 1, \ldots, N$ we have*

$$\dim_H \Lambda_i = \dim_B \Lambda_i = s_0 \ \textit{and} \ 0 < \mathcal{H}^{s_0}(\Lambda_i) < \infty$$

2.4 SELF-AFFINE SETS

Definition 2.15. Consider the following sets X, Y and vector spaces V, W. Let (X, V) and (Y, W) be two affine spaces over a same field. We say a map $f : X \to Y$ is an affine map, if there exists a linear map $m_f : V \to W$ such that,

$$m_f(x - y) = f(x) - f(y) \text{ for every } x, y \in X.$$

Definition 2.16. Let T be a $d \times d$ real valued matrix, then the singular value function $\varphi^t(T)$ of T can be defined,

$$\varphi^t(T) := \begin{cases} \alpha_1 \cdots \alpha_{k-1} \cdot \alpha_k^{t-(k-1)}, & \text{if } k-1 < t \leq k \leq d; \\ (\alpha_1 \cdots \alpha_d)^{\frac{t}{d}}, & \text{if } t \geq d, \end{cases}$$

where $\alpha_1 \geq \cdots \geq \alpha_d$ are the singular values of T.

For a finite word $\bar{\imath}$, denote $|\bar{\imath}|$ the length of $\bar{\imath}$.

For any finite word $\bar{\imath} = i_1 \cdot \ldots i_n$ we denote by $A_{\bar{\imath}}$ the finite product $A_{i_1} \cdot \ldots \cdot A_{i_n}$ for every $i = 1, \ldots, n$ and for every $n \in \mathbb{N}$.

Definition 2.17. We denote $s(A_1, \ldots, A_m)$ the affinity dimension of the self-affine IFS $\mathcal{F} = \{A_i x + t_i\}_{i=1}^m$ that is,

$$s(A_1, \ldots, A_m) = \inf \left\{ t > 0 : \sum_{m=0}^{\infty} \sum_{|\bar{\imath}|=m} \varphi^t(A_{\bar{\imath}}) < \infty \right\}.$$

Falconer defined the sub-additive pressure function $P : \mathbb{R}^+ \to \mathbb{R}$,

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\bar{\imath}|=n} \varphi^s(A_{\bar{\imath}}) \right), \tag{13}$$

The function system $\frac{1}{n} \log(\sum_{|\bar{\imath}|=n} \varphi^s(A_{\bar{\imath}}))$ gives the exponential growth rate of the sum in the definition of the affinity dimension corresponding to the natural covering cylinders, see [Fal88]. We know that, for all t the limit of this function system exists and finite. The affinity dimension $s(A_1, \ldots, A_n)$ what we defined in Definition is the unique zero of the sub-additive pressure $P(s)$.

In the following we show a few conditon, when the Hausdorff dimension will equal to the affinity dimension.

Theorem 2.18. *([Fal88]) For $m \geq 2$ let $\{A_1, \ldots, A_m\}$ be non-singular $d \times d$ matrices, such that their Euklidean norm satisfies,*

$$\|A_i\| < \frac{1}{2}, \qquad i = 1, \ldots, m.$$

For $t := (t_1, \ldots, t_m) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ define the m-parameter family of self-affine IFS on $\mathbb{R}^d$,

$$\mathcal{F}^t := \{A_i x + t_i\}_{i=1}^m.$$

Let Λ^t be the attractor of $\mathcal{F}^t$. Then for $\mathcal{L}^{md}$-almost all t we have

$$\dim_H(\Lambda) = \dim_B(\Lambda) = \min\{d, s(A_1, \ldots, A_m)\}.$$

Definition 2.19. Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a contracting IFS and Λ the attractor, then the Strong Separation Property holds for $\mathcal{F}$ if,

$$f_i(\Lambda) \cap f_j(\Lambda) = \emptyset \text{ for all } i \neq j.$$

Definition 2.20. A collection of matricies $\mathcal{A} = \{A_1, \ldots, A_m\}$ is strongly irreducible, if there is no finite collection of $V_1, \ldots, V_k$ of proper subspaces such that $A_i \left(\bigcup_{j=1}^k V_j \right) = \bigcup_{j=1}^k V_j$ for every $i = 1, \ldots, m$.

Theorem 2.21. *(Bárány, Hochman and Rapaport [BHR17]) Let* $\mathcal{F} = \{f_i(x) = A_i x + t_i\}_{i=1}^m$ *be a planar self-affine IFS, such that* $\mathcal{F}$ *satisfies the strong separation property and the collection of matrices* $\mathcal{A} = \{A_1, \ldots, A_m\}$ *is strongly irreducible. Then,*

$$\dim_H(\Lambda) = \dim_B(\Lambda) = s(A_1, \ldots, A_m).$$

3 SELF-AFFINE SETS WITH SINGULAR MATRICES ONLY

In this Chapter we show our first results about the connection between self-affine sets and graph directed iterated function systems. First, let us show an example for self-affine IFS having singular matrices.

Example 3.1. Let $\mathcal{F} = \{f_i(x) = A_i x + t_i\}_{i=1}^2$ be an IFS, such that

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, t_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, t_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

and $x \in [0,1]^2$. Denote $[0,1] := I_1$ and $[1,0] := I_2$, then

$$f_1(I_1) = [\frac{1}{2}, 1], \qquad f_1(I_2) = \{\frac{1}{2}\}$$
$$f_2(I_1) = \{\frac{1}{2}\}, \qquad f_2(I_2) = [\frac{1}{2}, 1].$$

To determine the attractor, we need to continue applying f_1 and f_2 on the intervals. By symmetry we can split the attractor into two parts, and we can investigate how it behaves on each axis. Figure 2 shows this behaviour on the **x** axis by each layer means an iteration.

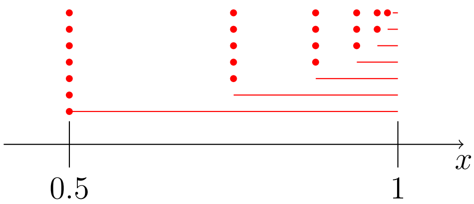

Figure 2: Iterates on x axis.

Let Λ_x and Λ_y be non-empty compact sets, such that

$$\Lambda_x = \bigcup_{n=1}^{\infty} \{(1 - 2^{-n}, 0)\} \cup \{(1,0)\}$$
$$\Lambda_y = \bigcup_{n=1}^{\infty} \{(0, 1 - 2^{-n})\} \cup \{(0,1)\} \tag{14}$$

Then by equation (11) the attractor

$$\Lambda = \Lambda_x \bigcup \Lambda_y. \tag{15}$$

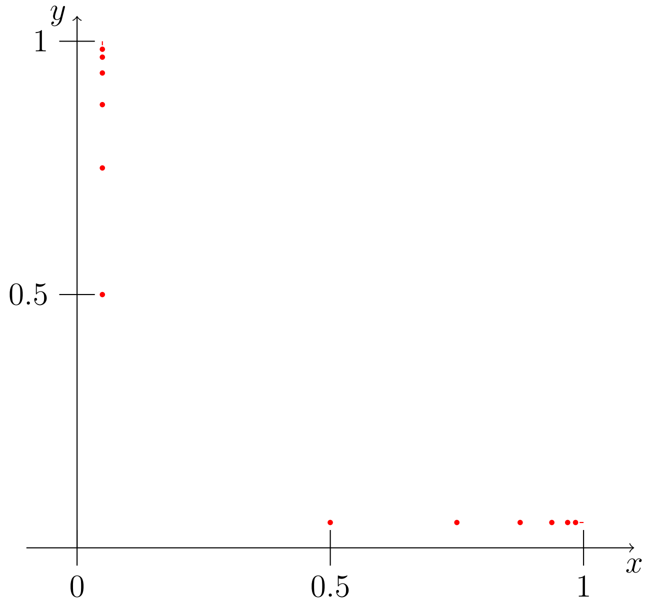

Figure 3: Attractor of IFS

We would like to illustrate the Mauldin Williams theorem, from previous chapter. By **Definition 2.11** the $B^{(s)}_{MW}$ matrix will be

$$B^{(s)}_{MW} = \begin{pmatrix} \frac{1}{2}^{s} & 0 \\ 0 & \frac{1}{2}^{s} \end{pmatrix}. \tag{16}$$

Then

$$\rho\left(B^{(s_0)}_{MW}\right) = 1 \text{ if and only if } \left(\frac{1}{2}\right)^{s_0} = 1 \text{ then } s_0 = 0. \tag{17}$$

Now we need to apply Mauldin and Williams **Theorem 2.14**, then we get

$$\dim_H(\Lambda) = s_0 = 0. \tag{18}$$

Unless typical cases the attractor is not a perfect set. Recall **Lemma 2.4**, property 1 says, if the attractor is a countable set, then its Hausdorff dimension is zero, exacly what we get.

3.1 SINGULAR CASE

In this section we show the dimension of an IFS when all the matrices of contracting functions are singular. Consider an IFS $\mathcal{F} = \{f_i = A_i x + t_i\}_{i=1}^{n}$ and for every A_i, $\det(A_i) = 0$.

Definition 3.2. Let A be a singular matrix such that V be a vector space, then

$$\|A|V\| = \sup_{x\in V} \frac{\|Ax\|}{\|x\|}$$

Let A_i be a matrix with $rank(A_i) = 1$ for any $i = 1, \dots, n \in \mathbb{N}$ and consider any vector $\underline{x} \in \mathbb{R}^2$ such that,

$$A_i = \begin{pmatrix} c_i \cdot a_i & c_i \cdot b_i \\ d_i \cdot a_i & d_i \cdot b_i \end{pmatrix} = \begin{pmatrix} c_i \\ d_i \end{pmatrix} \cdot \begin{pmatrix} a_i & b_i \end{pmatrix} \tag{19}$$

for some c_i, d_i constants. Then from equation (19) we easily get

$$Im(A_i) = span\left\langle \begin{pmatrix} c_i \\ d_i \end{pmatrix} \right\rangle \text{ and } Ker(A_i) = span\left\langle \begin{pmatrix} a_i \\ b_i \end{pmatrix} \right\rangle.$$

In the following we will se that, the image of a product $A_{\bar{\imath}}$, depends only on the first factor of the multiplication A_{i_1}.

Lemma 3.3. *Let $A_{\bar{\imath}}$ be an arbitrary product of the matrices $A_{i_1}, \dots, A_{i_n}$ for every $n \in \mathbb{N}$. Then,*

$$Im(A_{\bar{\imath}}) = Im(A_{i_1}) \text{ or } Im(A_{\bar{\imath}}) = \{0\}.$$

Proof. For $n = 1$,

$$A_{\bar{\imath}} = A_{i_1} \text{ then } Im(A_{\bar{\imath}}) = Im(A_{i_1}).$$

For $n = 2$,

$$\begin{aligned} A_{\bar{\imath}} &= A_{i_1} \cdot A_{i_2} = \\ &= \begin{pmatrix} c_{i_1} \cdot a_{i_2} (c_{i_2} \cdot a_{i_1} + b_{i_1} \cdot d_{i_2}) & c_{i_1} \cdot b_{i_2} (c_{i_2} \cdot a_{i_1} + b_{i_1} \cdot d_{i_2}) \\ d_{i_1} \cdot a_{i_2} (c_{i_2} \cdot a_{i_1} + b_{i_1} \cdot d_{i_2}) & d_{i_1} \cdot b_{i_2} (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2} \cdot c_{i_2}) \end{pmatrix}. \end{aligned}$$

The image of $A_{\bar{\imath}}$

$$\begin{aligned} A_{\bar{\imath}} \cdot \underline{x} &= \begin{pmatrix} c_{i_1} \cdot (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2}) \cdot (a_{i_2} \cdot x + b_{i_2} \cdot y) \\ d_{i_1} \cdot (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2}) \cdot (a_{i_2} \cdot x + b_{i_2} \cdot y) \end{pmatrix} = \\ &= (a_{i_1} \cdot c_{i_2} + b_{i_1} \cdot d_{i_2}) \cdot (a_{i_2} \cdot x + b_{i_2} \cdot y) \begin{pmatrix} c_{i_1} \\ d_{i_1} \end{pmatrix}. \end{aligned}$$

Then unless $Ker(A_{i_1}) = Im(A_{i_2})$,

$$Im(A_{\bar{\imath}}) = span\left\langle \begin{pmatrix} c_{i_1} \\ d_{i_1} \end{pmatrix} \right\rangle = Im(A_{i_1}).$$

For $|\bar{\imath}| = n$ we can split the product $A_{\bar{\imath}}$ into two parts

$$A_{\bar{\imath}} = A_{i_1} \cdot A_{|\bar{\imath}|-1}.$$

Then using the case $n = 2$ and induction we prove the lemma.

□

Recall the definition of $A_{\bar{\imath}}$ from page 6, but consider every matrix A_i are singular for every $i = 1, \ldots, n$.

Definition 3.4. Let $\left(B^{(s)}\right)_{i,j} = \left(b^{(s)}(i,j)\right)$ be an $n \times n$ matrix for every $n \in \mathbb{N}$, where

$$b^{(s)}(i,j) = \|A_i|Im(A_j)\|^s \quad \text{for every } i,j = 1,\ldots,n. \tag{20}$$

Lemma 3.5. *For every singular matrices A and B and for every subspace V*

$$\|AB|V\| = \|A|Im(B)\| \cdot \|A|V\|.$$

In particular for every finite word $\bar{\imath} = (i_1, \ldots, i_n)$,

$$\|A_{i_1}A_{i_2}\ldots A_{i_{n-1}}|Im(A_{i_n})\| = \|A_{i_1}|Im(A_{i_2})\| \ldots \|A_{i_{n-1}}|Im(A_{i_n})\|.$$

Proof. For every $v \in V$ and $\|v\| = 1$,

$$\begin{aligned} \|AB|V\| &= \|AB\underline{v}\| = \\ &= \begin{cases} 0, & \text{if } Ker(B) = V \\ \frac{\|AB\underline{v}\|}{\|B\underline{v}\|} \cdot \|B\underline{v}\| = \|A|Im(B)\| \cdot \|B|V\| & \text{if } Ker(B) \neq V \end{cases} \end{aligned} \tag{21}$$

If $Ker(B) = V$ then $\|B|V\| = 0$ and $\|A|Im(B)\| \cdot \|B|V\| = 0$. In general case, for every $\underline{v} \in Im(A_{i_n})$ and $\|\underline{v}\|$ if $A_{i_2} \cdots A_{i_{n-1}}\underline{v} \neq 0$,

$$\begin{aligned} &\|A_{i_1} \cdots A_{i_{n-1}}|Im(A_{i_n})\| = \|A_{i_1} \cdots A_{i_n}\underline{v}\| = \\ &= \|A_{i_1} \frac{A_{i_2}\ldots A_{i_{n-1}}\underline{v}}{\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\|}\| \cdot \|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\| \end{aligned}$$

We know that $\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\| \in Im(A_{i_2}\ldots A_{i_{n-1}})$ by **Lemma 3.5** $Im(A_{i_2}\ldots A_{i_{n-1}}) = Im(A_{i_2})$ then

$$\|A_{i_1} \frac{A_{i_2}\ldots A_{i_{n-1}}\underline{v}}{\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\|}\| = \|A_{i_1}|Im(A_{i_2})\|.$$

By induction we use the previous calculation for $\|A_{i_2}\ldots A_{i_{n-1}}\underline{v}\|$. □

Lemma 3.6. *Let $A_1, \dots, A_n$ be singular matrices of an IFS $\mathcal{F}$. Then an elemnt of the matrix $B^{(s)}$ to the power $m > 1$ is the following,*

$$((B^{(s)})^m)_{i,j} = \left(\sum_{|\bar{j}|=m-1} \|A_i \cdot A_{\bar{j}}|Im(A_j)\|^s \right) \quad \textit{for every } i, j = 1, \dots, n. \tag{22}$$

Proof. Using the rule of matrix multiplication and **Lemma 3.5**. □

Proposition 3.7. *Let $A_1, \dots, A_n$ be singular matrices of an IFS $\mathcal{F}$ and $\alpha(A_i)$ be maximum of the singular values of A_i. Then there exists constants $m, M > 0$ such that for every $k \geq 1$,*

$$m \cdot \|(B^{(s)})^k\|_1 \leq \sum_{|\bar{\imath}|=k+1} \alpha(A_{\bar{\imath}})^s \leq M \cdot \|(B^{(s)})^k\|_1. \tag{23}$$

Remark 3.8. In **Proposition 3.7** for a matrix A we use

$$\|A\|_1 = \sum_{i,j} |a_{ij}|.$$

Proof. Since for every $i = 1, \dots, n$ the matrices A_i are singular, then $\alpha(A_i) = \alpha_1(A_i)$. We know that, $\alpha_1(A_i) = \|A_i\|$. Observe that

$$\sum_{|\bar{\imath}|=k+1} \alpha_1(A_{\bar{\imath}}) = \sum_{i,j=1,\dots,n} \sum_{|\bar{j}|=k-1} \|A_i A_{\bar{j}}|Im(A_j)\| \cdot \|A_j\|. \tag{24}$$

Then denote $M = \max_j \{\|A_j\|\}$ and $m = \min_j \{\|A_j\|\}$. By using m and M in equation (24)

$$m \cdot \|(B^{(s)})^k\|_1 \leq \sum_{|\bar{\imath}|=k+1} \alpha_1(A_{\bar{\imath}})^s \leq M \cdot \|(B^{(s)})^k\|_1. \tag{25}$$

□

Proposition 3.9. *Let $A_1, \dots, A_n$ be singular matrices. Then,*

$$P(s) = \log\left(\rho(B^{(s)})\right) \quad \textit{for } 0 \leq s \leq 1. \tag{26}$$

Proof. By using inequality (23) and taking logarithm and dividing by n we have

$$\frac{1}{n} \log(\rho(B^{(s)})^n) \leq \frac{1}{n} \log\left(\sum_{|\bar{\imath}|=n+1} \alpha_1(A_{\bar{\imath}})^s \right) \leq \frac{1}{n} \log(\rho(B^{(s)})^n).$$

Then by Gelfand's formula $\|B^n\|^{1/n} \to \rho(B)$ as $n \to \infty$. Thus, we have

$$P(s) = \log(\rho(B^{(s)})). \tag{27}$$

□

Example 3.10. This example shows a connection between graph directed and IFS . Consider the following matrices

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix},\ A_3 = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix}$$

and the following contracting similarity transformations in two different cases.

I.

$$\varphi_1(\underline{x}) = A_1\underline{x} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix};$$

$$\varphi_2(\underline{x}) = A_2\underline{x} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix};$$

$$\varphi_3(\underline{x}) = A_3\underline{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

II.

$$\varphi_1(\underline{x}) = A_1\underline{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$\varphi_2(\underline{x}) = A_2\underline{x} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix};$$

$$\varphi_3(\underline{x}) = A_3\underline{x} + \begin{pmatrix} \delta \\ 0 \end{pmatrix};$$

for every $\underline{x} \in [0,1]^2$ and for some $1 > \delta > \frac{1}{2}$. Let $\mathcal{F} = \{\varphi_i\}_{i=1}^{3}$ be an IFS. If we investigate the Image spaces of the transformations, we observe

I.

$$Im(\varphi_1) = span\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$Im(\varphi_2) = span\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$Im(\varphi_3) = span\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

II.

$$Im(\varphi_1) = span\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

$$Im(\varphi_2) = span\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$Im(\varphi_3) = span\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle + \begin{pmatrix} \delta \\ 0 \end{pmatrix}$$

The following figures represent the connection between Graph-directed IFS and self-similar IFS Maps from the Graph-directed IFS have the form $f_{i,j} = \varphi_i|_{\varphi_j([0,1]^2)}$ for every $i, j = 1, 2, 3$, see in **Definition 2.6**.

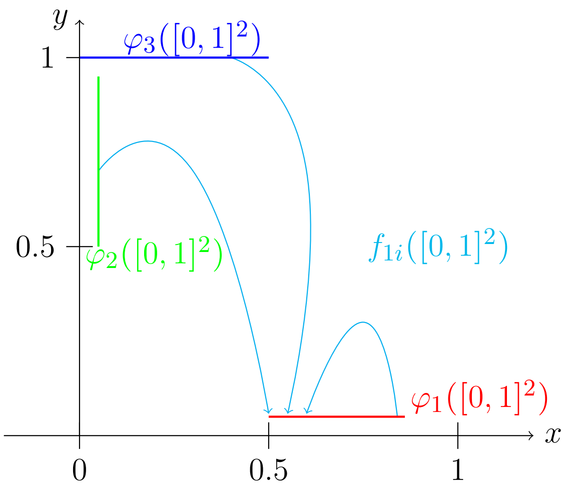

Figure 4: Graph of IFS in case I.

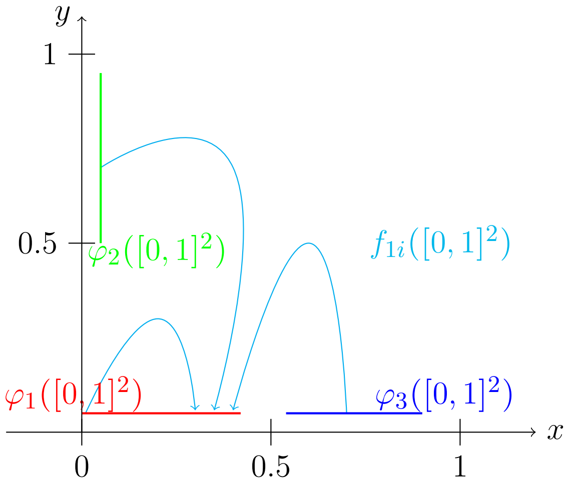

Figure 5: Graph of IFS in case II.

Investigate in both case the previously defined matrices $B^{(s)}$ see on page 11 and $B^{(s)}_{MW}$ see on page 5.

I.

$$B^{(s)} = \begin{pmatrix} a^s_{11} & a^s_{12} & a^s_{11} \\ a^s_{21} & a^s_{22} & a^s_{21} \\ b^s_{11} & b^s_{12} & b^s_{11} \end{pmatrix}$$

$$B^{(s)}_{MW} = \begin{pmatrix} a^s_{11} & a^s_{12} & a^s_{11} \\ a^s_{21} & a^s_{22} & a^s_{21} \\ b^s_{11} & b^s_{12} & b^s_{11} \end{pmatrix}$$

II.

$$B^{(s)} = \begin{pmatrix} a^s_{11} & a^s_{12} & a^s_{11} \\ a^s_{21} & a^s_{22} & a^s_{21} \\ b^s_{11} & b^s_{12} & b^s_{11} \end{pmatrix}$$

$$B^{(s)}_{MW} = \begin{pmatrix} a^s_{11} + b^s_{11} & a^s_{12} + b^s_{12} \\ a^s_{21} & a^s_{22} \end{pmatrix}$$

The previous example showed that the matrices $B^{(s)}$ and $B^{(s)}_{MW}$ are not necessarily equals. The difference between the matrices is, $B^{(s)}_{MW}$ can happen, that for an arbitrary image space there are more than one transformation. Because of this in $B^{(s)}_{MW}$ the corresponding rows and colums will collapse into a sum. Then, from $B^{(s)}$ to $B^{(s)}_{MW}$ we can construct an eigenvector for the same positive eigenvalue. By Perron-Frobenius theorem this vector is unique and positive, the spectral radius of the matrices should equal. This is necessary step which allows us to use the Mauldin-Williams theorem 2.14 for the matrix $B^{(s)}$.

Lemma 3.11. *Let* $\{\varphi_i(\underline{x}) = A_i\underline{x} + t_i\}_{i=1}^m$ *be an IFS for every* $\underline{x} \in \mathbb{R}^2$ *and for every* $m > 1$*. Consider the following matrices* $B_{MW}^{(s)}$ *and* $B^{(s)}$ *for the IFS. Then the spectral radius of the matrices are equal,*

$$\rho = \rho(B_{MW}^{(s)}) = \rho(B^{(s)}). \tag{28}$$

Proof. By definition $Im(\varphi_i) = Im(A_i) + t_i$ for every $i = 1, \ldots, m$. Let I be the set of all image spaces of φ_i which are distinct. In other words

$$I = \{Im(\varphi_i)\} := \{V_1, \ldots, V_M\}.$$

Furthermore, by definition $\mathcal{E}_{i,j} := \mathcal{E}_i = \{k : Im(\varphi_k) = V_i\}$. Since for every i, V_i is a hyperspace, we need to construct a set with subspaces to determine the norm of the matrix $B_{MW}^{(s)}$. For every V_i there is a unique W_i subspace in $\mathbb{R}^2$ such that for every $\underline{x}, \underline{y} \in V_i$, $\underline{x} - \underline{y} \in W_i$. Then the elements of the matrix $B_{MW}^{(s)}$ will be

$$\left(B_{MW}^{(s)}\right)_{i,j} = \sum_{k \in \mathcal{E}_i} \|A_i | W_j\|^s \quad \text{for every } i, j = 1, \ldots, M.$$

On the other hand, the elements of the matrix $B^{(s)}$ will be

$$\left(B^{(s)}\right)_{i,j} = \|A_i | Im(A_j)\|^s \quad \text{for every } i, j = 1, \ldots, m.$$

There exists a unique vector $\underline{v} \in \mathbb{R}^m$ such that $\|\underline{v}\| = 1$ and for every $i = 1, \ldots, m$, $v_i > 0$. The spectral radius of $\rho(B^{(s)}) = \rho$ and by the Perron Frobenius theorem,

$$B^{(s)}\underline{v} = \rho\underline{v}.$$

Now we construct a vector, by $\underline{v}$ then we will see this constructed vector is an eigenvector of $B_{MW}^{(s)}$.

Let $\underline{z} \in \mathbb{R}^M$ be a vector such that $z_j = \sum_{k \in \mathcal{E}_i} v_k$. Then

$$\left(B_{MW}^{(s)}\underline{v}\right)_i = \sum_{j=1}^M \sum_{k \in \mathcal{E}_i} \|A_k | W_j\|^s z_j = \sum_{j=1}^M \sum_{k \in \mathcal{E}_i} \sum_{l \in \mathcal{E}_j} \|A_k | W_j\|^s v_l =$$

$$\sum_{j=1}^M \sum_{k \in \mathcal{E}_i} \sum_{l \in \mathcal{E}_j} \|A_k | Im(A_l)\|^s v_l = \sum_{k \in \mathcal{E}_i} \sum_{j=1}^M \sum_{l \in \mathcal{E}_j} \|A_k | Im(A_l)\|^s v_l =$$

$$\sum_{k \in \mathcal{E}_i} \sum_{l=1}^m \|A_k | Im(A_l)\|^s v_l = \sum_{k \in \mathcal{E}_i} \rho v_k = \rho \underline{z}_i.$$

Then for every $z_i > 0$, $\underline{z}$ is an eigenvector of $B_{MW}^{(s)}$ with $\rho > 0$ eigenvalue. So, by Perron-Frobenius theorem

$$\rho = \rho(B_{MW}^{(s)}).$$

□

3.2 SEPARATION CONDITION

To proof that, the value of the sub-additive pressure $P(s)$ equals to the Hausdorff-dimension of the attractor, we need to state a separation condition for $\mathcal{F}$.

Lemma 3.12. *Let $\mathcal{F} = \{\varphi_i = A_i x + t_i\}_{i=1}^n$ be an IFS for every matrix A_i is singular. If the Strong Separation Property holds then s_0 is a unique solution for*

$$\rho(B^{(s_0)}) = 1 \textit{ and } \dim_H(\Lambda) = s_0. \tag{29}$$

Proof. If $\varphi_i \circ \varphi_j([0,1]^2) \cap \varphi_k \circ \varphi_l([0,1]^2) = \emptyset$ if $(i,j) \neq (k.l)$ for every $i,j,k,l = 1,\ldots,n$. Then by Mauldin-Williams theorem 2.14, s_0 is a unique solution for the equation

$$\rho(B_{MW}^{(s_0)}) = 1.$$

By **Lemma 3.11**, $\rho = \rho(B_{MW}^s) = \rho(B^{(s)})$. And using Mauldin-Williams theorem, s_0 is the unique solution for the equation $\rho(B^{(s_0)}) = 1$. Then

$$\dim_H(\Lambda) = s_0. \tag{30}$$

$\square$

4 MIXED CASE

In this section we will work with self-affine IFS which contains singular and regular matrices. We define the regular pressure P_{reg} and the singular pressure P_{sing}. The goal of this Chapter to show that the singularity dimension can be expressed by using P_{reg} and P_{sing}.

We extend the definition of $B^{(s)}$ from page 11, but in this case instead of singular matrices we have regular and singular too. Let Σ^*_{reg} be the set of all finite words which corresponds to only regular matrices, and let Σ_{sing} be the set of symbols corresponding to a singular matrices, and similarly let Σ^* be the set of every finite word. Let $B^{(s)}$ be an $n \times n$ matrix and $n = |\Sigma_{sing}|$, where

$$(B^{(s)})_{i,j} = \left(\sum_{\bar{\imath} \in \Sigma^*_{reg}} \|A_i A_{\bar{\imath}} | Im(A_j)\|^s \right)_{i,j \in \Sigma_{sing}} \tag{31}$$

Definition 4.1. We use the notation of

$$P_{sing}(s) = \log \left(\rho(B^{(s)}) \right) \tag{32}$$

$$P_{reg}(s) = \lim_{k \to \infty} \frac{1}{k} \log \left(\sum_{\bar{\jmath} \in \Sigma^*_{reg}} \varphi^s(A_{\bar{\jmath}}) \right), \quad \text{for } k \in \mathbb{N}. \tag{33}$$

Heuristically P_{reg} means we use only the regular matrices, from the definition of $B^{(s)}$ in equation (31) and $P_{sing}(s)$ is the pressure generated by $B^{(s)}$ from equation (31).

Definition 4.2. If $B^{(s)}$ is well defined, then s_{sing} will be the solution for the equation $\rho(B^{(s_{sing})}) = 1$ and s_{reg} be the root of $P_{reg}(s)$.

Proposition 4.3. *Let A_j be a regular matrix and $A_{\bar{\jmath}} = A_{j_1} \cdot \ldots \cdot A_{j_n}$ for every $j = 1, \ldots, n$ and $n \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists $c > 0$ such that*

$$c^{-1} \cdot e^{n(P_{reg}(s)+\varepsilon)} \leq \sum_{\bar{\imath} \in \Sigma_{reg}} \varphi^s(A_{\bar{\imath}}) \leq c \cdot e^{n(P_{reg}(s)+\varepsilon)}. \tag{34}$$

Proof. By definition of $P(s)$, there exists $N, n > 0$, for every $\varepsilon > 0$ if $n > N$ then,

$$\left| \frac{1}{n} \log \left(\sum_{|\bar{\imath}|=n} \varphi^s(A_{\bar{\imath}}) \right) - P(s) \right| < \varepsilon. \tag{35}$$

□

Lemma 4.4. *The matrix $B^{(s)}$ is well defined for every $s > s_{reg}$.*

Proof.
$$\sum_{\bar{\jmath}\in\Sigma^*_{reg}} \|A_i A_{\bar{\jmath}} | Im(A_j)\|^s \le \sum_{\bar{\jmath}\in\Sigma^*_{reg}} \|A_i A_{\bar{\jmath}}\|^s. \tag{36}$$

By the sub-additivity of the norm,

$$(36) \le \sum_{\bar{\jmath}\in\Sigma^*_{reg}} \|A_i\| \cdot \|A_{\bar{\jmath}}\|^s. \tag{37}$$

By **Proposition 4.3**,

$$(37) \le \|A_i\|^s \sum_{n=0}^{\infty} c \cdot e^{n(P_{reg}(s)+\varepsilon)} < \infty, \text{ if } P_{reg}(s) + \varepsilon < 0. \tag{38}$$

But there exists such ε if $s > s_{reg}$.

□

Lemma 4.5. *If the following conditions are holds,*

there exists $j_1 \neq j_2$ and $Im(A_{j_1}) \neq Im(A_{j_2})$
there exists $i_1 \neq i_2$ and $Ker(A_{i_1}) \neq Ker(A_{i_2})$

and there exists $c > 0$ for every $\bar{\jmath} \in \Sigma_{reg}$, there exists $i, j \in \Sigma_{sing}$

$$\|A_i A_{\bar{\jmath}} | Im(A_j)\| \ge c \cdot \|A_{\bar{\jmath}}\|. \tag{39}$$

Proof. For every $c > 0$ there exists $\bar{\jmath} \in \Sigma_{reg}$ and for every $i, j \in \Sigma_{sing}$ then

$$\|A_i \frac{A_{\bar{\jmath}}}{\|A_{\bar{\jmath}}\|} | Im(A_j)\| \le c. \tag{40}$$

If we choose $c = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$, then by compactness for $\bar{\jmath}_n$ there exists a subsequence n_k, $k \in \mathbb{N}$, such that

$$\frac{A_{\bar{\jmath}_{n_k}}}{\|A_{\bar{\jmath}_{n_k}}\|} \to \tilde{A}. \tag{41}$$

Then for every $i, j \in \Sigma_{sing}$ such $\|A_i \tilde{A} | Im(A_j)\| = 0$. If there exists $j_1 \neq j_2$ and $Im(A_{j_1}) \neq Im(A_{j_2})$, then $A_i \tilde{A} = 0$. It is possible, when $Ker(A_i) = Im(\tilde{A})$. We get a contradiction if there exists $i_1 \neq i_2$ and $Ker(A_{i_1}) \neq Ker(A_{i_2})$.

□

By Falconer **Theorem2.18** [Fal88]

$$s_0 = \inf\{s > 0 : \sum_{\bar{\jmath}\in\Sigma_*} \varphi^s(A_{\bar{\jmath}}) < \infty\}. \tag{42}$$

Proposition 4.6. *Consider* $s_0 = \inf\{s > 0 : \sum_{\bar{j}\in\Sigma_*} \varphi^s(A_{\bar{j}}) < \infty\}$, s_{reg} *and* s_{sing} *as defined on page 17, then*

$$s_0 = \begin{cases} s_{reg} \text{ if } s_{reg} > 1 \\ \min\{1, s_{sing}\} \text{ if } s_reg \leq 1. \end{cases}$$

Proof. For every $s > 1$

$$\sum_{\bar{\imath}\in\Sigma_*} \varphi^s(A_{\bar{\imath}}) = \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}})$$

If $s_0 > 1$ then $s_0 = s_{reg}$.
If $s_0 \leq 1$ for every $s \leq 1$

$$\sum_{\bar{\imath}\in\Sigma_*} \varphi^s(A_{\bar{\imath}}) \geq \sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}\in\Sigma_*} \varphi^s(A_i A_{\bar{\imath}} A_j) =$$

$$\sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}\in\Sigma_*} \|A_i A_{\bar{\imath}}|Im(A_j)\|^s \cdot \|A_j\|^s = \sum_{k=0}^{\infty} \sum_{i,j\in\Sigma_{sing}} \left((B^{(s)})^k\right)_{i,j} \cdot \|A_j\|^s.$$

$s > s_0$ then $s \geq s_{reg}$ and $s \geq s_{sing}$.
On the other hand

$$\sum_{\bar{\imath}\in\Sigma_*} \varphi^s(A_{\bar{\imath}}) = \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) + \sum_{i\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}_1} A_i A_{\bar{\imath}_2}) +$$
$$\sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \sum_{\bar{j}\in\Sigma_*} \varphi^s(A_{\bar{\imath}_1} A_i A_{\bar{j}} A_j A_{\bar{\imath}_2}) \leq$$
$$\leq \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) + \sum_{i\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}_1})\|A_i\|^s \varphi^s(A_{\bar{\imath}_2}) +$$
$$+ \sum_{i,j\in\Sigma_{sing}} \sum_{\bar{\imath}_1,\bar{\imath}_2\in\Sigma^*_{reg}} \sum_{\bar{j}\in\Sigma_*} \varphi^s(A_{\bar{\imath}_1})\|A_i A_{\bar{j}}|Im(A_j)\|^s \cdot \|A_j\|^s \varphi^s(A_{\bar{\imath}_2}) \leq$$
$$\leq \sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}}) + \left(\sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}})\right)^2 \left(\sum_{æ\in\Sigma_{sing}} \|A_j\|^s\right) +$$
$$+ \left(\sum_{\bar{\imath}\in\Sigma^*_{reg}} \varphi^s(A_{\bar{\imath}})\right)^2 \cdot \sum_{k=0}^{\infty} \sum_{i,j\in\Sigma_{sing}} \left((B^{(s)})^k\right)_{i,j} \cdot \|A_j\|^s$$

If $s > \max\{s_{reg}, s_{sing}\}$ then $s > s_0$. In summary if $s_0 < 1$ then

$$s_0 = \max\{s_{reg}, s_{sing}\} = s_{sing}. \tag{43}$$

If $s_0 = 1$ then $s_{sing} \geq 1$. □

4.1 SEPARATION CONDITION

In finite regular case, Bárány Hochman and Rapaport proved that, if the Strong Separation Condition holds, the Hausdorff dimension equal to the affinity dimension which is the root of the sub-additive pressure. In this section we give a condition when $s > 1$ and we have singular and regular matrices.

Lemma 4.7. *Assume that $s > 1$ then*

$$\dim_H(\Lambda) = s_{reg}. \tag{44}$$

Proof. I. $\dim_H(\Lambda) \geq s_{reg}$
By Bárány, Hochman and Rapaport Theorem 2.21, the Hausdorff dimension equal to the affinity dimension. By **Proposition 4.6** and Falconer **Theorem 2.18** the affinity dimension $s = s_{reg}$ if $s > 1$. Then

$$s_0 \leq \max\{s_{reg}, s_{sing}\} = s_{reg}.$$

II. $\dim_H(\Lambda) \leq s_{reg}$
Let $x \in \Lambda$ be a point. Then we can determine x as

$$x = \lim_{n \to \infty} \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(\underline{0}). \tag{45}$$

Then we have those x which can be represented by all regular matrices and those, which contains at least one singular. So we can define the attractor,

$$\Lambda = \Lambda_{reg} \bigcup \Lambda_{sing}. \tag{46}$$

where, $\Lambda_{sing} = \bigcup_{i \in Sing} \bigcup_{\bar{\jmath} \in Reg} \varphi_{\bar{\jmath}} \circ \varphi_i(\Lambda)$. We know that, every $\varphi_i(\Lambda)$ is contained in an interval. Let B be a ball as defined in **Theorem 2.8**, then $\varphi_i(B) \subset B$. So $\varphi_i(\Lambda) \subset \varphi_i(B)$. Since $\dim \varphi_i(B) = 1$, then

$$\dim_H(\Lambda_{sing}) = \sup_{i, \bar{\jmath} \in Reg} \dim_H \varphi_{\bar{\jmath}} \circ \varphi_i(\Lambda) \leq 1. \tag{47}$$

By the 7. property in **Lemma 2.4**

$$\dim_H(\Lambda) = \max\{dim_H(\Lambda_{reg}), \dim_H(\Lambda_{sing})\} = \max\{s_{reg}, 1\} = s_{reg}. \tag{48}$$

□

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