Budapest University of Technology and Economics
Institute of Mathematics
Department of Stochastics

Self-similar iterated function systems with non-distinct fixed points

TDK DISSERTATION

Author
Edina Szvák
MSc student

Supervisor
Dr. Balázs Bárány
Docent,
BME, Department of Stochastics

October 27, 2018

Contents

1 Preliminary **3**

2 Introduction of self-similar iterated function systems **5**

2.1 Definitions of self-similar IFS 5

2.2 The size of the attractor 8

2.3 Dimension theorems without separation condition 11

2.4 Dimension theorems with separation condition 11

3 The systems $\mathcal{S}_{\alpha,\beta,\gamma}$ **13**

4 Existence of forward separated systems **16**

5 The tools of calculating Hausdorff-dimension of the self-similar measure **24**

5.1 Conditional expectation 24

5.2 Conditional measure 25

5.3 Feng-Hu theorem 26

6 Hausdorff-dimension of a self-similar measure of a forward separated system $\mathcal{S}_{\alpha,\beta,\gamma}$ **29**

List of Figures **34**

Bibliography **34**

Acknowledgements

I would like to express my gratitude to my supervisor Balázs Bárány for his patience, motivation and enthusiasm. I am thankful for his invaluable support and encouraging. It brings me great pleasure for the opportunity to study from him. I am much obliged to him for his continuous assistance for a long time.

Besides my advisor, I would like to acknowledge the teaching of Károly Simon. I wish thank him for the beginning my studies regarding to iterated function systems.

I duly acknowledge the help, direct or indirect of my teachers. I grateful to them for sharing the knowlede with us. I am thankful for his continuous assistance for over a long time.

Chapter 1

Preliminary

Hutchinson showed that if the cylinder sets of a self-similar iterated function system (IFS) are disjoint, then the Hausdorff dimension of its attractor is equals with the similarity dimension. Also, he showed similar result for self-similar measures which belongs to such self-similar IFS for which some strong separation condition holds.

When the cylinder sets of an IFS has significant overlap, the dimension is difficult to understand, because we have to consider complicated overlapping system of cylinder sets.

Using transversality condition for a self-similar IFS family, then K. Simon, B. Solomyak and M. Urbanski calculated this dimensions for almost every paramaters of the IFS family. B. Bárány also proved almost everywhere results, when the self-similar IFS's have fix points that coincide.

Kamalutdinov and Tetenov studied twofold Cantor sets, which are very similar to the forward separated systems (Definition 3.3). In a system of a twofold Cantor set there are total overlaps. They have results for the properties of the attractor. They calculated the exact value of the Hausdorff dimension of twofold Cantor sets. This results are important, because they are the first not only almost everywhere statements for such IFS's for which its cylinder sets have significant overlaps. They do not mentioned about the self-similar measures of those systems.

Results of this dissertation

In this work we study self-similar IFS's on the interval $[0, 1]$ for which the so-called forward separated condition holds (Definition 3.3). In the considered IFS's there is also total overlap between the cylinder sets.

Using the argument of Kamalutdinov and Tetenov we proved that forward separated systems exist. The main result of this dissertation is Theorem 6.1, which states everywhere result for the Hausdorff dimension of a self-similar measure with respect to a forward separated system.

Theorem 6.1 Let $\alpha, \beta, \gamma \in (0, \frac{1}{9})$. Let $\mathcal{S} = \{S_1, S_2, S_3\}$ be a self-similar IFS on $[0,1]$ such that

$$\begin{gathered}\mathcal{S} = \{S_1, S_2, S_3\} \\ S_1(x) = \alpha x, \quad S_2(x) = \beta x, \quad S_3(x) = \gamma x + 1 - \gamma.\end{gathered} \tag{1.1}$$

Let K denote the attractor of $\mathcal{S}$. Moreover, we suppose that

$$\text{for every } m, n \in \mathbb{N}^+, \quad S_1^m S_3(K) \cap S_2^n S_3(K) = \emptyset. \tag{1.2}$$

The natural projection of $\mathcal{S}_{\alpha,\beta,\gamma}$ is $\Pi_{\alpha,\beta,\gamma}$. Let $\mu = (p_1, p_2, p_3)^{\mathbb{N}^+}$ be a Bernoulli measure on Σ for the probability vector $\mathbf{p} = (p_1, p_2, p_3)$. Let $\nu = \Pi_{\alpha,\beta,\gamma *}\mu = \mu \circ \Pi^{-1}_{\alpha,\beta,\gamma}$ be the self-similar measure on the attractor. Then the Hausdorff dimension of ν can be exactly determined.

The exact value of the dimension is in Chapter 6. To achieve this statement we used the theorem of Feng and Hu.

Chapter 2

Introduction of self-similar iterated function systems

In this chapter we would like to define the most fundamental notions and we collect the most important theorems concerning self-similar iterated function systems (IFS).

2.1 Definitions of self-similar IFS

Definition 2.1 *Let $m \geq 2$, $m \in \mathbb{Z}$ and $d \geq 1$, $d \in \mathbb{Z}$. We say that $\mathcal{S}$ is a self-similar iterated function system (IFS) on $\mathbb{R}^d$, if*

$$\mathcal{S} = \{S_1, \dots, S_m\}, \tag{2.1}$$

where i $S_i : \mathbb{R}^d \to \mathbb{R}^d$ is contracting similarity transformation with contraction ratio $0 < r_i < 1$ for alli. This means, that

$$\forall i \in \{1, \dots, m\} \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \quad \|S_i(\mathbf{x}) - S_i(\mathbf{y})\| = r_i \|\mathbf{x} - \mathbf{y}\|. \tag{2.2}$$

Frequently we use the notation $S_{i_1} \circ \cdots \circ S_{i_n} = S_{i_1,\dots,i_n}$.

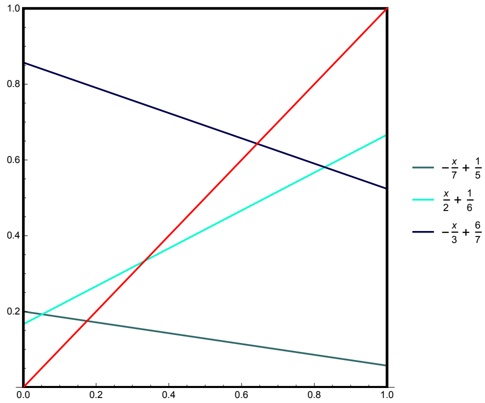

Figure 2.1: Example for a self-similar IFS on the line

Definition 2.2 *Let* $B = \overline{B}(0, R)$*, where* $R = \max_{1\le i\le m}\left\{\frac{\|S_i(\mathbf{0})\|}{1-r_i}\right\}$*. The set* Λ *is the attractor of the self-similar IFS* $\mathcal{S}$*, if*

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_n)\in\{1,\dots,m\}^n} S_{i_1,\dots,i_n}(B). \tag{2.3}$$

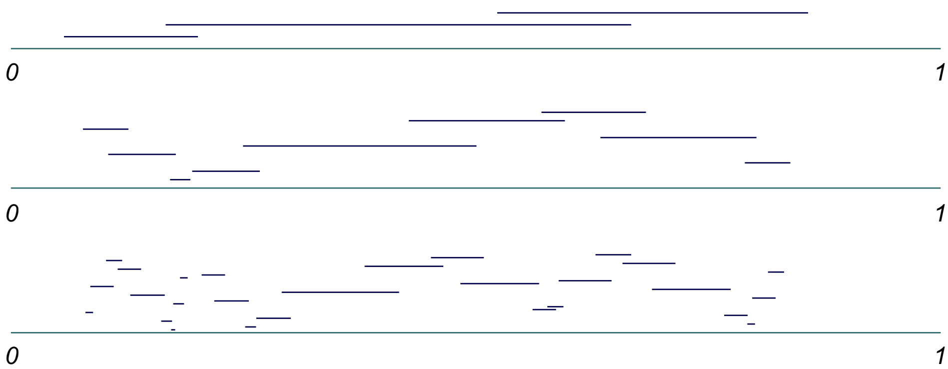

Figure 2.2: The first, second and third level cylinder sets of the IFS $\mathcal{S} = \{-\frac{x}{7}+\frac{1}{5}, \frac{x}{2}+\frac{1}{6}, -\frac{x}{3}+\frac{6}{7}\}$

Definition 2.3 *We call* $\Sigma = \{1,\dots,m\}^{\mathbb{N}}$ *the symbolic space of the IFS* $\mathcal{S}$ *defined in equation* (2.1).

On the symbolic space we use the following notation. If $\mathbf{i} = (i_1,\dots,i_k) \in \{1,\dots,m\}^k$ and $\mathbf{j} \in \{1,\dots,m\}^l$, then let $\mathbf{i} * \mathbf{j} = (i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_l)$. Denote $\mathbf{i}^2 = \mathbf{i} * \mathbf{i}$ and $\mathbf{i}^k = \mathbf{i}^{k-1} * \mathbf{i}$. This definition is also proper for $l = \infty$.

Let us denote the set of all finite length word by $\Sigma^* = \bigcup_{k=1}^{\infty}\{1,\dots,m\}^k$.

We denote the left shift on the symbolic space with $\sigma : \Sigma \to \Sigma$ for all $\mathbf{j} = (j_1, j_2, \dots) \in \Sigma$ $\sigma(\mathbf{j}) = (j_2, j_3, \dots)$.

Definition 2.4 *The map* Π *is the natural projection of the IFS* $\mathcal{S}$, *if*

$$\Pi : \Sigma \to \Lambda \quad \Pi(\mathbf{i}) = \lim_{n\to\infty} S_{i_1,\ldots,i_n}(\mathbf{0}), \tag{2.4}$$

where $\mathbf{i} = (i_1, i_2, \ldots) \in \Sigma$.

It is easy to see that

$$\Lambda = \Pi(\Sigma). \tag{2.5}$$

Theorem 2.5 (Hutchinson) *The* Λ *attractor of the IFS* $\mathcal{S}$ (2.1) *is the only non-empty compact set solution of the following equation on sets*

$$X = \bigcup_{i=1}^{m} S_i(X), \tag{2.6}$$

where X *is the variable.*

The proof can be found in [2].

Definition 2.6 *Let* $\Sigma = \{1, \ldots, m\}^{\mathbb{N}^+}$ *and* $\mathbf{i} = (i_1, \ldots i_k) \in \{1, \ldots, m\}^k$, *then the set*

$$[i_1, \ldots, i_k] = \{\mathbf{j} \in \Sigma : j_1 = i_1, \ldots j_k = i_k\} \tag{2.7}$$

is called a cylinder set.

Let $\mathbf{p} = (p_1, \ldots, p_m)$ be a probability vector. Then, let $\mu = \mathbf{p}^{\mathbb{N}}$ be the infinite product measure or Bernolli measure on Σ. That is

$$\mu([i_1, \ldots, i_k]) = p_{i_1} \ldots p_{i_k}, \tag{2.8}$$

where $(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$. Using Kolmogorov's extension theorem, we can see that there exists a unique μ Borel measure on Σ defined on the σ-algebra generated by the cylinder sets and for which the equation (2.8) holds.

Definition 2.7 *Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *be a probability vector. We say that* ν *is a self-similar measure or invariant measure of the self-similar IFS* $\mathcal{S}$ *with the probabilty vector* $\mathbf{p}$, *if* ν *is the following push-down measure*

$$\nu(E) = \Pi_* \mathbf{p}^{\mathbb{N}}(E) = \mathbf{p}^{\mathbb{N}} \circ \Pi^{-1}(E). \tag{2.9}$$

Theorem 2.8 *Let* $\mathbf{p} = (p_1, \ldots, p_m)$ *a probabilty vector and* $\mathcal{S}$ *is a self-similar IFS in the form* (2.1)*. Then* ν *self-similar measure of* $\mathcal{S}$ *with the probabilty vector* $\mathbf{p}$ *if it is the only* ν *Borel probabilty measure on* $\mathbb{R}^d$ *for which*

$$\nu = \sum_{k=1}^{m} p_k (\nu \circ S_k^{-1}) \tag{2.10}$$

holds.

The proof can be found in [2].

2.2 The size of the attractor

Most of the time the attractor has zero Lebesgue measure, thus we need some definition to be able to compare the size of sets with zero Lebesgue measure.

Definition 2.9 *Let* $t \geq 0$*. The measure* $\mathcal{H}^t$ *is called the t-dimensional Hausdorff measure on* $\mathbb{R}^d$*, if it is the restriction of the following outer measure for the* σ*-algebra of the measurable sets. Let*

$$\mathcal{H}^t(E) = \lim_{\delta \to 0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t \,:\, E \subseteq \bigcup_{i=1}^{\infty} A_i,\, |A_i| \leq \delta \right\} \right\} = \lim_{\delta \to 0} \mathcal{H}^t_\delta(E), \tag{2.11}$$

where $A \subseteq \mathbb{R}^d$ $|A|$ *is the diameter of the set* A*.*

Remark 2.10 *The limit in the equation* (2.11) *is exists, because the function*

$$\delta \mapsto \inf \left\{ \sum_{i=1}^{\infty} |A_i|^t \,:\, E \subseteq \bigcup_{i=1}^{\infty} A_i,\, |A_i| \leq \delta \right\} \tag{2.12}$$

is monoton decreasing.

Now, let us introduce some basic facts regarding to Hausdorff measure.

Theorem 2.11 *For every* $t > 0$*, all Borel set in* $\mathbb{R}^d$ *is measurable with respect to the t-dimensional Hausdorff measure.*

Theorem 2.12 *For every* $n \in \mathbb{N}^+$*, there exists* $c \in \mathbb{R}^+$ *such that for all Borel set* $B \subseteq \mathbb{R}^n$ $\mathcal{H}^n(B) = c\mathcal{L}^n(B)$ *hold.*

Lemma 2.13 *For every Borel set* $B \subseteq \mathbb{R}^d$ *and every* $0 \leq \alpha < \beta$*, we have the following implications:*

(i) $\mathcal{H}^\alpha(B) < \infty \Longrightarrow \mathcal{H}^\beta(B) = 0$

(ii) $\mathcal{H}^\beta(B) > 0 \Longrightarrow \mathcal{H}^\alpha(B) = \infty$

Definition 2.14 *By Lemma 2.13 we can define the Hausdorff dimension of a* $B \subseteq \mathbb{R}^d$ *Borel set by*

$$\dim_H(B) = \inf_{t\geq 0}\{\mathcal{H}^t(B) = 0\} = \sup_{t\geq 0}\{\mathcal{H}^t(B) = \infty\}. \tag{2.13}$$

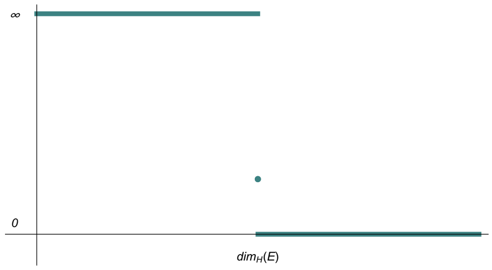

Figure 2.3: The definition of the Hausdorff dimension.

Definition 2.15 *If* $\mathcal{S} = \{S_1, \dots, S_m\}$ *is a* C^1 *IFS, then the value of upper and lower Lyapunov exponents in* $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma$ *is defined respectively by*

$$\begin{aligned} \overline{\lambda}(\mathbf{i}) &= \limsup_{n\to\infty}\left(-\frac{1}{n}\log\left\|S'_{i_1 i_2\dots i_n}(\Pi(\sigma^n\mathbf{i}))\right\|\right), \\ \underline{\lambda}(\mathbf{i}) &= \liminf_{n\to\infty}\left(-\frac{1}{n}\log\left\|S'_{i_1 i_2\dots i_n}(\Pi(\sigma^n\mathbf{i}))\right\|\right). \end{aligned} \tag{2.14}$$

When $\overline{\lambda}(\mathbf{i}) = \underline{\lambda}(\mathbf{i})$*, then the common value is denoted by* $\lambda(\mathbf{i})$ *and we call it the Lyapunov exponent of the system* $\mathcal{S}$ *at the point* $\mathbf{i} \in \Sigma$*.*

Definition 2.16 *If* $\mathcal{S}$ *is a* C^1 *IFS and* μ *is a Bernoulli measure on* Σ*, then we call the system* $\mathcal{S}$ *is* μ*-conformal, if* $\lambda(\mathbf{i})$ *exists for* μ*-almost every* $\mathbf{i} \in \Sigma$*.*

Definition 2.17 *Suppose that* ν *is a Borel probability measure on* $\mathbb{R}^d$*, then the definition of upper and lower local dimension of* ν *at* $x \in \mathbb{R}^d$ *is respectively*

$$\begin{aligned} \overline{\dim}_\nu(x) &= \limsup_{r\to 0}\frac{\log\nu(B(x,r))}{\log r}, \\ \underline{\dim}_\nu(x) &= \liminf_{r\to 0}\frac{\log\nu(B(x,r))}{\log r}, \end{aligned} \tag{2.15}$$

where $B(x, r)$ *denotes the open ball of radius* r *centered at* x*. If* $\overline{\dim}_\nu(x) = \underline{\dim}_\nu(x)$*, then the common value is denoted by* $\dim_\nu(x)$ *and we call it the local dimension of* ν *at* x*.*

Definition 2.18 *We can also define the Hausdorff dimension of a Borel probability measure ν on $\mathbb{R}^d$ with*

$$\dim_H(\nu) = \inf\{\dim_H(E) \,:\, \nu(E^c) = 1\}. \tag{2.16}$$

Theorem 2.19 *If ν is a Borel probability measure on $\mathbb{R}^d$ with compact support, then* $\dim_H(\nu) = \operatorname{ess\,sup}\{\underline{\dim}_\nu(x) : x \in \mathbb{R}^d\} = \inf\{\alpha : \nu(\{x : \underline{\dim}_\nu(x) \le \alpha\}) = 1\}$

Lemma 2.20 *If $\mathcal{S} = \{S_1, \dots, S_m\}$ is a self-similar IFS and μ is a σ invariant, ergodic Borel probability measure on Σ, then $\mathcal{S}$ is μ-conformal.*

Proof: Let $\phi_n : \Sigma \to \mathbb{R}$ such that for $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma$ $\quad \phi_n(\mathbf{i}) = -\frac{1}{n} \log \left\| S'_{i_1 i_2 \dots i_n}(\Pi(\sigma^n \mathbf{i}))\right\|$. Using $\mathcal{S}$ is self-similar and the chain rule, we get $\left\| S'_{i_1 i_2 \dots i_n}(\Pi(\sigma^n \mathbf{i}))\right\| = \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_n}$. Thus

$$\phi_n(\mathbf{i}) = -\frac{1}{n} \sum_{k=1}^{n} \log(\lambda_{i_k}) = \frac{1}{n} \sum_{k=1}^{n} \psi(\sigma^{k-1}\mathbf{i}), \tag{2.17}$$

where $\psi(\mathbf{i}) = -\log(\lambda_{i_1})$. Using Birkhoff ergodic theorem, we get

$$\lim_{n\to\infty} \phi_n(\mathbf{i}) = \int_\Sigma \psi(\mathbf{i}) \mathrm{d}\mu(\mathbf{i}) \text{ for } \mu\text{-almost every } \mathbf{i} \in \Sigma. \tag{2.18}$$

Thus λ is a constant μ-almost everywhere. So $\mathcal{S}$ is μ-conformal. □

Lemma 2.21 *If $\mathcal{S} = \{S_1, \dots, S_m\}$ is a self-similar IFS. The Lipschitz constant of S_i is λ_i. Assume μ is a Bernoulli measure on Σ for the probability vector $\mathbf{p} = (p_1, \dots, p_m)$. Then $\mathcal{S}$ is μ-conformal and*

$$\int_\Sigma \lambda(\mathbf{i}) \mathrm{d}\mu(\mathbf{i}) = -\sum_{k=1}^{m} p_k \log(\lambda_k). \tag{2.19}$$

Proof: It is a well-known fact that if μ is a Bernoulli measure on Σ, then it is σ invariant and ergodic, thus due to the previous lemma $\mathcal{S}$ is μ-conformal. Using the argument in the previous proof, we can see that

$$\int_\Sigma \lambda(\mathbf{i}) \mathrm{d}\mu(\mathbf{i}) = \int_\Sigma \psi(\mathbf{i}) \mathrm{d}\mu(\mathbf{i}) = -\sum_{k=1}^{m} p_k \log(\lambda_k). \tag{2.20}$$

□

2.3 Dimension theorems without separation condition

Definition 2.22 *We call s the similarity dimension of the self-similar IFS defined in* (2.1)*, if s is the solution of*

$$\sum_{i=1}^{m} r_i^s = 1. \tag{2.21}$$

Theorem 2.23 *Let $\mathcal{S}$ be a self-similar IFS on $\mathbb{R}^d$, defined in* (2.1)*. Let Λ be the attractor of $\mathcal{S}$ and s is the similarity dimension of $\mathcal{S}$. Then*

$$\dim_H(\Lambda) \leq s. \tag{2.22}$$

The proof can be found in [2].

Theorem 2.24 *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a self-similar IFS on $\mathbb{R}^d$. The vector $\mathbf{r} = (r_1, \ldots, r_m)$ contains the contraction ratios of $\mathcal{S}$.The ν is the invariant measure of $\mathcal{S}$ with the probabilty vector $\mathbf{p} = (p_1, \ldots, p_m)$. Then we have*

$$\dim_H(\nu) \leq \frac{-\sum_{i=1}^{m} p_i \log p_i}{-\sum_{i=1}^{m} p_i \log r_i} = \frac{h_{\mathbf{p}}}{\chi_{\mathbf{r}}^{\mathbf{p}}}. \tag{2.23}$$

The proof can be found in [2].

2.4 Dimension theorems with separation condition

In the special case, when the cylinder sets satify certain separation condition we are able to estimate the Hausdorff dimension of the attractor of such IFS. Moreover, in this case we can study the self-similar measure of the IFS.

Definition 2.25 *The Strong Separation Property (SSP) holds for the self-similar IFS $\mathcal{S}$ defined in* (2.1)*, if*

$$\forall i \neq j \quad S_i(\Lambda) \cap S_j(\Lambda) = \emptyset. \tag{2.24}$$

Definition 2.26 *The Open Set Condition (OSC) holds for the self-similar IFS $\mathcal{S}$ defined in* (2.1)*, if*

$$\exists V \subseteq \mathbb{R}^d \text{ open set } V \neq \emptyset \quad \forall i\, S_i(V) \subseteq V \text{ and } \forall i \neq j\, S_i(V) \cap S_j(V) = \emptyset. \tag{2.25}$$

Theorem 2.27 (Moran, Hutchinson) *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a self-similar IFS on $\mathbb{R}^d$ for which the OSC holds. We denote the attractor of $\mathcal{S}$ with Λ and the similarity dimension of $\mathcal{S}$ with s. Then,*

$$\dim_H(\Lambda) = s. \tag{2.26}$$

The proof can be found in [2].

Theorem 2.28 *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a self-similar IFS on $\mathbb{R}^d$ for which the OSC holds. The vector $\mathbf{r} = (r_1, \ldots, r_m)$ contains the contraction ratios of $\mathcal{S}$.The ν is the invariant measure of $\mathcal{S}$ with the probabilty vector $\mathbf{p} = (p_1, \ldots, p_m)$. Then we have*

$$\dim_H(\nu) = \frac{-\sum_{i=1}^m p_i \log p_i}{-\sum_{i=1}^m p_i \log r_i} = \frac{h_{\mathbf{p}}}{\chi_{\mathbf{r}}^{\mathbf{p}}}. \tag{2.27}$$

The proof can be found in [2].

Remark 2.29 *In the case, when we do not know any separation condition holds for the self-similar IFS $\mathcal{S}$ the values s and $\frac{h_{\mathbf{p}}}{\chi_{\mathbf{r}}^{\mathbf{p}}}$ in Theorem 2.27 and 2.28 is only an upper bound on the Hausdorff dimension.*

Chapter 3

The systems $\mathcal{S}_{\alpha,\beta,\gamma}$

We study a family of self-similar iterated function systems (IFS) on the interval $[0,1]$ such that there is total overlap and for which some separation condition holds.

Kamalutdinov and Tetenov in [3] studied similar iterated function systems, which called twofold Cantor set.

We follow their argument with similar statements in this chapter.

Definition 3.1 *Let $\alpha,\beta,\gamma \in (0,1)$ arbitrary. Then $\mathcal{S}_{\alpha,\beta,\gamma}$ is a system of contractive similarities such that*

$$\begin{gathered}\mathcal{S}_{\alpha,\beta,\gamma} = \{S_1, S_2, S_3\} \\ S_1(x) = \alpha x, \quad S_2(x) = \beta x, \quad S_3(x) = \gamma x + 1 - \gamma\end{gathered} \tag{3.1}$$

B. Bárány has already considered the Hausdorff dimension of the attractor of the system introduced in Definition 3.1. He showed this result for Lebesgue almost every $\alpha, \beta, \gamma(0, \frac{1}{2})$.

Let $K_{\alpha,\beta,\gamma}$ be the attractor of the system $\mathcal{S}_{\alpha,\beta,\gamma}$. Let $L_{\alpha,\beta,\gamma} = S_1(K_{\alpha,\beta,\gamma}) \cup S_2(K_{\alpha,\beta,\gamma})$ and $R_{\alpha,\beta,\gamma} = S_3(K_{\alpha,\beta,\gamma})$.
It is easy to see that $K_{\alpha,\beta,\gamma} = L_{\alpha,\beta,\gamma} \cup R_{\alpha,\beta,\gamma}$.

We denote the symbolic space of $\mathcal{S}_{\alpha,\beta,\gamma}$ with $\Sigma = \{1,2,3\}^{\mathbb{N}^+}$.
Let $\Pi_{\alpha,\beta,\gamma} : \Sigma \to K_{\alpha,\beta,\gamma}$ be the natural projection of the system $\mathcal{S}_{\alpha,\beta,\gamma}$.

First, we consider some obvious properties of the systems $\mathcal{S}_{\alpha,\beta,\gamma}$:

Lemma 3.2 *If $\alpha,\beta,\gamma \in (0,\frac{1}{2})$, then:*

(i) $S_1 \circ S_2 = S_2 \circ S_1$,

(ii) for all $i \in \{1,2\}$ and every $m,n \in \mathbb{N}$ with $m \neq n$, $S_i^m(R_{\alpha,\beta,\gamma}) \cap S_i^n(R_{\alpha,\beta,\gamma}) = \emptyset$,

(iii) for all $m,n \in \mathbb{N}$, $S_1^m S_2^n(K_{\alpha,\beta,\gamma}) \subseteq S_1^m(K_{\alpha,\beta,\gamma}) \cap S_2^n(K_{\alpha,\beta,\gamma})$,

(iv) $K_{\alpha,\beta,\gamma}\backslash\{0\} = \bigcup_{n,m=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma})$.

Proof:

(i) For every $x \in [0,1]$ $\quad S_1(S_2(x)) = \alpha(\beta x) = \beta(\alpha x) = S_2(S_1(x))$.

(ii) We prove only for $i = 1$, the case $i = 2$ is similar. Let $m, n \in \mathbb{N}$ $\quad m > n$. $R_{\alpha,\beta,\gamma} \subseteq (\frac{1}{2}, 1)$, thus $S_1^m(R_{\alpha,\beta,\gamma}) \subseteq (\frac{1}{2}\alpha^m, \alpha^m)$ and $S_1^n(R_{\alpha,\beta,\gamma}) \subseteq (\frac{1}{2}\alpha^n, \alpha^n)$. Since we can see that the right endpoint of one interval is smaller than the left endpoint of the other interval that is $\alpha^m = \alpha \cdot \alpha^{m-1} < \frac{1}{2}\alpha^{m-1} \leq \frac{1}{2}\alpha^n$.

(iii) Let $m, n \in \mathbb{N}$, then $S_1^m(K_{\alpha,\beta,\gamma}) \subseteq K_{\alpha,\beta,\gamma}$ and $S_2^n(K_{\alpha,\beta,\gamma}) \subseteq K_{\alpha,\beta,\gamma}$. So, we conclude that $S_2^n S_1^m(K_{\alpha,\beta,\gamma}) \subseteq S_2^n(K_{\alpha,\beta,\gamma})$ and $S_1^m S_2^n(K_{\alpha,\beta,\gamma}) \subseteq S_1^m(K_{\alpha,\beta,\gamma})$. Using commutativity, which is property (i) we get the statements.

(iv) Consider the natural projection $\Pi_{\alpha,\beta,\gamma}$ of $\mathcal{S}_{\alpha,\beta,\gamma}$. The map $\Pi_{\alpha,\beta,\gamma}$ is surjective. It is easy to see that

$$\Pi^{-1}_{\alpha,\beta,\gamma}\Big(\bigcup_{m,n=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma})\Big) = \{\mathbf{i} \in \Sigma : \exists k \quad i_k = 3\}. \tag{3.2}$$

For those $\mathbf{i} \in \Sigma$ such that there is no k for which $i_k = 3$, then the image of $\mathbf{i}$ is 0.

□

Using Theroem 2.23, we can conclude that the dimension of $K_{\alpha,\beta,\gamma}$ is less than $\frac{1}{2}$ if $\alpha, \beta, \gamma \in (0, \frac{1}{9})$.

Definition 3.3 *We call the system $\mathcal{S}_{\alpha,\beta,\gamma}$ forward separated, if $\alpha, \beta, \gamma \in (0, \frac{1}{9})$ and*

$$\forall m, n \in \mathbb{N} \quad m, n > 0 \quad S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) = \emptyset. \tag{3.3}$$

We denote the disjoint union with $\sqcup$.

Lemma 3.4 *The system $\mathcal{S}_{\alpha,\beta,\gamma}$ is forward separated if and only if*

$$K_{\alpha,\beta,\gamma}\backslash\{0\} = \bigsqcup_{n,m=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma}), \tag{3.4}$$

where $\sqcup$ denotes the disjoint union.

Proof: ($\Rightarrow$)First, we assume that $\mathcal{S}_{\alpha,\beta,\gamma}$ is forward separated. Let $(m_1, n_1) \neq (m_2, n_2)$, then

$$\begin{aligned} S_1^{m_1} S_2^{n_1}(R_{\alpha,\beta,\gamma}) &= S_1^{min\{m_1,m_2\}} S_2^{min\{n_1,n_2\}}(S_1^{k_1} S_2^{l_1}(R_{\alpha,\beta,\gamma})) \\ S_1^{m_2} S_2^{n_2}(R_{\alpha,\beta,\gamma}) &= S_1^{min\{m_1,m_2\}} S_2^{min\{n_1,n_2\}}(S_1^{k_2} S_2^{l_2}(R_{\alpha,\beta,\gamma})) \end{aligned} \tag{3.5}$$

hold. At least one of k_1, k_2 is zero and one of l_1, l_2 is zero. So if we use the forward separated property we get the statement.
($\Leftarrow$) Now, assume that $K_{\alpha,\beta,\gamma}\backslash\{0\} = \bigsqcup_{n,m=0}^{\infty} S_1^m S_2^n(R_{\alpha,\beta,\gamma})$ holds. Then we can get the statement by using the conditon for the indeces $(m, 0)$ and $(0, n)$.

□

Lemma 3.5 *If the system $\mathcal{S}_{\alpha,\beta,\gamma}$ is forward separated, then for every*

$$m, n \in \mathbb{N} \quad S_1^m(K_{\alpha,\beta,\gamma}) \cap S_2^n(K_{\alpha,\beta,\gamma}) = S_1^m S_2^n(K_{\alpha,\beta,\gamma}). \tag{3.6}$$

Proof: Using the above results, we get

$$S_1^m(K_{\alpha,\beta,\gamma}) \cap S_2^n(K_{\alpha,\beta,\gamma})\backslash\{0\} = S_1^m(\bigcup_{k,l=0}^{\infty} S_1^k S_2^l(R_{\alpha,\beta,\gamma})) \cap S_2^n(\bigcup_{k,l=0}^{\infty} S_1^k S_2^l(R_{\alpha,\beta,\gamma})) =$$

$$= \bigcup_{k,l=0}^{\infty} S_1^{k+m} S_2^{l+n}(R_{\alpha,\beta,\gamma}) = S_1^m S_2^n(\bigcup_{k,l=0}^{\infty} S_1^k S_2^l(R_{\alpha,\beta,\gamma})) = S_1^m S_2^n(K_{\alpha,\beta,\gamma})\backslash\{0\}.$$

In the first and last equation we use Lemma 3.2 (iv) point and in the second equation we use Lemma 3.4. □

Chapter 4

Existence of forward separated systems

Kamalutdinov and Tetenov proved that twofold Cantor sets exist in [3]. In this whole section we follow their arguement with similar statements.

Due to requirement of completeness we take over the same proof of this Theorem from [3].

Theorem 4.1 (General Position Theorem [3]) *Let $(D, d_D), (L_1, d_{L_1}), (L_2, d_{L_2})$ be compact metric spaces and let $\varphi_i(\xi, x) : D \times L_i \to \mathbb{R}^n$ for $i \in \{1, 2\}$ be continuous functions. If these functions satisfies:*

(i) The functions φ_i are α-Hölder with respect to x which is

$$\text{there exists } \alpha > 0 \text{ for all } i \in \{1,2\} \text{ there exists } C_i > 0 \text{ for all } \xi \in D \text{ for all } x, y \in L_i$$
$$\|\varphi_i(\xi, x) - \varphi_i(\xi, y))\| \leq C_i d_{L_i}(x, y)^\alpha,$$

where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^n$.

(ii) Let $\Phi : D \times L_1 \times L_2 \to \mathbb{R}^n \quad \Phi(\xi, x_1, x_2) = \varphi_1(\xi, x_1) - \varphi_2(\xi, x_2)$ such that

$$\begin{gathered}\text{there exist } M > 0 \text{ for all } \xi, \xi' \in D \text{ for all } x_1 \in L_1 \text{ for all } x_2 \in L_2 \\ \|\Phi(\xi, x_1, x_2) - \Phi(\xi', x_1, x_2)\| \geq M d_D(\xi, \xi').\end{gathered} \tag{4.1}$$

Then the set $\Delta = \{\xi \in D : \varphi_1(\xi, L_1) \cap \varphi_2(\xi, L_2) \neq \emptyset\}$ is a compact in D and

$$\dim_H(\Delta) \leq \frac{\dim_H(L_1 \times L_2)}{\alpha}. \tag{4.2}$$

Proof: Let $\tilde{\Delta} = \{(\xi, x_1, x_2) \in D \times L_1 \times L_2 : \varphi_1(\xi, x_1) = \varphi_2(\xi, x_2)\} = \{(\xi, x_1, x_2) \in D \times L_1 \times L_2 : \Phi(\xi, x_1, x_2) = 0\}$ be the set of those parameters where $\varphi_1(\xi, L_1)$ and $\varphi_2(\xi, L_2)$ intersects. Then $\Delta = \text{proj}_D\left(\tilde{\Delta}\right)$. Let $L = L_1 \times L_2$ and $\Delta_L = \text{proj}_L\left(\tilde{\Delta}\right)$.

The map Φ is a continuous map and $\tilde{\Delta} = \Phi^{-1}(\{0\})$, thus $\tilde{\Delta}$ is closed. Then $\tilde{\Delta}$ is closed in a compact metric space, so it is compact. The projection is continuous, thus Δ is also compact.

The functions $\mathrm{proj}_D : \tilde{\Delta} \to \Delta$ and $\mathrm{proj}_L : \tilde{\Delta} \to \Delta_L$ are surjective. Moreover, proj_L is also injective, because if exist $(\xi, x_1, x_2) \neq (\xi', x_1', x_2') \in \tilde{\Delta}$ such that $\mathrm{proj}_L(\xi, x_1, x_2) = \mathrm{proj}_L(\xi', x_1', x_2')$, then $x_1' = x_1$, $x_2' = x_2$ and $\xi' \neq \xi$. By the definition of $\tilde{\Delta}$ $\Phi(\xi, x_1, x_2) = \Phi(\xi', x_1, x_2) = 0$ and this is contradicts with the second assumption. So proj_L is injective, thus it is invertible.

Let $g = \mathrm{proj}_D \circ {\mathrm{proj}_L}^{-1} : \Delta_L \to \Delta$. This is surjective. Let $g(x_1, x_2) = \xi$ and $g(x_1', x_2') = \xi'$. Then $\Phi(\xi, x_1, x_2) = 0$ and $\Phi(\xi', x_1', x_2') = 0$.

$$M \cdot d_D(\xi, \xi') \leq \|\Phi(\xi, x_1, x_2) - \Phi(\xi', x_1, x_2)\| = \|\Phi(\xi', x_1', x_2') - \Phi(\xi', x_1, x_2)\| \leq$$
$$\|\varphi_1(\xi', x_1') - \varphi_1(\xi', x_1)\| + \|\varphi_2(\xi', x_2') - \varphi_2(\xi', x_2)\| \leq C\left(d_{L_1}(x_1, x_1')^\alpha + d_{L_2}(x_2, x_2')^\alpha\right),$$

where $C = \max\{C_1, C_2\}$. In the first inequation we use the (ii) assumption, the next inequation is triangle inequality and the last inequation is the Hölder continuity in (i). $\square$

Also for the completeness we take over the same proof of thefollwing theorem from [3].

Lemma 4.2 (Displacement theorem) *Let $\mathcal{S} = \{S_1, \dots, S_m\}$ and $\tilde{\mathcal{S}} = \{\tilde{S}_1, \dots, \tilde{S}_m\}$ be two iterated function systems on $\mathbb{R}^n$. We denote the natural projection of $\mathcal{S}$ with $\Pi : \Sigma \to \mathbb{R}^n$ and the natural projection of $\tilde{\mathcal{S}}$ with $\tilde{\Pi} : \Sigma \to \mathbb{R}^n$, where $\Sigma = \{1, \dots, m\}^{\mathbb{N}^+}$ is the symbolic space. Let $V \subseteq \mathbb{R}^n$ be a compact set such that for every $i \in \{1, \dots, m\}$, $S_i(V) \subseteq V$ and $\tilde{S}_i(V) \subseteq V$. Then*

$$\forall \mathbf{i} = (i_1, i_2, \dots) \in \Sigma \quad \left\|\Pi(\mathbf{i}) - \tilde{\Pi}(\mathbf{i})\right\| \leq \frac{\delta}{1-p}, \tag{4.3}$$

where

$$\begin{gathered}\delta = \max\{\left\|S_i(x) - \tilde{S}_i(x)\right\| : i \in \{1, \dots, m\}, \quad x \in V\} \text{ and} \\ p = \max_{1 \leq i \leq m}\{\max\{\mathrm{Lip}(S_i), \mathrm{Lip}(\tilde{S}_i)\}\}.\end{gathered} \tag{4.4}$$

Proof: Let $\mathbf{i} \in \Sigma$ arbitrary. We can conclude

$$\begin{aligned} \left\| \Pi(\mathbf{i}) - \tilde{\Pi}(\mathbf{i}) \right\| &= \left\| S_{i_1}(\Pi(\sigma\mathbf{i})) - \tilde{S}_{i_1}(\tilde{\Pi}(\sigma\mathbf{i})) \right\| \leq \\ \leq \left\| S_{i_1}(\Pi(\sigma\mathbf{i})) - S_{i_1}(\tilde{\Pi}(\sigma\mathbf{i})) \right\| &+ \left\| S_{i_1}(\tilde{\Pi}(\sigma\mathbf{i})) - \tilde{S}_{i_1}(\tilde{\Pi}(\sigma\mathbf{i})) \right\| \leq \\ &\leq p \left\| \Pi(\sigma\mathbf{i}) - \tilde{\Pi}(\sigma\mathbf{i}) \right\| + \delta. \end{aligned} \tag{4.5}$$

Using the above inequation n times, then we get

$$\left\| \Pi(\mathbf{i}) - \tilde{\Pi}(\mathbf{i}) \right\| \leq p^n \left\| \Pi(\sigma^n\mathbf{i}) - \tilde{\Pi}(\sigma^n\mathbf{i}) \right\| + \delta \cdot \sum_{i=0}^{n-1} p^i. \tag{4.6}$$

If $n \to \infty$, then we get $\left\| \Pi(\mathbf{i}) - \tilde{\Pi}(\mathbf{i}) \right\| \leq \frac{\delta}{1-p}$, because V is compact. □

Let $\Sigma = \{1, \ldots, m\}^{\mathbb{N}^+}$ and $a \in (0, 1)$. We can construct a metric space (Σ, ρ_a) with metric ρ_a. We define $\forall \mathbf{i}, \mathbf{j} \in \Sigma \quad s(\mathbf{i}, \mathbf{j}) = \min\{k \geq 1 : i_k \neq j_k\}$, then let $\rho_a(\mathbf{i}, \mathbf{j}) = a^{s(\mathbf{i},\mathbf{j})}$. It is a well known fact that the metric space (Σ, ρ_a) is compact.

Lemma 4.3 *Let (Σ, ρ_a) be a metric space as above. Then*

$$\dim_H(\Sigma) = -\frac{\log(m)}{\log(a)}. \tag{4.7}$$

Proof: Let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be a set of $\Sigma \to \Sigma$ functions.For all $k \in \{1, \ldots, m\}$ and for all $\mathbf{i} \in \Sigma$

$$G_k(\mathbf{i}) = (k, i_1, i_2, \ldots). \tag{4.8}$$

Then every G_k is a contractive function with Lipschitz constant a, so $\mathcal{G}$ is an IFS. The attractor of $\mathcal{G}$ is Σ. $\mathcal{G}$ satisfies the strong separation property, so the Hausdorff-dimension of its attractor is equal to the similarity dimension of the IFS. It is also a well known fact. The proof can be found in [2]. Thus $\dim_H(\Sigma) = -\frac{\log(m)}{\log(a)}$. □

So using this fact for the symbolic space of the system $\mathcal{S}_{\alpha,\beta,\gamma}$. Remind that $\Sigma = \{1, 2, 3\}^{\mathbb{N}^+}$, then

$$\dim_H(\Sigma) < \frac{1}{2} \text{ in the metric } \rho_a \iff a \in \left(0, \frac{1}{9}\right). \tag{4.9}$$

Lemma 4.4 *Let $\alpha, \beta, \gamma < a$ and $a \in \left(0, \frac{1}{9}\right)$. Then $\Pi_{\alpha,\beta,\gamma}$ natural projection of the system $\mathcal{S}_{\alpha,\beta,\gamma}$ is 1-Lipschitz with respect to the metric space (Σ, ρ_a).*

Proof: Let $\mathbf{i}, \mathbf{j} \in \Sigma$ with $s(\mathbf{i}, \mathbf{j}) = k$. Then $\rho_a(\mathbf{i}, \mathbf{j}) = a^k$ and $i_1 = j_1, \dots, i_k = j_k$, thus $\Pi_{\alpha,\beta,\gamma}(\mathbf{i}), \Pi_{\alpha,\beta,\gamma}(\mathbf{j}) \in S_{i_1 \dots i_k}(K_{\alpha,\beta,\gamma})$. The diameter of $S_{i_1 \dots i_k}(K_{\alpha,\beta,\gamma})$ is $\mathrm{Lip}(S_{i_1}) \cdot \dots \cdot \mathrm{Lip}(S_{i_k})$, which is strictly smaller than a^k. So

$$|\Pi_{\alpha,\beta,\gamma}(\mathbf{i}) - \Pi_{\alpha,\beta,\gamma}(\mathbf{j})| < a^k = \rho_a(\mathbf{i}, \mathbf{j}). \tag{4.10}$$

□

Lemma 4.5 *Let $m, n \in \mathbb{N}^+$. $\alpha, \beta, \gamma \in \left(0, \frac{1}{9}\right)$ and consider the system $\mathcal{S}_{\alpha,\beta,\gamma}$. If $S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \neq \emptyset$, then $\frac{8}{9} \leq \frac{\alpha^m}{\beta^n} \leq \frac{9}{8}$.*

Proof: If $\alpha, \beta, \gamma \in \left(0, \frac{1}{9}\right)$, then $R_{\alpha,\beta,\gamma} \subseteq [\frac{8}{9}, 1]$. Thus $S_1^m(R_{\alpha,\beta,\gamma}) \subseteq [\frac{8}{9}\alpha^m, \alpha^m]$ and $S_2^n(R_{\alpha,\beta,\gamma}) \subseteq [\frac{8}{9}\beta^n, \beta^n]$. The intersection can not happen if $\alpha^m < \frac{8}{9}\beta^n$ or $\beta^n < \frac{8}{9}\alpha^m$. Thus we do not have intersection if $\frac{\alpha^m}{\beta^n} < \frac{8}{9}$ or $\frac{\alpha^m}{\beta^n} > \frac{9}{8}$. So $\frac{8}{9} \leq \frac{\alpha^m}{\beta^n} \leq \frac{9}{8}$. □

Lemma 4.6 *Let $m, n \in \mathbb{N}^+$ and $\beta, \gamma \in (0, \frac{1}{9})$ be fixed. We denote*

$$D_{m,n}(\beta, \gamma) = \left\{ \alpha \in \left(0, \frac{1}{9}\right) : \frac{8}{9} \leq \frac{\alpha^m}{\beta^n} \leq \frac{9}{8} \right\}. \tag{4.11}$$

Let $\varphi_i : D_{m,n}(\beta, \gamma) \times \Sigma \to \mathbb{R}$ for $i = 1, 2$. We define

$$\forall \alpha \in D_{m,n}(\beta, \gamma) \quad \forall \mathbf{i} \in \Sigma \quad \varphi_1(\alpha, \mathbf{i}) = \Pi_{\alpha,\beta,\gamma}((1)^m * (3) * \mathbf{i}) = S_1^m S_3(\Pi_{\alpha,\beta,\gamma}(\mathbf{i})),$$
$$\forall \alpha \in D_{m,n}(\beta, \gamma) \quad \forall \mathbf{i} \in \Sigma \quad \varphi_2(\alpha, \mathbf{i}) = \Pi_{\alpha,\beta,\gamma}((2)^n * (3) * \mathbf{i}) = S_2^n S_3(\Pi_{\alpha,\beta,\gamma}(\mathbf{i})),$$

where $\Pi_{\alpha,\beta,\gamma}$ is the natural projection of $\mathcal{S}_{\alpha,\beta,\gamma}$. Then for every $\alpha, \alpha' \in D_{m,n}(\beta, \gamma)$ and for every $\mathbf{i}, \mathbf{j} \in \Sigma$

$$|\varphi_1(\alpha, \mathbf{i}) - \varphi_2(\alpha, \mathbf{j}) - \varphi_1(\alpha', \mathbf{i}) + \varphi_2(\alpha', \mathbf{j})| \geq M |\alpha - \alpha'|, \tag{4.12}$$

where $M(m, n, \beta, \gamma) > 0$ constant.

Proof: Let $\alpha, \alpha' \in D_{m,n}(\beta, \gamma)$ and $\mathbf{i}, \mathbf{j} \in \Sigma$ be arbitrary. We introduce the notation $\mathcal{S} = \mathcal{S}_{\alpha,\beta,\gamma} = \{S_1, S_2, S_3\}$, $\mathcal{S}' = \mathcal{S}_{\alpha',\beta,\gamma} = \{S_1', S_2', S_3'\}$, let $\Pi = \Pi_{\alpha,\beta,\gamma}$ and $\Pi' = \Pi_{\alpha',\beta,\gamma}$. Then $S_2' = S_2$ and $S_3' = S_3$.

Let $\alpha < \alpha'$ and $\delta = |\alpha' - \alpha|$, then using Lagrange mean value theorem

$$m\alpha^{m-1} \leq \frac{\alpha'^m - \alpha^m}{\alpha' - \alpha} = \frac{|\alpha'^m - \alpha^m|}{\delta} \leq m\alpha'^{m-1}. \tag{4.13}$$

We defined $\delta = |\alpha' - \alpha|$ and using displacement Theorem 4.2 for $\mathcal{S}$ and $\mathcal{S}'$, then we get

$$\text{for every } \mathbf{i} \in \Sigma \quad |\Pi(\mathbf{i}) - \Pi'(\mathbf{i})| \leq \frac{9}{8}\delta. \tag{4.14}$$

Consider the difference that we have to estimate

$$\begin{aligned}
&\varphi_1(\alpha, \mathbf{i}) - \varphi_1(\alpha', \mathbf{i}) + \varphi_2(\alpha', \mathbf{j}) - \varphi_2(\alpha, \mathbf{j}) = \\
&= S_1^m S_3(\Pi(\mathbf{i})) - S_1'^m S_3'(\Pi'(\mathbf{i})) + S_2'^n S_3'(\Pi'(\mathbf{j})) - S_2^n S_3(\Pi(\mathbf{j})) = \\
&= S_1^m S_3(\Pi(\mathbf{i})) - S_1'^m S_3(\Pi'(\mathbf{i})) + S_2^n S_3(\Pi'(\mathbf{j})) - S_2^n S_3(\Pi(\mathbf{j})) = \\
&= \underbrace{S_1^m S_3(\Pi(\mathbf{i})) - S_1^m S_3(\Pi'(\mathbf{i}))}_{A} + \underbrace{S_1^m S_3(\Pi'(\mathbf{i})) - S_1'^m S_3(\Pi'(\mathbf{i}))}_{B} + \underbrace{S_2^n S_3(\Pi'(\mathbf{j})) - S_2^n S_3(\Pi(\mathbf{j}))}_{C}.
\end{aligned}$$

We will use the estimate

$$|A + B + C| \geq |B| - |A + C| \geq |B| - |A| - |C|, \tag{4.15}$$

where first we use the reversed triangle inewquality and second the triangle inequality.

Consider $|A|$ part of the above calculation

$$|A| = |S_1^m S_3(\Pi(\mathbf{i})) - S_1^m S_3(\Pi'(\mathbf{i}))| = \alpha^m \gamma\, |\Pi(\mathbf{i}) - \Pi'(\mathbf{i})| \leq \frac{9}{8}\alpha^m \gamma \delta,$$

where in the inequation we use (4.14).

The next part is

$$|B| = |S_1^m S_3(\Pi'(\mathbf{i})) - S_1'^m S_3(\Pi'(\mathbf{i}))| = |\alpha^m - \alpha'^m|\, |S_3(\Pi'(\mathbf{i}))| \geq \frac{8}{9} m\alpha^{m-1}\delta,$$

where in the inequation we use (4.13).

The last part is

$$|C| = |S_2^n S_3(\Pi'(\mathbf{j})) - S_2^n S_3(\Pi(\mathbf{j}))| = \beta^n \gamma\, |\Pi(\mathbf{j}) - \Pi'(\mathbf{j})| \leq \frac{9}{8}\beta^n \gamma \delta,$$

where in the inequation we use (4.14).

Now estimate

$$|B|-|A| \geq \left(\frac{8m}{9\alpha}-\frac{9}{8}\gamma\right)\alpha^m\delta \geq \left(8-\frac{9}{8}\right)\alpha^m\delta \geq \left(8-\frac{9}{8}\right)\frac{8}{9}\beta^n\delta > 6\beta^n\delta, \tag{4.16}$$

where in the second inequation we use $\gamma < 1$, $m \geq 1$, $\alpha < \frac{1}{9}$.

The following

$$|C| \leq \frac{9}{8}\gamma\beta^n\delta < \beta^n\delta \tag{4.17}$$

is true, beacuse $\gamma < \frac{1}{9}$. Thus

$$|\varphi_1(\alpha,\mathbf{i})-\varphi_2(\alpha,\mathbf{j})-\varphi_1(\alpha',\mathbf{i})+\varphi_2(\alpha',\mathbf{j})| \geq 5\beta^n\,|\alpha'-\alpha|\,, \tag{4.18}$$

so $M = 5\beta^n$. □

Lemma 4.7 *Let $m,n \in \mathbb{N}^+$ and $\beta,\gamma \in (0,\frac{1}{9})$. Then the set*

$$\Delta_{m,n}(\beta,\gamma) = \left\{\alpha \in \left(0,\frac{1}{9}\right) : S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \neq \emptyset\right\} \tag{4.19}$$

is closed in $(0,\frac{1}{9})$ and $\mathcal{L}(\Delta_{m,n}(\beta,\gamma)) = 0$, where $\mathcal{L}$ means the Lebesgue measure on $\mathbb{R}$.

Proof: Let $\varepsilon > 0$ be such that $\frac{1}{9}-\varepsilon > \beta,\gamma$. Then $E_{m,n}(\beta,\gamma) = D_{m,n}(\beta,\gamma) \cap [\varepsilon,\frac{1}{9}-\varepsilon]$ is a closed interval in $\mathbb{R}$, so it is compact. We consider the compact metric space (Σ,ρ_a), where $\Sigma = \{1,2,3\}^{\mathbb{N}^+}$ and $a = \frac{1}{9}-\varepsilon$.

Let $\varphi_i : E_{m,n}(\beta,\gamma)\times\Sigma \to \mathbb{R}$ for $i = 1,2$. We define

$$\begin{aligned}\varphi_1(\alpha,\mathbf{i}) &= \Pi_{\alpha,\beta,\gamma}((1)^m * (3) * \mathbf{i}) = S_1^m S_3(\Pi_{\alpha,\beta,\gamma}(\mathbf{i})),\\ \varphi_2(\alpha,\mathbf{i}) &= \Pi_{\alpha,\beta,\gamma}((2)^n * (3) * \mathbf{i}) = S_2^n S_3(\Pi_{\alpha,\beta,\gamma}(\mathbf{i})).\end{aligned} \tag{4.20}$$

Let

$$\Xi^{\varepsilon}_{m,n}(\beta,\gamma) = \Delta_{m,n}(\beta,\gamma) \cap \left[\varepsilon,\frac{1}{9}-\varepsilon\right].$$

For an α the $S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \neq \emptyset$ holds if and only if there exist $\mathbf{i},\mathbf{j} \in \Sigma$ such that $\varphi_1(\alpha,\mathbf{i}) = \varphi_2(\alpha,\mathbf{j})$, thus

$$\Xi^{\varepsilon}_{m,n}(\beta,\gamma) = \{\alpha \in E_{m,n}(\beta,\gamma) : \varphi_1(\alpha,\Sigma) \cap \varphi_2(\alpha,\Sigma) \neq \emptyset\}. \tag{4.21}$$

Using Lemma 4.4 one can see that φ_i is Hölder continuous with respect to the second variable for $i = 1, 2$. Applying Lemma 4.6, we get that the conditions of the General Position Theorem 4.1 holds. Using General Position Theorem 4.1, then get

$$\dim_H(\Xi^{\varepsilon}_{m,n}(\beta,\gamma)) \leq \dim_H(\Sigma \times \Sigma) \leq 2\dim_H(\Sigma) < 1, \tag{4.22}$$

the last inequation is true because of the equation (4.9). So $\mathcal{L}(\Xi^{\varepsilon}_{m,n}(\beta,\gamma)) = 0$. Moreover,

$$\Delta_{m,n}(\beta,\gamma) = \bigcup_{k=1}^{\infty} \Xi^{1/k}_{m,n}(\beta,\gamma), \tag{4.23}$$

thus the continuity of measure yields that $\mathcal{L}(\Delta_{m,n}(\beta,\gamma)) = 0$.

General Position Theorem 4.1 implies that $\Xi^{\varepsilon}_{m,n}(\beta,\gamma)$ is closed for every $\varepsilon > 0$, so $\Delta_{m,n}(\beta,\gamma)$ is also closed. □

Lemma 4.8 *Let $m, n \in \mathbb{N}^+$ be arbitrary. Then the set*

$$\tilde{\Delta}_{m,n} = \left\{ (\alpha,\beta,\gamma) \in \left(0, \frac{1}{9}\right)^3 : S_1^m(R_{\alpha,\beta,\gamma}) \cap S_2^n(R_{\alpha,\beta,\gamma}) \neq \emptyset \right\} \tag{4.24}$$

is closed in $(0, \frac{1}{9})^3$ and $\mathcal{L}^3(\tilde{\Delta}_{m,n}) = 0$, that is its Lebesgue measure is zero in $\mathbb{R}^3$.

Proof: Let $I = (0, \frac{1}{9})^3$ and $\Psi : I \times \Sigma^2 \to \mathbb{R}$ be such that

$$\begin{aligned} \Psi(\alpha,\beta,\gamma,\mathbf{i},\mathbf{j}) &= \Pi_{\alpha,\beta,\gamma}((1)^m * (3) * \mathbf{i}) - \Pi_{\alpha,\beta,\gamma}((2)^n * (3) * \mathbf{j}) = \\ &= S_1^m S_3(\Pi_{\alpha,\beta,\gamma}(\mathbf{i})) - S_2^n S_3(\Pi_{\alpha,\beta,\gamma}(\mathbf{j})). \end{aligned} \tag{4.25}$$

Then Ψ is a continuous function and $\tilde{\Delta}_{m,n} = \mathrm{proj}_I\left(\Psi^{-1}(\{0\})\right)$. Because Ψ is continuous $\Psi^{-1}(\{0\})$ is closed, and because Σ is compact $\tilde{\Delta}_{m,n}$ is closed. This is because if $x \in I$ is an accumulation point of $\mathrm{proj}_I\left(\Psi^{-1}(\{0\})\right)$, then there exists an $(x, y) \in I \times \Sigma^2$ such that is an accumulation point of $\Psi^{-1}(\{0\})$. Consider the integral

$$\begin{aligned}
\mathcal{L}^3(\tilde{\Delta}_{m,n}) &= \iiint\limits_{(0,\frac{1}{9})^3} \mathbb{1}_{\tilde{\Delta}_{m,n}}(\alpha,\beta,\gamma)\,\mathrm{d}\mathcal{L}^3(\alpha,\beta,\gamma) = \\
&= \iint\limits_{(0,\frac{1}{9})^2} \int\limits_{(0,\frac{1}{9})} \mathbb{1}_{\tilde{\Delta}_{m,n}}(\alpha,\beta,\gamma)\,\mathrm{d}\mathcal{L}(\alpha)\mathrm{d}\mathcal{L}^2(\beta,\gamma) = \\
&= \iint\limits_{(0,\frac{1}{9})^2} \int\limits_{(0,\frac{1}{9})} \mathbb{1}_{\Delta_{m,n}(\beta,\gamma)}(\alpha)\,\mathrm{d}\mathcal{L}(\alpha)\mathrm{d}\mathcal{L}^2(\beta,\gamma) = \\
&= \iint\limits_{(0,\frac{1}{9})^2} \mathcal{L}(\Delta_{m,n}(\beta,\gamma))\mathrm{d}\mathcal{L}^2(\beta,\gamma) = \iint\limits_{(0,\frac{1}{9})^2} 0\mathrm{d}\mathcal{L}^2(\beta,\gamma) = 0,
\end{aligned} \tag{4.26}$$

where we use Fubini's theorem in the second equality and the fourth equality we use Lemma 4.7. □

Theorem 4.9 *Let $J = \left(0, \frac{1}{9}\right)^3$. We define*

$$\Omega = \{(\alpha,\beta,\gamma) \in J : \mathcal{S}_{\alpha,\beta,\gamma} \textit{ is a forward separated system}\}. \tag{4.27}$$

Then $\mathcal{L}^3(J/\Omega) = 0$ and J/Ω is uncountable and dense in J.

Proof: The set

$$J/\Omega = \bigcup_{m,n=1}^{\infty} \tilde{\Delta}_{m,n}\ , \text{ thus } \mathcal{L}^3(J/\Omega) \le \sum_{m,n=1}^{\infty} \mathcal{L}^3(\tilde{\Delta}_{m,n}) = 0, \tag{4.28}$$

where we use Lemma 4.8 in the last equality. Thus $\mathcal{L}^3(J/\Omega) = 0$.

If $\alpha^m = \beta^n$ for $m, n \in \mathbb{N}^+$, then $\mathcal{S}_{\alpha,\beta,\gamma}$ is not a forward separated system, so

$$\tilde{\Delta} = \left\{(\alpha,\beta,\gamma) \in J : \frac{\log\alpha}{\log\beta} \in \mathbb{Q}\right\} \subseteq J/\Omega. \tag{4.29}$$

Then for every $z \in (0, \frac{1}{9})$ the set

$$\{(\alpha,\beta,\gamma) \in \tilde{\Delta} : \gamma = z\} = \bigcup_{q\in\mathbb{Q}^+} \left\{(\alpha, f_q(\alpha), z) : \alpha \in \left(0, \frac{1}{9}\right), \quad f_q(x) = x^q\right\} \tag{4.30}$$

is union of smooth curves. From this one can easily see that $\tilde{\Delta}$ is dense and uncountable. This implies J/Ω is also dense and uncountable. □

Chapter 5

The tools of calculating Hausdorff-dimension of the self-similar measure

We would like to calculate the Hausdorff dimension of the self-similar measure of the forward separated system $\mathcal{S}_{\alpha,\beta,\gamma}$ and for this we require the following statements.

5.1 Conditional expectation

Definition 5.1 *Let $\mathcal{G} \subseteq \mathcal{B}$ be an arbitrary σ-algebra. Let $\varphi \in L^1(Z, \mathcal{B}, \mu)$, then the function $\psi \in L^1(Z, \mathcal{B}, \mu)$ is the conditional expectation of φ with respect to the σ-algebra $\mathcal{G}$, if*

(i) ψ is $\mathcal{G}$-measurable,

(ii) for every $G \in \mathcal{G}$

$$\int_Z \varphi(x)\mathbb{1}_G(x)\,\mathrm{d}\mu(x) = \int_Z \psi(x)\mathbb{1}_G(x)\,\mathrm{d}\mu(x). \tag{5.1}$$

Theorem 5.2 *Let $\mathcal{G} \subseteq \mathcal{B}$ be an arbitrary σ-algebra. If ψ and $\tilde{\psi}$ are conditional expectations of the function $\varphi \in L^1(Z, \mathcal{B}, \mu)$ with respect to $\mathcal{G}$, then $\psi(z) = \tilde{\psi}(z)$ for μ-almost every $z \in Z$.*

We denote the conditional expectation of $\varphi \in L^1(Z, \mathcal{B}, \mu)$ with respect to $\mathcal{G}$ with $\mathbb{E}_\mu(\varphi|\mathcal{G})$.

5.2 Conditional measure

The proof of the statements that contained in this section can be found in [4].

Let Z be a compact metric space. We consider the probability space $(Z, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra of Z and μ is a probability measure on Z.

Moreover, let $\mathcal{F}$ be a σ-algebra such that there exists some $E_1, E_2, \dots \in \mathcal{B}$ for which

$$\mathcal{F} = \bigvee_{i=1}^{\infty} \{E_i, Z/E_i\}, \tag{5.2}$$

where $\vee$ denotes the generated σ-algebra. Indeed, if $\mathcal{A}_i \subseteq \mathcal{B}$ is a σ-algebra for all $i = 1, 2, \dots,$ then $\bigvee\limits_{i=1}^{\infty} \mathcal{A}_i$ is the generated σ-algebra by $\bigcup\limits_{i=1}^{\infty} \mathcal{A}_i$.

Definition 5.3 *The $\mathcal{P} \subseteq \mathcal{B}$ is a partition of Z, if for every $P_1 \neq P_2 \in \mathcal{P}$ $\quad P_1 \cap P_2 = \emptyset$ and $\bigcup\limits_{P \in \mathcal{P}} P = Z$.*

Let $\mathcal{P}$ be a partition of Z. Then for $z \in Z$ the set $\mathcal{P}(z)$ denotes those $\mathcal{P}(z) \in \mathcal{P}$ such that $z \in \mathcal{P}(z)$.

For every $n = 1, 2, \dots$ let $\mathcal{P}_n$ be a partition of Z such that

$$\sigma(\mathcal{P}_n) = \bigvee_{i=1}^{n} \{E_i, Z/E_i\}, \tag{5.3}$$

where $\sigma(\mathcal{A})$ denotes the generated sigma algebra by $\mathcal{A}$.

Definition 5.4 *The set $\{\mu_z\}_{z \in Z}$ of Borel probability measures on Z is a system of conditional measures of μ with respect to the σ-algebra $\mathcal{F}$, if*

(i) for every $E \in \mathcal{F},\ z \in E \quad \mu_z(E) = 1$ holds for μ-almost every $z \in Z$,

(ii) for every bounded measurable function $\varphi : Z \to \mathbb{R}$ the function $z \mapsto \int\limits_Z \varphi \, \mathrm{d}\mu_z$ is $\mathcal{F}$-measurable and

$$\int\limits_Z \varphi(x) \, \mathrm{d}\mu(x) = \int\limits_Z \int\limits_Z \varphi(x) \, \mathrm{d}\mu_z(x) \ \mathrm{d}\mu(z). \tag{5.4}$$

Theorem 5.5 *If $\{\mu_z\}_{z \in Z}$ and $\{\nu_z\}_{z \in Z}$ are two systems of condtional measures of μ with respect to $\mathcal{F}$, then $\mu_z = \nu_z$ for μ-almost every $z \in Z$.*

The proof is in [4].

Theorem 5.6 *The limit of the measures*

$$\mu_z^{\mathcal{F}} = \lim_{n\to\infty} \frac{\mu|_{\mathcal{P}_n(z)}}{\mu(\mathcal{P}_n(z))} \text{ exists for } \mu\text{-almost every } z \in Z, \tag{5.5}$$

where the limit is meant in the weak-star topology.

Moreover, the set $\{\mu_z^{\mathcal{F}}\}_{z\in Z}$ *is a system of conditional measures of* μ *with respect to the* σ*-algebra* $\mathcal{F}$*.*

The proof can be found in [4].

Theorem 5.7 *Let* $\varphi : Z \to \mathbb{R}$ *is bounded and measurable, then the function*

$$\begin{gathered}\Phi : Z \to \mathbb{R} \text{ for which} \\ \Phi(z) = \int_Z \varphi(x) \mathrm{d}\mu_z(x) \text{ for } \mu\text{-almost every } z \in Z\end{gathered} \tag{5.6}$$

is the conditional expectation of φ *with respect to* $\mathcal{F}$*, thus* $\mathbb{E}_\mu(\varphi|\mathcal{F}) = \Phi$*.*

The proof is in [4].

5.3 Feng-Hu theorem

First, we introduce some notations, which we will use in this section. Let $(X, \mathcal{F}, m)$ be a probability space. Let $\xi \subseteq \mathcal{A}$ be a countable partition of X. Denote with $\mathcal{A} \subseteq \mathcal{F}$ an arbitrary σ-algebra. Then $I_m(\xi|\mathcal{A})$ denotes the conditional information of the partition ξ given by $\mathcal{A}$, which is

$$I_m(\xi|\mathcal{A})(x) = -\sum_{E\in\xi} \mathbb{1}_E(x) \log[\, \mathbb{E}_m(\mathbb{1}_E|\mathcal{A})(x)\,]. \tag{5.7}$$

The conditional entropy of ξ given $\mathcal{A}$ is defined by the following formula

$$H_m(\xi|\mathcal{A}) = \int I_m(\xi|\mathcal{A}) \mathrm{d}m. \tag{5.8}$$

Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be an IFS on $\mathbb{R}^d$. The attractor of $\mathcal{S}$ is Λ and $\Sigma = \{1, \ldots, m\}^{\mathbb{N}^+}$ is the symbolic space. The natural projection of $\mathcal{S}$ is $\Pi : \Sigma \to \Lambda$. We denote the left shift on Σ with σ.

Let $(\Sigma, \mathcal{C}, \mu)$ be a probability space, where $\mathcal{C}$ is the σ-algera generated by the cylinder sets. The Borel probability measure μ on Σ is such that μ is invariant for σ. We define the measure ν on Λ with $\nu = \Pi_*\mu = \mu \circ \Pi^{-1}$. $\mathcal{P} = \{[1], \ldots, [m]\}$ is a Borel partition of Σ and γ denotes the Borel σ-algebra on $\mathbb{R}^d$.

We define the projection entropy of μ under Π to the IFS $\mathcal{S}$ as

$$h_\Pi(\sigma, \mu) = H_\mu(\mathcal{P}|\sigma^{-1}\Pi^{-1}\gamma) - H_\mu(\mathcal{P}|\Pi^{-1}\gamma). \tag{5.9}$$

The Theorem 2.8 in [1] states the following:

Theorem 5.8 (Feng-Hu) *Assume that μ is a σ-invariant ergodic Bernoulli probabilty measure on Σ. $\mathcal{S}$ is a μ-conformal IFS and $\nu = \Pi_*\mu = \mu \circ \Pi^{-1}$, then*

$$\dim_\nu(x) = \frac{h_\Pi(\sigma, \mu)}{\int \lambda \mathrm{d}\mu} \text{ for } \nu\text{-almost every } x \in \Lambda. \tag{5.10}$$

Lemma 5.9 *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be an IFS. If μ is a Bernoulli measure on Σ for the probability vector $\mathbf{p} = (p_1, \ldots, p_m)$, then*

$$H_\mu(\mathcal{P}|\sigma^{-1}\Pi^{-1}\gamma) = -\sum_{k=1}^{m} p_k \log(p_k). \tag{5.11}$$

Proof: The definition of conditional entropy and conditional information is contained in equation (5.8) and (5.7) respectively. Let $[k] \in \mathcal{P}$, then the generated σ-algebra by the function $\mathbb{1}_{[k]}$ is $\sigma(\mathbb{1}_{[k]}) = \{[k], \Sigma/[k]\} \subseteq \sigma(\mathcal{P})$, where $\sigma(\mathcal{P})$ is the generated σ-algebra by $\mathcal{P}$. We can easily check that $\sigma(\mathcal{P})$ and $\sigma^{-1}\Pi^{-1}\gamma$ are independent σ-algebras. We know that if $\varphi \in L^1(\Sigma, \mathcal{C}, \mu)$, $\sigma(\varphi)$ and $\mathcal{G}$ are independent σ-algebras, then $\mathbb{E}_\mu(\varphi|\mathcal{G}) = \mathbb{E}_\mu(\varphi) = \int \varphi \mathrm{d}\mu$. Thus $\mathbb{E}_\mu(\mathbb{1}_{[k]}|\sigma^{-1}\Pi^{-1}\gamma) = \mu([k]) = p_k$. So we can conclude that $H_\mu(\mathcal{P}|\sigma^{-1}\Pi^{-1}\gamma) = -\sum_{k=1}^{m} p_k \log(p_k)$. □

Theorem 5.10 (Feng-Hu theorem in a special case) *We assume that $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a self-similar IFS and the Lipschitz constant of S_i is λ_i. μ is a Bernoulli measure on Σ for the probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ and $\nu = \Pi_*\mu = \mu \circ \Pi^{-1}$, then*

$$\dim_H(\nu) = \frac{-\sum_{k=1}^{m} p_k \log(p_k) - H_\mu(\mathcal{P}|\Pi^{-1}\gamma)}{-\sum_{k=1}^{m} p_k \log(\lambda_k)}. \tag{5.12}$$

Proof: The proof is using Lemma 2.21, Lemma 5.9 and Theorem 2.19 in the Feng-Hu Theorem 5.8. □

Chapter 6

Hausdorff-dimension of a self-similar measure of a forward separated system $\mathcal{S}_{\alpha,\beta,\gamma}$

Theorem 6.1 *Let $\mathcal{S}_{\alpha,\beta,\gamma} = \{S_1, S_2, S_3\}$ be a forward separated system. (See Definition 3.1 and 3.3). The symbolic space is $\Sigma = \{1, 2, 3\}^{\mathbb{N}^+}$ and the natural projection of $\mathcal{S}_{\alpha,\beta,\gamma}$ is $\Pi_{\alpha,\beta,\gamma}$. Let $\mu = (p_1, p_2, p_3)^{\mathbb{N}^+}$ be a Bernoulli measure on Σ for the probability vector $\mathbf{p} = (p_1, p_2, p_3)$. And let $\nu = \Pi_{\alpha,\beta,\gamma *}\mu = \mu \circ \Pi^{-1}_{\alpha,\beta,\gamma}$ is the self-similar measure on the attractor with respect to $\mathbf{p}$. Then the Hausdorff dimension of ν can be exactly determined as*

$$\dim_H(\nu) = \frac{-(p_1 \log(p_1) + p_2 \log(p_2) + p_3 \log(p_3)) + \phi(p_1, p_2, p_3)}{-(p_1 \log(\alpha) + p_2 \log(\beta) + p_3 \log(\gamma))}, \tag{6.1}$$

where

$$\Phi(p_1, p_2, p_3) = \sum_{k=1}^{\infty} \left(\sum_{m=1}^{k} \binom{k-1}{m-1} \log\left(\frac{m}{k}\right) p_1^m p_2^{k-m} p_3 + \sum_{m=0}^{k-1} \binom{k-1}{m} \log\left(\frac{k-m}{k}\right) p_1^m p_2^{k-m} p_3 \right).$$

Proof: We introduce easier notation for further use. We denote $\mathcal{S}_{\alpha,\beta,\gamma}$ with $\mathcal{S}$ and $\Pi_{\alpha,\beta,\gamma}$ is Π. The $K_{\alpha,\beta,\gamma}$ attractor of $\mathcal{S}$ is K. Let $R = S_3(K)$ and $L = S_1(K) \cup S_2(K)$.

The system $\mathcal{S}$ is self-similar IFS and the measure μ is a Bernoulli measure on the symbolic space Σ. Thus, we can use the special form of Feng-Hu Theorem 5.10. So the only thing that we have to calculate is $H_\mu(\mathcal{P}|\Pi^{-1}\gamma)$, where $\mathcal{P} = \{[k] : k = 1, 2, 3\}$ is a partition of Σ and γ is the Borel σ-algebra on $[0, 1] \subseteq \mathbb{R}$.

To the calculation of $H_\mu(\mathcal{P}|\Pi^{-1}\gamma)$, we need

$$I_\mu(\mathcal{P}|\Pi^{-1}\gamma)(\mathbf{i}) = -\sum_{k=1}^{3} \mathbb{1}_{[k]}(\mathbf{i}) \log(\mathbb{E}_\mu(\mathbb{1}_{[k]}|\Pi^{-1}\gamma))(\mathbf{i}). \tag{6.2}$$

The σ-algebra γ can be generated by countable many finite partition. Let

$$\mathcal{P}_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) : 0 \le k \le 2^n - 1 \right\}, \tag{6.3}$$

this is a finite partition of $[0,1]$ and $\bigvee_{i=1}^{\infty} \mathcal{P}_i$ is the Borel σ-algebra on $[0,1]$. Using this fact and Theorem 5.6, then we get that $\{\mu_\mathbf{i}\}_{\mathbf{i}\in\Sigma}$ system of conditional measures with respect to the σ-algebra $\Pi^{-1}\gamma$ exists. Using Theorem 5.7, we get

$$\mathbb{E}_\mu(\mathbb{1}_{[k]}|\Pi^{-1}\gamma)(\mathbf{i}) = \int_\Sigma \mathbb{1}_{[k]}(\mathbf{j})\, \mathrm{d}\mu_\mathbf{i}(\mathbf{j}) = \mu_\mathbf{i}([k]) = \begin{cases} \mu_\mathbf{i}([i_1]), & \text{if } i_1 = k \\ 0, & \text{if } i_1 \neq k \end{cases}. \tag{6.4}$$

Using the above observation, one get

$$H_\mu(\mathcal{P}|\Pi^{-1}\gamma) = -\int_\Sigma \log(\mu_\mathbf{i}([i_1]))\mathrm{d}\mu(\mathbf{i}). \tag{6.5}$$

Using Theorem 5.6, we conclude that

$$\mu_\mathbf{i} = \lim_{n\to\infty} \frac{\mu|_{\Pi^{-1}(\mathcal{P}_n(\Pi(\mathbf{i})))}}{\mu(\Pi^{-1}(\mathcal{P}_n(\Pi(\mathbf{i}))))}, \tag{6.6}$$

where limit is meant in the weak-star topology. We know the property of weak-star convergence, that if ν_n, ν are Borel probability measures on the compact metric space X and $\lim_{n\to\infty} \nu_n = \nu$ in weak-star sense, then for all $U \subseteq X$ open and $Z \subseteq X$ closed $\liminf_{n\to\infty} \nu_n(U) \ge \nu(U)$ and $\limsup_{n\to\infty} \nu_n(Z) \le \nu(Z)$ hold. Because $[k] \subseteq \Sigma$ is open and closed, then

$$\mu_\mathbf{i}([i_1]) = \lim_{n\to\infty} \frac{\mu(\Pi^{-1}(\mathcal{P}_n(\Pi(\mathbf{i}))) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_n(\Pi(\mathbf{i}))))}. \tag{6.7}$$

For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ enough large such that $\mathcal{P}_n(\Pi(\mathbf{i})) \subseteq B(\Pi(\mathbf{i}), \varepsilon)$.

We define for all $m,n=0,1,\dots$ the set

$$H(m,n)=\{\mathbf{i}=(i_1,i_2,\dots)\in\Sigma\,:\,i_l\neq 3\quad \forall l=1,\dots m+n,\\ i_{m+n+1}=3,\ |\{1\leq k\leq m+n:i_k=1\}|=m\}\,. \tag{6.8}$$

We can see that $\Pi(H(m,n))=S_1^mS_2^nS_3(K)=S_1^mS_2^n(R)$.

Let $\mathbf{i}\in H(m,n)\subseteq\Sigma$. Then $\Pi(\mathbf{i})\neq 0$, because $i_{m+n+1}=3$, thus $\Pi(\mathbf{i})\in S_1^mS_2^nS_3(K)\subseteq[a(m,n),1]$, where $a(m,n)=\frac{1}{2}(min\{\alpha,\beta\})^{m+n}$

If $(\max\{\alpha,\beta\})^{k+l}<\frac{a(m,n)}{2}$, then the sets $S_1^kS_2^l(R)\subseteq(0,\frac{a(m,n)}{2})$. Thus there exist $M(m,n)\in\mathbb{N}$ such that if $k+l>M(m,n)$, then $S_1^kS_2^l(R)\subseteq(0,\frac{a(m,n)}{2})$.

We introduce

$$\mathcal{H}=\left\{S_1^kS_2^l(R)\,:\,k+l\leq M(m,n),\quad k,l=0,1,\dots\right\}. \tag{6.9}$$

We can notice that $S_1^mS_2^n(R)\in\mathbb{H}$. The set $\mathcal{H}$ is finite and the elements are disjoint compact sets, thus there exists $\varepsilon_1>0$ such that

$$\forall H_1\neq H_2\in\mathcal{H}\quad N_{\varepsilon_1}(H_1)\cap N_{\varepsilon_1}(H_2)=\emptyset, \tag{6.10}$$

where $N_\varepsilon(H)$ means the ε neighbourhood of the set H.

Using $K/\{0\}=\bigcup\limits_{k,l=0}^{\infty}S_1^kS_2^l(R)$, then there exists ε_2 such that

$$\forall k,l=0,1\dots\quad N_{\varepsilon_2}(S_1^mS_2^n(R))\cap N_{\varepsilon_2}(S_1^kS_2^l(R))=\emptyset, \tag{6.11}$$

Thus we can conclude that $B(\Pi(\mathbf{i}),\varepsilon_2)\cap K\subseteq S_1^mS_2^n(R)$. For every $\varepsilon>0$ there exists $n\in\mathbb{N}$ enough large such that $\mathcal{P}_n(\Pi(\mathbf{i}))\subseteq B(\Pi(\mathbf{i}),\varepsilon)$. So there exist $N(n,m)\in\mathbb{N}$ such that for all $N(m,n)<z\quad\mathcal{P}_z(\Pi(\mathbf{i}))\subseteq B(\Pi(\mathbf{i}),\varepsilon_2)$. Moreover, for all $N(m,n)<z\quad\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))\subseteq H(m,n)$.

Let $z>N(m,n)$ be fix, then

$$\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i})))=E\times T_z,\text{ where}$$
$$E=\{(i_1,\dots,i_{m+n+1})\,:\,i_{m+n+1}=3,\ |\{k:i_k=1\}|=m,\ |\{k:i_k=2\}|=n\}$$
$$T_z=\{(j_1,j_2,\dots)\in\{1,2,3\}^{\mathbb{N}^+}\,:\,\exists\mathbf{k}\in\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i})))\quad j_{n+m+1}=k_{n+m+1},\,j_{n+m+2}=k_{n+m+2},\dots\}.$$

The above equation is true, because if $\mathbf{k}\in\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i})))$, then if permute the first $n+m$

coordinates and in this way we get $\mathbf{k}'$, then $\Pi(\mathbf{k}) = \Pi(\mathbf{k}')$. Then

$$\frac{\mu(E \times T_z \cap [i_1])}{\mu(E \times T_z)} = \frac{\mu((E \cap [i_1]) \times T_z)}{\mu(E \times T_z)} = \frac{\mu([E \cap [i_1]\,])\mu(T_z)}{\mu([E])\mu(T_z)} = \frac{\mu([E \cap [i_1]\,])}{\mu([E])}, \tag{6.12}$$

where $[E] = \{\mathbf{i} \in \Sigma : \exists \mathbf{j} \in E \quad \exists \mathbf{k} \in \Sigma \quad \mathbf{i} = \mathbf{j} * \mathbf{k}\}$.

Because μ is a Bernoulli measure, then

$$\mu(E) = \frac{(m+n)!}{m!n!} p_1^m p_2^n p_3. \tag{6.13}$$

Suppose that $i_1 = 1$, then

$$\mu(E \cap [i_1]) = \frac{(m+n-1)!}{(m-1)!n!} p_1^m p_2^n p_3. \tag{6.14}$$

In this case for large z

$$\frac{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))))} = \frac{\mu(E \times T_z \cap [i_1])}{\mu(E \times T_z)} = \frac{m}{m+n}. \tag{6.15}$$

Suppose that $i_1 = 2$, then

$$\mu(E \cap [i_1]) = \frac{(m+n-1)!}{m!(n-1)!} p_1^m p_2^n p_3. \tag{6.16}$$

In this case for large z

$$\frac{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))))} = \frac{\mu(E \times T_z \cap [i_1])}{\mu(E \times T_z)} = \frac{n}{m+n}. \tag{6.17}$$

Suppose that $i_1 = 3$, then for enough large z we get $\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))) \cap [3] = \Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i})))$, thus for large z

$$\frac{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))) \cap [i_1])}{\mu(\Pi^{-1}(\mathcal{P}_z(\Pi(\mathbf{i}))))} = 1. \tag{6.18}$$

From the above observations, if $\mathbf{i} \in H(m,n)$ then

$$\mu_{\mathbf{i}}([i_1]) = \begin{cases} \frac{m}{m+n}, & \text{if } i_1 = 1, \\ \frac{n}{m+n}, & \text{if } i_1 = 2, \\ 1, & \text{if } i_1 = 3. \end{cases} \tag{6.19}$$

Using Kolmogorov 0-1 law we get

$$\mu(\bigcup_{m,n=0}^{\infty} H(m,n)) = 1. \tag{6.20}$$

The integral that we want to calculate is

$$\int_{\Sigma} \log(\mu_{\mathbf{i}}([i_1]))\mathrm{d}\mu(\mathbf{i}) =$$

$$= \sum_{k=1}^{\infty}\sum_{m=1}^{k} \log\left(\frac{m}{k}\right) \frac{(k-1)!}{(m-1)!(k-m)!} p_1^m p_2^{k-m} p_3 + \sum_{k=1}^{\infty}\sum_{m=0}^{k-1} \log\left(\frac{k-m}{k}\right) \frac{(k-1)!}{m!(k-m-1)!} p_1^m p_2^{k-m} p_3,$$

where we use combinatorics calculation. □

List of Figures

2.1 Example for a self-similar IFS on the line 6
2.2 The first, second and third level cylinder sets of the IFS $\mathcal{S} = \{-\frac{x}{7} + \frac{1}{5}, \frac{x}{2} + \frac{1}{6}, -\frac{x}{3} + \frac{6}{7}\}$ 6
2.3 The definition of the Hausdorff dimension. 9

Bibliography

[1] Feng, D.-J., and Hu, H. Dimension theory of iterated function systems. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences 62*, 11 (2009), 1435–1500.

[2] Hutchinson, J. E. Fractals and self-similarity. *Indiana Univ. Math. J. 30*, 5 (1981), 713–747.

[3] Kamalutdinov, K., and Tetenov, A. Twofold cantor sets in r. *arXiv preprint arXiv:1802.03872* (2018).

[4] Rokhlin, V. A. *On the fundamental ideas of measure theory.* Series 1. Trans. Amer. Math. Soc., 1962.