COVERING PROPERTIES OF IDEALS

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Abstract. M. Elekes proved that any infinite-fold cover of a σ-finite measure space by a sequence of measurable sets has a subsequence with the same property such that the set of indices of this subsequence has density zero. Thanks to this theorem he gave a new proof for the random-indestructibility of the density zero ideal. He asked about other variants of this theorem.

We present some negative results and discuss the category case when the set of indices of the required subsequence should be in a fixed ideal $\mathcal{J}$ on ω. We introduce the notion of the $\mathcal{J}$-covering property of a pair $(\mathcal{A}, I)$ where $\mathcal{A}$ is a σ-algebra on X and $I \subseteq \mathcal{P}(X)$ is an ideal. We investigate connections between this property and forcing-indestructibility of ideals. Also, we study the $\mathcal{J}$-covering property for the pairs $(\mathrm{Borel}(X\times Y), I\otimes K)$ where X, Y are Polish spaces, and $(\mathcal{P}(\omega), \mathcal{I})$ where $\mathcal{I}$ is an ideal on ω.

1. Introduction

We will discuss the following result due to Elekes [4].

Theorem 1.1. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $(A_n)_{n\in\omega}$ be a sequence of sets from $\mathcal{A}$ that covers μ-almost every $x \in X$ infinitely many times. Then there exists a set $M \subseteq \omega$ of asymptotic density zero such that $(A_n)_{n\in M}$ also covers μ-almost every $x \in X$ infinitely mamy times.*

Using this result Elekes gave a nice new proof for the fact that the density zero ideal is random-indestructible. He asked about other variants of this theorem.

For an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$, we assume that $\omega \notin \mathcal{I}$ and $\mathrm{Fin} \subseteq \mathcal{I}$ where Fin stands for the ideal of finite subsets of ω. An ideal $\mathcal{I}$ is *tall* if each infinite subset of ω contains an infinite element of $\mathcal{I}$. Clearly, an ideal $\mathcal{J}$ on ω is tall iff its dual filter $\mathcal{J}^*$ does not have a *pseudointersection*, that is a set $X \in [\omega]^\omega$ such that $X \subseteq^* Y$ for each $Y \in \mathcal{J}^*$ where $X \subseteq^* Y$ means that $X \setminus Y$ is finite.

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Each ideal on ω can be treated as a subset of the Cantor space 2^ω via the standard bijection between 2^ω and $\mathcal{P}(\omega)$, so we can talk about *Borel, F_σ, analytic, meager...* ideals.

If I is an ideal on a set X then let $I^* = \{X \setminus A : A \in I\}$ its dual filter and let $I^+ = \mathcal{P}(X) \setminus I$ the set of *I-positive* subsets of X. If $Y \subseteq X$ and $Y \in I^+$, then the *restriction of I to Y* is the following ideal on Y: $I \restriction Y = \{Y \cap A : A \in I\}$.

More informations about ideals on ω can be found in [8].

Elekes discovered that these covering properties have an effect on forcing indestructibility of ideals. Assume $\mathcal{J}$ is a tall ideal on ω and $\mathbb{P}$ is a forcing notion. We say that *$\mathcal{J}$ is $\mathbb{P}$-indestructible* if $\Vdash_{\mathbb{P}} \exists\ A \in \mathcal{J}\ |\dot{X} \cap A| = \aleph_0$ for each $\mathbb{P}$-name $\dot{X}$ for an infinite subset of ω, i.e. in $V^{\mathbb{P}}$ the ideal generated by $\mathcal{J}$ is tall. This property has been widely studied for years. Understanding deeper the connection between these covering properties and forcing indestructibility, we recall some important notions and results.

The *Katětov-order on ideals*: if $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω then $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ iff there is a function $f\colon \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$.

The *G_δ-closure* of a set $A \subseteq 2^{<\omega}$ (or $\omega^{<\omega}$) is

$$[A] = \big\{f \in 2^\omega \text{ (or } \omega^\omega) : \exists^\infty\ n\ f \restriction n \in A\big\}.$$

The *trace ideal* of a σ-ideal I on 2^ω (or on ω^ω) is

$$\mathrm{tr}(I) = \big\{A \subseteq 2^{<\omega} \text{ (or } \omega^{<\omega}) : [A] \in I\big\}.$$

If I is a σ-ideal on a Polish space X, then $\mathbb{P}_I$ denotes the forcing notion $\mathrm{Borel}(X) \setminus I$ partially ordered by the reverse inclusion. Properties of forcing notions of the form $\mathbb{P}_I$ is a central topic of the theory of forcing methods. For more details and notions, for instance the property *continuous readings of names* (CRN), see [10] or [5].

The following theorem shows the crucial role of the Katětov-order in forcing indestructibility of ideals:

Theorem 1.2. (see [5]) *Let I be a σ-ideal on 2^ω or on ω^ω, and let $\mathcal{J}$ be a tall ideal on ω. Assume furthermore that $\mathbb{P}_I$ is proper and has the CRN. Then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible if, and only if $\mathcal{J} \not\leq_{\mathrm{K}} \mathrm{tr}(I) \restriction X$ for any $X \in \mathrm{tr}(I)^+$.*

In some classical cases we know a little bit more: we do not need to investigate $\mathcal{J} \not\leq_{\mathrm{K}} \mathrm{tr}(I) \restriction X$ for any $X \in \mathrm{tr}(I)^+$ only $\mathcal{J} \not\leq_{\mathrm{K}} \mathrm{tr}(I)$ because of homogeneous properties of $\mathrm{tr}(I)$ (for more details see [8, Section 2.1.1.]).

For example:

(i) An ideal $\mathcal{J}$ on ω is Cohen-indestructible iff $\mathcal{J} \not\leq_{\mathrm{K}} \mathrm{tr}(\mathcal{M})$ where $\mathcal{M}$ is the σ-ideal of meager subsets of 2^ω (or of ω^ω or of $\mathbb{R}$).

(ii) An ideal $\mathcal{J}$ is random-indestructible iff $\mathcal{J} \not\leq_{\mathrm{K}} \mathrm{tr}(\mathcal{N})$ where $\mathcal{N}$ is the σ-ideal of null subsets of 2^ω (or of ω^ω or of $\mathbb{R}$).

(iii) An ideal $\mathcal{J}$ is Sacks-indestructible iff $\mathcal{J} \not\leq_{\mathrm{K}} \mathrm{tr}([2^\omega]^{\leq\omega})$.

For other types of general characterization theorems see [3].

We will need the following ideals on ω:

Let $\mathcal{ED}$ be the *eventually different* ideal, that is

$$\mathcal{ED} = \Big\{A \subseteq \omega \times \omega : \limsup_{n\to\infty} |(A)_n| < \infty\Big\}$$

where $(A)_n = \{m \in \omega : (n,m) \in A\}$. Let $\Delta = \{(n,m) \in \omega \times \omega : m \leq n\}$ and $\mathcal{ED}_{\mathrm{fin}} = \mathcal{ED} \restriction \Delta$. These two ideals are tall F_σ non P-ideals.

At last, the Fubini-product of Fin by itself

$$\mathrm{Fin} \otimes \mathrm{Fin} = \big\{A \subseteq \omega \times \omega : \forall^\infty\, n\ |(A)_n| < \infty\big\}.$$

This is a tall $F_{\sigma\delta\sigma}$ non P-ideal.

Furthermore, we will use the *Katětov-Blass order* on ideals: $\mathcal{J}_0 \leq_{\mathrm{KB}} \mathcal{J}_1$ iff there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}_1$ for each $A \in \mathcal{J}_0$.

In Section 2 we introduce a general covering property of a pair $(\mathcal{A}, I)$ with respect to an ideal on ω where $\mathcal{A}$ is a σ-algebra and I is an ideal on its underlying set. This property is a natural generalization of the interaction between the pair (measurable sets, ideal of measure zero sets) and the density zero ideal proved by Elekes. We give some negative results showing that in certain cases the respective a.e.-subcovers do not exist on ω^ω. And at last, we present the general effect of the covering property on forcing indestructibility of ideals.

In Section 3 we prove general positive results for the category case which answers a question from [4], and we present some examples which show that our implications are not reversible.

In Section 4 we describe a class of ideals I on $\mathbb{R}$ for which the $\mathcal{J}$-covering property for $(\mathrm{Borel}(\mathbb{R}), I)$ fails in a strong fashion.

In Section 5 we investigate $\mathcal{J}$-covering properties of the pair $(\mathrm{Borel}(\mathbb{R}^2), \mathcal{N} \otimes \mathcal{M})$ and a stronger hypothesis (Z-uniform $\mathcal{J}$-covering property) as well.

In Section 6 we investigate $\mathcal{J}$-covering properties of the pair $(\mathcal{P}(\omega), \mathcal{I})$ where $\mathcal{I}$ is an ideal on ω. It has turned out that $\mathcal{J}$-(ultra)filters play an important role in this situation.

2. The $\mathcal{J}$-covering property

We can consider the following abstract setting.

Definition 2.1. Let X be an arbitrary set and $I \subseteq \mathcal{P}(X)$ be an ideal of subsets of X. We say that a sequence $(A_n)_{n\in\omega}$ of subsets of X is an *I-a.e. infinite-fold cover of X* if

$$\big\{x \in X : \{n \in \omega : x \in A_n\} \text{ is finite}\big\} \in I\ , \quad \text{i.e.} \quad \limsup_{n\in\omega} A_n \in I^*.$$

Of course, the sequence (A_n) above can be indexed by any countable infinite set. Assume furthermore that given a σ-algebra $\mathcal{A}$ of subsets of X and an ideal $\mathcal{J}$ on ω. We say that the pair $(\mathcal{A}, I)$ has the *$\mathcal{J}$-covering property* if for every I-a.e. infinite-fold cover $(A_n)_{n\in\omega}$ of X by sets from $\mathcal{A}$, there is a set $S \in \mathcal{J}$ such that $(A_n)_{n\in S}$ is also an I-a.e. infinite-fold cover of X.

Clearly, in the previous definition it is enough to check infinite-fold covers instead of I-a.e. infinite-fold covers. Observe that, if $I_1 \subseteq I_2$ and $(\mathcal{A}, I_1)$ possesses the $\mathcal{J}$-covering property, then also $(\mathcal{A}, I_2)$ possesses it.

In this context, Elekes' theorem says that if $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, then $(\mathcal{A}, \mathcal{N}_\mu)$ has the $\mathcal{Z}$-covering property where

$$\mathcal{Z} = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\right\}$$

is the density zero ideal (a tall $F_{\sigma\delta}$ P-ideal) and $\mathcal{N}_\mu = \{H \in \mathcal{A} : \mu(H) = 0\}$ (more precisely, $\mathcal{N}_\mu$ is the ideal generated by null sets because we assume that I is an ideal on the underlying set).

Notice that if $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property, then $(\mathcal{A}[I], I)$ also has this property where $\mathcal{A}[I]$ is the "I-completion of $\mathcal{A}$", that is

$$\mathcal{A}[I] = \big\{B \subseteq X : \exists\, A \in \mathcal{A}\ A \triangle B \in I\big\}.$$

For instance, if $(\mathrm{Borel}(\mathbb{R}), \mathcal{N})$ has the $\mathcal{J}$-covering property where $\mathcal{N}$ is the σ-ideal of Lebesgue null sets, then $(\mathrm{LM}(\mathbb{R}), \mathcal{N})$ also has this property where $\mathrm{LM}(\mathbb{R})$ is the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Similarly in the category case, it is enough to prove a $\mathcal{J}$-covering property for Borel sets, then it holds for the σ-algebra of sets with the Baire property as well.

Furthermore, if $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property then for all $Y \in \mathcal{A} \setminus I$ the pair $(\mathcal{A} \restriction Y, I \restriction Y)$ also has this property where of course, $\mathcal{A} \restriction Y = \{Y \cap A : A \in \mathcal{A}\}$ is the restricted σ-algebra.

Clearly, if $\mathcal{J}$ is not tall, then there is no $(\mathcal{A}, I)$ with the $\mathcal{J}$-covering property.

We give another motivation to this notion as well. First we reformulate the $\mathcal{J}$-covering property:

Fact 2.2. *$(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property ($X = \bigcup \mathcal{A}$) if and only if, for every $(\mathcal{A}, \mathrm{Borel}([\omega]^\omega))$-measurable function $F \colon X \to [\omega]^\omega$, there is an $S \in \mathcal{J}$ such that $\{x \in X : |F(x) \cap S| < \omega\} \in I$.*

The $\mathcal{J}$-covering property can be seen as "analytic uniformity" in the following sense: If $\mathcal{J}$ is a tall ideal on ω then the *star-uniformity of* $\mathcal{J}$ is the following cardinal:

$$\mathsf{non}^*(\mathcal{J}) = \min\left\{|\mathcal{H}| : \mathcal{H} \subseteq [\omega]^\omega \text{ and } \nexists\, A \in \mathcal{J}\ \forall\, H \in \mathcal{H}\ |A \cap H| = \omega\right\}.$$

Clearly $(\mathcal{P}(X), \{\emptyset\})$ has the $\mathcal{J}$-covering property iff $|X| < \mathsf{non}^*(\mathcal{J})$.

We present two easy negative results. First we show that, in Elekes' theorem applied to Lebesgue measure, one cannot use the summable ideal $\mathcal{I}_{1/n} = \{A \subseteq \omega \colon \sum_{n\in A} \frac{1}{n+1} < \infty\} (\subseteq \mathcal{Z})$ instead of $\mathcal{Z}$.

Example 2.3. We will show that $(\mathrm{Borel}(\mathbb{R}), \mathcal{N})$ does not have the $\mathcal{I}_{1/n}$-covering property in a strong sense. First consider interval $(0,1)$ and a fixed infinite-fold cover $(A_n)_{n\in\omega}$ of $(0,1)$ of the form $A_n = (a_n, b_n)$, $b_n - a_n = \frac{1}{n+1}$. Then for each $S \in \mathcal{I}_{1/n}$ we have $\sum_{n\in S} \lambda(A_n) < \infty$ where λ stands for Lebesgue measure on $\mathbb{R}$. Hence by the Borel-Cantelli lemma, $\lambda(\limsup_{n\in S} A_n) = 0$. Fix a homeomorphism h from $(0,1)$ onto $\mathbb{R}$ of class C^1. Then $(h[A_n])_{n\in\omega}$ is an open infinite-fold cover of $\mathbb{R}$. Since h is absolutely continuous, we have $\lambda(\limsup_{n\in S} h[A_n]) = 0$, which gives the desired claim.

This example motivates the following question which will be discussed in Section 4 (see Example 4.1 and Theorem 4.2 below).

Question 2.4. Assume X is a Polish space and $(\mathrm{Borel}(X), I)$ does not have the $\mathcal{J}$-covering property. Does there exist an infinite-fold Borel cover $(A_n)_{n\in\omega}$ of X such that $\limsup_{n\in S} A_n \in I$ for all $S \in \mathcal{J}$?

Let us denote $\mathcal{K}_\sigma$ be the σ-ideal on ω^ω generated by compact sets. We will use the fact that an $H \subseteq \omega^\omega$ is in $\mathcal{K}_\sigma$ iff there is an $h \in \omega^\omega$ such that $H \subseteq \{x \in \omega^\omega : x \leq^* h\}$ where $x \leq^* h$ means that the set $\{n \in \omega : x(n) > h(n)\}$ is finite.

Example 2.5. Consider the following infinite-fold cover $(A_n)_{n\in\omega}$ of ω^ω by F_σ sets: Let $B = \{x \in \omega^\omega : \forall^\infty\, n\ x(n) = 0\}$ and $A_n = \{y \in \omega^\omega : y(n) \neq 0\} \cup B$ for $n \in \omega$. It is easy to see that if $X \in [\omega]^\omega$ with $\omega \setminus X \in [\omega]^\omega$, then $\omega^\omega \setminus \limsup_{n\in X} A_n$ is dense, uncountable, and does not belong to $\mathcal{K}_\sigma$.

In particular, there is no ideal $\mathcal{J}$ on ω such that $(\mathrm{Borel}(\omega^\omega), I)$ has the $\mathcal{J}$-covering property if $I = [\omega^\omega]^{\leq\omega}$, NWD (the ideal of nowhere dense sets), or $\mathcal{K}_\sigma$. Consequently (by the Borel isomorphism theorem), given an uncountable Polish space X, $(\mathrm{Borel}(X), [X]^{\leq\omega})$ does not have the $\mathcal{J}$-covering property for any ideal $\mathcal{J}$ on ω.

How could we conclude a $\mathcal{J}_1$-covering property from a $\mathcal{J}_0$-covering property? In special cases we can do it by the following easy observation:

Fact 2.6. *Assume $\mathcal{J}_0 \leq_{\mathrm{KB}} \mathcal{J}_1$ and that $(\mathcal{A}, I)$ has the $\mathcal{J}_0$-covering property. Then $(\mathcal{A}, I)$ has the $\mathcal{J}_1$-covering property as well.*

The next observation shows a connection between trace ideals and covering properties.

Proposition 2.7. *Assume I is a σ-ideal on 2^ω (or on ω^ω) and $(\mathrm{Borel}(2^\omega), I)$ has the $\mathcal{J}$-covering property. Then $\mathcal{J} \not\leq_{\mathrm{K}} \mathrm{tr}(I) \restriction X$ for any $X \in \mathrm{tr}(I)^+$.*

Proof. Assume on the contrary that $\mathcal{J} \leq_{\mathrm{K}} \mathrm{tr}(I) \restriction X$ for some $X \in \mathrm{tr}(I)^+$. Let $(A_n)_{n\in\omega}$ be the following infinite-fold cover of $[X]$:

$$A_n = \big\{x \in [X] : \exists\, k \in \omega\ (x \restriction k \in X \text{ and } f(x \restriction k) = n)\big\}.$$

If $S \in \mathcal{J}$ then $\limsup_{n\in S} A_n = [f^{-1}[S]] \in \mathrm{tr}(I)$, a contradiction. □

Using Theorem 1.2 we obtain the following:

Corollary 2.8. *Let I is a σ-ideal on 2^ω (or on ω^ω), and $\mathcal{J}$ is a tall ideal on ω. Assume furthermore that $\mathbb{P}_I$ is proper and has the CRN. If $(\mathrm{Borel}(2^\omega), I)$ has the $\mathcal{J}$-covering property, then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible.*

We do not need to use Theorem 1.2, the trace ideal or the CRN property in this result. The following theorem is a natural generalization of Elekes' result about random-indestructibility of $\mathcal{Z}$.

Theorem 2.9. *Let X be a Polish space, I be a σ-ideal on X, and assume that $\mathbb{P}_I$ is proper. If $(\mathrm{Borel}(X), I)$ has the $\mathcal{J}$-covering property, then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible.*

Proof. Assume on the contrary that $\dot{Y}$ is a $\mathbb{P}_I$-name for an infinite subset of ω, i.e. $\Vdash_{\mathbb{P}_I} \dot{Y} \in [\omega]^\omega$ and $B \Vdash_{\mathbb{P}_I} \forall\, A \in \mathcal{J}\ |\dot{Y} \cap A| < \omega$ for some $B \in \mathbb{P}_I$. Then there are a $C \in \mathbb{P}_I$, $C \subseteq B$, and a Borel function $f : C \to [\omega]^\omega$ (coded in the ground model) such that $C \Vdash_{\mathbb{P}_I} f(\dot{r}_{\mathrm{gen}}) = \dot{Y}$ where $\dot{r}_{\mathrm{gen}}$ is a name for the generic real (see [10, Prop. 2.3.1]). For each $n \in \omega$ let

$$Y_n = f^{-1}[\{S \in [\omega]^\omega : n \in S\}] \in \mathrm{Borel}(X).$$

Then $(Y_n)_{n\in\omega}$ is an infinite-fold cover of C (by Borel sets) because $x \in Y_n \iff n \in f(x)$ and $|f(x)| = \omega$. Using the $\mathcal{J}$-covering property of $(\mathrm{Borel}(X) \restriction C, I \restriction C)$ we can choose an $A \in \mathcal{J}$ such that $(Y_n)_{n\in A}$ is an I-a.e. infinite-fold cover of C, that is $|f(x) \cap A| = \omega$ for I-a.e. $x \in C$, i.e. $\{x \in C : |f(x) \cap A| < \omega\} \in I$, so $C \Vdash_{\mathbb{P}_I} |f(\dot{r}_{\mathrm{gen}}) \cap A| = \omega$, and consequently, $C \Vdash_{\mathbb{P}_I} |\dot{Y} \cap A| = \omega$, a contradiction. □

3. Around the category case

If X is a Polish space, then let $\mathcal{M}(X)$ be the σ-ideal of meager subsets of X.

Theorem 3.1. $(\mathrm{Borel}(X), \mathcal{M}(X))$ *has the* $\mathcal{ED}_{\mathrm{fin}}$*-covering property for each Polish space* X*.*

Proof. Let $(A_{(n,m)})_{(n,m)\in\Delta}$ be an infinite-fold cover of X by sets with the Baire property. Without loss of generality, we can assume that all $A_{(n,m)}$'s are open and nonempty.

Enumerate $\{U_k : k \in \omega\}$ a base of X. We will define by recursion a sequence $(n_k, m_k)_{k\in\omega}$ of elements of Δ. First, pick $(n_0, m_0) \in \Delta$ such that $A_{(n_0,m_0)} \cap U_0 \neq \emptyset$. Assume (n_i, m_i) are done for $i < k$. Then we can choose an $(n_k, m_k) \in \Delta$ such that $n_k \neq n_i$ for $i < k$ and $A_{(n_k,m_k)} \cap U_k \neq \emptyset$. We obtain the desired set $S = \{(n_k, m_k) : k \in \omega\} \in \mathcal{ED}_{\mathrm{fin}}$.

For every $k \in \omega$, the set $\bigcup_{i\geq k} A_{(n_i,m_i)}$ is dense and open. Consequently,

$$\limsup_{(n,m)\in S} A_{(n,m)} = \bigcap_{k\in\omega} \bigcup_{i\geq k} A_{(n_i,m_i)}$$

is a dense G_δ set. Hence it is residual. □

Using the Fact 2.6 and Theorem 2.9 we obtain the following:

Corollary 3.2. *If* $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathfrak{J}$*, then* $(\mathrm{Borel}(X), \mathcal{M}(X))$ *has the* $\mathfrak{J}$*-covering property for each Polish space* X*, and hence* $\mathfrak{J}$ *is Cohen-indestructible.*

Note that $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathfrak{J}$ holds for a quite big class of ideals, namely for analytic P-ideals. An ideal $\mathcal{I}$ on ω is called a *P-ideal* whenever for every sequence of sets $E_n \in \mathcal{I}$, $n \in \omega$, there is a set $E \in \mathcal{I}$ such that $E_n \subseteq^* E$.

Analytic P-ideals can be characterized by using submeasures on ω. A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure on* ω iff $\varphi(\emptyset) = 0$, $\varphi(A) \leq \varphi(B)$ for $A \subseteq B \subseteq \omega$, $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for $A, B \subseteq \omega$, and $\varphi(\{n\}) < \infty$ for $n \in \omega$. A submeasure φ is *lower semicontinuous* (lsc in short) iff $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for each $A \subseteq \omega$. Note that if φ is an lsc submeasure on ω then it is σ-subadditive, i.e. $\varphi(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \varphi(A_n)$ holds for $A_n \subseteq \omega$. We assign an ideal to an lsc submeasure φ as follows

$$\mathrm{Exh}(\varphi) = \Big\{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\Big\}.$$

$\mathrm{Exh}(\varphi)$ is an $F_{\sigma\delta}$ P-ideal or equal to $\mathcal{P}(\omega)$. It is straightforward to see that $\mathrm{Exh}(\varphi)$ is tall iff $\lim_{n\to\infty} \varphi(\{n\}) = 0$.

Theorem 3.3. ([9, Thm. 3.1.]) *If* $\mathfrak{J}$ *is an analytic P-ideal then* $\mathfrak{J} = \mathrm{Exh}(\varphi)$ *for some lsc submeasure* φ*.*

Therefore each analytic P-ideal is $F_{\sigma\delta}$ so it is a Borel subset of 2^ω.

Proposition 3.4. *$\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathcal{J}$ holds for each tall analytic P-ideal $\mathcal{J}$.*

Proof. Let $\mathcal{J} = \mathrm{Exh}(\varphi)$ for some lsc submeasure φ. For $k \in \omega$ let $d(k) = \min\{\ell_0 \in \omega : \forall\, \ell \geq \ell_0\ \varphi(\{\ell\}) < 2^{-k}\}$. We can choose a strictly increasing sequence $(n_k)_{k\in\omega} \in \omega^\omega$ such that $d(k+1) - d(k) \leq n_k$. Let $f\colon \omega \to \omega$ be any one-to-one function with the property $f[d(k+1)\setminus d(k)] \subseteq \{(n_k, m) : m \leq n_k\}$. Then f shows that $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathcal{J}$. □

Now, by Corollary 3.2 and Proposition 3.4 we obtain

Corollary 3.5. $(\mathrm{Borel}(X), \mathcal{M}(X))$ *has the $\mathcal{J}$-covering property for each Polish space X and for each tall analytic P-ideal $\mathcal{J}$.*

One can ask if the implications in Corollary 3.2 could be equivalences. The answer is no by the following examples.

Example 3.6. Fin $\otimes$ Fin is Cohen-indestructible but $(\mathrm{Borel}(\omega^\omega), \mathcal{M})$ does not have the Fin $\otimes$ Fin-covering property.

Cohen-indestructibility of Fin $\otimes$ Fin: Is easy to see that a forcing notion $\mathbb{P}$ destroys Fin $\otimes$ Fin iff $\mathbb{P}$ adds dominating reals.

$(\mathrm{Borel}(\omega^\omega), \mathcal{M})$ does not have the Fin$\otimes$Fin-covering property: Enumerate $\{s^n_m : m \in \omega\}$ the elements of $\omega^{<\omega}$ with first element n. Consider the following infinite-fold cover of ω^ω: $A_{(n,m)} = \{x \in \omega^\omega : s^n_m \subseteq x\}$ for $(n, m) \in \omega \times \omega$. It is trivial to see that there is no $S \in \mathrm{Fin} \otimes \mathrm{Fin}$ such that $(A_{(n,m)})_{(n,m)\in S}$ is an $\mathcal{M}$-a.e. infinite-fold cover of ω^ω.

Example 3.7. $\mathcal{ED}$ is also Cohen-indestructible: Let $\mathbb{C} = (2^{<\omega}, \supseteq)$ be the Cohen forcing and assume that $\dot{X}$ is a $\mathbb{C}$-name for an infinite subset of $\omega \times \omega$ such that $\Vdash_{\mathbb{C}} \exists^\infty n\ (\dot{X})_n \neq \emptyset$ (because else $\dot{X}$ cannot destroy $\mathcal{ED}$). Enumerate $\mathbb{C} = \{p_n : n \in \omega\}$. By recursion on $n \in \omega$ we will define $A = \{(m_n, k_n) : n \in \omega\} \subseteq \omega \times \omega$ (in the ground model). Assume (m_ℓ, k_ℓ) is done for $\ell < n$. Then we can choose a $q_n \leq p_n$, an $m_n > m_{n-1}$, and a k_n such that $q_n \Vdash k_n \in (\dot{X})_{m_n}$. Clearly $A \in \mathcal{ED}$.

We show that $\Vdash_{\mathbb{C}} |A \cap \dot{X}| = \omega$. Assume on the contrary that $p \Vdash \forall\, n \geq N$ $(A)_n \cap (\dot{X})_n = \emptyset$ for some $p \in \mathbb{C}$ and $N \in \omega$. Then $p = p_n$ for some n and we can assume that $n \geq N$. Then $m_n \geq n$ and $q_n \Vdash k_n \in (A)_{m_n} \cap (\dot{X})_{m_n}$, a contradiction.

Of course, $(\mathrm{Borel}(\omega^\omega), \mathcal{M})$ does not have the $\mathcal{ED}$-covering property because $\mathcal{ED} \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$ (and we can use Example 3.6).

It would be nice to know a characterization of forcing notions which destroy $\mathcal{ED}$ (similar to the characterization in the case of Fin $\otimes$ Fin):

Question 3.8. Is it true that a forcing notion $\mathbb{P}$ destroys $\mathcal{ED}$ iff $\mathbb{P}$ adds an eventually different real, i.e. a real $r \in \omega^\omega \cap V^{\mathbb{P}}$ such that $|f \cap r| < \omega$ for each $f \in \omega^\omega \cap V$? (The "if" part trivially holds.)

In the case of the first implication of Corollary 3.2 we have only consistent counterexamples. Recall that a sequence $\mathcal{T} = (T_\alpha)_{\alpha<\gamma}$ in $[\omega]^\omega$ is a *tower* if it is $\subseteq^*$-descending (i.e. $T_\beta \subseteq^* T_\alpha$ if $\alpha < \beta < \gamma$), and it has no pseudointersection. The *tower number* $\mathfrak{t}$ is the smallest cardinality of a tower, and $\mathfrak{c}$ stands for the continuum.

Theorem 3.9. *Assume* $\mathfrak{t} = \mathfrak{c}$ *and* $|\mathcal{A}| \leq \mathfrak{c}$ *then there is no smallest element of* $\{\mathcal{J} : (\mathcal{A}, I)$ *has the* $\mathcal{J}$*-covering property*$\}$ *in the Katětov-Blass order.*

Proof. If $\{\mathcal{J} : (\mathcal{A}, I)$ has the $\mathcal{J}$-covering property$\} = \emptyset$ then we are done. If $(\mathcal{A}, I)$ has the $\mathcal{J}_0$-covering property then we will construct a $\mathcal{J}$ such that $\mathcal{J}_0 \not\leq_{\mathrm{KB}} \mathcal{J}$ but $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property.

Enumerate $(f_\alpha)_{\alpha<\mathfrak{c}}$ all finite-to-one functions from ω to ω, and enumerate $((A^\alpha_n)_{n\in\omega} : \alpha < \mathfrak{c})$ the infinite-fold covers of $X = \bigcup \mathcal{A}$ by sets from $\mathcal{A}$. By recursion on $\mathfrak{c}$ we will define a $\subseteq^*$-increasing sequence $(S_\xi)_{\alpha<\mathfrak{c}}$ of infinite and co-infinite subsets of ω and the ideal $\mathcal{J}$ generated by this sequence will be as required.

Assume $(S_\xi)_{\xi<\alpha}$ is done for some $\alpha < \mathfrak{c}$. Because of our assumption on $\mathfrak{t}$ we can choose an infinite and co-infinite S'_α such that $S_\xi \subseteq^* S'_\alpha$ for each $\xi < \alpha$. The set $f_\alpha[\omega \setminus S'_\alpha]$ contains an infinite element E of $\mathcal{J}_0$. We want to guarantee that $f^{-1}_\alpha[E] \notin \mathcal{J}$ because then f_α can not witness $\mathcal{J}_0 \leq_{\mathrm{KB}} \mathcal{J}$. Let $H = f^{-1}_\alpha[E] \setminus S'_\alpha \in [\omega]^\omega$.

Consider the αth cover $(A^\alpha_n)_{n\in\omega}$. If $(A_n)_{n\in\omega\setminus H}$ is an I-a.e. infinite-fold cover of X, then let $S_\alpha = S'_\alpha \cup (\omega \setminus H)$.

If not, then

$$C = \big\{x \in X : \{n \in \omega \setminus H : x \in A^\alpha_n\} \text{ is finite}\big\} \notin I.$$

By using our assumption $(\mathcal{A}[I] \restriction C, I \restriction C)$ and $(A^\alpha_n \cap C)_{n\in H}$ (with a copy of $\mathcal{J}_0$ on H) we can choose an infinite $H' \subseteq H$ such that $H \setminus H'$ is also infinite and $(A^\alpha_n \cap C)_{n\in H'}$ is an $I \restriction C$-a.e. infinite-fold cover of C.

Finally, let $S_\alpha = S'_\alpha \cup (\omega \setminus H) \cup H'$. It is easy to see from the construction that $\mathcal{J}$ is as required. □

Corollary 3.10. *If* $\mathfrak{t} = \mathfrak{c}$ *then there are ideals* $\mathcal{J}_0$ *and* $\mathcal{J}_1$ *such that* $\mathcal{Z} \not\leq_{\mathrm{KB}} \mathcal{J}_0$ *and* $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}_1$ *but* $(\mathrm{Borel}(2^\omega), \mathcal{N})$ *has the* $\mathcal{J}_0$*-covering property and* $(\mathrm{Borel}(2^\omega), \mathcal{M})$ *has the* $\mathcal{J}_1$*-covering property.*

Without $\mathfrak{t} = \mathfrak{c}$ we can use a simple forcing construction.

Theorem 3.11. *After adding ω_1 Cohen-reals by finite support iteration there is an ideal $\mathcal{J}$ such that $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}$ (in particular, $\mathcal{Z} \not\leq_{\mathrm{KB}} \mathcal{J}$) but* $(\mathrm{Borel}(2^\omega), \mathcal{N})$ *and* $(\mathrm{Borel}(2^\omega), \mathcal{M})$ *have the $\mathcal{J}$-covering property.*

Proof. Let $(c_\alpha)_{\alpha<\omega_1}$ be the sequence of generic Cohen-reals in 2^ω, $C_\alpha = c_\alpha^{-1}[\{1\}] \subseteq \omega$, and let $\mathcal{J}$ be the ideal generated by these sets. $\mathcal{J}$ is a proper ideal because it is well-known that $\{C_\alpha : \alpha < \omega_1\}$ is an independent system of subsets of ω.

To show that $(\mathrm{Borel}(2^\omega), \mathcal{N})$ has the $\mathcal{J}$-covering property in the extension, it is enough to see that if $(A_n)_{n\in\omega}$ is an infinite-fold cover of 2^ω by Borel sets in a ground model V, then $(A_n)_{n\in C}$ is an $\mathcal{N}$-a.e. infinite-fold cover of 2^ω in $V[C]$ where $C \subseteq \omega$ is a Cohen-real over V.

Clearly, it is enough to prove that $V[C] \models \lambda\big(\bigcup_{n\in C\setminus k} A_n\big) = 1$ for each $k \in \omega$ (because then $V[C] \models \lambda\big(\limsup_{n\in C} A_n\big) = 1$). Let $p \in \mathbb{C} = (2^{<\omega}, \supseteq)$, $k \in \omega$, and $\varepsilon < 1$. We can assume that $|p| \geq k$. Then there is an $m \geq |p|$ such that $\lambda\big(\bigcup_{n\in m\setminus|p|} A_n\big) > \varepsilon$ so if $q : m \to 2$, $q \restriction |p| = p$, and $q \restriction (m \setminus |p|) \equiv 1$, then $q \leq p$ and $q \Vdash \lambda\big(\bigcup_{n\in \dot{C}\setminus k} A_n\big) > \varepsilon$.

To show that $(\mathrm{Borel}(2^\omega), \mathcal{M})$ has the $\mathcal{J}$-covering property in the extension, it is enough to prove that if $(A_n)_{n\in\omega}$ is an infinite-fold cover of 2^ω by open sets in V, then $(A_n)_{n\in C}$ is an $\mathcal{M}$-a.e. infinite-fold cover of 2^ω in $V[C]$. By a simple density argument $V[C] \models$"$\bigcup_{n\in C\setminus k} A_n$ is dense open" for each $k \in \omega$, so $V[C] \models$"$\limsup_{n\in C} A_n$ is residual."

To show that $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}$, it is enough to see that if $f \in \Delta^\omega \cap V[(c_\xi)_{\xi<\alpha}]$ is a finite-to-one function for some $\alpha < \omega_1$, then there is an $A \in \mathcal{ED}_{\mathrm{fin}} \cap V[(c_\xi)_{\xi\leq\alpha}]$ such that $f^{-1}[A]$ cannot be covered by finitely many of C_ξ's ($\xi < \omega_1$). Simply let A be a Cohen function in $\prod_{n\in\omega}(n+1)$, i.e. the graph of a Cohen-function in Δ, for example if $c'_\alpha \in \omega^\omega$ is a Cohen-real over $V[(c_\xi)_{\xi<\alpha}]$ then $A = \big\{(n,k) \in \Delta : c'_\alpha(n) \equiv k \mod (n+1)\big\}$ is suitable. Using the presentation of this iteration by finite partial functions from $\omega_1 \times \omega$ to 2, we are done by a simple density argument. □

Question 3.12. Does there exists an (analytic) (P-)ideal $\mathcal{J}$ in ZFC such that $\mathcal{Z} \not\leq_{\mathrm{KB}} \mathcal{J}$ but $(\mathrm{Borel}(2^\omega), \mathcal{N})$ has the $\mathcal{J}$-covering property? (Remark: Recently Sz. Głąb found a Borel non P-ideal with these properties, manuscript in preparation.)

Does there exist Katětov-Blass-smallest ideal in the family of all analytic (or Borel) ideals $\mathcal{J}$ such that $(\mathrm{Borel}(2^\omega), \mathcal{N})$ has the $\mathcal{J}$-covering property?

Similarly, one can ask the analogous question for the meager ideal with $\mathcal{ED}_{\mathrm{fin}}$ instead of $\mathcal{Z}$; and about the existence of Katětov-Blass-smallest ideal in the corresponding family as well.

4. When the $\mathcal{J}$-covering property "strongly" fails

In this section we give a positive answer for Question 2.4 in a special case. First of all, we present a counterexample:

Example 4.1. Consider $X = (-1, 1)$ and let an ideal I on X consist of sets $A \subseteq X$ such that $A \cap (-1, 0]$ is meager and $A \cap (0, 1)$ is of Lebesgue measure zero. Using Example 2.3 and Corollary 3.5 observe that $(\mathrm{Borel}(X), I)$ yields the negative answer to Question 2.4 with $\mathcal{J} = \mathcal{I}_{1/n}$. However, this question remains interesting if we restrict it to pairs $(\mathrm{Borel}(\mathbb{R}), I)$ where I is a translation invariant ideal on $\mathbb{R}$. In this case, we describe a class of ideas I which yields a positive answer to Question 2.4 provided $\mathcal{J}$ is a P-ideal.

Let $\mathbb{Q}$ stand for the set of rational numbers. For $A, B \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ we write $A + x = \{a + x : a \in A\}$ and $A + B = \{a + b : A \in A \text{ and } b \in B\}$. We say that an ideal I on a Polish space X is a *ccc ideal* if every disjoint subfamily of $\mathrm{Borel}(X) \setminus I$ is countable.

Theorem 4.2. *Assume that I is a translation invariant ccc σ-ideal on $\mathbb{R}$ fulfilling the condition*

$$\mathbb{Q} + A \in I^* \text{ for each } A \in \mathrm{Borel}(\mathbb{R}) \setminus I. \tag{1}$$

Fix a P-ideal $\mathcal{J}$ on ω. If $(\mathrm{Borel}(\mathbb{R}), I)$ does not have the $\mathcal{J}$-covering property, then there exists an infinite-fold Borel cover $(A'_n)_{n\in\omega}$ of $\mathbb{R}$ with $\limsup_{n\in S} A'_n \in I$ for all $S \in \mathcal{J}$.

Proof. Fix an infinite-fold Borel cover $(A_n)_{n\in\omega}$ of $\mathbb{R}$ such that $\limsup_{n\in S} A_n \notin I^*$ for all $S \in \mathcal{J}$. We will show that there is a Borel set $B \subseteq \mathbb{R}$ with $B \notin I$ and $(\limsup_{n\in S} A_n) \cap B \in I$ for all $S \in \mathcal{J}$. Suppose it is not the case. So, in particular (when $B = \mathbb{R}$), we find $S_0 \in \mathcal{J}$ with $X_0 := \limsup_{n\in S_0} A_n \notin I$. Then by transfinite recursion we define sequences $(S_\alpha)_{\alpha<\gamma}$ and $(X_\alpha)_{\alpha<\gamma}$ with $S_\alpha \in \mathcal{J}$ and $X_\alpha := (\limsup_{n\in S_\alpha} A_n) \setminus \bigcup_{\beta<\alpha} X_\beta \notin I$ (when $B = \mathbb{R} \setminus \bigcup_{\beta<\alpha} X_\beta \notin I$). Since I is ccc, this construction stops at a stage $\gamma < \omega_1$ with $\bigcup_{\alpha<\gamma} \limsup_{n\in S_\alpha} A_n = \bigcup_{\alpha<\gamma} X_\alpha \in I^*$. Since I is a P-ideal, there is $S \in \mathcal{J}$ which almost contains each S_α for $\alpha < \gamma$. Then $\limsup_{n\in S} A_n \in I^*$ which contradics our supposition.

So, fix a Borel set $B \notin I$ such that $(\limsup_{n\in S} A_n) \cap B \in I$ for all $S \in \mathcal{J}$. Let $\mathbb{Q} = \{q_k : k \in \omega\}$. Define $B_0 := B$ and $B_k := (q_k + B) \setminus \bigcup_{i<k} B_i$ for $k \in \omega$. Then put $A'_n := \bigcup_{k\in\omega}((q_k + A_n) \cap B_k)$ for $n \in \omega$. Since $(A_n)_{n\in\omega}$ is an infinite-fold cover of $\mathbb{R}$, we have $\limsup_{n\in\omega}((q_k + A_n) \cap B_k) = B_k$ for all

$k \in \omega$. Note that $(A'_n)_{n\in\omega}$ is an I-a.e. infinite-fold cover of $\mathbb{R}$ since

$$\limsup_{n\in\omega} A'_n \supseteq \bigcup_{k\in\omega} \limsup_{n\in\omega}((q_k + A_n) \cap B_k) = \mathbb{Q} + B$$

and $\mathbb{Q} + B \in I^*$ by (1). Finally, let $S \in \mathcal{J}$. Since I is translation invariant and $(\limsup_{n\in S} A_n) \cap B \in I$, we have $\limsup_{n\in S}((q_k + A_n) \cap B_k) \in I$ for all $k \in \omega$. Since I is a σ-ideal and B_k's are pairwise disjoint, it follows that

$$\limsup_{n\in S} A'_n = \bigcup_{k\in\omega} \limsup_{n\in S}((q_k + A_n) \cap B_k) \in I.$$

Of course, we can modify (A'_n) to be an infinite-fold cover of $\mathbb{R}$. □

Theorem 4.2 can be generalized to any Polish group G with $\mathbb{Q}$ replaced by a countable dense subset of G. Condition (1) is related to the Steinhaus property, for details see [2]. Note that $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy (1) with $\mathbb{Q}$ replaced by any dense subset of $\mathbb{R}$ (resp. $\mathbb{R}^2$).

5. Consequences for intersections and Fubini products

Having positive results on the $\mathcal{J}$-covering property for measure and category (Elekes' theorem and Theorem 3.1) one can ask, for which ideals $\mathcal{J}$ on ω, the pair $(\mathrm{Borel}(\mathbb{R}), \mathcal{M} \cap \mathcal{N})$ has the $\mathcal{J}$-covering property. A similar question concerns the pairs $(\mathrm{Borel}(\mathbb{R}^2), \mathcal{N} \otimes \mathcal{M})$ and $(\mathrm{Borel}(\mathbb{R}^2), \mathcal{M} \otimes \mathcal{N})$. For information on $\mathcal{N} \otimes \mathcal{M}$ and $\mathcal{M} \otimes \mathcal{N}$, see e.g. [1].

Recall that for an ideal $\mathcal{J}$ on ω its *star-additivity* is given by

$$\mathsf{add}^*(\mathcal{J}) = \min\big\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{J} \text{ and } \forall\, A \in \mathcal{J}\ \exists\, H \in \mathcal{H}\ H \not\subseteq^* A\big\}.$$

As to intersections of ideals, we have the following general fact whose proof is strightforward.

Fact 5.1. *Let $\mathcal{J}$ be an ideal on ω and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. If $\kappa < \mathsf{add}^*(\mathcal{J})$ and $\{I_\alpha : \alpha < \kappa\}$ is a family of ideals such that $(\mathcal{A}, I_\alpha)$ has the $\mathcal{J}$-covering property for all $\alpha < \kappa$, then $(\mathcal{A}, \bigcap_{\alpha<\kappa} I_\alpha)$ has the $\mathcal{J}$-covering property.*

In particular, by Elekes' theorem and Corollary 3.2 we obtain

Corollary 5.2. *If $\mathcal{Z} \leq_{\mathrm{KB}} \mathcal{J}$ then $(\mathrm{Borel}(\mathbb{R}), \mathcal{M} \cap \mathcal{N})$ has the $\mathcal{J}$-covering property.*

Now, let us start to study the behaviour of Fubini products of ideals in the aspect of the $\mathcal{J}$-covering property. For an ideal I on a Polish space X, we assume that $X \notin I$, and either $I = \{\emptyset\}$ or I contains all finite subsets of X. Assume that I and K are ideals on uncountable Polish spaces X and

Y, respectively. For $A \subseteq X \times Y$ and $x \in X$ let $(A)_x = \{y \in Y : (x, y) \in A\}$. Recall that the *Fubini product* of I and K is defined as follows

$$I \otimes K = \{A \subseteq X \times Y : \{x \in X : (A)_x \in K\} \in I^*\}.$$

Let $Z \subseteq X$ and let $\mathcal{J}$ be an ideal on ω. We say (cf. Definition 2.1) that $(\mathrm{Borel}(Y), K)$ has the *Z-uniform $\mathcal{J}$-covering property* whenever for every infinite-fold Borel cover $(A_n)_{n\in\omega}$ of $Z \times Y$ there exists an $S \in \mathcal{J}$ such that for all $x \in Z$, $((A_n)_x)_{n\in S}$ is a K-a.e. infinite-fold cover of Y.

Note that for any fixed $x \in X$, the $\{x\}$-uniform $\mathcal{J}$-covering property coincides with the $\mathcal{J}$-covering property. Next observe that, if $|Z| < \mathsf{add}^*(\mathcal{J})$ then the $\mathcal{J}$-covering property of $(\mathrm{Borel}(Y), K)$ implies its Z-uniform $\mathcal{J}$-covering property. Indeed, let $(A_n)_{n\in\omega}$ be an infinite-fold Borel cover of $Z \times Y$. For each $x \in Z$ pick an $S_x \in \mathcal{J}$ such that $((A_n)_x))_{n\in S_x}$ is a K-a.e. infinite-fold cover of Y. Since $|Z| < \mathsf{add}^*(\mathcal{J})$ we can find an $S \in \mathcal{J}$ such that $S_x \subseteq^* S$ for all $x \in Z$. Then for all $x \in Z$, $((A_n)_x)_{n\in S}$ is a K-a.e. infinite-fold cover of Y.

Below, we will keep all assumptions about $\mathcal{J}$, K and I.

Proposition 5.3. *Let* $(\mathrm{Borel}(X \times Y), I \otimes K)$ *have the $\mathcal{J}$-covering property. Then*

(i) $(\mathrm{Borel}(X), I)$ *has the $\mathcal{J}$-covering property,*
(ii) $(\mathrm{Borel}(Y), K)$ *has the Z-uniform $\mathcal{J}$-covering property for some* $Z \in I^*$.

Proof. (i) Let $(B_n)_{n\in\omega}$ be an infinite-fold Borel cover of X. Then $(B_n \times Y)_{n\in\omega}$ is an infinite-fold Borel cover of $X \times Y$. Pick $S \in \mathcal{J}$ such that $(B_n \times Y)_{n\in S}$ is an $I \otimes K$-a.e. infinite-fold cover of $X \times Y$. Then $(B_n)_{n\in S}$ is an I-a.e. infinite-fold cover of X.

(ii) If $(A_n)_{n\in\omega}$ is an infinite-fold Borel cover of $X \times Y$ then by the assumption there is an $S \in \mathcal{J}$ such that $(A_n)_{n\in S}$ is an $I \otimes K$-a.e. infinite-fold cover of $X \times Y$. Then there is a set $Z \in I^*$ such that $((A_n)_x)_{n\in S}$ is a K-a.e. infinite-fold cover of $Z \times Y$. □

By Proposition 5.3(i) and Example 2.5 we obtain

Corollary 5.4. *Assume* $I = [\omega^\omega]^{\le\omega}$, NWD, *or* $\mathcal{K}_\sigma$. *Then* $(\mathrm{Borel}(\omega^\omega \times Y), I \otimes K)$ *does not have the $\mathcal{J}$-covering property for any ideal $\mathcal{J}$ on ω and any ideal K on Y.*

Corollary 5.5.

(a) *If* $\mathcal{ED}_{\text{fin}} \leq_{\text{KB}} \mathcal{J}$ *then* $(\text{Borel}(Y), \mathcal{M}(Y))$ *has the* C*-uniform* $\mathcal{J}$*-covering property for some* $C \in (\mathcal{M}(X))^*$ *(where* X *and* Y *are arbitrary Polish spaces).*

(b) *If* $\mathcal{Z} \leq_{\text{KB}} \mathcal{J}$ *then* $(\text{Borel}(\mathbb{R}), \mathcal{N})$ *has the* C*-uniform* $\mathcal{J}$*-covering property for some* $C \in \mathcal{N}^*$*.*

Proof. (a): By Fact 2.6 we may assume that $\mathcal{J} = \mathcal{ED}_{\text{fin}}$. From Theorem 3.1 it follows that $(\text{Borel}(X \times Y), \mathcal{M}(X \times Y))$ has the $\mathcal{J}$-covering property. Since the ideals $\mathcal{M}(X \times Y)$ and $\mathcal{M}(X) \otimes \mathcal{M}(Y)$ intersected with $\text{Borel}(X \times Y)$ are the same, the asssertion is a consequence of Proposition 5.3(ii). The proof of (b) is analogous. $\square$

Theorem 5.6. *The pair* $(\text{Borel}(\mathbb{R}^2), \mathcal{N} \otimes \mathcal{M})$ *has the* $\mathcal{J}$*-covering property for each ideal* $\mathcal{J}$ *on* ω *with* $\mathcal{Z} \leq_{\text{KB}} \mathcal{J}$*. Consequently,* $(\text{Borel}(\mathbb{R}), \mathcal{M})$ *has the* C*-uniform* $\mathcal{J}$*-covering property for some* $C \in \mathcal{N}^*$*.*

Proof. By Fact 2.6 we may assume that $\mathcal{J} = \mathcal{Z}$. Let $(A_n)_{n\in\omega}$ be an $\mathcal{N} \otimes \mathcal{M}$-a.e. infinite-fold Borel cover of $\mathbb{R}^2$. By [1, Prop. 4], for each Borel set G in $\mathbb{R}^2$ there is a Borel set H with open sections such that $G \triangle H \in \mathcal{N} \otimes \mathcal{M}$. So, we may assume that all sections $(A_n)_x$ for $n \in \omega$ and $x \in \mathbb{R}$ are open. There exists a Borel set $B \in \mathcal{N}^*$ such that $\limsup_{n\in\omega}(A_n)_x$ is residual for all $x \in B$. Fix a base $\{U_k : k \in \omega\}$ of open sets in $\mathbb{R}$. For all $n, k \in \omega$ define

$$D_{nk} := \{x \in \mathbb{R} : (A_n)_x \cap U_k \neq \emptyset\}.$$

Since $(A_n)_x \cap U_k \neq \emptyset$ iff $(A_n)_x \cap U_k \notin \mathcal{M}$, the sets D_{nk} are Borel (see [7, 22.22]). Observe that for all $k \in \omega$ we have $B \subseteq \limsup_{n\in\omega} D_{nk}$ since for each $x \in B$ there are infinitely many A_n's such that $(\{x\} \times U_k) \cap A_n \neq \emptyset$. By the Elekes theorem, for each $k \in \omega$, pick an $S_k \in \mathcal{Z}$ and a Borel set $B_k \in \mathcal{N}^*$ such that $B_k \subseteq \limsup_{n\in S_k} D_{nk}$. Since $\mathcal{Z}$ is a P-ideal, we can pick an $S \in \mathcal{Z}$ such that $S_k \subseteq^* S$ for all $k \in \omega$. Put $C := \bigcap_{k\in\omega} B_k$. Then $C \in \mathcal{N}^*$ and $C \subseteq \bigcap_{k\in\omega} \limsup_{n\in S} D_{nk}$. Fix an $x \in C$. Then for all $k \in \omega$ and infinitely many indices $n \in S$, we have $(A_n)_x \cap U_k \neq \emptyset$. It follows that $\limsup_{n\in S}(A_n)_x$ is a residual G_δ set for all $x \in C$. Hence $(A_n)_{n\in S}$ is an $\mathcal{N} \otimes \mathcal{M}$-a.e. infinite-fold cover of $\mathbb{R}^2$. This yields the first part of the assertion. The rest is clear (cf. Proposition 5.3 (ii)). $\square$

Question 5.7. Is the analogous result true for $\mathcal{M} \otimes \mathcal{N}$?

6. $\mathcal{J}$-covering properties of $(\mathcal{P}(\omega), \mathcal{I})$

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω where $\mathcal{J}$ is tall. We will say that $\mathcal{I}$ *has the* $\mathcal{J}$*-covering property* whenever $(\mathcal{P}(\omega), \mathcal{I})$ has the $\mathcal{J}$-covering property.

It is trivial that $\mathsf{non}^*(\mathcal{J}) > \omega$ iff $\mathcal{J}$ is *ω-hitting*, that is, for every sequence $(X_n)_{n\in\omega}$ in $[\omega]^\omega$, there is an $A \in \mathcal{J}$ such that $|X_n \cap A| = \omega$ for each $n \in \omega$. We will use a weaker version of this property: An ideal $\mathcal{J}$ on ω is *weakly ω-hitting* if for each sequence $(X_n)_{n\in\omega}$ in $[\omega]^\omega$ there is an $A \in \mathcal{J}$ such that $\{n \in \omega : |X_n \cap A| = \omega\}$ is infinite.

This property is really weaker than $\mathsf{non}^*(\mathcal{J}) > \omega$ by Lemma 6.2(4). Moreover, it is easy to see the following characterization:

Proposition 6.1. *$\mathcal{J}$ is weakly ω-hitting if, and only if $\mathcal{J} \not\leq_{\mathrm{KB}} \mathrm{Fin} \otimes \mathrm{Fin}$.*

By the following easy result, in the characterization of "$\mathcal{I}$ has the $\mathcal{J}$-covering property" the interesting case is when $\mathcal{J}$ is weakly ω-hitting but not ω-hitting.

Lemma 6.2. *Assume $\mathcal{J}$ is a tall ideal on ω. Then*

1. *$\mathcal{J}$ is ω-hitting iff all ideals have the $\mathcal{J}$-covering property.*
2. *If $\mathcal{J}$ is not weakly ω-hitting, then there is no ideal with the $\mathcal{J}$-covering property.*
3. *If $\mathcal{J}$ is weakly hitting and all tall ideals have the $\mathcal{J}$-covering property, then $\mathcal{J}$ is ω-hitting.*
4. *If $\mathcal{J}$ is weakly ω-hitting but not ω-hitting, then there is a tall ideal $\mathcal{J}_0$ such that (up to isomorphism) $\mathcal{J}$ is contained in $\mathcal{J}_0 \otimes \mathrm{Fin}$. And all ideals of this form are weakly ω-hitting but not ω-hitting.*

Proof. (1): It is trivial by Fact 2.2 that if $\mathcal{J}$ is ω-hitting, then all ideals have the $\mathcal{J}$-covering property. Conversely, clearly Fin has the $\mathcal{J}$-covering property iff $\mathcal{J}$ is ω-hitting.

(2): Let $(X_n)_{n\in\omega}$ witness that $\mathcal{J}$ is not weakly ω-hitting and let $F(n) = X_n$. Then $\{n \in \omega : |F(n) \cap A| = \omega\}$ is finite for each $A \in \mathcal{J}$, so by Fact 2.2, F shows that $\mathcal{I}$ does not have the $\mathcal{J}$-covering property for all $\mathcal{I}$ (because finite sets cannot be in $\mathcal{I}^*$).

(3): Assume on the contrary that $\mathcal{J}$ is not ω-hitting witnessed by the sequence $(X_n)_{n\in\omega}$. For each $A \in \mathcal{J}$ let $E_A = \{n \in \omega : |X_n \cap A| = \omega\}$. Clearly, E_A is co-infinite and $E_{A\cup B} = E_A \cup E_B$ for $A, B \in \mathcal{J}$, so these sets generate an ideal $\mathcal{I}$ on ω. Moreover, it is trivial to see that $\mathcal{I} = \{E_A : A \in \mathcal{J}\}$. $\mathcal{I}$ is tall because of the weak ω-hitting property of $\mathcal{J}$. If $F(n) = X_n$ then $\{n \in \omega : |F(n) \cap A| = \omega\} \in \mathcal{I}$ for each $A \in \mathcal{J}$, so $\mathcal{I}$ does not have the $\mathcal{J}$-covering property, a contradiction.

(4): Let $\mathcal{J}_0 = \mathcal{I}$ from the proof of (3). (Of course, we can assume that X_n's are pairwise disjoint and $\bigcup_{n\in\omega} X_n = \omega$.)

Assume $\mathcal{J}_0$ is a tall ideal. The columns in $\omega\times\omega$ show that $\mathcal{J}_0\otimes\mathrm{Fin}$ is not ω-hitting. To prove the weak ω-hitting property, fix a sequence $X_n \in [\omega\times\omega]^\omega$ $(n\in\omega)$. We can assume the followings:

(i) if there is a k such that $|X_n \cap (\{k\}\times\omega)| = \omega$, then $X_n \subseteq \{k\}\times\omega$;

(ii) if $n \neq m$ and $X_n \cap (\{k\}\times\omega)$ and $X_m \cap (\{k\}\times\omega)$ are finite for all $k\in\omega$ then $\{k\in\omega : (X_n)_k \neq \emptyset\} \cap \{k\in\omega : (X_m)_k \neq \emptyset\} = \emptyset$.

If $B = \{n\in\omega : \exists\, k\in\omega\ X_n \subseteq \{k\}\times\omega\}$ is finite, then let $A = \bigcup_{n\in\omega\setminus B} X_n \in \mathcal{J}_0\otimes\mathrm{Fin}$. If $|B| = \omega$ then let $B' \subseteq B$, $|B'| = \omega$, $B' \in \mathcal{J}_0$, and let $A = B'\times\omega \in \mathcal{J}_0\otimes\mathrm{Fin}$. Clearly, the set $\{n\in\omega : |X_n \cap A| = \omega\}$ is infinite. □

In particular, if $\mathcal{J}$ is weakly ω-hitting but not ω-hitting, then there is a tall ideal $\mathcal{I}$ which does not have the $\mathcal{J}$-covering property, so it is natural to ask the following: Does there exist an ideal $\mathcal{I}$ with the $\mathcal{J}$-covering property in this case?

In the next theorem we characterize ideals with the $\mathcal{J}_0\otimes\mathrm{Fin}$-covering property. First we recall an important notion: Assume $\mathcal{J}$ is an ideal on ω. Then a filter $\mathcal{F}$ is a $\mathcal{J}$-*(ultra)filter* if for each function $f:\omega\to\omega$ there is an $X\in\mathcal{F}$ such that $f[X]\in\mathcal{J}$ (or equivalently, there is an $A\in\mathcal{J}$ such that $f^{-1}[A]\in\mathcal{F}$). For combinatorial properties of $\mathcal{J}$-filters and investigation of their existence see for example [6] and J. Flašková's other publications.

Theorem 6.3. *Let $\mathcal{J}_0$ be a tall ideal. Then an ideal $\mathcal{I}$ has the $\mathcal{J}_0\otimes\mathrm{Fin}$-covering property iff $\mathcal{I}^*$ is a $\mathcal{J}_0$-filter.*

Proof. Assume that $\mathcal{I}$ has the $\mathcal{J}_0\otimes\mathrm{Fin}$-covering property and let $f:\omega\to\omega$ be arbitrary. Let $F(n) = \{f(n)\}\times\omega$. Then there is an $A\in\mathcal{J}_0\otimes\mathrm{Fin}$ such that $X = \{n\in\omega : |F(n)\cap A| = \omega\}\in\mathcal{I}^*$. We can assume that A is of the form $A_0\times\omega$ for some $A_0\in\mathcal{J}_0$. Clearly $f^{-1}[A_0] = X\in\mathcal{I}^*$, so $f[X]\in\mathcal{J}_0$.

Conversely, assume that $\mathcal{I}^*$ is a $\mathcal{J}_0$-filter and let $F:\omega\to[\omega\times\omega]^\omega$ be arbitrary. We can assume the followings:

(i) if there is a k such that $|F(n)\cap(\{k\}\times\omega)| = \omega$, then $F(n) = \{k\}\times\omega$;

(ii) if $n\neq m$ and $F(n)\cap(\{k\}\times\omega)$ and $F(m)\cap(\{k\}\times\omega)$ are finite for all $k\in\omega$ then $\{k\in\omega : (F(n))_k\neq\emptyset\}\cap\{k\in\omega : (F(m))_k\neq\emptyset\} = \emptyset$.

Let $X = \{n\in\omega : \exists\, k_n\in\omega\ F(n) = \{k_n\}\times\omega\}$. If $X\in\mathcal{I}$ then we do not have to deal with it. If $X\in\mathcal{I}^+$ then let $f:X\to\omega$, $f(n) = k_n$. Clearly $(\mathcal{I}\restriction X)^*$ is also a $\mathcal{J}_0$-filter, so there is an $A_0\in\mathcal{J}_0$ such that $X\setminus f^{-1}[A_0]\in\mathcal{I}$.

It is easy to see that

$$A = \big(A_0\times\omega\big)\cup\bigcup_{n\in\omega\setminus X} F(n)\in\mathcal{J}_0\otimes\mathrm{Fin}$$

and $\{n\in\omega : |F(n)\cap A| = \omega\}\in\mathcal{I}^*$. □

Question 6.4. Is there any similarly easy and reasonable characterization of ideals with the $\mathcal{J}$-covering property for an arbitrary (weakly ω-hitting but not ω-hitting) $\mathcal{J}$?

References

[1] M. Balcerzak, S. Głąb: *Measure-category properties of Borel plane sets and Borel functions of two variables*, Acta Math. Hung. **126** (2009), pages 241–252.

[2] A. Bartoszewicz, M. Filipczak, T. Natkaniec: *On Smital properties*, Topology Appl. **158** (2011), pages 2066–2075.

[3] J. Brendle and S. Yatabe: *Forcing indestructibility of MAD families*, Ann. Pure Appl. Logic **132** (2005), no. 2-3, pages 271–312.

[4] M. Elekes: *A covering theorem and the random-indesctructibility of the density zero ideal*, to appear in Real Anal. Exchange, available at http://www.renyi.hu/ emarci/

[5] M. Hrušák, J. Zapletal: *Forcing with qoutients*, Arch. Math. Logic **47** (2008), pages 719–739.

[6] J. Flašková: *Ultrafilters and small sets*, PhD thesis, Charles University, Czech Republic, 2006.

[7] A.S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.

[8] D. Meza-Alcántara: *Ideals and filters on countable sets*, PhD thesis, Universidad Nacional Autónoma México, México, 2009.

[9] S. Solecki: *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), pages 51–72.

[10] J. Zapletal: *Forcing idealized*, Cambridge Tracts in Mathematics **174**, Cambridge University Press, Cambridge, 2008.

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