CARDINAL COEFFICIENTS ASSOCIATED TO CERTAIN ORDERS ON IDEALS

PIOTR BORODULIN-NADZIEJA AND BARNABÁS FARKAS

Abstract. We study cardinal invariants connected to certain classical orderings on the family of ideals on ω. We give topological and analytic characterizations of these invariants using the idealized version of Fréchet-Urysohn property and, in a special case, using sequential properties of the space of finitely-supported probability measures with the weak* topology. We investigate consistency of some inequalities between these invariants and classical ones, and other related combinatorial questions. At last, we discuss maximality properties of almost disjoint families related to certain ordering on ideals.

1. Introduction

Recall that an infinite set $P \subseteq \omega$ is a *pseudo-intersection* of a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ if $P \subseteq^* A$ (which means that $P \setminus A$ is finite) for every $A \in \mathcal{A}$.

The *pseudo-intersection number* $\mathfrak{p}$ is the minimal size of a family $\mathcal{A}$ with the *strong finite intersection property* (SFIP in short, i.e. $\bigcap \mathcal{A}'$ is infinite for every nonempty finite $\mathcal{A}' \subseteq \mathcal{A}$) without a pseudo-intersection.

One can consider other ideals than $\mathrm{Fin} = [\omega]^{<\omega}$ in the above definitions. It was done in various ways in many papers. We always assume that each ideal $\mathcal{I}$ on ω contains Fin and $\omega \notin \mathcal{I}$. By $\mathcal{I}^*$ we denote the filter dual to $\mathcal{I}$, i.e. $\mathcal{I}^* = \{\omega \setminus I \colon I \in \mathcal{I}\}$, and by $\mathcal{I}^+$ the family of all sets outside $\mathcal{I}$, i.e. $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$. The analogous notation is used also for filters. For an ideal $\mathcal{I}$ on ω one can define the analogs of the pseudo-intersection number for $\mathcal{I}$ as e.g.

- the minimal cardinality of a family of elements of the dual filter $\mathcal{I}^*$ which does not have a pseudo-intersection in $\mathcal{I}^*$ ($\mathfrak{p}(\mathcal{I}^*)$, see [5]; $\mathsf{add}^*(\mathcal{I})$, see [8]);
- the minimal cardinality of a family of elements of the dual filter $\mathcal{I}^*$ which does not have a pseudo-intersection ($\chi\mathfrak{p}(\mathcal{I}^*)$, see [5]; $\mathsf{cov}^*(\mathcal{I})$, see [8]) ;
- the minimal cardinality of a family $\mathcal{A}$ with the $\mathcal{I}$-strong finite intersection property (every finite subfamily has an intersection outside $\mathcal{I}$) without a set outside $\mathcal{I}$ which is almost included (in the sense of $\mathcal{I}$) in every member of $\mathcal{A}$ ($\mathfrak{p}_{\mathcal{I}}$, see [9]; $\mathfrak{p}(\mathcal{I})$, see [4]).

The aim of this paper is to present another way of generalizing the pseudo-intersection number. This generalization is, we believe, quite natural, particularly in the context of topological motivations.

2000 *Mathematics Subject Classification.* 03E17, 03E35, 03E75.

Key words and phrases. Katětov-order, analytic ideals, cardinal invariants of the continuum, almost disjoint families, Frechet-Urysohn property, weak* topology.

The first author was partially supported by the grant N N201 418939 from the Polish Ministry of Science and Higher Education and the grant 2351/W/IM/10 from University of Wrocław. The second author was supported by Hungarian National Foundation for Scientific Research grant nos. 68262 and 77476.

In Section 2 we recall some classical definitions and theorems about ideals on ω which will be used later in the paper. We introduce here also the main definition of our paper: a generalization of pseudo-intersection number with respect to different orders on ideals. So, we define the *Katětov-intersection number*, $\mathfrak{p}_{\mathrm{K}}(\mathcal{I})$ for any ideal $\mathcal{I}$: the minimal character of an ideal $\mathcal{J}$ with $\mathcal{J} \not\leq_{\mathrm{K}} \mathcal{I}$. We define analogously $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I})$ for Katětov-Blass order and $\mathfrak{p}_{1\text{–}1}(\mathcal{I})$ for the variant of Katětov-Blass order in which we consider only one-to-one functions.

The idea of these coefficients came from considerations which are far away, at least outwardly, from the combinatorics of ω. This is explained in Section 3 which is devoted to applications of the defined coefficients in topology and functional analysis. We show that $\mathfrak{p}_{\mathrm{K}}(\mathcal{I})$ is the smallest weight of a (locally) countable space which is not $\mathcal{I}$-Fréchet-Urysohn. Furthermore, we show the connection between $\mathfrak{p}_{\mathrm{K}}(\mathcal{Z})$ and certain sequential properties of finitely-supported probability measures.

In Section 4 we present a couple of results concerning inequalities between our coefficients and the classical ones. We show that $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) \leq \mathfrak{b}$ for each meager ideal $\mathcal{I}$. Then we discuss the values of these invariants in the Cohen-model. At last in this section, we present a model of $2^{\omega_1} = 2^{\omega}$ in which $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) \leq \omega_1$ for each ideal $\mathcal{I}$.

In Section 5 we discuss the existence of MAD families which cannot be permuted into a fixed ideal under cardinal assumptions and under Martin's Axiom for σ-centered posets. This section is inspired by the question if the almost-disjointness number generalized with respect to the one-to-one order is well-defined.

2. Preliminaries and basic definitions

We say that an ideal $\mathcal{I}$ on ω is *analytic* (*Borel, F_σ, meager, null, etc.*) if it is analytic (Borel, F_σ, meager, null, etc.) as a subset of 2^ω. An ideal $\mathcal{I}$ is a *P-ideal* if for every $\{A_n : n \in \omega\} \subseteq \mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for every n. An ideal on ω is *tall* if its dual filter does not have a pseudo-intersection, i.e. each infinite $X \subseteq \omega$ contains an infinite element of the ideal. The following families are well-known examples of tall analytic P-ideals: the *density zero ideal*: $\mathcal{Z} = \big\{A \subseteq \omega \colon \lim_{n\to\infty} \frac{|A\cap n|}{n} = 0\big\}$, and the *summable ideal*: $\mathcal{I}_{\frac{1}{n}} = \big\{A \subseteq \omega \colon \sum_{n\in A} \frac{1}{n+1} < \infty\big\}$.

Sierpiński proved that if an ideal is measurable, then it has measure zero; and if it has the Baire-property, then it is meager. In particular, all analytic ideals are meager null sets. Furthermore, there is a nice characterization of meager ideals (and filters):

Proposition 2.1. ([2, Proposition 9.4]) *An ideal $\mathcal{I}$ on ω is meager if, and only if $\mathcal{I}^*$ is feeble which means that there is a partition (P_n) of ω into finite sets such that $\{n \in \omega \colon X \cap P_n = \emptyset\}$ is finite for each $X \in \mathcal{I}^*$, i.e. $\{n \in \omega \colon P_n \subseteq A\}$ is finite for each $A \in \mathcal{I}$.*

We will use the characterizations of F_σ ideals and analytic P-ideals as ideals associated to submeasures. A *lower semicontinuous (LSC) submeasure* on ω is a function $\varphi \colon \mathcal{P}(\omega) \to [0, \infty]$ such that

(0) $\varphi(\emptyset) = 0$;
(1) $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for each $A, B \subseteq \omega$;
(2) $\varphi(\{n\}) < \infty$ for each $n \in \omega$;
(3) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for each $A \subseteq \omega$.

We will use the notation $\|A\|_\varphi = \lim_{n\to\infty} \varphi(A\setminus n)$ for $A \subseteq \omega$. Clearly, $\|A\cup B\|_\varphi \le \|A\|_\varphi + \|B\|_\varphi$ if $A, B \subseteq \omega$ but it does not necessarily hold for infinitely many sets. With every LSC submeasure we can associate two ideals defined by

$$\mathrm{Fin}(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\},$$
$$\mathrm{Exh}(\varphi) = \{A \subseteq \omega \colon \|A\|_\varphi = 0\}.$$

It is easy to see that $\mathrm{Fin}(\varphi)$ is an F_σ (i.e. $\mathbf{\Sigma}^0_2$) ideal and $\mathrm{Exh}(\varphi)$ is an $F_{\sigma\delta}$ (i.e. $\mathbf{\Pi}^0_3$) P-ideal if they are not equal to $\mathcal{P}(\omega)$, and $\mathrm{Exh}(\varphi) \subseteq \mathrm{Fin}(\varphi)$. From now on, if we discuss $\mathrm{Fin}(\varphi)$ (resp. $\mathrm{Exh}(\varphi)$), we always assume that $\mathrm{Fin}(\varphi) \neq \mathcal{P}(\omega)$ ($\mathrm{Exh}(\varphi) \neq \mathcal{P}(\omega)$). Hence, without loss of generality we can assume that $\|\omega\|_\varphi = 1$. Note that $\mathrm{Exh}(\varphi)$ is tall iff $\lim_{n\to\infty} \varphi(\{n\}) = 0$.

We will use the following characterization due to Mazur and Solecki:

Theorem 2.2. ([11] and [14]) *If $\mathcal{I}$ is an ideal on ω, then*

- *$\mathcal{I}$ is an F_σ ideal iff $\mathcal{I} = \mathrm{Fin}(\varphi)$ for some LSC submeasure φ;*
- *$\mathcal{I}$ is an analytic P-ideal iff $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some LSC submeasure φ;*
- *$\mathcal{I}$ is an F_σ P-ideal iff $\mathcal{I} = \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$ for some LSC submeasure φ.*

Using the representation of analytic P-ideals of the form $\mathrm{Exh}(\varphi)$ and Proposition 2.1, it is easy to prove that analytic P-ideals are meager sets (without using Sierpiński's result).

If $f\colon X \to Y$, $A \subseteq X$, and $\mathcal{A} \subseteq \mathcal{P}(X)$ then denote $f[A] = \{f(a)\colon a \in A\} \subseteq Y$ and $f''[\mathcal{A}] = \{f[A]\colon A \in \mathcal{A}\} \subseteq \mathcal{P}(Y)$. Similarly, we will use $f^{-1}[B]$ and $(f^{-1})''[\mathcal{B}]$ for $B \subseteq Y$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$.

Recall the definition of the *Katětov order* on the family of ideals on ω:

$$\mathcal{I} \le_{\mathrm{K}} \mathcal{J} \text{ iff there is an } f\colon \omega \to \omega \text{ such that } A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}.$$

Of course, we can use the Katětov-order (and other orders) for filters as well: $\mathcal{F} \le_{\mathrm{K}} \mathcal{G}$ iff $\mathcal{F}^* \le_{\mathrm{K}} \mathcal{G}^*$.

Several deep results were proved about the Katětov and other classical partial orders and preorders on ideals, see e.g. [9] and [12]. We will need the following properties of the Katětov-order:

Proposition 2.3. ([12, Proposition 1.7.2] and A. Blass (private communication)) *The Katětov-order is $\mathfrak{c}^+$-downward and $\mathfrak{c}^+$-upward directed, that is, every family of ideals on ω with cardinality at most $\mathfrak{c}$ has both a $\le_{\mathrm{K}}$-lower bound and a $\le_{\mathrm{K}}$-upper bound.*

Proof. Assume $\{\mathcal{I}_\alpha\colon \alpha < \mathfrak{c}\}$ is a family of ideals on ω. We will show that it has a $\le_{\mathrm{K}}$-lower bound $\mathcal{I}$ and a $\le_{\mathrm{K}}$-upper bound $\mathcal{J}$. Let $\{A_\alpha\colon \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ be an almost disjoint family on ω (i.e. $|A_\alpha \cap A_\beta| < \omega$ for each $\alpha < \beta < \mathfrak{c}$) and fix bijections $e_\alpha\colon \omega \to A_\alpha$. The ideal $\mathcal{I}$ generated by $\bigcup\{e''_\alpha[\mathcal{I}_\alpha]\colon \alpha < \mathfrak{c}\}$ is then a Katětov-lower bound of $\{\mathcal{I}_\alpha\colon \alpha < \mathfrak{c}\}$ because e_α witnesses $\mathcal{I} \le_{\mathrm{K}} \mathcal{I}_\alpha$ for each α.

Now, let $\{f_\alpha\colon \alpha < \mathfrak{c}\} \subseteq \omega^\omega$ be an independent family of functions, that is, for every $\alpha_0, \dots, \alpha_{k-1} < \mathfrak{c}$ and $n_0, \dots, n_{k-1} \in \omega$ there is some $x \in \omega$ such that $f_{\alpha_i}(x) = n_i$ for all $i < k$ (see [6, Theorem 3] for the proof of the existence of such a family). Let $\mathcal{J}$ be the ideal generated by the family $\bigcup\{(f_\alpha^{-1})''[\mathcal{I}_\alpha]\colon \alpha < \mathfrak{c}\}$. Notice that no finite union of elements from the family covers ω, so $\omega \notin \mathcal{J}$. Indeed, if $A_i \in \mathcal{I}_{\alpha_i}$ and $n_i \in \omega \setminus A_i$ for $i < k \in \omega$, then there is an $x \in \omega$ such that $f_{\alpha_i}(x) = n_i$

for each $i < k$ so $x \notin \bigcup\{f_{\alpha_i}^{-1}[A_i] \colon i < k\}$. Clearly, $\mathcal{I}_\alpha \leq_{\mathrm{K}} \mathcal{J}$ is witnessed by f_α for each α. □

Proposition 2.4. *Meager filters are cofinal in the Katětov-order.*

Proof. Let $\mathcal{F}$ be an arbitrary filter on ω. Fix a partition (P_n) of ω into finite sets such that $|P_n| = n$, $P_n = \{p^n_k \colon k < n\}$. Let $\mathcal{G}$ be the filter generated by the sets $\widetilde{F} = \{p^n_k \colon k \in F \cap n\}$ for $F \in \mathcal{F}$. The filter $\mathcal{G}$ is meager because $\widetilde{F} \cap P_n \neq \emptyset$ for each $n > \min(F)$ and because of Proposition 2.1. Now, the function $g \colon \omega \to \omega$ defined by $g(p^n_k) = k$ witnesses that $\mathcal{F} \leq_{\mathrm{K}} \mathcal{G}$. □

The *character of a filter* $\mathcal{F}$, denoted by $\chi(\mathcal{F})$, is the minimal cardinality of a family generating $\mathcal{F}$. Similarly, the character of an ideal $\mathcal{I}$ is the character of its dual filter. The following theorem reveals some properties of the characters of nonmeager filters. Denote by $\mathfrak{b}$ the unbounding number, i.e. the minimal cardinality of a set $B \subseteq \omega^\omega$ that is $\leq^*$-unbounded where $f \leq^* g$ iff $\{n \in \omega \colon f(n) > g(n)\}$ is finite.

Theorem 2.5. (R. C. Solomon [15] and P. Simon [unpublished], see [2, Theorem 9.10]) *If an ideal (or filter) has character less than* $\mathfrak{b}$*, then it is meager but there is a nonmeager ideal (filter) generated by* $\mathfrak{b}$ *sets.*

Recall the definition of the *Katětov-Blass order*: $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ if there is a finite-to-one $f \colon \omega \to \omega$ such that $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$. We will use one more preorder stronger than the Katětov-Blass:

Definition 2.6. (*One-to-one order*) For ideals $\mathcal{I}$ and $\mathcal{J}$ let $\mathcal{I} \leq_{1\text{–}1} \mathcal{J}$ if there is a one-to-one $f \colon \omega \to \omega$ such that $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.

One can think about one more natural order here, defined in the same way as $\leq_{1\text{–}1}$ but with "bijection" instead of "one-to-one function". However, the following fact shows that it would not give us anything new.

Proposition 2.7. *If* $\mathcal{J}$ *strictly extends* Fin *then* $\mathcal{I} \leq_{1\text{–}1} \mathcal{J}$ *if and only if there is a permutation* $g \colon \omega \to \omega$ *such that* $A \in \mathcal{I} \Rightarrow g^{-1}[A] \in \mathcal{J}$*.*

Proof. The "if" part is trivial. Conversely, assume f is one-to-one and $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$. We can modify f on an infinite element B of $\mathcal{J}$ to be a permutation g such that $g \upharpoonright (\omega \setminus B) \equiv f \upharpoonright (\omega \setminus B)$ and $g[B] = f[B] \cup (\omega \setminus \mathrm{ran}(f))$. Then g is as required. □

Using the above proposition, $\mathcal{I} \leq_{1\text{–}1} \mathcal{J} (\neq \mathrm{Fin})$ means that $\mathcal{I}$ can be permuted into $\mathcal{J}$ (by g^{-1}). Clearly, $\mathrm{Fin} \leq_{1\text{–}1} \mathcal{J}$ for each $\mathcal{J}$, and $\mathcal{J}$ is maximal in this order iff $\mathcal{J}$ is a prime ideal. There is no largest element in this order because there are $2^{\mathfrak{c}}$ many prime ideals but only $\mathfrak{c}$ many permutations.

For all unexplained terminology concerning cardinal invariants we refer the reader to [2].

We will finish this section with the main definitions of the paper. We will start with a general one.

Definition 2.8. For a partial order (or simply a relation) $\sqsubseteq$ on ideals on ω the $\sqsubseteq$-intersection number of an ideal $\mathcal{I}$ on ω is

$$\mathfrak{p}_{\sqsubseteq}(\mathcal{I}) = \min\{\chi(\mathcal{J}) \colon \mathcal{J} \not\sqsubseteq \mathcal{I}\}$$

provided there is an ideal $\mathcal{J}$ such that $\mathcal{J} \not\sqsubseteq \mathcal{I}$.

The $\sqsubseteq$-intersection number of a filter $\mathcal{F}$ is $\mathfrak{p}_{\sqsubseteq}(\mathcal{F}) = \mathfrak{p}_{\sqsubseteq}(\mathcal{F}^*)$.

We will be interested only in the *Katětov-intersection number* (which we will denote for simplicity by $\mathfrak{p}_{\mathrm{K}}(\mathcal{I})$), *Katětov-Blass-intersection number* ($\mathfrak{p}_{\mathrm{KB}}(\mathcal{I})$), and *one-to-one-intersection number* ($\mathfrak{p}_{1\text{–}1}(\mathcal{I})$).

We list some immediate facts.

Proposition 2.9.

(a) $\mathfrak{p} \leq \mathfrak{p}_{1\text{–}1}(\mathcal{I}) \leq \mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) \leq \mathfrak{p}_{\mathrm{K}}(\mathcal{I})$;
(b) *if* $\mathcal{I} \sqsubseteq \mathcal{J}$, *then* $\mathfrak{p}_{\sqsubseteq}(\mathcal{I}) \leq \mathfrak{p}_{\sqsubseteq}(\mathcal{J})$ *for any partial order (or preorder)* $\sqsubseteq$;
(c) $\mathfrak{p}_{\mathrm{K}}(\mathcal{I}) = \mathfrak{p}$ *for any* $\mathcal{I}$ *which is not tall.*

The last part of the above proposition explains why $\mathfrak{p}_{\mathrm{K}}$ and other cardinal coefficients defined above in a sense generalize the pseudo-intersection number. In fact, we can indicate also the generalization of the notion of pseudo-intersection itself in this context. Assume $\mathcal{I}$ is an ideal on ω. For an injective sequence $\bar{x} = (x_n) \in \omega^\omega$, *the copy of* $\mathcal{I}$ *on* $\bar{x}$ is the ideal

$$\mathcal{I}(\bar{x}) = \{A \subseteq \omega \colon \{n \in \omega \colon x_n \in A\} \in \mathcal{I}\}.$$

Let $\mathcal{F}$ be a family with SFIP (or simply a filter) and assume that $\mathcal{I}$ is an ideal. We say that an injective sequence $\bar{x} = (x_n) \in \omega^\omega$ is an *$\mathcal{I}$-intersection* of $\mathcal{F}$ if $\omega \setminus F \in \mathcal{I}(\bar{x})$ for each $F \in \mathcal{F}$. In other words, a set $X = \mathrm{ran}(\bar{x})$ is an $\mathcal{I}$-intersection of $\mathcal{F}$ if we can reorder the elements of X in such a way that elements of $\mathcal{F}$ are in the copy of $\mathcal{I}^*$ on the rearranged X. Notice that $\bar{x}$ is a Fin-intersection of $\mathcal{F}$ iff $\mathrm{ran}(\bar{x})$ is a pseudo-intersection of $\mathcal{F}$. Plainly, $\mathfrak{p}_{1\text{–}1}(\mathcal{I})$ is the minimal cardinality of a family with SFIP without an $\mathcal{I}$-intersection.

3. Analytic background of the problem

Orders on ideals have gained some attention recently, mainly because it turned out to be a useful tool in investigating properties of forcings of the form $\mathcal{P}(\omega)/\mathcal{I}$ and Mathias forcings $\mathbb{M}(\mathcal{I})$, where $\mathcal{I}$ is an ideal (see e.g. [9] or [10]). In this section we will show that the Katětov-intersection number has applications in certain topological and analytical considerations.

All topological spaces in what follows are Hausdorff. The *weight* of a space X (denoted by $\mathrm{w}(X)$) is the minimal cardinality of a base of topology of X. Recall that a topological space is *Fréchet-Urysohn* (FU) if for every subset A of this space and every $x \in \overline{A}$ there is a sequence in A converging to x. The definition of the pseudo-intersection number can be reformulated in topological terms: the pseudo-intersection number is the smallest weight of a (locally) countable (even completely regular or normal) space which is not FU (it is a special case of Theorem 3.2).

We can generalize the FU property for ideals using the notion of $\mathcal{I}$-convergency. A sequence (x_n) in a space X *$\mathcal{I}$-converges* to x if

$$\forall\, U \text{ open } \big(x \in U \Rightarrow \ \{n \in \omega \colon x_n \notin U\} \in \mathcal{I}\big).$$

Definition 3.1. Let $\mathcal{I}$ be an ideal on ω. A space X satisfies the *$\mathcal{I}$-Fréchet-Urysohn* ($\mathcal{I}$-FU) condition if for every $A \subseteq X$ and every $x \in \overline{A}$ there is a sequence in A $\mathcal{I}$-converging to x.

Clearly, if $\mathcal{I}$ is not tall, then the $\mathcal{I}$-FU condition is equivalent to the (Fin-)FU condition.

Theorem 3.2. $\mathfrak{p}_{\mathrm{K}}(\mathcal{I})$ *is the smallest weight of a countable space which is not* $\mathcal{I}$*-FU.*

Proof. Let $\mathcal{F}$ be a filter, $\chi(\mathcal{F}) = \mathfrak{p}_{\mathrm{K}}(\mathcal{I})$, and $\mathcal{F} \not\leq_{\mathrm{K}} \mathcal{I}^*$. Let $X = \omega \cup \{\mathcal{F}\}$ be equipped with the topology inherited from the Stone space of the Boolean algebra generated by $\mathcal{F}$, that is

(a) subsets of ω are open;
(b) $U \cup \{\mathcal{F}\}$ is an open neighborhood of $\mathcal{F}$ iff $U \cap \omega \in \mathcal{F}$.

Clearly, $\mathrm{w}(X) = \chi(\mathcal{F})$ and $\mathcal{F} \in \overline{\omega}$. We claim that there is no sequence (x_n) in ω which $\mathcal{I}$-converges to $\mathcal{F}$. Assume (x_n) is a sequence in ω and let $f \in \omega^\omega$, $f(n) = x_n$. Using the assumption $\mathcal{F} \not\leq_{\mathrm{K}} \mathcal{I}^*$ we deduce that there is a $V \in \mathcal{F}$ such that $f^{-1}[V] \notin \mathcal{I}^*$. Consider the open neighborhood $U = V \cup \{\mathcal{F}\}$ of $\mathcal{F}$. Then $\{n \colon x_n \notin U\} = \omega \setminus f^{-1}[V] \notin \mathcal{I}$ so (x_n) does not $\mathcal{I}$-converge to $\mathcal{F}$.

Conversely, let X be a countable space with $\mathrm{w}(X) < \mathfrak{p}_{\mathrm{K}}(\mathcal{I})$, let $A \subseteq X$, and $x \in \overline{A} \setminus A$. Then the family $\{A \cap U \colon U$ is an open neighborhood of $x\}$ forms a filter-base on A. Let $\mathcal{F}$ be the generated filter. Since $\chi(\mathcal{F}) \leq \mathrm{w}(X)$ we know that $\mathcal{F} \leq_{\mathrm{K}} \mathcal{I}^*$ is witnessed by a function $f \in \omega^\omega$.

We claim that the sequence defined by $x_n = f(n)$ $\mathcal{I}$-converges to x. Let U be an open neighborhood of x. Then $\{n \colon x_n \notin U\} = \omega \setminus f^{-1}[A \cap U] \in \mathcal{I}$ and we are done. □

We can give another characterization of $\mathfrak{p}_{\mathrm{K}}(\mathcal{I})$ in the special case when $\mathcal{I} = \mathcal{Z}$. Recall that a completely regular space X can be seen as a closed subspace of the space of Borel probability measures $P(X)$ on X with the weak* topology.

The subbase of this topology is given by the following sets:

$$\mathcal{U}_{f,\varepsilon}(\mu) = \left\{ \nu \in P(X) \colon \left| \int_X f \mathrm{d}\mu - \int_X f \mathrm{d}\nu \right| < \varepsilon \right\}$$

where $\mu \in P(X)$, $f \in C_b(X) = \{$bounded continuous real-valued functions on $X\}$, and $\varepsilon > 0$. Recall that in this topology (μ_n) converges to μ if and only if

$$\int_X f \mathrm{d}\mu_n \to \int_X f \mathrm{d}\mu$$

for every $f \in C_b(X)$.

The embedding $X \to P(X)$ is given by $x \mapsto \delta_x$ where δ_x is the Dirac-measure concentrated on x. We will use the notation $A^\delta = \{\delta_y \colon y \in A\}$ for $A \subseteq X$.

Denote by $\mathrm{conv}\,(A)$ the convex hull of A for $A \subseteq P(X)$. We will be interested in $\mathrm{conv}\,(X^\delta)$, i.e. in the probability measures with finite support.

Definition 3.3. We say that X satisfies the *convex Fréchet-Urysohn condition* if for every $A \subseteq X$, if $x \in \overline{A}$ then there is a sequence in $\mathrm{conv}\,(A^\delta)$ which converges to δ_x.

In [13, Theorem 1] the following result was proved:

Theorem 3.4. *Assume X is compact. A measure $\mu \in P(X)$ is a weak* limit of measures of finite support if and only if μ has a uniformly distributed sequence (x_k) in X, that is*

$$\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k<n} \delta_{x_k}.$$

Remark 3.5.

(1) In [13] this theorem was formulated only for compact spaces, but the proof presented there does not use the assumption of compactness and the assertion is true for every (completely regular) topological space.
(2) It is clear from the proof of this theorem that if $\mu_n \in \operatorname{conv}(X^\delta)$ and $\mu_n \to \mu$, then the sequence (x_k) can be chosen from $\bigcup_{n<\omega} \operatorname{supp}(\mu_n)$.
(3) Using this theorem, a space X is convex FU iff if for every $A \subseteq X$, if $x \in \overline{A}$ then there is a sequence (x_k) in A such that $\delta_x = \lim_{n\to\infty} \frac{1}{n}\sum_{k<n} \delta_{x_k}$.

Theorem 3.6. *Assume X is completely regular. Then X satisfies the convex FU condition if and only if X satisfies the $\mathcal{Z}$-FU condition.*

Proof. Assume X satisfies the convex FU condition and let $A \subseteq X$, $x \in \overline{A} \setminus A$. Then, according to Remark 3.5 there is a sequence (x_k) in A such that $\delta_x = \lim_{n\to\infty} \frac{1}{n}\sum_{k<n} \delta_{x_k}$.

We claim that (x_k) $\mathcal{Z}$-converges to x. Assume on the contrary that there is an open neighborhood U of x such that $H = \{n \colon x_n \notin U\} \notin \mathcal{Z}$. Using complete regularity of X, there is a continuous $f \colon X \to [0,1]$ such that $f(x) = 0$ but $f \restriction \overline{\{x_n \colon n \in H\}} \equiv 1$. Then by the assumption on (x_k) we have

$$\int_X f \mathrm{d}\left(\frac{1}{n}\sum_{k<n} \delta_{x_k}\right) = \frac{1}{n}\sum_{k<n} f(x_k) \to f(x) = 0$$

but $\frac{1}{n}\sum_{k<n} f(x_k) \geq \frac{|H\cap n|}{n}$ for each n, a contradiction because $H \notin \mathcal{Z}$.

Conversely, assume that X is $\mathcal{Z}$-FU and let $A \subseteq X$, $x \in \overline{A} \setminus A$. Then there is a sequence (x_k) in A which $\mathcal{Z}$-converges to x.

We claim that $\delta_x = \lim_{n\to\infty} \frac{1}{n}\sum_{k<n} \delta_{x_k}$. Let $f \in C_b(X)$, $|f| \leq c$, $\varepsilon > 0$ and $x \in U$ be an open set such that $|f(x) - f(y)| < \varepsilon$ for each $y \in U$. If $H = \{k \colon x_k \in U\} \in \mathcal{Z}^*$, then

$$\left| f(x) - \frac{1}{n}\sum_{k<n} f(x_k) \right| = \frac{1}{n}\left| \sum_{k\in n\cap H} \big(f(x) - f(x_k)\big) + \sum_{k\in n\setminus H} \big(f(x) - f(x_k)\big) \right| \leq$$

$$\leq \frac{1}{n}\big(|H\cap n|\cdot\varepsilon + |n\setminus H|\cdot 2c\big) \to \varepsilon \text{ if } n \to \infty.$$

Because ε was arbitrary, we are done. □

So, in a sense we can call $\mathfrak{p}_K(\mathcal{Z})$ the *convex pseudo-intersection number*.

The idea of the cardinal invariant $\mathfrak{p}_{\mathrm{K}}(\mathcal{I})$ came from certain analytic considerations contained in [3], where authors were exploring a problem if there is a Mazur space without the Gelfand-Phillips property. The Gelfand-Phillips condition is widely used in functional analysis. The Mazur property is a certain condition weaker than reflexivity used in the theory of Pettis integrability (for the detailed discussion about these properties, confront [3]). It is known that there is a Gelfand-Phillips space without the Mazur property. It is still open if every Mazur space is Gelfand-Phillips.

In [3] the following question connected to the above considerations was raised.

Problem 3.7. Is there a minimally generated Boolean algebra $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ such that no ultrafilter on $\mathfrak{A}$ has a pseudo-intersection but for every ultrafilter $\mathcal{F}$ on $\mathfrak{A}$ we have $\mathcal{F} \leq_{\mathrm{K}} \mathcal{Z}^*$?

In [3] it was shown that if there is such a Boolean algebra and this Boolean algebra is dense in $\mathcal{P}(\omega)$, then there is a space which is Mazur but not Gelfand-Phillips [1]. Briefly speaking, the minimal generation implies that every measure on $\mathfrak{A}$ is in the sequential closure of measures of finite support. Since $\mathcal{F} \leq_{\mathrm{K}} \mathcal{Z}^*$ for every ultrafilter $\mathcal{F}$ on $\mathfrak{A}$, every measure of finite support is a limit of measures finitely supported on ω (cf. Remark 3.5 and Theorem 3.6 above) and so every measure is in the sequential closure of measures finitely supported on ω. This property simplifies the form of functionals on the space of measures on $\mathfrak{A}$ and in this way it can be used to achieve Mazur property. In [3] it was also proved that if $\mathfrak{p}_{\mathrm{K}}(\mathcal{Z}) > \mathfrak{h}$, then there is a Boolean algebra as described above.

In the next section we will show that consistently there is an ideal (unfortunately, not $\mathcal{Z}$) with the above property (see Theorem 4.7).

Problem 3.8. Do there exist reasonable topological characterizations of $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I})$ and $\mathfrak{p}_{1\text{–}1}(\mathcal{I})$?

4. Consistency results

The cardinal $\sup\{\mathfrak{p}_{\mathrm{K}}(\mathcal{I})\colon \mathcal{I}$ is an ideal on $\omega\} \leq \mathfrak{c}$ is the smallest cardinal κ such that there is no $\leq_{\mathrm{K}}$-upper bound of all ideals generated by at most κ many elements. First we show that this supremum is determined by cardinal exponentiation.

Proposition 4.1. *$\mathfrak{p}_{\mathrm{K}}(\mathcal{I}) \leq \kappa$ for each ideal $\mathcal{I}$ if and only if $2^\kappa > 2^\omega$.*

Proof. First, we prove the "if" part. Let $\{(A^0_\alpha, A^1_\alpha)\colon \alpha < \kappa\} \subseteq \big([\omega]^\omega\big)^2$ be an independent system, that is $A^1_\alpha = \omega \setminus A^0_\alpha$ for each α, and if $D \in [\kappa]^{<\omega}$ and $f\colon D \to 2$ then $|\bigcap\{A^{f(\alpha)}_\alpha\colon \alpha \in D\}| = \omega$ (see [2, Proposition 8.9]).

For an $F \in 2^\kappa$ let $\mathcal{I}_F$ be the ideal generated by $\{A^{F(\alpha)}_\alpha\colon \alpha < \kappa\}$. Observe that $\chi(\mathcal{I}_F) = \kappa$ for every F. Suppose for a contradiction that there is an ideal $\mathcal{I}$ such that $\mathcal{I}_F \leq_{\mathrm{K}} \mathcal{I}$ for each F witnessed by $g_F \in \omega^\omega$. Since $2^\kappa > \mathfrak{c}$ there are distinct $F_0, F_1 \in 2^\kappa$ such that $g_{F_0} = g_{F_1}$. Let α be such that $F_0(\alpha) \neq F_1(\alpha)$. Then $A^{F_0(\alpha)}_\alpha \cup A^{F_1(\alpha)}_\alpha = \omega$. Consequently

$$g^{-1}_{F_0}[A^{F_0(\alpha)}_\alpha] \cup g^{-1}_{F_1}[A^{F_1(\alpha)}_\alpha] = g^{-1}_{F_0}[A^{F_0(\alpha)}_\alpha] \cup g^{-1}_{F_0}[A^{F_1(\alpha)}_\alpha] = g^{-1}_{F_0}[A^{F_0(\alpha)}_\alpha \cup A^{F_1(\alpha)}_\alpha] = \omega$$

so $\omega \in \mathcal{I}$, a contradiction.

Now, we prove the converse implication. Assume that $2^\kappa = 2^\omega$. Then the family of ideals generated by at most κ elements has cardinality $\mathfrak{c}$. Using Proposition 2.3, this family has a $\leq_{\mathrm{K}}$-upper bound $\mathcal{I}$ and so $\mathfrak{p}_{\mathrm{K}}(\mathcal{I}) > \kappa$, a contradiction. □

We have an easy upper bound for $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I})$ if $\mathcal{I}$ is meager:

Proposition 4.2. *If $\mathcal{I}$ is meager, then $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) \leq \mathfrak{b}$.*

Proof. It is enough to show that if $\mathcal{I}$ is meager and $\mathcal{J} \leq_{\mathrm{KB}} \mathcal{I}$, then $\mathcal{J}$ is also meager because then we can use Theorem 2.5 (there is a nonmeager ideal of character $\mathfrak{b}$).

Assume the partition (P_n) witnesses that $\mathcal{I}$ is meager (see Proposition 2.1), and assume $f\colon \omega \to \omega$ is finite-to-one and witnesses $\mathcal{J} \leq_{\mathrm{KB}} \mathcal{I}$. We can define a partition (P'_n) of $\mathrm{ran}(f)$ into finite sets by recursion on n such that $\forall\, n \in \omega$ $\exists\, k \in \omega$ $P_k \subseteq f^{-1}[P'_n]$. Then (P'_n) witnesses that $\mathcal{J} \restriction \mathrm{ran}(f)$ is meager and

[1] In fact, in [3] it was shown for the case of one-to-one order but it can be immediately generalized.

it clearly implies that $\mathcal{J}$ is meager too because of the natural homeomorphism $\mathcal{P}(\omega) \to \mathcal{P}(\mathrm{ran}(f)) \times \mathcal{P}(\omega \setminus \mathrm{ran}(f))$. □

The assumption on meagerness of the ideal and the use of finite-to-one functions are necessary because of part (a) and part (b) of the following theorem. Denote by $\mathbb{C}_\alpha$ the standard Cohen forcing which adds α many Cohen reals and let $\mathbb{C} = \mathbb{C}_1$. It is well-known that if $\kappa > \omega$, then $V^{\mathbb{C}_\kappa} \models \mathfrak{b} = \omega_1$.

Theorem 4.3. *Assume* GCH. *Then in $V^{\mathbb{C}_{\omega_2}}$ the following hold:*

(a) *there is a filter $\mathcal{F}$ with $\mathfrak{p}_{1\text{–}1}(\mathcal{F}) = \omega_2$;*
(b) *there is a meager filter $\mathcal{G}$ with $\mathfrak{p}_{\mathrm{K}}(\mathcal{G}) = \omega_2$;*
(c) *$\mathfrak{p}_{\mathrm{K}}(\mathcal{I}) = \omega_1$ for all F_σ ideals and analytic P-ideals.*

Proof. (a): We interpret $\mathbb{C}_{\omega_2}$ as the ω_2 stage finite support iteration of $\mathbb{C}$ where now let $\mathbb{C}$ be the set of finite injective sequences from ω ordered by reverse inclusion. A trivial density argument shows that if $\dot{c}$ is the generic (Cohen-)real, then $\Vdash_{\mathbb{C}}$"$\dot{c}$ is a permutation on ω."

Notice that for every subfamily of $[\omega]^\omega$ in $V^{\mathbb{C}_{\omega_2}}$ of size ω_1, a nice name of it appears already in some $V^{\mathbb{C}_\alpha}$ for $\alpha < \omega_2$. Additionally, there are $\omega_2^{\omega_1} = \omega_2$ such families in $V^{\mathbb{C}_{\omega_2}}$ so we can fix an enumeration $\{\dot{\mathcal{F}}_\alpha \colon \alpha < \omega_2\}$ of all names of bases of filters of cardinality ω_1 in $V^{\mathbb{C}_{\omega_2}}$ in such a way that $\dot{\mathcal{F}}_\alpha \in V^{\mathbb{C}_\alpha}$ for each α.

If $\dot{c}_\alpha \in V^{\mathbb{C}_{\alpha+1}}$ is the α's Cohen-real, then $|\dot{c}_\alpha[X] \cap Y| = \omega$ for each $X, Y \in [\omega]^\omega \cap V^{\mathbb{C}_\alpha}$: if $X = \{x_n \colon n \in \omega\}$, then $D_k = \{p \in \mathbb{C} \colon \exists\, n \geq k\ p(x_n) \in Y\}$ is dense in $\mathbb{C}$ for each $k \in \omega$.

We show that

$$V^{\mathbb{C}_{\omega_2}} \models \text{“}\bigcup \left\{\dot{c}_\alpha''[\dot{\mathcal{F}}_\alpha] \colon \alpha < \omega_2\right\} \text{ forms a base of a filter.”}$$

Indeed, consider $\alpha < \beta < \omega_2$ and $F \in \dot{\mathcal{F}}_\alpha$, $G \in \dot{\mathcal{F}}_\beta$. Since $\dot{c}_\alpha[F] \in V^{\mathbb{C}_{\alpha+1}} \subseteq V^{\mathbb{C}_\beta}$ is infinite, the set $\dot{c}_\beta[G] \cap \dot{c}_\alpha[F]$ is also infinite. By induction we can show that every finite subfamily of this family has infinite intersection.

Clearly, the filter $\mathcal{F}$ generated by this family satisfies $\mathfrak{p}_{1\text{–}1}(\mathcal{F}) = \omega_2$.

(b) follows from part (a), Proposition 2.9 and Proposition 2.4.

(c): Now let $\mathbb{C}_{\omega_2}$ be the set of finite functions from $\omega_2 \times \omega$ to 2 ordered by reverse inclusion. Let $\mathcal{J}$ be the ideal generated by the first ω_1 Cohen-reals, i.e. by $\{c_\alpha^{-1}[\{1\}] \colon \alpha < \omega_1\}$ where $c_\alpha \colon \omega \to 2$ is the α's Cohen-real. We show that $\mathcal{J}$ witnesses part (c), i.e. for each $\mathcal{I}$ as in the theorem $\mathcal{J} \not\leq_{\mathrm{K}} \mathcal{I}$.

Case 1: Let $\mathcal{I} = \mathrm{Fin}(\varphi)$ be an F_σ ideal. Assume G is a $(V, \mathbb{C}_{\omega_2})$-generic filter and $f \in \omega^\omega$ in $V[G]$. Then there is a countable $H \subseteq \omega_2$ such that both $\varphi \restriction \mathrm{Fin}$ and f are in $V[G \cap \mathbb{C}_H]$ where $\mathbb{C}_H$ is the Cohen-forcing which adds Cohen-reals indexed by elements of H. If $\alpha \in \omega_1 \setminus H$ then c_α is Cohen over $V[G \cap \mathbb{C}_H]$ so it is enough to show that

$$D_n = \left\{p \in \mathbb{C} \colon \varphi\left(f^{-1}[p^{-1}[\{1\}]]\right) > n\right\}$$

is dense in $\mathbb{C}$ for each $n \in \omega$ because then f cannot witness $\mathcal{J} \leq_{\mathrm{K}} \mathcal{I}$ in the extension. Assume $q \in \mathbb{C}$ is defined on an initial segment. If $\varphi(f^{-1}[|q|]) = \infty$, then we are done, because f cannot show any Katětov-reduction (so we do not have to deal with D_n). If not, then $f^{-1}[|q|] \in \mathrm{Fin}(\varphi)$ so we can choose a large enough $\ell > |q|$ such that $\varphi(f^{-1}[\ell \setminus |q|]) > n$. Define $p \in \mathbb{C}$ by $p \restriction |q| = q$ and $p \restriction [|q|, \ell) \equiv 1$. Then $p \leq q$ and $p \in D_n$.

Case 2: Assume $\mathcal{I} = \mathrm{Exh}(\varphi)$ is an analytic P-ideal, $\|\omega\|_\varphi = 1$. Similarly to the previous case it is enough to show that

$$D_n = \left\{p \in \mathbb{C}\colon \varphi\left(f^{-1}[p^{-1}[\{1\}]] \setminus n\right) > 0.5\right\}$$

is dense in $\mathbb{C}$ for each n. Assume $q \in \mathbb{C}$ is defined on an initial segment. If $\|f^{-1}[|q|]\|_\varphi > 0$ then there is nothing to worry about. If not, then we can choose a large enough $\ell > |q|$ such that $\varphi(f^{-1}[\ell \setminus |q|] \setminus n) > 0.5$. Let p be chosen as in Case 1. $\square$

We list here some related questions:

Problem 4.4.

- Is $\mathfrak{p}_{\mathrm{K}}(\mathcal{I}) \leq \mathfrak{b}$ for each analytic (P-)ideal $\mathcal{I}$?
- Is $\mathfrak{p}_{1\text{–}1}(\mathcal{I}) = \mathfrak{p}_{\mathrm{KB}}(\mathcal{I})$ for each ideal $\mathcal{I}$?
- Is $\mathfrak{p} < \mathfrak{p}_{1\text{–}1}(\mathcal{I})$ (or at least $\mathfrak{p} < \mathfrak{p}_{\mathrm{KB}}(\mathcal{I})$) consistent for some meager (or even analytic (P-)) ideal $\mathcal{I}$? Also, for the purposes described in Section 3, the consistency of $\mathfrak{h} < \mathfrak{p}_{\mathrm{K}}(\mathcal{Z})$ is particularly interesting.
- Is $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) < \mathfrak{b}$ (or at least $\mathfrak{p}_{1\text{–}1}(\mathcal{I}) < \mathfrak{b}$) consistent for some tall ideal $\mathcal{I}$?

In Proposition 4.1 we showed that $2^\kappa > 2^\omega$ implies that $\mathfrak{p}_{\mathrm{K}}(\mathcal{I}) \leq \kappa$ and thus $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) \leq \kappa$ for each ideal $\mathcal{I}$, and that for the Katětov-order the converse implication also holds. Now, we show that $\forall\ \mathcal{I}\ \mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) \leq \omega_1$ does not imply $2^{\omega_1} > 2^\omega$. We present a model of $2^{\omega_1} = 2^\omega$ in which $\mathfrak{p}_{\mathrm{KB}}(\mathcal{F}) \leq \omega_1$ for each filter $\mathcal{F}$.

Moreover, in this model $\leq_{\mathrm{KB}}$ (so $\leq_{1\text{–}1}$ too) will not be upward directed on filters generated by ω_1 sets (it clearly implies that $\mathfrak{p}_{\mathrm{KB}}(\mathcal{F}) \leq \omega_1$ for each filter $\mathcal{F}$).

Theorem 4.5. *It is consistent with* ZFC *that $2^{\omega_1} = 2^\omega$ is arbitrary large and the Katětov-Blass-order is not upward directed on filters generated by ω_1 sets. In particular, $\mathfrak{p}_{\mathrm{KB}}(\mathcal{F}) \leq \omega_1$ for each filter $\mathcal{F}$.*

Proof. Let $2^\omega = 2^{\omega_1}$ be arbitrary. We will construct two filters $\mathcal{F}$ and $\mathcal{G}$ generated by $\subseteq^*$-descending sequences $\{\dot{X}_\alpha\colon \alpha < \omega_1\}$ and $\{\dot{Y}_\alpha\colon \alpha < \omega_1\}$ inductively in a model obtained by an ω_1 stage finite-support iteration of σ-centered forcing notions $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta)_{\alpha \leq \omega_1, \beta < \omega_1}$.

It is well-known that a $\leq \mathfrak{c}$-step finite-support iteration of σ-centered forcing notions is σ-centered, in particular ccc, so it does not collapse cardinals. It will be trivial that $|\mathbb{P}_{\omega_1}| = \mathfrak{c}$ so $\mathfrak{c}^{V^{\mathbb{P}_{\omega_1}}} = \mathfrak{c}^V$ and $(2^{\omega_1})^{V^{\mathbb{P}_{\omega_1}}} = (2^{\omega_1})^V$.

At the α's stage (in $V^{\mathbb{P}_\alpha}$) we have initial segments $\{X_\xi\colon \xi < \alpha\}$ and $\{Y_\xi\colon \xi < \alpha\}$. Let X'_α and Y'_α be pseudo-intersections of these sequences. We want to add $\dot{X}_\alpha \in [X'_\alpha]^\omega$ and $\dot{Y}_\alpha \in [Y'_\alpha]^\omega$ such that $|f^{-1}[\dot{X}_\alpha] \cap g^{-1}[\dot{Y}_\alpha]| < \omega$ for each pairs $(f,g) \in V^{\mathbb{P}_\alpha}$ of finite-to-one functions from ω to ω. Then in the final extension $\mathcal{F}$ and $\mathcal{G}$ cannot have a common upper bound in $\leq_{\mathrm{KB}}$.

Let $X = X'_\alpha$ and $Y = Y'_\alpha$, and let $\mathbb{Q} = \dot{\mathbb{Q}}_\alpha$ be the following forcing notion: $(s,t,\varrho) \in \mathbb{Q}$ if $s \in [X]^{<\omega}$, $t \in [Y]^{<\omega}$, and ϱ is a finite partial function from $\mathrm{FO} \times \mathrm{FO}$ to ω such that $|f^{-1}[s] \cap g^{-1}[t]| \leq \varrho(f,g)$ for every $(f,g) \in \mathrm{dom}(\varrho)$ (where FO denotes the set of all finite-to-one functions from ω to ω). Define the order in the following way: $(s,t,\varrho) \leq (s',t',\varrho')$ if $s \supseteq s'$, $t \supseteq t'$, $\mathrm{dom}(\varrho) \supseteq \mathrm{dom}(\varrho')$, and $\varrho(f,g) \leq \varrho'(f,g)$ for each $(f,g) \in \mathrm{dom}(\varrho')$.

Clearly, it is a partial order. It is also σ-centered: fix $s \in [X]^{<\omega}$, $t \in [Y]^{<\omega}$, and consider conditions $(s,t,\varrho_i) \in \mathbb{Q}$ for $i < n \in \omega$. Let ϱ be the following partial function: $\mathrm{dom}(\varrho) = \bigcup\{\mathrm{dom}(\varrho_i)\colon i < n\}$ and $\varrho(f,g) = \min\{\varrho_i(f,g)\colon i < n, (f,g) \in \mathrm{dom}(\varrho_i)\}$. Then $(s,t,\varrho) \leq (s,t,\varrho_i)$ for every $i < n$.

Let $\dot{A}$ and $\dot{B}$ be $\mathbb{Q}$-names for the union of the first and respectively second coordinates of the conditions in the generic filter. We claim that these sets can serve as $\dot{X}_\alpha$ and $\dot{Y}_\alpha$.

$\Vdash_{\mathbb{Q}}$"$\dot{A}$ is infinite": The set $D_n = \{p \in \mathbb{Q}\colon |s^p| > n\}$ is dense in $\mathbb{Q}$ for each n (where $p = (s^p, t^p, \varrho^p)$) because if $p \in \mathbb{Q}$ is arbitrary, then s^p can be extended by any element of the infinite set $\omega \setminus \bigcup\{f''[g^{-1}[t]]\colon (f,g) \in \mathrm{dom}(\varrho)\}$.

Similarly, $\Vdash_{\mathbb{Q}}$"$\dot{B}$ is infinite."

At last, we have to show that $E_{(f,g)} = \{p \in \mathbb{Q}\colon (f,g) \in \mathrm{dom}(\varrho^p)\}$ is dense in $\mathbb{Q}$. Any $p \in \mathbb{Q}$ can be extended by adding (f,g) to $\mathrm{dom}(\varrho^p)$ and choosing $\varrho(f,g)$ to be large enough. □

Problem 4.6. Is it consistent with ZFC that $2^{\omega_1} = 2^\omega$ and $\mathfrak{p}_{\mathrm{KB}}(\mathcal{I}) \leq \omega_1$ (or $\mathfrak{p}_{1\text{–}1}(\mathcal{I}) \leq \omega_1$) for each ideal $\mathcal{I}$ but $\leq_{\mathrm{KB}}$ (respectively $\leq_{1\text{–}1}$) is upward directed on ideals generated by ω_1 elements?

Note that if it is possible for $\leq_{1\text{–}1}$, then in such a model $\mathfrak{c} \geq \omega_3$: It is easy to see that if $\leq_{1\text{–}1}$ is upward directed on ideals generated by ω_1 sets, then any ω_2 ideals with character ω_1 have an upper bound in $\leq_{1\text{–}1}$. If there would be only $2^\omega = 2^{\omega_1} = \omega_2$ many ideals with character ω_1, then they would form a $\leq_{1\text{–}1}$-bounded set, i.e. there would be an ideal $\mathcal{I}$ with $\mathfrak{p}_{1\text{–}1}(\mathcal{I}) > \omega_1$.

Remark 4.7. In [3, Theorem 7.4] the authors proved that consistently there is a Boolean algebra $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ such that all ultrafilters on $\mathfrak{A}$ are meager but none of them has a pseudo-intersection. Using Theorem 4.3(a) and the fact that $\mathfrak{h} = \omega_1$ in the Cohen model, we can mimic this proof to show a similar result. Namely, we can prove that, consistently, there is an ideal $\mathcal{I}$ and a Boolean algebra $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ such that for each ultrafiler $\mathcal{F}$ on $\mathfrak{A}$ there is no pseudo-intersection of $\mathcal{F}$ but $\mathcal{F} \leq_{\mathrm{K}} \mathcal{I}^*$.

5. Permuting MAD families into ideals

The pseudo-intersection number is not the only cardinal coefficient which can be generalized in the way presented in the paper. E.g. we can easily define the analog of the tower number. Recall that a tower is a family with SFIP whose elements can well-ordered by $\subseteq^*$. Define e.g.

$$\mathfrak{t}_{1\text{–}1}(\mathcal{I}) = \min\{\chi(\mathrm{fr}(\mathcal{T}))\colon \mathcal{T} \text{ is a tower and } \mathrm{fr}(\mathcal{T}) \not\leq_{1\text{–}1} \mathcal{I}^*\},$$

where $\mathcal{I}$ is an ideal and $\mathrm{fr}(\mathcal{T})$ is the filter generated by $\mathcal{T}$. As in the case of $\mathfrak{p}_{1\text{–}1}$, we have $\mathfrak{t}_{1\text{–}1}(\mathrm{Fin}) = \mathfrak{t}$. However, in general this coefficient may be not well defined, i.e. maybe the family of all (filters generated by) towers are $\leq_{1\text{–}1}$-bounded. E.g. consider the filter $\mathcal{F}$ from Theorem 4.3(a). Since in the Cohen-model there are no towers of character ω_2 [Kunen, unpublished] and every filter generated by ω_1 sets is one-to-one-below $\mathcal{F}$, every tower is one-to-one–below $\mathcal{F}$, and $\mathfrak{t}_{1\text{–}1}(\mathcal{F}^*)$ in undefined.

Similarly, we can define the cardinal coefficient $\mathfrak{a}_{1\text{–}1}(\mathcal{I})$ analogous to the almost-disjointness number $\mathfrak{a}$. An infinite family $\mathcal{A} = \{A_\alpha\colon \alpha < \lambda\} \subseteq [\omega]^\omega$ is *almost disjoint* (AD) if $A_\alpha \cap A_\beta$ is finite for every $\alpha \neq \beta$. $\mathcal{A}$ is *maximal almost disjoint* (MAD) family if for every $X \in [\omega]^\omega$ there is $\alpha < \lambda$ such that $A_\alpha \cap X$ is infinite, i.e. $\mathcal{A}$ is $\subseteq$-maximal among AD families. For an almost disjoint family $\mathcal{A}$ denote by $\mathrm{id}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. Equivalently, an almost disjoint family is maximal if the filter dual to $\mathrm{id}(\mathcal{A})$ does not have a pseudo-intersection. The *almost disjointness number* $\mathfrak{a}$ is the minimal cardinality of a MAD family.

Definition 5.1. Let $\mathcal{I}$ be an ideal on ω. An almost disjoint family $\mathcal{A}$ is *$\mathcal{I}$-maximal* if $\mathrm{id}(\mathcal{A}) \not\leq_{1\text{–}1} \mathcal{I}$.

Using Proposition 2.7, if $\mathcal{I} \neq \mathrm{Fin}$ then an AD family is $\mathcal{I}$-maximal iff it cannot be permuted into $\mathcal{I}$.

Clearly, an AD family is Fin-maximal iff it is a MAD family. Furthermore, if $\mathcal{I} \leq_{1\text{–}1} \mathcal{J}$ and an AD family is $\mathcal{J}$-maximal, then it is $\mathcal{I}$-maximal as well. In particular, each $\mathcal{I}$-maximal AD family is a MAD family. From now on we will use the phrase "$\mathcal{I}$-maximal MAD family." It is trivial that if $\mathcal{I}$ is not tall, then each MAD family is $\mathcal{I}$-maximal.

As before, let

$$\mathfrak{a}_{1\text{–}1}(\mathcal{I}) = \min\{\chi(\mathrm{id}(\mathcal{A}))\colon \mathcal{A} \text{ is } \mathcal{I} - \text{maximal}\}.$$

This section is devoted to study when the above cardinal coefficient is well-defined, i.e. when there is an $\mathcal{I}$-maximal MAD family.

Note that it is easy to see that $\mathrm{id}(\mathcal{A})$ is meager for each AD family $\mathcal{A}$.

Recall that $\mathsf{add}^*(\mathcal{I})$ is one of the generalizations of $\mathfrak{p}$ mentioned in Introduction.

Proposition 5.2. $\mathsf{add}^*(\mathcal{I}) = \mathfrak{c}$ *implies that there is an $\mathcal{I}$-maximal MAD family.*

Proof. Fix an enumeration $\{f_\alpha\colon \omega \leq \alpha < \mathfrak{c}\}$ of injective sequences of natural numbers. We will construct the desired MAD family inductively. Start with a disjoint partition (A_n) of ω into infinite sets and assume we have constructed all A_ξ's for $\xi < \alpha < \mathfrak{c}$.

If for some $\xi < \alpha$ we have $f_\alpha^{-1}[A_\xi] \notin \mathcal{I}$, then take $A_\alpha = A_\xi$.

If not, then consider the family $\{f_\alpha^{-1}[A_\xi]\colon \xi < \alpha\} \subseteq \mathcal{I}$. Using the assumption $\mathsf{add}^*(\mathcal{I}) = \mathfrak{c}$ we can find a set $B \in \mathcal{I}$ such that $f_\alpha^{-1}[A_\xi] \subseteq^* B$ for every $\xi < \alpha$. Let $A_\alpha = f_\alpha''[\omega \setminus B]$. Then $\{A_\xi\colon \xi \leq \alpha\}$ is an AD family, and $f_\alpha^{-1}[A_\alpha] \in \mathcal{I}^*$.

In this way we will construct an AD family $\mathcal{A} = \{A_\alpha\colon \alpha < \mathfrak{c}\}$ such that $\mathrm{id}(\mathcal{A}) \not\leq_{1\text{–}1} \mathcal{I}$. $\square$

In generalizing Proposition 5.2 we have to be careful. It is easy to construct an almost disjoint family which can be extended only by a set from a given ideal $\mathcal{I}$. Indeed, consider $\omega = A \cup B$, where $A \in \mathcal{I}$ and define a MAD family $\mathcal{B} = \{B_\alpha\colon \alpha < \mathfrak{c}\}$ on B and a non-maximal almost disjoint family $\mathcal{A} = \{A_\alpha\colon \alpha < \mathfrak{c}\}$ on A. The family defined by $\{A_\alpha \cup B_\alpha\colon \alpha < \mathfrak{c}\}$ can be extended only by sets from $\mathcal{I}$.

We will show that under Martin's Axiom for σ-centered posets (i.e. $\mathfrak{p} = \mathfrak{c}$, see [1]) there are $\mathcal{I}$-maximal MAD families for each F_σ ideal and analytic P-ideal $\mathcal{I}$. Recall that this axiom does not imply that $\mathsf{add}(\mathcal{N}) = \mathfrak{c}$ (see [7, 522S]) and $\mathrm{add}^*(\mathcal{I}) = \mathrm{add}(\mathcal{N})$ for a lot of tall analytic P-ideals (e.g. for tall summable and tall density ideals, see [8, Theorem 2.2]).

Theorem 5.3. *Let $\mathcal{I}$ be a tall F_σ ideal or a tall analytic P-ideal. Then* MA(σ-centered) *implies that there is an $\mathcal{I}$-maximal MAD family.*

Proof. Let $\mathcal{I}$ be a tall analytic P-ideal and fix an enumeration $\{f_\alpha\colon \omega \leq \alpha < \mathfrak{c}\}$ of injective sequences of natural numbers. As in the proof of Proposition 5.2, we construct the desired MAD family inductively. Start with a disjoint partition (A_n) of ω into infinite sets and assume we have constructed all A_ξ's for $\xi < \alpha < \mathfrak{c}$.

As before, if for some $\xi < \alpha$ we have $f_\alpha^{-1}[A_\xi] \notin \mathcal{I}$, then let $A_\alpha = A_\xi$. If not, consider the almost disjoint family $\mathcal{A}$ of sets $f_\alpha^{-1}[A_\xi] \in \mathcal{I}$ for $\xi < \alpha$.

We claim that $\mathsf{MA}(\sigma\text{-centered})$ implies that $\mathcal{A}$ can be extended to an AD family by a set C from $\mathcal{I}^+$. It would be enough because then we can let $A_\alpha = f''_\alpha[C]$ and proceed as in the proof of Proposition 5.2.

Let $\mathcal{I}$ be a tall F_σ ideal or a tall analytic P-ideal and assume that $\mathcal{A} \subseteq \mathcal{I}$ is an AD family. Then we have to find a σ-centered forcing notion $\mathbb{P}(\mathcal{A})$ such that in $V^{\mathbb{P}(\mathcal{A})}$ the family $\mathcal{A}$ can be extended by an $\mathcal{I}$-positive set.

Let $\mathbb{P}(\mathcal{A})$ be the natural forcing notion that extends $\mathcal{A}$ with a new element. Namely, let $p = (n^p, s^p, \mathcal{B}^p) \in \mathbb{P}(\mathcal{A})$ iff $n^p \in \omega$, $s^p \subseteq n$, and $\mathcal{B}^p \in [\mathcal{A}]^{<\omega}$. We say that $p \leq q$ iff $n^p \geq n^q$, $s^p \cap n^q = s^q$, and $(s^p \setminus s^q) \cap \bigcup \mathcal{B}^q = \emptyset$.

It is easy to see that $\mathbb{P}(\mathcal{A})$ is σ-centered and that the sets $D_k = \{p \in \mathbb{P}(\mathcal{A})\colon |s^p| > k\}$ and $D_A = \{p \in \mathbb{P}(\mathcal{A})\colon A \in \mathcal{B}^p\}$ are dense in $\mathbb{P}(\mathcal{A})$ for each $k \in \omega$ and $A \in \mathcal{A}$. Consequently, if $\dot{S}$ is a $\mathbb{P}(\mathcal{A})$-name such that $\Vdash_{\mathbb{P}(\mathcal{A})} \dot{S} = \bigcup\{s^p \colon p \in \dot{G}\}$ (where $\dot{G}$ is the canonical name of the generic filter), then $\Vdash_{\mathbb{P}(\mathcal{A})} \dot{S} \in [\omega]^\omega$ and $\Vdash_{\mathbb{P}(\mathcal{A})} |\dot{S} \cap A| < \omega$ for each $A \in \mathcal{A}$.

We have to show that $\Vdash_{\mathbb{P}(\mathcal{A})} \dot{S} \in \mathcal{I}^+$. We have two cases:

Case I: $\mathcal{I} = \mathrm{Fin}(\varphi)$ is a tall F_σ ideal. We will show that $\Vdash_{\mathbb{P}(\mathcal{A})} \varphi(\dot{S}) = \infty$. It is enough to prove that $E_k = \{p \in \mathbb{P}(\mathcal{A})\colon \varphi(s^p) > k\}$ is dense in $\mathbb{P}(\mathcal{A})$ for each $k \in \omega$. Fix a $p \in \mathbb{P}(\mathcal{A})$. Since $\mathcal{A} \subseteq \mathcal{I}$ and φ is subadditive we have

$$\varphi\Big(\omega \setminus \Big(n^p \cup \bigcup \mathcal{B}^p\Big)\Big) = \infty$$

so by the LSC property of φ we can find a finite $F \subseteq \omega \setminus \big(n^p \cup \bigcup \mathcal{B}^p\big)$ such that $\varphi(F) > k$. If $q = (\max(F) + 1, s^p \cup F, \mathcal{B}^p)$, then $q \in E_k$ and $q \leq p$. We are done.

Case II: $\mathcal{I} = \mathrm{Exh}(\varphi)$ is a tall analytic P-ideal, $\|\omega\|_\varphi = 1$. We will show that $\Vdash_{\mathbb{P}(\mathcal{A})} \|\dot{S}\|_\varphi = 1$. It is enough to prove that $H_k^\varepsilon = \{p \in \mathbb{P}(\mathcal{A})\colon \varphi(s^p \setminus k) > \varepsilon\}$ is dense in $\mathbb{P}(\mathcal{A})$ for each $\varepsilon < 1$ and $k \in \omega$. To see this, fix a condition $p \in \mathbb{P}(\mathcal{A})$. Since $\mathcal{A} \subseteq \mathcal{I}$ and $\|\cdot\|_\varphi$ is subadditive we conclude that

$$\varphi\Big(\omega \setminus \Big(n^p \cup k \cup \bigcup \mathcal{B}^p\Big)\Big) \geq \Big\|\omega \setminus \Big(n^p \cup k \cup \bigcup \mathcal{B}^p\Big)\Big\|_\varphi = 1$$

so by the LSC property of φ we can find a finite $F \subseteq \omega \setminus \big(n^p \cup k \cup \bigcup \mathcal{B}^p\big)$ such that $\varphi(F) > \varepsilon$. If $q = (\max(F) + 1, s^p \cup F, \mathcal{B}^p)$, then $q \in H_k^\varepsilon$ and $q \leq p$. The proof is complete. $\square$

We finish with some related questions:

Problem 5.4.

- Does there exist an $\mathcal{I}$-maximal MAD family for a tall (analytic) ideal $\mathcal{I}$ in ZFC?
- Is it consistent with ZFC that there is no $\mathcal{I}$-maximal MAD family for some (nice) $\mathcal{I}$?
- Is it consistent with $\mathsf{ZFC} + \neg\mathsf{CH}$ that there are $\mathcal{I}$-maximal MAD families for each ideal?

6. Acknowledgements

A part of the paper was born in a fertile atmosphere of the Winterschool on Abstract Analysis (Section Topology) 2010 in Hejnice, Czech Republic. The authors want to thank the occupants of the smokers room in Hejnice, particularly Prof. Petr

Simon and Prof. Bohuslav Balcar, for interesting discussions about the subject of this paper. We would also like to thank Prof. Andreas Blass for his valuable remarks on the Katětov-order.

References

1. M. G. Bell, *On the combinatorial principle p(c)*, Fund. Math. **114** (1981), no. 2, 149–157. MR 643555 (83e:03077)
2. A. Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of Set Theory (M. Foreman and A. Kanamori, eds.), Springer, 2010, pp. 395–491.
3. P. Borodulin-Nadzieja and G. Plebanek, *On sequential properties of Banach spaces, spaces of measures and densities*, Czechoslovak Math. J. **60(135)** (2010), no. 2, 381–399. MR 2657956
4. J. Brendle, *Cardinal invariants of analytic quotient*, Slides for ESI workshop on large cardinals and descriptive set theory, Vienna, Jun 14–27 2009.
5. J. Brendle and S. Shelah, *Ultrafilters on ω — their ideals and their cardinal characteristics*, Transactions of the American Mathematical Society **351** (1999), 2643–2674, math.LO/9710217.
6. R. Engelking and M. Karłowicz, *Some theorems of set theory and their topological consequences*, Fund. Math. **57** (1965), 275–285. MR 0196693 (33 #4880)
7. D.H. Fremlin, *Measure theory: Topological measure spaces v. 4*, Torres Fremlin, 2003.
8. F. Hernández-Hernández and M. Hrušák, *Cardinal invariants of analytic P-ideals*, Canad. J. Math. **59** (2007), no. 3, 575–595. MR 2319159 (2008c:03051)
9. M. Hrušák, *Combinatorics of filters and ideals*, preprint.
10. M. Hrušák and H. Minami, *Mathias-Prikry and Laver-Prikry type forcing*, preprint.
11. K. Mazur, *F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$*, Fund. Math. **138** (1991), no. 2, 103–111. MR 1124539 (92g:06019)
12. D. Meza-Alcántara, *Ideals and filters on countable sets*, Ph.D. thesis, Universidad Nacional Autónoma México, 2009.
13. H. Niederreiter, *On the existence of uniformly distributed sequences in compact spaces*, Compositio Math. **25** (1972), 93–99.
14. S. Solecki, *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), no. 1-3, 51–72. MR 1708146 (2000g:03112)
15. R. C. Solomon, *Families of sets and functions*, Czechoslovak Math. J. **27(102)** (1977), no. 4, 556–559. MR 0457218 (56 #15429)

Instytut Matematyczny, Uniwersytet Wrocławski
E-mail address: pborod@math.uni.wroc.pl

Institute of Mathematics, Budapest University of Technology and Economics
E-mail address: barnabasfarkas@gmail.com