Combinatorics of analytic P-ideals and related forcing problems
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Introduction
The study of ideals on natural numbers (ω) and on the reals (ω^ω or 2^ω) has become a central topic of infinite combinatorics and forcing problems in the past few years. My research is focused on a nice but large enough class of ideals on ω.

An ideal I on ω containing the ideal of finite sets fin = {ω < ω} is analytic if I ⊆ P(ω) ⊇ 2^ω is an analytic set in the usual product topology of the Cantor-set. I is a P-ideal if for each countable C ⊆ ω there is an A ∈ C such that I ⊆ A for each I ∈ C, where A ⊆ B if A is finite. I is tall (or dense) if each infinite subset of ω contains an infinite element of I.

The density zero ideal and the summing ideal:
Z = \{A ⊆ ω : \lim_{n→ω} |A ∩ n|/n = 0\},
T_{fin} = \{A ⊆ ω : \lim_{n→ω} |A ∩ n|/n + 1 < 1\}
are classical tall analytic P-ideals.
A function ϕ : P(ω) → [0, ω] is a lower semicontinuous (lsc) submeasure on ω if
(i) ϕ(∅) = 0 and ϕ({n}) < ω for each n
(ii) ϕ(A) ≤ ϕ(B) for A ⊆ B ⊆ ω
(iii) ϕ(A ∪ B) ≤ ϕ(A) + ϕ(B) for A, B ⊆ ω
(iv) ϕ(A) = lim_{n→ω} ϕ(A ∩ n) for each A

Note that an lsc submeasure on ω is σ-subadditive as well (i.e., ϕ(∪_{n∈N} A_n) ≤ \sum_{n∈N} ϕ(A_n)). We assign two ideals to an lsc submeasure ϕ as follows
Exh(ϕ) = \{A ⊆ ω : \lim_{n→ω} ϕ(A ∩ n) = 0\},
Fin(ϕ) = \{A ⊆ ω : ϕ(A) < ω\}
Exh(ϕ) is a F_σ P-ideal, and Fin(ϕ) is an F_σ ideal.

The almost-disjointness number
We say that A, B ∈ P^n = P(ω)/I are I-almost-disjoint if A ∩ B ∈ I. The almost-disjointness number of an ideal I, a(I(ZP)) is the minimum of cardinals of infinite (uncountable) maximal I-almost disjoint subsets of Z^+, for example a = a(fin) = a(fin), ω = a(ω), and ω < a(T_{fin}). In general, ω < a(I) holds for each F_σ P-ideal I.

Theorem. ([3]) \leq a(I) for any analytic P-ideal I, and a(ZP) ≤ a for each density ideal ZP.

Problem. Does a(I) \leq a for each hold for a analytic P-ideal I?

Towers in the dual filters
A sequence \{A_n : 0 < n < ω\} ⊆ [ω]ω is a tower if it is \subseteq \sigma-descending, i.e. A_n \subseteq A_m for all 0 ≤ n < m, and it has no pseudointersection, i.e. a set X ∈ [ω]ω such that X \subseteq A_n for each n < ω.
Let F be a filter on ω. A tower \{A_n : 0 < n < ω\} is a tower in F if A_n ∈ F for each n < ω. Denote T = \{ω : A ∈ T\} as the dual filter of T.

Theorem. ([3]) After adding ω. Cohen-reals (to any model) there is a tower in T^ω with height ω. for each tall analytic P-ideal I.

Versions of Hechler's theorem
Theorem. (Hechler’s original theorem [5]) Let (Q, ≤) be a partial ordered set such that each countable subset of Q has a strict upper bound in Q. Then there is a ccc forcing notion P such that in V^P a cofinal subset of (ω^ω, ≤) is order isomorphic to (Q, ≤).

Let N be the ideal of measure zero subsets of the reals and M be the ideal of meager subsets of the reals.

Theorem. ([11], [2]) Let (Q, ≤) be as above. Then there is a ccc forcing notion P such that in V^P a cofinal subset of (M, ≤) (resp. (N, ≤)) is order isomorphic to (Q, ≤).

Theorem. (Farkas) Let (Q, ≤) be as above. Then there is a ccc forcing notion P such that in V^P a cofinal subset of (T, ≤) is order isomorphic to (Q, ≤) for each tall analytic P-ideal I from V.

T-bounding and I-dominating
A supported relation (or simply relation) is a triple R = (A, R, B) where R ⊆ A × B, dom(R) = A, ran(R) = B, and we always assume that for each b ∈ B there is an a ∈ A such that (a, b) \notin R. R is Borel if A, B ⊆ ω^ω and R ⊆ (ω^ω)^2 are Borel sets.

Let R_1 = (A_1, R_1, B_1) and R_2 = (A_2, R_2, B_2) be (Borel) supported relations. A pair of (Borel) functions φ : A_1 → A_2, ψ : B_2 → B_1 is a (Borel) Galois-Tyke connection from R_1 to R_2, (φ, ψ) : R_1 \succeq R_2 if (a_1, ψ(b_2)) ∈ R_1 whenever (φ(a_1), b_2) ∈ R_2.

Theorem. ([3], [4], [7]) (ω^ω, ≤, ω^ω) \succeq (ω^ω, ≤, ω^ω) \succeq (\mathcal{T}, ≤, \mathcal{T}) \succeq (\mathcal{T}, ≤, \mathcal{S}) \succeq (\mathcal{N}, ≤, \mathcal{N}) where f ≤ g if |n : f(n) > g(n)| ∈ I; as usual * stands for ≤_I; T is a tall analytic P-ideal (totalness is in the second connection); S = \{n ∈ ω : n ≤ n\} is the set of slaloms; and if f ∈ ω^ω and S in S then f ≤ S if \forall n \in S(n).

Let R = (A, R, B) be a Borel relation.
A forcing notion P is R-bounding if
Π ∃ a ∈ A ∃ V \mathcal{N} \mathcal{V} ∃ b ∈ B ∨ a ∈ A \in R,
and P is R-dominating if
Π ∃ b ∈ B ∨ a ∈ A ∃ V \mathcal{N} \mathcal{V} ∃ a ∈ A \in R.

Almost all classical bounding or dominating-like properties of forcing notions can be written in this form. Examples: A forcing notion P is \omega^ω-bounding (resp. adds dominating real) iff P is (\omega^ω, ≤, ω^ω)-bounding (resp. -dominating); P has the Sacks property iff P is (ω^ω, ≤, S)-bounding; and Π ∃ b \mathcal{N} \mathcal{V} ∃ a \mathcal{N} \mathcal{V} \mathcal{N} \mathcal{V} has measure zero iff P is (\mathcal{N}, ≤, \mathcal{N})-dominating etc.

The following observation shows that we can translate questions about implications between properties of forcing notions to combinatorial ones: Assume R_1 \succeq R_2 then if P is R_2-bounding / dominating then it is R_1-bounding / dominating as well.

Using the above mentioned Borel GT connections we obtain for example that the Sacks property (resp. adding a slalom capturing all ground model reals) implies the Z-bounding (resp. -dominating) property which implies the \omega^ω-bounding property (resp. adding dominating real) for each tall analytic P-ideal I.

The converse of the second implication is false (see the Random and resp. the Hechler forcing). The converse of the first implication is a more difficult question.

Theorem. ([3] using [4]) P is \mathcal{Z}-bounding iff P has the Sacks property.

Problem. Does the I-bounding (or I-dominating) property imply adding slalom capturing all ground model reals? Does the \mathcal{Z}-bounding (or I-dominating) property imply adding slalom capturing all ground model reals?

References

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