

MORE ON CARDINAL INVARIANTS OF ANALYTIC P-IDEALS

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ABSTRACT. Given an ideal \mathcal{I} on ω let $\mathfrak{a}(\mathcal{I})$ ($\bar{\mathfrak{a}}(\mathcal{I})$) be minimum of the cardinalities of infinite (uncountable) maximal \mathcal{I} -almost disjoint subsets of $[\omega]^\omega$. We show that $\mathfrak{a}(\mathcal{I}_h) > \omega$ if \mathcal{I}_h is a summable ideal; but $\mathfrak{a}(\mathcal{Z}_{\bar{\mu}}) = \omega$ for any tall density ideal $\mathcal{Z}_{\bar{\mu}}$ including the density zero ideal \mathcal{Z} . On the other hand, you have $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$ for any analytic P -ideal \mathcal{I} , and $\bar{\mathfrak{a}}(\mathcal{Z}_{\bar{\mu}}) \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\bar{\mu}}$.

For each ideal \mathcal{I} on ω denote $\mathfrak{b}_{\mathcal{I}}$ and $\mathfrak{d}_{\mathcal{I}}$ the unbounding and dominating numbers of $\langle \omega^\omega, \leq_{\mathcal{I}} \rangle$ where $f \leq_{\mathcal{I}} g$ iff $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$. We show that $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$ for each analytic P -ideal \mathcal{I} .

Given a Borel ideal \mathcal{I} on ω we say that a poset \mathbb{P} is \mathcal{I} -*bounding* iff $\forall x \in \mathcal{I} \cap V^{\mathbb{P}} \exists y \in \mathcal{I} \cap V x \subseteq y$. \mathbb{P} is \mathcal{I} -*dominating* iff $\exists y \in \mathcal{I} \cap V^{\mathbb{P}} \forall x \in \mathcal{I} \cap V x \subseteq^* y$.

For each analytic P -ideal \mathcal{I} if a poset \mathbb{P} has the Sacks property then \mathbb{P} is \mathcal{I} -bounding; moreover if \mathcal{I} is tall as well then the property \mathcal{I} -bounding/ \mathcal{I} -dominating implies ω^ω -bounding/adding dominating reals, and the converses of these two implications are false.

Using David Fremlin's results we prove that if \mathbb{P} adds a slalom capturing all ground model reals then \mathbb{P} is \mathcal{I} -dominating for each analytic P -ideal \mathcal{I} , and that a poset \mathbb{P} is \mathcal{Z} -bounding iff it has the Sacks property.

1. INTRODUCTION

In this paper we investigate some properties of some cardinal invariants associated with analytic P -ideals. Moreover we analyze related “bounding” and “dominating” properties of forcing notions.

Let us denote \mathfrak{fin} the Frechet ideal on ω , i.e. $\mathfrak{fin} = [\omega]^{<\omega}$. Further we always assume that if \mathcal{I} is an ideal on ω then the ideal is *proper*, i.e. $\omega \notin \mathcal{I}$, and $\mathfrak{fin} \subseteq \mathcal{I}$, so especially \mathcal{I} is *non-principal*. Write $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{I}^* = \{\omega \setminus X : X \in \mathcal{I}\}$.

An ideal \mathcal{I} on ω is *analytic* if $\mathcal{I} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$ is an analytic set in the usual product topology. \mathcal{I} is a *P-ideal* if for each countable $\mathcal{C} \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that $Y \subseteq^* X$ for each $Y \in \mathcal{C}$, where $A \subseteq^* B$ iff $A \setminus B$ is finite. \mathcal{I} is *tall* (or *dense*) if each infinite subset of ω contains an infinite element of \mathcal{I} .

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A function $\varphi : \mathcal{P}(\omega) \rightarrow [0, \infty]$ is a *submeasure on ω* iff $\varphi(X) \leq \varphi(Y)$ for $X \subseteq Y \subseteq \omega$, $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for $X, Y \subseteq \omega$, and $\varphi(\{n\}) < \infty$ for $n \in \omega$. A submeasure φ is *lower semicontinuous* iff $\varphi(X) = \lim_{n \rightarrow \infty} \varphi(X \cap n)$ for each $X \subseteq \omega$. A submeasure φ is *finite* if $\varphi(\omega) < \infty$. Note that if φ is a lower semicontinuous submeasure on ω then $\varphi(\bigcup_{n \in \omega} A_n) \leq \sum_{n \in \omega} \varphi(A_n)$ holds as well for $A_n \subseteq \omega$. We assign the *exhaustive ideal* $\text{Exh}(\varphi)$ to a submeasure φ as follows

$$\text{Exh}(\varphi) = \{X \subseteq \omega : \lim_{n \rightarrow \infty} \varphi(X \setminus n) = 0\}.$$

Solecki, [So, Theorem 3.1], proved that an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an analytic P -ideal or $\mathcal{I} = \text{Exh}(\varphi)$ for some lower semicontinuous finite submeasure. Therefore each analytic P -ideal is $F_{\sigma\delta}$ (i.e. Π_3^0) so a Borel subset of 2^ω . It is straightforward to see that if φ is a lower semicontinuous finite submeasure on ω then the ideal $\text{Exh}(\varphi)$ is tall iff $\lim_{n \rightarrow \infty} \varphi(\{n\}) = 0$.

Let \mathcal{I} be an ideal on ω . A family $\mathcal{A} \subseteq \mathcal{I}^+$ is \mathcal{I} -almost-disjoint (\mathcal{I} -AD in short), if $A \cap B \in \mathcal{I}$ for each $\{A, B\} \in [\mathcal{A}]^2$. An \mathcal{I} -AD family \mathcal{A} is an \mathcal{I} -MAD family if for each $X \in \mathcal{I}^+$ there exists an $A \in \mathcal{A}$ such that $X \cap A \in \mathcal{I}^+$, i.e. \mathcal{A} is \subseteq -maximal among the \mathcal{I} -AD families.

Denote $\mathfrak{a}(\mathcal{I})$ the minimum of the cardinalities of infinite \mathcal{I} -MAD families. In Theorem 2.2 we show that $\mathfrak{a}(\mathcal{I}_h) > \omega$ if \mathcal{I}_h is a summable ideal; but $\mathfrak{a}(\mathcal{Z}_{\bar{\mu}}) = \omega$ for any tall density ideal $\mathcal{Z}_{\bar{\mu}}$ including the *density zero ideal*

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

On the other hand, if you define $\bar{\mathfrak{a}}(\mathcal{I})$ as minimum of the cardinalities of uncountable \mathcal{I} -MAD families then you have $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$ for any analytic P -ideal \mathcal{I} , and $\bar{\mathfrak{a}}(\mathcal{Z}_{\bar{\mu}}) \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\bar{\mu}}$ (see Theorems 2.6 and 2.8).

In Theorem 3.1 we prove under CH the existence of an uncountable Cohen-indestructible \mathcal{I} -MAD families for each analytic P -ideal \mathcal{I} .

A sequence $\langle A_\alpha : \alpha < \kappa \rangle \subset [\omega]^\omega$ is a *tower* if it is \subseteq^* -descending, i.e. $A_\beta \subseteq^* A_\alpha$ if $\alpha \leq \beta < \kappa$, and it has no *pseudointersection*, i.e. a set $X \in [\omega]^\omega$ such that $X \subseteq^* A_\alpha$ for each $\alpha < \kappa$. In Section 4 we show it is consistent that the continuum is arbitrarily large and for each tall analytic P -ideal \mathcal{I} there is towers of height ω_1 whose elements are in \mathcal{I}^* .

Given an ideal \mathcal{I} on ω if $f, g \in \omega^\omega$ write $f \leq_{\mathcal{I}} g$ if $\{n \in \omega : f(n) > g(n)\} \in \mathcal{I}$. As usual let $\leq^* = \leq_{\text{fin}}$. The unbounding and dominating numbers of the partially ordered set $\langle \omega^\omega, \leq_{\mathcal{I}} \rangle$, denoted by $\mathfrak{b}_{\mathcal{I}}$ and $\mathfrak{d}_{\mathcal{I}}$ are defined in the natural way, i.e. $\mathfrak{b}_{\mathcal{I}}$ is the minimal size of a $\leq_{\mathcal{I}}$ -unbounded family, and $\mathfrak{d}_{\mathcal{I}}$ is the minimal size of a $\leq_{\mathcal{I}}$ -dominating family. By these notations $\mathfrak{b} = \mathfrak{b}_{\text{fin}}$ and $\mathfrak{d} = \mathfrak{d}_{\text{fin}}$. In Section 5 we show that $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$ for each analytic P -ideal \mathcal{I} . We also prove, in Corollary 6.8, that for any analytic P -ideal \mathcal{I} a poset \mathbb{P} is $\leq_{\mathcal{I}}$ -bounding iff it is ω^ω -bounding, and \mathbb{P} adds $\leq_{\mathcal{I}}$ -dominating reals iff it adds dominating reals.

In Section 6 we introduce the \mathcal{I} -bounding and \mathcal{I} -dominating properties of forcing notions for Borel ideals: \mathbb{P} is \mathcal{I} -*bounding* iff any element of $\mathcal{I} \cap V^{\mathbb{P}}$ is contained in some element of $\mathcal{I} \cap V$; \mathbb{P} is \mathcal{I} -*dominating* iff there is an element in $\mathcal{I} \cap V^{\mathbb{P}}$ which mod-finite contains all elements of $\mathcal{I} \cap V$.

In Theorem 6.2 we show that for each tall analytic P-ideal \mathcal{I} if a forcing notion is \mathcal{I} -bounding then it is ω^ω -bounding, and if it is \mathcal{I} -dominating then it adds dominating reals. Since the random real forcing is not \mathcal{I} -bounding for each tall summable and tall density ideal \mathcal{I} by Proposition 6.3, the converse of the first implication is false. Since a σ -centered forcing can not be \mathcal{I} -dominating for a tall analytic P-ideal \mathcal{I} by Theorem 6.4, the standard dominating real forcing \mathbb{D} witnesses that the converse of the second implication is also false.

We prove in Theorem 6.5 that the Sacks property implies the \mathcal{I} -bounding property for each analytic P-ideal \mathcal{I} .

Finally, based on a theorem of Fremlin we show that the \mathcal{Z} -bounding property is equivalent to the Sacks property.

2. AROUND THE ALMOST DISJOINTNESS NUMBER OF IDEALS

For any ideal \mathcal{I} on ω denote $\mathfrak{a}(\mathcal{I})$ the minimum of the cardinalities of infinite \mathcal{I} -MAD families.

To start the investigation of this cardinal invariant we recall the definition of two special classes of analytic P-ideals: the density ideals and the summable ideals (see [Fa]).

Definition 2.1. Let $h : \omega \rightarrow \mathbb{R}^+$ be a function such that $\sum_{n \in \omega} h(n) = \infty$. The *summable ideal corresponding to h* is

$$\mathcal{I}_h = \left\{ A \subseteq \omega : \sum_{n \in A} h(n) < \infty \right\}.$$

Let $\langle P_n : n < \omega \rangle$ be a decomposition of ω into pairwise disjoint nonempty finite sets and let $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$ be a sequences of probability measures, $\mu_n : \mathcal{P}(P_n) \rightarrow [0, 1]$. The *density ideal generated by $\vec{\mu}$* is

$$\mathcal{Z}_{\vec{\mu}} = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \mu_n(A \cap P_n) = 0 \right\}.$$

A summable ideal \mathcal{I}_h is tall iff $\lim_{n \rightarrow \infty} h(n) = 0$; and a density ideal $\mathcal{Z}_{\vec{\mu}}$ is tall iff

$$(\dagger) \quad \lim_{n \rightarrow \infty} \max_{i \in P_n} \mu_n(\{i\}) = 0.$$

Clearly the density zero ideal \mathcal{Z} is a tall density ideal, and the summable and the density ideals are proper ideals.

Theorem 2.2. (1) $\mathfrak{a}(\mathcal{I}_h) > \omega$ for any summable ideal \mathcal{I}_h .

(2) $\mathfrak{a}(\mathcal{Z}_{\vec{\mu}}) = \omega$ for any tall density ideal $\mathcal{Z}_{\vec{\mu}}$.

Proof. (1): We show that if $\{A_n : n < \omega\} \subseteq \mathcal{I}_h^+$ is \mathcal{I} -AD then there is $B \in \mathcal{I}_h^+$ such that $B \cap A_n \in \mathcal{I}$ for $n \in \omega$.

For each $n \in \omega$ let $B_n \subseteq A_n \setminus \cup\{A_m : m < n\}$ be finite such that $\sum_{i \in B_n} h(i) > 1$, and put

$$B = \cup\{B_n : n \in \omega\}.$$

(2): Write $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$ and μ_n concentrates on P_n . By (\dagger) we have $\lim_{n \rightarrow \infty} |P_n| = \infty$.

Now for each n we can choose $k_n \in \omega$ and a partition $\{P_{n,k} : k < k_n\}$ of P_n such that

- (a) $\lim_{n \rightarrow \infty} k_n = \infty$,
- (b) if $k < k_n$ then $\mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$.

Put $A_k = \cup\{P_{n,k} : k < k_n\}$ for each $k \in \omega$. We show that $\{A_k : k \in \omega\}$ is a $\mathcal{Z}_{\vec{\mu}}$ -MAD family.

If $k_n > k$ then $\mu_n(A_k \cap P_n) = \mu_n(P_{n,k}) \geq \frac{1}{2^{k+1}}$. Since for an arbitrary k for all but finitely many n we have $k_n > k$ it follows that

$$\limsup_{n \rightarrow \infty} \mu_n(A_k \cap P_n) = \limsup_{n \rightarrow \infty} \mu_n(P_{n,k}) \geq \limsup_{n \rightarrow \infty} \frac{1}{2^{k+1}} = \frac{1}{2^{k+1}} > 0,$$

thus $A_k \in \mathcal{Z}_{\vec{\mu}}^+$.

Assume that $X \in \mathcal{Z}_{\vec{\mu}}^+$. Pick $\varepsilon > 0$ with $\limsup_{n \rightarrow \infty} \mu_n(X \cap P_n) > \varepsilon$. For a large enough k we have $\frac{1}{2^{k+1}} < \frac{\varepsilon}{2}$ so if $k < k_n$ then

$$\mu_n(P_n \setminus \cup\{P_{n,i} : i \leq k\}) \leq \frac{1}{2^{k+1}} < \frac{\varepsilon}{2}.$$

So for each large enough n there is $i_n \leq k$ such that $\mu_n(X \cap P_{n,i_n}) > \frac{\varepsilon}{2^{(k+1)}}$. Then $i_n = i$ for infinitely many n , so $\limsup_{n \rightarrow \infty} \mu_n(X \cap A_i) \geq \frac{\varepsilon}{2^{(k+1)}}$, and so $X \cap A_i \in \mathcal{Z}_{\vec{\mu}}^+$. \square

This Theorem gives new proof of the following well-known fact:

Corollary 2.3. *The density zero ideal \mathcal{Z} is not a summable ideal.*

Given two ideals \mathcal{I} and \mathcal{J} on ω write $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$ (see [Ru]) iff there is a function $f : \omega \rightarrow \omega$ such that

$$\mathcal{I} = \{I \subseteq \omega : f^{-1}I \in \mathcal{J}\},$$

and write $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$ (see [LaZh]) iff there is a finite-to-one function $f : \omega \rightarrow \omega$ such that

$$\mathcal{I} = \{I \subseteq \omega : f^{-1}I \in \mathcal{J}\}.$$

The following Observations imply that there are \mathcal{I} -MAD families of cardinality \mathfrak{c} for each analytic P-ideal \mathcal{I} .

Observation 2.4. *Assume that \mathcal{I} and \mathcal{J} are ideals on ω , $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$ witnessed by a function $f : \omega \rightarrow \omega$. If \mathcal{A} is an \mathcal{I} -AD family then $\{f^{-1}A : A \in \mathcal{A}\}$ is a \mathcal{J} -AD family.*

Observation 2.5. *$\text{fin} \leq_{\text{RB}} \mathcal{I}$ for any analytic P-ideal \mathcal{I} .*

Proof. Let $\mathcal{I} = \text{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ on ω . Since $\omega \notin \mathcal{I}$ we have $\lim_{n \rightarrow \infty} \varphi(\omega \setminus n) = \varepsilon > 0$. Hence by the lower semicontinuous property of φ for each $n > 0$ there is $m > n$ such that $\varphi([n, m]) > \varepsilon/2$.

So there is a partition $\{I_n : n < \omega\}$ of ω into finite pieces such that $\varphi(I_n) > \varepsilon/2$ for each $n \in \omega$. Define the function $f : \omega \rightarrow \omega$ by the stipulation $f''I_n = \{n\}$. Then f witnesses $\text{fin} \leq_{\text{RB}} \mathcal{I}$. \square

For any analytic P-ideal \mathcal{I} denote $\bar{\mathfrak{a}}(\mathcal{I})$ the minimum of the cardinalities of uncountable \mathcal{I} -MAD families.

Clearly $\mathfrak{a}(\mathcal{I}) > \omega$ implies $\mathfrak{a}(\mathcal{I}) = \bar{\mathfrak{a}}(\mathcal{I})$, especially $\mathfrak{a}(\mathcal{I}_h) = \bar{\mathfrak{a}}(\mathcal{I}_h)$ for summable ideals.

Theorem 2.6. $\bar{\mathfrak{a}}(\mathcal{Z}_{\vec{\mu}}) \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\vec{\mu}}$.

Proof. Let $f : \omega \rightarrow \omega$ be the finite-to-one function defined by $f^{-1}\{n\} = P_n$ where $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$ and $\mu_n : \mathcal{P}(P_n) \rightarrow [0, 1]$. Specially f witnesses $\text{fin} \leq_{\text{RB}} \mathcal{Z}_{\vec{\mu}}$.

Let \mathcal{A} be an uncountable (fin-)MAD family. We show that $f^{-1}[\mathcal{A}] = \{f^{-1}A : A \in \mathcal{A}\}$ is a $\mathcal{Z}_{\vec{\mu}}$ -MAD family.

By Observation 2.4, $f^{-1}[\mathcal{A}]$ is a $\mathcal{Z}_{\vec{\mu}}$ -AD family.

To show the maximality let $X \in \mathcal{Z}_{\vec{\mu}}^+$ be arbitrary, $\limsup_{n \rightarrow \infty} \mu_n(X \cap P_n) = \varepsilon > 0$. Thus

$$J = \{n \in \omega : \mu_n(X \cap P_n) > \varepsilon/2\}$$

is infinite. So there is $A \in \mathcal{A}$ such that $A \cap J$ is infinite.

Then $f^{-1}A \in f^{-1}[\mathcal{A}]$ and $X \cap f^{-1}A \in \mathcal{Z}_{\vec{\mu}}^+$ because there are infinitely many n such that we have $P_n \subseteq f^{-1}A$ and $\mu_n(X \cap P_n) > \varepsilon/2$. \square

Problem 2.7. Does $\bar{\mathfrak{a}}(\mathcal{I}) \leq \mathfrak{a}$ hold for each analytic P-ideal \mathcal{I} ?

Theorem 2.8. $\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$ provided that \mathcal{I} is an analytic P-ideal.

Remark. If $\mathcal{X} \subset [\omega]^\omega$ is an infinite almost disjoint family then there is a tall ideal \mathcal{I} such that \mathcal{X} is \mathcal{I} -MAD. So the Theorem above does not hold for an arbitrary tall ideal on ω .

Proof. $\mathcal{I} = \text{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ .

Let \mathcal{A} be an uncountable \mathcal{I} -AD family of cardinality smaller than \mathfrak{b} . We show that \mathcal{A} is not maximal.

There exists an $\varepsilon > 0$ such that the set

$$\mathcal{A}_\varepsilon = \{A \in \mathcal{A} : \lim_{n \rightarrow \infty} \varphi(A \setminus n) > \varepsilon\}$$

is uncountable. Let $\mathcal{A}' = \{A_n : n \in \omega\} \subseteq \mathcal{A}_\varepsilon$ be a set of pairwise distinct elements of \mathcal{A}_ε . We can assume that these sets are pairwise disjoint. For each $A \in \mathcal{A} \setminus \mathcal{A}'$ choose a function $f_A \in \omega^\omega$ such that

$$(*_A) \varphi((A \cap A_n) \setminus f_A(n)) < 2^{-n} \text{ for each } n \in \omega.$$

Using the assumption $|\mathcal{A}| < \mathfrak{b}$ there exists a strictly increasing function $f \in \omega^\omega$ such that $f_A \leq^* f$ for each $A \in \mathcal{A} \setminus \mathcal{A}'$. For each n pick $g(n) > f(n)$ such that $\varphi(A_n \cap [f(n), g(n))) > \varepsilon$, and let

$$X = \bigcup_{n \in \omega} (A_n \cap [f(n), g(n))).$$

Clearly $X \in \mathcal{Z}_{\bar{\mu}}^+$ because for each $n < \omega$ there is m such that $A_m \cap [f(m), g(m)) \subseteq X \setminus n$ and so $\varphi(X \setminus n) \geq \varphi(A_m \cap [f(m), g(m))) > \varepsilon$, i.e. $\lim_{n \rightarrow \infty} \varphi(X \setminus n) \geq \varepsilon$.

We have to show that $X \cap A \in \mathcal{Z}_{\bar{\mu}}$ for each $A \in \mathcal{A}$. If $A = A_n$ for some n then $X \cap A = X \cap A_n = A_n \cap [f(n), g(n))$, i.e. the intersection is finite.

Assume now that $A \in \mathcal{A} \setminus \mathcal{A}'$. Let $\delta > 0$. We show that if k is large enough then $\varphi((X \cap A) \setminus k) < \delta$.

There is $N \in \omega$ such that $2^{-N+1} < \delta$ and $f_A(n) \leq f(n)$ for each $n \geq N$.

Let k be so large that k contains the finite set $\bigcup_{n < N} [f(n), g(n))$.

Now $(X \cap A) \setminus k = \bigcup_{n \in \omega} (A_n \cap A \cap [f(n), g(n))) \setminus k$ and $(A_n \cap A \cap [f(n), g(n))) \setminus k = \emptyset$ if $n < N$ so

$$\begin{aligned} (X \cap A) \setminus k &= \bigcup_{n \geq N} (A_n \cap A \cap [f(n), g(n))) \setminus k \subseteq \\ &\bigcup_{n \geq N} ((A_n \cap A) \setminus f(n)) \subseteq \bigcup_{n \geq N} ((A_n \cap A) \setminus f_A(n)). \end{aligned}$$

Thus by $(*_A)$ we have

$$\varphi((X \cap A) \setminus k) \leq \sum_{n \geq N} \varphi(A_n \cap A \setminus f_A(n)) \leq \sum_{n \geq N} \frac{1}{2^n} = 2^{-N+1} < \delta.$$

□

3. COHEN-INDESTRUCTIBLE \mathcal{I} -MAD FAMILIES

If φ is a lower semicontinuous finite submeasure on ω then clearly φ is determined by $\varphi \upharpoonright [\omega]^{<\omega}$. Using this observation one can define forcing indestructibility of \mathcal{I} -MAD families for an analytic P-ideal \mathcal{I} . The following Theorem is a modification of Kunen's proof for existence of Cohen-indestructible MAD family from CH (see [Ku] Ch. VIII Th. 2.3.).

Theorem 3.1. *Assume CH. For each analytic P-ideal \mathcal{I} then there is an uncountable Cohen-indestructible \mathcal{I} -MAD family.*

Proof. We will define the uncountable Cohen-indestructible \mathcal{I} -MAD family $\{A_\xi : \xi < \omega_1\} \subseteq \mathcal{I}^+$ by recursion on $\xi \in \omega_1$. The family $\{A_\xi : \xi < \omega_1\}$ will be fin-AD as well. Our main concern is that we do have $\mathfrak{a}(\mathcal{I}) > \omega$ so it is not automatic that $\{A_\eta : \eta < \xi\}$ is not maximal for $\xi < \omega_1$.

Denote \mathbb{C} the Cohen forcing. Let $\mathcal{I} = \text{Exh}(\varphi)$ be an analytic P-ideal. Let $\{\langle p_\xi, \dot{X}_\xi, \delta_\xi \rangle : \omega \leq \xi < \omega_1\}$ be an enumeration of all triples $\langle p, \dot{X}, \delta \rangle$ such

that $p \in \mathbb{C}$, \dot{X} is a nice name for a subset of ω , and δ is a positive rational number.

Write $\varepsilon = \lim_{n \rightarrow \infty} \varphi(\omega \setminus n) > 0$. Partition ω into infinite sets $\{A_m : m < \omega\}$ such that $\lim_{n \rightarrow \infty} \varphi(A_m \setminus n) = \varepsilon$ for each $m < \omega$.

Assume $\xi \geq \omega$ and we have $A_\eta \in \mathcal{I}^+$ for $\eta < \xi$ such that $\{A_\eta : \eta < \xi\}$ is a fin-AD so especially an \mathcal{I} -AD family.

Claim: There is $X \in \mathcal{I}^+$ such that $|X \cap A_\zeta| < \omega$ for $\zeta < \xi$.

Proof of the Claim. Write $\xi = \{\zeta_i : i < \omega\}$. Recursion on $j \in \omega$ we can choose $x_j \in [A_{\ell_j}]^{<\omega}$ for some $\ell_j \in \omega$ such that

- (i) $\varphi(x_j) \geq \varepsilon/2$,
- (ii) $x_j \cap (\cup_{i \leq j} A_{\zeta_i}) = \emptyset$.

Assume that $\{x_i : i < j\}$ is chosen. Pick $\ell_j \in \omega \setminus \{\zeta_i : i < j\}$. Let $m \in \omega$ such that $A_{\ell_j} \cap \cup\{A_{\zeta_i} : i \leq j\} \subseteq m$. Since $\varphi(A_{\ell_j} \setminus m) \geq \varepsilon$ there is $x_j \in [A_{\ell_j} \setminus m]^{<\omega}$ with $\varphi(x_j) \geq \varepsilon/2$.

Let $X = \cup\{x_j : j < \omega\}$. Then $|A_\zeta \cap X| < \omega$ for $\zeta < \xi$ and $\lim_{n \rightarrow \infty} \varphi(X \setminus n) \geq \varepsilon/2$. \square

If p_ξ does not force (a) and (b) below then let A_ξ be X from the Claim.

- (a) $\lim_{n \rightarrow \infty} \check{\varphi}(\dot{X}_\xi \setminus n) > \check{\delta}_\xi$,
- (b) $\forall \eta < \xi \ \dot{X}_\xi \cap \check{A}_\eta \in \mathcal{I}$.

Assume $p_\xi \Vdash (a) \wedge (b)$. Let $\{B_k^\xi : k \in \omega\} = \{A_\eta : \eta < \xi\}$ and $\{p_k^\xi : k \in \omega\} = \{p' \in \mathbb{C} : p' \leq p_\xi\}$ be enumerations. Clearly for each $k \in \omega$ we have

$$p_k^\xi \Vdash \lim_{n \rightarrow \infty} \check{\varphi}((\dot{X}_\xi \setminus \cup\{\check{B}_l^\xi : l \leq \check{k}\}) \setminus n) > \check{\delta}_\xi,$$

so we can choose a $q_k^\xi \leq p_k^\xi$ and a finite $a_k^\xi \subseteq \omega$ such that $\varphi(a_k^\xi) > \delta_\xi$ and $q_k^\xi \Vdash \check{a}_k^\xi \subseteq (\dot{X}_\xi \setminus \cup\{\check{B}_l^\xi : l \leq \check{k}\}) \setminus \check{k}$. Let $A_\xi = \cup\{a_k^\xi : k \in \omega\}$. Clearly $A_\xi \in \mathcal{I}^+$ and $\{A_\eta : \eta \leq \xi\}$ is a fin-AD family.

Thus $\mathcal{A} = \{A_\xi : \xi < \omega_1\} \subseteq \mathcal{I}^+$ is a fin-AD family.

We show that \mathcal{A} is a Cohen-indestructible \mathcal{I} -MAD. Assume otherwise there is a ξ such that $p_\xi \Vdash \lim_{n \rightarrow \infty} \check{\varphi}(\dot{X}_\xi \setminus n) > \check{\delta}_\xi \wedge \forall \eta < \omega_1 \ \dot{X}_\xi \cap \check{A}_\eta \in \mathcal{I}$, specially $p_\xi \Vdash (a) \wedge (b)$. There is a $p_k^\xi \leq p_\xi$ and an N such that $p_k^\xi \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{N}) < \check{\delta}_\xi$. We can assume $k \geq N$, so $p_k^\xi \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{k}) < \check{\delta}_\xi$. By the choice of q_k^ξ and a_k^ξ we have $q_k^\xi \Vdash \check{a}_k^\xi \subseteq (\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{k}$, so $q_k^\xi \Vdash \check{\varphi}((\dot{X}_\xi \cap \check{A}_\xi) \setminus \check{k}) > \check{\delta}_\xi$, contradiction. \square

4. TOWERS IN \mathcal{I}^*

Let \mathcal{I} be an ideal on ω . A \subseteq^* -decreasing sequence $\langle A_\alpha : \alpha < \kappa \rangle$ is a *tower* in \mathcal{I}^* if (a) it is a tower (i.e. there is no $X \in [\omega]^\omega$ with $X \subseteq^* A_\alpha$ for $\alpha < \kappa$), and (b) $A_\alpha \in \mathcal{I}^*$ for $\alpha < \kappa$. Under CH it is straightforward to construct towers in \mathcal{I}^* for each tall analytic P-ideal \mathcal{I} . The existence of such towers is consistent with $2^\omega > \omega_1$ as well by the Theorem 4.2 below. Denote \mathbb{C}_α the standard forcing adding α Cohen reals by finite conditions.

Lemma 4.1. *Let $\mathcal{I} = \text{Exh}(\varphi)$ be a tall analytic P-ideal in the ground model V . Then there is a set $X \in V^{\mathbb{C}_1} \cap \mathcal{I}$ such that $|X \cap S| = \omega$ for each $S \in [\omega]^\omega \cap V$.*

Proof. Since \mathcal{I} is tall we have $\lim_{n \rightarrow \infty} \varphi(\{n\}) = 0$. Fix a partition $\langle I_n : n \in \omega \rangle$ of ω into finite intervals such that $\varphi(\{x\}) < \frac{1}{2^n}$ for $x \in I_{n+1}$ (we can not say anything about $\varphi(\{x\})$ for $x \in I_0$). Then $X' \in \mathcal{I}$ whenever $|X' \cap I_n| \leq 1$ for each n .

Let $\{i_k^n : k < k_n\}$ be the increasing enumeration of I_n . Our forcing \mathbb{C} adds a Cohen real $c \in \omega^\omega$ over V . Let

$$X_\alpha = \{i_k^n : c(n) \equiv k \pmod{k_n}\} \in V^{\mathbb{C}} \cap \mathcal{I}.$$

A trivial density argument shows that $|X_\alpha \cap S| = \omega$ for each $S \in V \cap [\omega]^\omega$. \square

Theorem 4.2. $\Vdash_{\mathbb{C}_{\omega_1}}$ "There exists a tower in \mathcal{I}^* for each tall analytic P-ideal \mathcal{I} ."

Proof. Let V be a countable transitive model and G be a \mathbb{C}_{ω_1} -generic filter over V . Let $\mathcal{I} = \text{Exh}(\varphi)$ be a tall analytic P-ideal in $V[G]$ with some lower semicontinuous finite submeasure φ on ω . There is a $\delta < \omega_1$ such that $\varphi \upharpoonright [\omega]^{<\omega} \in V[G_\delta]$ where $G_\delta = G \cap \mathbb{C}_\delta$, so we can assume $\varphi \upharpoonright [\omega]^{<\omega} \in V$.

Work in $V[G]$ recursion on ω_1 we construct the tower $\bar{A} = \langle A_\alpha : \alpha < \omega_1 \rangle$ in \mathcal{I}^* such that $\bar{A} \upharpoonright \alpha \in V[G_\alpha]$.

Because \mathcal{I} contains infinite elements we can construct in V a sequence $\langle A_n : n \in \omega \rangle$ in \mathcal{I}^* which is strictly \subseteq^* -descending, i.e. $|A_n \setminus A_{n+1}| = \omega$ for $n \in \omega$. Assume $\langle A_\xi : \xi < \alpha \rangle$ are done.

Since \mathcal{I} is a P-ideal there is $A'_\alpha \in \mathcal{I}^*$ with $A'_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$.

By lemma 4.1 there is a set $X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}$ such that $X_\alpha \cap S \neq \emptyset$ for each $S \in [\omega]^\omega \cap V[G_\alpha]$.

Let $A_\alpha = A'_\alpha \setminus X_\alpha \in V[G_{\alpha+1}] \cap \mathcal{I}^*$ so $S \not\subseteq^* A_\alpha$ for any $S \in V[G_\alpha] \cap [\omega]^\omega$. Hence $V[G] \models \langle A_\alpha : \alpha < \omega_1 \rangle$ is a tower in \mathcal{I}^* . \square

Problem 4.3. Do there exist towers in \mathcal{I}^* for some tall analytic P-ideal \mathcal{I} in ZFC?

5. UNBOUNDING AND DOMINATING NUMBERS OF IDEALS

A *supported relation* (see [Vo]) is a triple $\mathcal{R} = (A, R, B)$ where $R \subseteq A \times B$, $\text{dom}(R) = A$, $\text{ran}(R) = B$, and we always assume that for each $b \in B$ there is an $a \in A$ such that $\langle a, b \rangle \notin R$.

The *unbounding and dominating numbers of \mathcal{R}* :

$$\mathfrak{b}(\mathcal{R}) = \min\{|A'| : A' \subseteq A \wedge \forall b \in B \ A' \not\subseteq R^{-1}\{b\}\},$$

$$\mathfrak{d}(\mathcal{R}) = \min\{|B'| : B' \subseteq B \wedge A = R^{-1}B'\}.$$

For example $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$. Note that $\mathfrak{b}(\mathcal{R})$ and $\mathfrak{d}(\mathcal{R})$ are defined for each \mathcal{R} , but in general $\mathfrak{b}(\mathcal{R}) \leq \mathfrak{d}(\mathcal{R})$ does not hold.

We recall the definition of Galois-Tukey connection of relations.

Definition 5.1. ([Vo]) Let $\mathcal{R}_1 = (A_1, R_1, B_1)$ and $\mathcal{R}_2 = (A_2, R_2, B_2)$ be supported relations. A pair of functions $\phi : A_1 \rightarrow A_2$, $\psi : B_2 \rightarrow B_1$ is a *Galois-Tukey connection from \mathcal{R}_1 to \mathcal{R}_2* , in notation $(\phi, \psi) : \mathcal{R}_1 \preceq \mathcal{R}_2$ if $a_1 R_1 \psi(b_2)$ whenever $\phi(a_1) R_2 b_2$. In a diagram:

$$\begin{array}{ccc} \psi(b_2) \in B_1 & \xleftarrow{\psi} & B_2 \ni b_2 \\ R_1 & \longleftarrow & R_2 \\ a_1 \in A_1 & \xrightarrow{\phi} & A_2 \ni \phi(a_1) \end{array}$$

We write $\mathcal{R}_1 \preceq \mathcal{R}_2$ if there is a Galois-Tukey connection from \mathcal{R}_1 to \mathcal{R}_2 . If $\mathcal{R}_1 \preceq \mathcal{R}_2$ and $\mathcal{R}_2 \preceq \mathcal{R}_1$ also hold then we say \mathcal{R}_1 and \mathcal{R}_2 are *Galois-Tukey equivalent*, in notation $\mathcal{R}_1 \equiv \mathcal{R}_2$.

Fact 5.2. *If $\mathcal{R}_1 \preceq \mathcal{R}_2$ then $\mathfrak{b}(\mathcal{R}_1) \geq \mathfrak{b}(\mathcal{R}_2)$ and $\mathfrak{d}(\mathcal{R}_1) \leq \mathfrak{d}(\mathcal{R}_2)$.*

Theorem 5.3. *If $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$ then $(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \equiv (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)$.*

Proof. Fix a finite-to-one function $f : \omega \rightarrow \omega$ witnessing $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$.

Define $\phi, \psi : \omega^\omega \rightarrow \omega^\omega$ as follows:

$$\begin{aligned} \phi(x)(i) &= \max(x'' f^{-1}\{i\}), \\ \psi(y)(j) &= y(f(j)). \end{aligned}$$

We prove two claims.

Claim 5.3.1. $(\phi, \psi) : (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$.

Proof of the claim. We show that if $\phi(x) \leq_{\mathcal{I}} y$ then $x \leq_{\mathcal{J}} \psi(y)$. Indeed, $I = \{i : \phi(x)(i) > y(i)\} \in \mathcal{I}$. Assume that $f(j) = i \notin I$. Then $\phi(x)(i) = \max(x'' f^{-1}\{i\}) \leq y(i)$. Since $y(i) = \psi(y)(j)$, so

$$x(j) \leq \max(x'' f^{-1}\{f(j)\}) \leq y(f(j)) = \psi(y)(j)$$

Since $f^{-1}I \in \mathcal{J}$ this yields $x \leq_{\mathcal{J}} \psi(y)$. \square

Claim 5.3.2. $(\psi, \phi) : (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \preceq (\omega^\omega, \leq_{\mathcal{J}}, \omega^\omega)$.

Proof of the claim. We show that if $\psi(y) \leq_{\mathcal{J}} x$ then $y \leq_{\mathcal{I}} \phi(x)$. Assume on the contrary that $y \not\leq_{\mathcal{I}} \phi(x)$. Then $A = \{i \in \omega : y(i) > \phi(x)(i)\} \in \mathcal{I}^+$. By definition of ϕ , we have $A = \{i : y(i) > \max(x'' f^{-1}\{i\})\}$.

Let $B = f^{-1}A \in \mathcal{J}^+$. For $j \in B$ we have $f(j) \in A$ and so

$$\psi(y)(j) = y(f(j)) > \phi(x)(f(j)) = \max(x'' f^{-1}\{f(j)\}) \geq x(j).$$

Hence $\psi(y) \not\leq_{\mathcal{J}} x$, contradiction. \square

These claims prove the statement of the Theorem, so we are done. \square

By Fact 5.2 we have:

Corollary 5.4. *If $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$ holds then $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}_{\mathcal{J}}$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}_{\mathcal{J}}$.*

By Observation 2.5 this yields:

Corollary 5.5. *If \mathcal{I} is an analytic P -ideal then $(\omega^\omega, \leq^*, \omega^\omega) \equiv (\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega)$, and $\mathfrak{b}_{\mathcal{I}} = \mathfrak{b}$ and $\mathfrak{d}_{\mathcal{I}} = \mathfrak{d}$.*

6. \mathcal{I} -BOUNDING AND \mathcal{I} -DOMINATING FORCING NOTIONS

Definition 6.1. Let \mathcal{I} be a Borel ideal on ω . A forcing notion \mathbb{P} is \mathcal{I} -*bounding* if

$$\Vdash_{\mathbb{P}} \forall A \in \mathcal{I} \exists B \in \mathcal{I} \cap V \ A \subseteq B;$$

\mathbb{P} is \mathcal{I} -*dominating* if

$$\Vdash_{\mathbb{P}} \exists B \in \mathcal{I} \forall A \in \mathcal{I} \cap V \ A \subseteq^* B.$$

Theorem 6.2. *Let \mathcal{I} be a tall analytic P -ideal. If \mathbb{P} is \mathcal{I} -bounding then \mathbb{P} is ω^ω -bounding as well; if \mathbb{P} is \mathcal{I} -dominating then \mathbb{P} adds dominating reals.*

Proof. Assume that $\mathcal{I} = \text{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ . For $A \in \mathcal{I}$ let

$$d_A(n) = \min\{k \in \omega : \varphi(A \setminus k) < 2^{-n}\}.$$

Clearly if $A \subseteq B \in \mathcal{I}$ then $d_A \leq d_B$.

It is enough to show that $\{d_A : A \in \mathcal{I}\}$ is cofinal in $\langle \omega^\omega, \leq^* \rangle$. Let $f \in \omega^\omega$. Since \mathcal{I} is a tall ideal we have $\lim_{k \rightarrow \infty} \varphi(\{k\}) = 0$ but $\lim_{m \rightarrow \infty} (\omega \setminus m) = \varepsilon > 0$. Thus for all but finite $n \in \omega$ we can choose a finite set $A_n \subseteq \omega \setminus f(n)$ such that $2^{-n} \leq \varphi(A_n) < 2^{-n+1}$ so $A = \cup\{A_n : n \in \omega\} \in \mathcal{I}$ and $f \leq^* d_A$.

Why? We can assume if $k \geq f(n)$ then $\varphi(\{k\}) < 2^{-n}$. Let n be so large such that $2^{-n} < \varepsilon$. Now if there is no a suitable A_n then $\varphi(\omega \setminus f(n)) \leq 2^{-n} < \varepsilon$, contradiction. \square

The converse of the first implication of Theorem 6.2 is not true by the following Proposition.

Proposition 6.3. *The random forcing is not \mathcal{I} -bounding for any tall summable and tall density ideal \mathcal{I} .*

Proof. Denote \mathbb{B} the random forcing and λ the Lebesgue-measure.

If $\mathcal{I} = \mathcal{I}_h$ is a tall summable ideal then we can chose pairwise disjoint sets $H(n) \in [\omega]^\omega$ such that $\sum_{l \in H(n)} h(l) = 1$ and $\max\{h(l) : l \in H(n)\} < 2^{-n}$ for each $n \in \omega$. Let $H(n) = \{l_k^n : k \in \omega\}$. For each n fix a partition $\{[B_k^n] : k \in \omega\}$ of \mathbb{B} such that $\lambda(B_k^n) = h(l_k^n)$ for each $k \in \omega$. Let \dot{X} be a \mathbb{B} -name such that $\Vdash_{\mathbb{B}} \dot{X} = \{\check{l}_k^n : [B_k^n] \in \dot{G}\}$. Clearly $\Vdash_{\mathbb{B}} \dot{X} \in \mathcal{I}_h$. \dot{X} shows that \mathbb{B} is not \mathcal{I}_h -bounding.

Assume on the contrary that there is a $[B] \in \mathbb{B}$ and an $A \in \mathcal{I}_h$ such that $[B] \Vdash \dot{X} \subseteq \check{A}$. There is an $n \in \omega$ such that

$$\sum_{l_k^n \in A} \lambda(B_k^n) = \sum_{l_k^n \in A} h(l_k^n) < \lambda(B).$$

Choose a k such that $l_k^n \notin A$ and $[B_k^n] \wedge [B] \neq [\emptyset]$. We have $[B_k^n] \wedge [B] \Vdash \check{l}_k^n \in \dot{X} \setminus \check{A}$, contradiction.

If $\mathcal{I} = \mathcal{Z}_{\bar{\mu}}$ is a tall density ideal then for each n fix a partition $\{[B_k^n] : k \in P_n\}$ of \mathbb{B} such that $\lambda(B_k^n) = \mu_n(\{k\})$ for each k . Let \dot{X} be a \mathbb{B} -name such that $\Vdash_{\mathbb{B}} \dot{X} = \{\check{k} : [\check{B}_k^n] \in \dot{G}\}$. Clearly $\Vdash_{\mathbb{B}} \dot{X} \in \mathcal{Z}_{\bar{\mu}}$. \dot{X} shows that \mathbb{B} is not $\mathcal{Z}_{\bar{\mu}}$ -bounding.

Assume on the contrary that there is a $[B] \in \mathbb{B}$ and an $A \in \mathcal{Z}_{\bar{\mu}}$ such that $[B] \Vdash \dot{X} \subseteq \dot{A}$. There is an $n \in \omega$ such that

$$\sum_{k \in A \cap P_n} \lambda(B_k^n) = \mu_n(A \cap P_n) < \lambda(B).$$

Choose a $k \in P_n \setminus A$ such that $[B_k^n] \wedge [B] \neq [\emptyset]$. We have $[B_k^n] \wedge [B] \Vdash \check{k} \in \dot{X} \setminus \dot{A}$, contradiction. \square

The converse of the second implication of Theorem 6.2 is not true as well: the Hechler forcing is a counterexample according to the following Theorem.

Theorem 6.4. *If \mathbb{P} is σ -centered then \mathbb{P} is not \mathcal{I} -dominating for any tall analytic P-ideal \mathcal{I} .*

Proof. Assume that $\mathcal{I} = \text{Exh}(\varphi)$ for some lower semicontinuous finite submeasure φ . Let $\varepsilon = \lim_{n \rightarrow \infty} \varphi(\omega \setminus n) > 0$.

Let $\mathbb{P} = \cup\{C_n : n \in \omega\}$ where C_n is centered for each n . Assume on the contrary that $\Vdash_{\mathbb{P}} \dot{X} \in \mathcal{I} \wedge \forall A \in \mathcal{I} \cap V \ A \subseteq^* \dot{X}$ for some \mathbb{P} -name \dot{X} .

For each $A \in \mathcal{I}$ choose a $p_A \in \mathbb{P}$ and a $k_A \in \omega$ such that

$$(o) \quad p_A \Vdash \dot{A} \setminus \check{k}_A \subseteq \dot{X} \wedge \varphi(\dot{X} \setminus \check{k}_A) < \varepsilon/2.$$

For each $n, k \in \omega$ let $\mathcal{C}_{n,k} = \{A \in \mathcal{I} : p_A \in C_n \wedge k_A = k\}$, and let $B_{n,k} = \bigcup \mathcal{C}_{n,k}$. We show that for each n and k

$$\varphi(B_{n,k} \setminus k) \leq \varepsilon/2.$$

Assume indirectly $\varphi(B_{n,k} \setminus k) > \varepsilon/2$ for some n and k . There is a k' such that $\varphi(B_{n,k} \cap [k, k']) > \varepsilon/2$ and there is a finite $\mathcal{D} \subseteq \mathcal{C}_{n,k}$ such that $B_{n,k} \cap [k, k'] = (\cup \mathcal{D}) \cap [k, k']$. Choose a common extension q of $\{p_A : A \in \mathcal{D}\}$. Now we have $q \Vdash \cup\{A \setminus \check{k} : A \in \dot{\mathcal{D}}\} \subseteq \dot{X}$ and so

$$q \Vdash \varepsilon/2 < \varphi(\check{B}_{n,k} \cap [\check{k}, \check{k}']) = \varphi((\cup \check{\mathcal{D}}) \cap [\check{k}, \check{k}']) \leq \varphi(\dot{X} \cap [\check{k}, \check{k}']) \leq \varphi(\dot{X} \setminus \check{k}),$$

which contradicts (o).

So for each n and k the set $\omega \setminus B_{n,k}$ is infinite, so $\omega \setminus B_{n,k}$ contains an infinite $D_{n,k} \in \mathcal{I}$. Let $D \in \mathcal{I}$ such that $D_{n,k} \subseteq^* D$ for each $n, k \in \omega$.

Then there is no n, k such that $D \subseteq^* B_{n,k}$. Contradiction. \square

By this Theorem and by Lemma 4.1 the Cohen forcing is neither \mathcal{I} -dominating nor \mathcal{I} -bounding for any tall analytic P-ideal \mathcal{I} .

Finally in the rest of the paper we compare the Sacks property and the \mathcal{I} -bounding property.

Theorem 6.5. *If \mathbb{P} has the Sacks property then \mathbb{P} is \mathcal{I} -bounding for each analytic P-ideal \mathcal{I} .*

Proof. Let $\mathcal{I} = \text{Exh}(\varphi)$. Assume $\Vdash_{\mathbb{P}} \dot{X} \in \mathcal{I}$. Let $d_{\dot{X}}$ be a \mathbb{P} -name for an element of ω^ω such that $\Vdash_{\mathbb{P}} d_{\dot{X}}(\check{n}) = \min\{k \in \omega : \varphi(\dot{X} \setminus k) < 2^{-\check{n}}\}$. We know that \mathbb{P} is ω^ω -bounding. If $p \Vdash d_{\dot{X}} \leq \check{f}$ for some strictly increasing $f \in \omega^\omega$ then by the Sacks property there is a $q \leq p$ and a slalom $S : \omega \rightarrow [[\omega]^{<\omega}]^{<\omega}$, $|S(n)| \leq n$ such that

$$q \Vdash \forall^\infty n \dot{X} \cap [f(n), f(n+1)) \in S(n).$$

Now let

$$A = \bigcup_{n \in \omega} \{D \in S(n) : \varphi(D) < 2^{-n}\}.$$

$A \in \mathcal{I}$ because $\varphi(A \setminus f(n)) \leq \sum_{k \geq n} \varphi(A \cap [f(k), f(k+1))) \leq \sum_{k \geq n} \frac{k}{2^k}$. Clearly $q \Vdash \dot{X} \subseteq^* \dot{A}$. \square

A supported relation $\mathcal{R} = (A, R, B)$ is called *Borel-relation* iff there is a Polish space X such that $A, B \subseteq X$ and $R \subseteq X^2$ are Borel sets. Similarly a Galois-Tukey connection $(\phi, \psi) : \mathcal{R}_1 \preceq \mathcal{R}_2$ between Borel-relations is called *Borel GT-connection* iff ϕ and ψ are Borel functions. To be Borel-relation and Borel GT-connection is absolute for transitive models containing all relevant codes.

Some important Borel-relation:

(A): $(\mathcal{I}, \subseteq, \mathcal{I})$ and $(\mathcal{I}, \subseteq^*, \mathcal{I})$ for a Borel ideal \mathcal{I} .

(B): Denote Slm the set of *slaloms on ω* , i.e. $S \in \text{Slm}$ iff $S : \omega \rightarrow [\omega]^{<\omega}$ and $|S(n)| = 2^n$ for each n . Let \sqsubseteq and \sqsubseteq^* be the following relations on $\omega^\omega \times \text{Slm}$:

$$f \sqsubseteq^{(*)} S \iff \forall^{(\infty)} n \in \omega f(n) \in S(n).$$

The supported relations $(\omega^\omega, \sqsubseteq, \text{Slm})$ and $(\omega^\omega, \sqsubseteq^*, \text{Slm})$ are Borel-relations.

(C): Denote ℓ_1^+ the set of positive summable series. Let \leq be the coordinate-wise and \leq^* the almost everywhere coordinate-wise ordering on ℓ_1^+ . $(\ell_1^+, \leq, \ell_1^+)$ and $(\ell_1^+, \leq^*, \ell_1^+)$ are Borel-relations.

Definition 6.6. Let $\mathcal{R} = (A, R, B)$ be a Borel-relation. A forcing notion \mathbb{P} is *\mathcal{R} -bounding* if

$$\Vdash_{\mathbb{P}} \forall a \in A \exists b \in B \cap V a R b;$$

\mathcal{R} -dominating if

$$\Vdash_{\mathbb{P}} \exists b \in B \forall a \in A \cap V a R b.$$

For example the property \mathcal{I} -bounding/dominating is the same as $(\mathcal{I}, \subseteq^*$, $\mathcal{I})$ -bounding/dominating.

We can reformulate some classical properties of forcing notions:

ω^ω -bounding	\equiv	$(\omega^\omega, \leq^{(*)}, \omega^\omega)$ -bounding
adding dominating reals	\equiv	$(\omega^\omega, \leq^*, \omega^\omega)$ -dominating
Sacks property	\equiv	$(\omega^\omega, \sqsubseteq^{(*)}, \text{Slm})$ -bounding
adding a slalom capturing all ground model reals	\equiv	$(\omega^\omega, \sqsubseteq^*, \text{Slm})$ -dominating

If $\mathcal{R} = (A, R, B)$ is a supported relation then let $\mathcal{R}^\perp = (B, \neg R^{-1}, A)$ where $b(\neg R^{-1})a$ iff not aRb . Clearly $(\mathcal{R}^\perp)^\perp = \mathcal{R}$ and $\mathfrak{b}(\mathcal{R}) = \mathfrak{d}(\mathcal{R}^\perp)$. Now if \mathcal{R} is a Borel-relation then \mathcal{R}^\perp is a Borel-relation too, and a forcing notion is \mathcal{R} -bounding iff it is not \mathcal{R}^\perp -dominating.

Fact 6.7. *Assume $\mathcal{R}_1 \preceq \mathcal{R}_2$ are Borel-relations with Borel GT-connection and \mathbb{P} is a forcing notion. If \mathbb{P} is \mathcal{R}_2 -bounding/dominating then \mathbb{P} is \mathcal{R}_1 -bounding/dominating.*

By Corollary 5.5 this yields

Corollary 6.8. *For each analytic P-ideal \mathcal{I} (1) a poset \mathbb{P} is $\leq_{\mathcal{I}}$ -bounding iff it is ω^ω -bounding, (2) forcing with a poset \mathbb{P} adds $\leq_{\mathcal{I}}$ -dominating reals iff this forcing adds dominating reals.*

Fremlin's two results have interesting corollaries for us.

Lemma 6.9. ([Fr] 524C) *Let (P, \leq) be a partially ordered set such that $\sup\{p, q\}$ is defined for all $p, q \in P$. Suppose d is a metric on P such that (P, d) is a polish space and $\sup\{\cdot, \cdot\} : P^2 \rightarrow P$ is uniformly continuous. Then $(P, \leq, P) \preceq (\ell_1^+, \leq, \ell_1^+)$ with a Borel GT-connection.*

Corollary 6.10. *There is a Borel GT-connection $(\mathcal{I}, \subseteq, \mathcal{I}) \preceq (\ell_1^+, \leq, \ell_1^+)$ for each analytic P-ideal \mathcal{I} .*

Proof. Let $\mathcal{I} = \text{Exh}(\varphi)$ where φ is a finite lower semicontinuous submeasure on ω with $\varphi(A) = 0$ iff $A = \emptyset$. One can show that (\mathcal{I}, \subseteq) with $d(A, B) = \varphi(A \Delta B)$ satisfy the requirements of Lemma 6.9. \square

Note that there is no any Galois-Tukey connection from $(\ell_1^+, \leq, \ell_1^+)$ to $(\mathcal{Z}, \subseteq, \mathcal{Z})$ so they are not GT-equivalent (see [LoVe] Th. 7.).

Theorem 6.11. ([Fr] 524I) *There is a Borel GT-connection $(\ell_1^+, \leq^*, \ell_1^+) \equiv (\omega^\omega, \sqsubseteq^*, \text{Sln})$.*

Corollary 6.12. *If \mathbb{P} adds a slalom capturing all ground model reals then \mathbb{P} is \mathcal{I} -dominating for each analytic P-ideal \mathcal{I} .*

Proof. By Fact 6.7 and Theorem 6.11 adding slalom is the same as $(\ell_1^+, \leq^*, \ell_1^+)$ -dominating. Let \dot{x} be a \mathbb{P} -name such that $\Vdash_{\mathbb{P}} \dot{x} \in \ell_1^+ \wedge \forall y \in \ell_1^+ \cap V$ $y \leq^* \dot{x}$. Moreover let \dot{X} be a \mathbb{P} -name such that $\Vdash_{\mathbb{P}} \dot{X} = \{z \in \ell_1^+ : |z \setminus \dot{x}| < \omega, \forall n (z(n) \neq \dot{x}(n) \Rightarrow z(n) \in \omega)\}$. Let $(\phi, \psi) : (\mathcal{I}, \subseteq, \mathcal{I}) \preceq (\ell_1^+, \leq, \ell_1^+)$ be a Borel GT-connection. Now if \dot{A} is a \mathbb{P} -name such that $\Vdash_{\mathbb{P}} \forall z \in \dot{X} \psi(z) \subseteq^* \dot{A}$ then \dot{A} shows that \mathbb{P} is \mathcal{I} -dominating. \square

Denote \mathbb{D} the dominating forcing and LOC the Localization forcing. Because of the complicated GT-connections it seems hard to construct an \mathcal{I} -dominating set in V^{LOC} . Despite LOC it is much more simple to show directly that the two-step iteration $\mathbb{D} * \text{LOC}$ is \mathcal{I} -dominating.

Observation 6.13. *If \mathcal{I} is an arbitrary analytic P-ideal then two step iteration $\mathbb{D} * \text{LOC}$ is \mathcal{I} -dominating.*

Indeed, let $\mathcal{I} \in V \subseteq M \subseteq N$ be transitive models, $d \in M \cap \omega^\omega$ be strictly increasing and dominating over V , and $S \in N$, $S : \omega \rightarrow [[\omega]^{<\omega}]^{<\omega}$, $|S(n)| \leq n$ a slalom which captures all reals from M . Now if

$$X_n = \cup\{A \in S(n) \cap \mathcal{P}([d(n), d(n+1)) : \varphi(A) < 2^{-n}\}$$

then it is easy to see that $Y \subseteq^* \cup\{X_n : n \in \omega\} \in \mathcal{I} \cap N$ for each $Y \in V \cap \mathcal{I}$.

Problem 6.14. Does \mathcal{Z} -dominating (or \mathcal{I} -dominating) imply adding slaloms?

We will use the following deep result of Fremlin to prove Theorem 6.16.

Theorem 6.15. ([Fr] 526G) *There is a family $\{P_f : f \in \omega^\omega\}$ of Borel subsets of ℓ_1^+ such that the following hold:*

- (i) $\ell_1^+ = \cup\{P_f : f \in \omega^\omega\}$,
- (ii) if $f \leq g$ then $P_f \subseteq P_g$,
- (iii) $(P_f, \leq, \ell_1^+) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z})$ with a Borel GT-connection for each f .

Theorem 6.16. \mathbb{P} is \mathcal{Z} -bounding iff \mathbb{P} has the Sacks property.

Proof. Let $\{P_f : f \in \omega^\omega\}$ be a family satisfying (i), (ii), and (iii) in Theorem 6.15, and fix Borel GT-connections $(\phi_f, \psi_f) : (P_f, \leq, \ell_1^+) \preceq (\mathcal{Z}, \subseteq, \mathcal{Z})$ for each $f \in \omega^\omega$. Assume \mathbb{P} is \mathcal{Z} -bounding and $\Vdash_{\mathbb{P}} \dot{x} \in \ell_1^+$. \mathbb{P} is ω^ω -bounding by Theorem 6.2 so using (ii) we have $\Vdash_{\mathbb{P}} \ell_1^+ = \cup\{P_f : f \in \omega^\omega \cap V\}$. We can choose a \mathbb{P} -name \dot{f} for an element of $\omega^\omega \cap V$ such that $\Vdash_{\mathbb{P}} \dot{x} \in P_{\dot{f}}$. By \mathcal{Z} -bounding property of \mathbb{P} there is a \mathbb{P} -name \dot{A} for an element of $\mathcal{Z} \cap V$ such that $\Vdash_{\mathbb{P}} \phi_{\dot{f}}(\dot{x}) \subseteq \dot{A}$, so $\Vdash_{\mathbb{P}} \dot{x} \leq \psi_{\dot{f}}(\dot{A}) \in \ell_1^+ \cap V$. So we have \mathbb{P} is $(\ell_1^+, \leq^{(*)}, \ell_1^+)$ -bounding. By Theorem 6.11 and Fact 6.7 \mathbb{P} has the Sacks property.

The converse implication was proved in Theorem 6.5. □

Problem 6.17. Does the \mathcal{I} -bounding property imply the Sacks property for each tall analytic P-ideal \mathcal{I} ?

REFERENCES

- [Fa] Farah, Ilijas: *Analytic quotients: theory of liftings for quotient over analytic ideals on the integers*, *Memoirs of the American Mathematical Society* **148** (2000), no 702, 177 pp.
- [Fr] Fremlin, David H.: *Measure Theory. Set-theoretic Measure Theory*. Torres Fremlin, Colchester, England, 2004. Available at <http://www.essex.ac.uk/maths/staff/fremlin/mt.html>
- [Ku] Kunen, Kenneth: *Set Theory, An Introduction to Independence Proofs*, North Holland, Amsterdam, New York, Oxford 1980
- [LaZh] Laffamme, Claude and Zhu, Jian-Ping: *The Rudin-Blass ordering of ultrafilters*, *J. Symbolic Logic* **63** (1998), no. 2, pages 584-592
- [LoVe] Louveau, A. and Veličković, B.: *Analytic ideals and cofinal types*, *Ann. Pure Appl. Logic* **99** (1999), pages 171-195
- [Ru] Rudin, M. E.: *Partial orders on the types of $\beta\omega$* , *Trans. Amer. Math. Soc.* **155** (1971), pages 353-362

- [So] Solecki, Sławomir: *Analytic P-ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), pages 51-72
- [Vo] Vojtáš, Peter: *Generalized Galois-Tukey-connections between explicit relations on classical objects of real analysis*, Set theory of the reals (Ramat Gan, 1991), pages 619-643, Israel Math. Conf. Proc., 6, Bar-Ilan Univ., Ramat Gan, 1993.

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