Fat Arcs and Spheres for Bounding Curves and Solving Polynomial Systems

Szilvia Béla

Joint work with Bert Jüttler

MŰEGYETEM 1782

Department of Geometry

Doctoral Program Computational Mathematics

Numerical Analysis and Symbolic Computation

This work was supported by the Austrian Science Fund (FWF) under grant W1214/DK3.

FWF Der Wissenschaftsfonds.

Introduction

Modeling algebraic objects is an essential ingredient of free-form surface visualization and numerical simulations. Therefore various methods exist to approximate or to isolate the solution set of algebraic systems [3, 4, 5]. They are using symbolic, numeric or combined techniques in order to compute the solutions. We present here a technique to generate bounding regions for one- or zero-dimensional solution sets of multivariate polynomial systems.

The problem:
Find the intersection of algebraic hyper-surfaces defined by m polynomial equations in an axis-aligned box Ω

$$\begin{cases} f_1(\mathbf{x}) = 0, \\ \quad\vdots \\ f_m(\mathbf{x}) = 0, \end{cases} \qquad \mathbf{x} \in \Omega = \times_{i=1}^{n}[\alpha_i, \beta_i] \subset \mathbb{R}^n.$$

The zero set of $f_1, \dots f_m$:

$$\mathcal{Z}(f_1, \dots f_m) = \{\mathbf{x} \,:\, \forall i = 1, \dots, m \quad f_i(\mathbf{x}) = 0\} \cap \Omega.$$

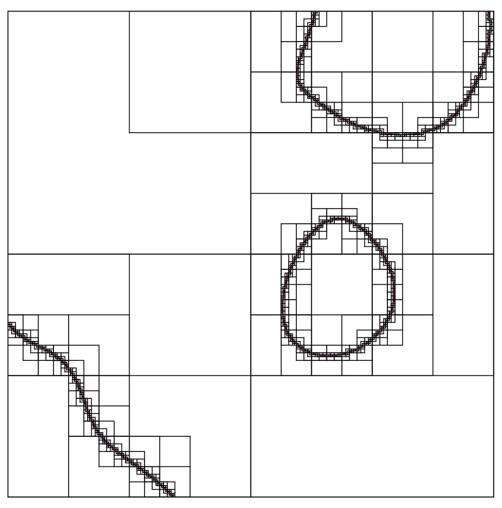
We consider the cases when $\mathcal{Z}$ has dimension zero or one.

Subdivision technique

Subdivision algorithms are based on the "divide and conquer" paradigm. These algorithms decompose the problem into several sub-problems. The decomposition terminates if suitable approximating primitives can be generated in each sub-problem. We use techniques, which compute in axis-aligned boxes and provide information only about the solution set in this computational domain.

Main ingredients of subdivision algorithms:

- exclusion test: detect sub-domains without solution
  $\hookrightarrow$ reject or subdivide the domain
- inclusion test: detect sub-domains with certain type of solution sets
  $\hookrightarrow$ further approximation or subdivide the domain
- termination criterion



Essential tool: The multivariate polynomials are given by their Bernstein-Bézier tensor-product representation with respect to the domain $\Omega \subset \mathbb{R}^n$. This representation form provides stable computations and simple error bounds.

Fat arcs and spheres

We combine the standard subdivision technique with a local bounding region generation method. Fat arcs as bounding primitives were introduced by Sederberg [6]. They consist of an approximating circular arc with some finite thickness. We generalize this definition in order to bound implicitly defined curves and surfaces.

Definition: A **fat arc/sphere** is defined by a circular arc/spherical patch (median arc/sphere) $\mathcal{S} \subset \mathbb{R}^n$ and a thickness $\varrho \in \mathbb{R}$.

$$\mathcal{F}(\mathcal{S}, \varrho) = \{\mathbf{x} \,:\, \exists \mathbf{x}_0 \in \mathcal{S}, \quad |\mathbf{x} - \mathbf{x}_0| \le \varrho\} \cap \Omega.$$

Computation with arcs and spheres has several advantages. Our aim was to exploit these advantages for bounding region generation and real-root approximation.

Main advantages of arcs and spheres:

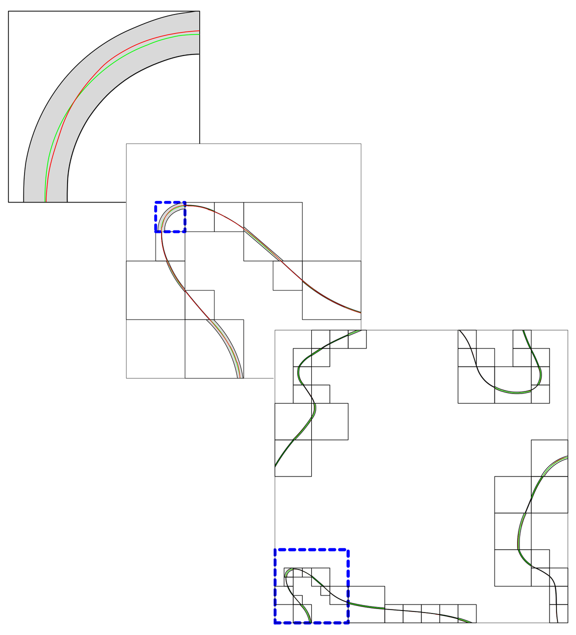
- higher convergence rate
- exact parametric and implicit representation form
- simple computations (solving linear and quadratic equations) to find intersections
- simple offset generation

Bounding regions

In order to generate fat arcs/spheres as bounding primitives we reformulate the algebraic system. A new set of polynomials $\hat{F} = \{\hat{f}_1, \dots, \hat{f}_m\}$ is computed. Each polynomial has a special Hessian-matrix in the center of the computational domain. Therefore the quadratic Taylor-expansion of the polynomial $\hat{f}_i$ is the equation of a sphere

$$s_i = T^2_{\mathbf{c}}(\hat{f}_i).$$

Algebraic curve approximation



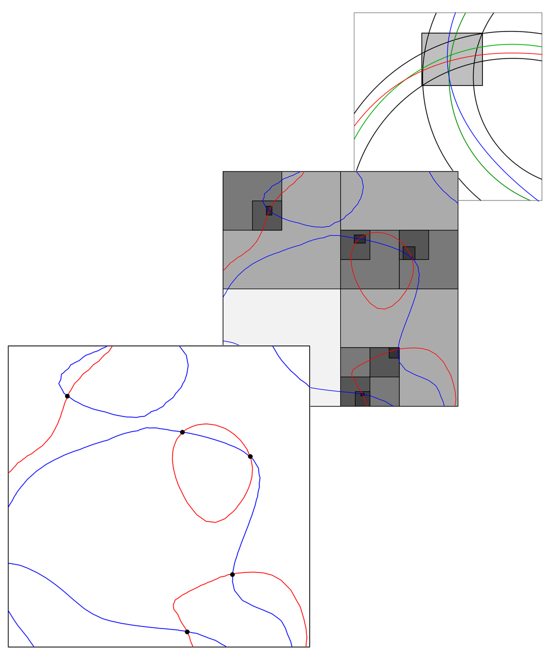
- A **median arc** is defined by

$$\mathcal{S} = \{\mathbf{x} \,:\, i = 1 \dots n-1,\ s_i(\mathbf{x}) = 0\} \cap \Omega$$

  to approximate the algebraic set $\mathcal{Z} \cap \Omega$.
- In order to compute the fat arc thickness we bound the one-sided Hausdorff-distance of the curves on the domain:

$$\mathrm{HD}_\Omega(\mathcal{Z}, \mathcal{S}) \le \varrho.$$

Real root approximation



- A **median sphere** is defined by

$$\mathcal{S}_i = \{\mathbf{x} \,:\, s_i(\mathbf{x}) = 0\} \cap \Omega$$

  to approximate the zero set of each new polynomial $\hat{f}_i$.
- We bound the distance of the polynomials using the Bernstein–Bézier-norm:

$$\delta_i = \mathrm{d}(\hat{f}_i, s_i) = \|\hat{f}_i(\mathbf{x}) - s_i(\mathbf{x})\|^{\Omega}_{_{\mathrm{BB}}}.$$

- The min-max box around the intersection of the fat spheres bounds the algebraic set.

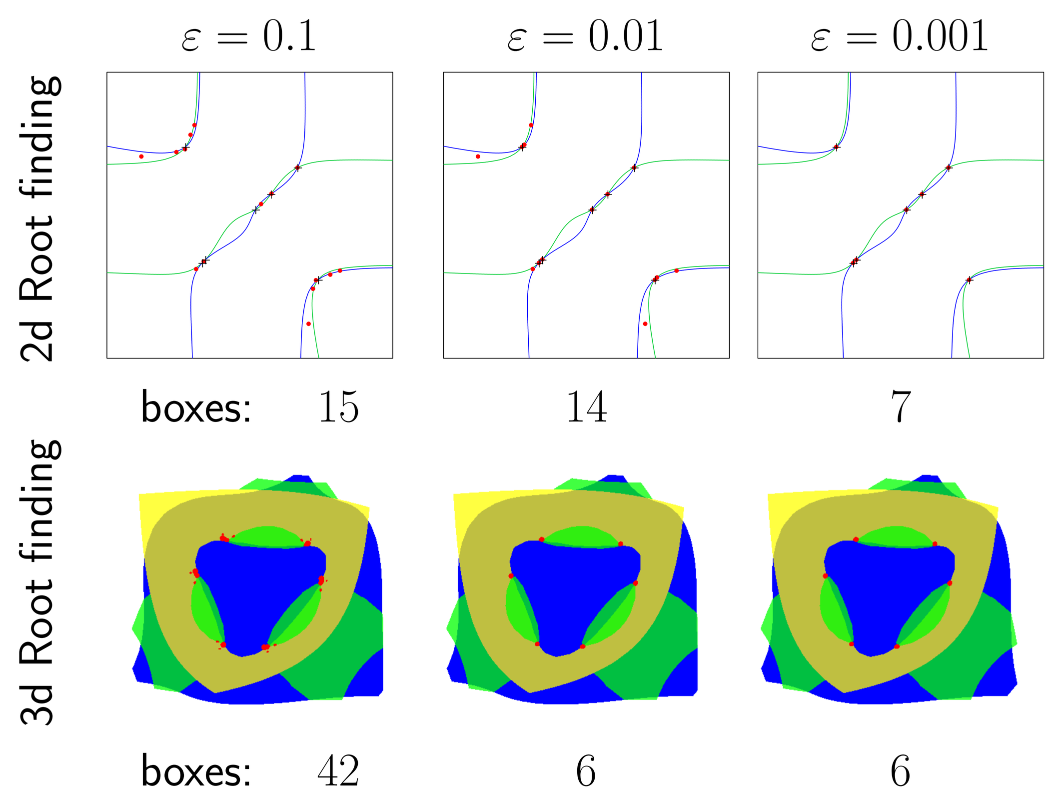
Theorem:

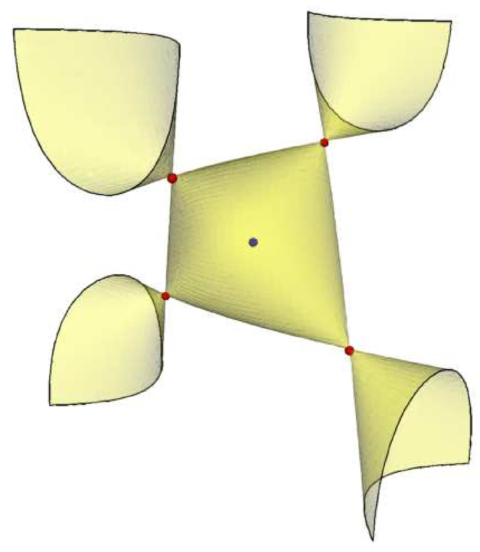
(a) Fat arcs converge with order three to single segments of regular curves.

(b) Fat sphere generation as domain reduction technique generates sub-domains, which converge with order three to single roots.

Numerical experiments indicate that fat sphere generation combined with iterative subdivision provide super-linear convergence rate to double roots of a polynomial system.

Root finding: examples and application





Application in 3d - Ordinary singularities

Conclusion

We presented a new family of algorithms to approximate implicitly defined algebraic curves and real roots of polynomial systems. These methods are based on the geometrical properties of polynomial systems. They generate sequences of bounding regions, which converge with order three to the regular solution set of a multivariate polynomial system.

References

[1] Sz. Béla and B. Jüttler. Fat arcs for implicitly defined curves. In *Mathematical Methods for Curves and Surfaces*, volume 5862 of *Lecture Notes in Computer Science*, pages 26–40. Springer, 2010.

[2] Sz. Béla and B. Jüttler. Approximating algebraic space curves by circular arcs. In M.-L. Mazure et al., editors, *Curves and Surfaces – Avignon 2010*, Lecture Notes in Computer Science. Springer, submitted. DK Report 2010-12.

[3] M. Elkadi and B. Mourrain. Symbolic-numeric tools for solving polynomial equations and applications. In I.Z. Emiris and A. Dickenstein, editors, *Algorithms and Computation in Mathematics*, volume 14, pages 125–168. Springer-Verlag, 2005.

[4] B. Mourrain and J.-P. Pavone. Subdivision methods for solving polynomial equations. *Journal of Symbolic Computation*, 44(3):292–306, 2009.

[5] N. M. Patrikalakis and T. Maekawa. *Shape interrogation for computer aided design and manufacturing*. Springer, 2002.

[6] T. W. Sederberg, S. C. White, and A. K. Zundel. Fat arcs: a bounding region with cubic convergence. *Comput. Aided Geom. Des.*, 6(3):205–218, 1989.