Combinatorics of analytic P-ideals and related forcing problems

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Introduction

The study of ideals on natural numbers (ω) and on the reals (ω^ω or 2^ω) has become a central topic of infinite combinatorics and forcing theory in the past few years. My research is focused on a nice but large enough class of ideals on ω.

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An ideal $\mathcal{I}$ on ω containing the ideal of finite sets $\mathrm{fin} = [\omega]^{<\omega}$ is *analytic* if $\mathcal{I} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$ is an analytic set in the usual product topology of the Cantor-set. $\mathcal{I}$ is a *P-ideal* if for each countable $\mathcal{C} \subseteq \mathcal{I}$ there is an $A \in \mathcal{I}$ such that $I \subseteq^* A$ for each $I \in \mathcal{C}$, where $A \subseteq^* B$ iff $A \setminus B$ is finite. $\mathcal{I}$ is *tall* (or *dense*) if each infinite subset of ω contains an infinite element of $\mathcal{I}$.

The *density zero ideal* and the *summable ideal*:

$$\mathcal{Z} = \Big\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\Big\},$$

$$\mathcal{I}_{1/n} = \Big\{A \subseteq \omega : \sum_{n \in A} \frac{1}{n+1} < \infty\Big\}$$

are classical tall analytic P-ideals.

A function $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ is a *lower semicontinuous (lsc) submeasure on* ω if

(i) $\varphi(\emptyset) = 0$ and $\varphi(\{n\}) < \infty$ for each n

(ii) $\varphi(A) \leq \varphi(B)$ for $A \subseteq B \subseteq \omega$

(iii) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for $A, B \subseteq \omega$

(iv) $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for each A

Note that an lsc submeasure on ω is σ-subadditive as well (i.e. $\varphi(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \varphi(A_n)$). We assign two ideals to an lsc submeasure φ as follows

$$\mathrm{Exh}(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\},$$

$$\mathrm{Fin}(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}.$$

$\mathrm{Exh}(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, and $\mathrm{Fin}(\varphi)$ is an F_σ ideal.

Theorem. ([6],[7]) *Let $\mathcal{I}$ be an ideal on ω.*

- *$\mathcal{I}$ is an F_σ ideal iff $\mathcal{I} = \mathrm{Fin}(\varphi)$ for some lsc submeasure φ.*
- *$\mathcal{I}$ is an analytic P-ideal iff $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lsc submeasure φ.*
- *$\mathcal{I}$ is an F_σ P-ideal iff $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ for some lsc submeasure φ.*

The almost-disjointness number

We say that $A, B \in \mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ are *$\mathcal{I}$-almost-disjoint* if $A \cap B \in \mathcal{I}$. The *almost-disjointness number* of an ideal $\mathcal{I}$, $\mathfrak{a}(\mathcal{I})$ ($\bar{\mathfrak{a}}(\mathcal{I})$) is the minimum of cardinalities of infinite (uncountable) maximal $\mathcal{I}$-almost disjoint subsets of $\mathcal{I}^+$. For example $\mathfrak{a} = \mathfrak{a}(\mathrm{fin})(= \bar{\mathfrak{a}}(\mathrm{fin}))$, $\omega = \mathfrak{a}(\mathcal{Z})$, and $\omega < \mathfrak{a}(\mathcal{I}_{1/n})$. In general, $\omega < \mathfrak{a}(\mathcal{I})$ holds for each F_σ P-ideal $\mathcal{I}$.

Theorem. ([3]) *$\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$ for any analytic P-ideal $\mathcal{I}$, and $\bar{\mathfrak{a}}(\mathcal{Z}_{\vec{\mu}}) \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\vec{\mu}}$.*

Problem. Does $\bar{\mathfrak{a}}(\mathcal{I}) \leq \mathfrak{a}$ hold for each analytic P-ideal $\mathcal{I}$?

Towers in the dual filters

A sequence $\{A_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ is a *tower* if it is $\subseteq^*$-descending, i.e. $A_\beta \subseteq^* A_\alpha$ if $\alpha \leq \beta < \kappa$, and it has no *pseudointersection*, i.e. a set $X \in [\omega]^\omega$ such that $X \subseteq^* A_\alpha$ for each $\alpha < \kappa$.

Let $\mathcal{F}$ be a filter on ω. A tower $\{A_\alpha : \alpha < \kappa\}$ is a *tower in* $\mathcal{F}$ if $A_\alpha \in \mathcal{F}$ for each $\alpha < \kappa$. Denote $\mathcal{I}^* = \{\omega \setminus A : A \in \mathcal{I}\}$ the dual filter of $\mathcal{I}$.

Theorem. ([3]) *After adding ω_1 Cohen-reals (to any model) there is a tower in $\mathcal{I}^*$ with height ω_1 for each tall analytic P-ideal $\mathcal{I}$.*

Theorem. (Brendle and Farkas) *It is consistent with* ZFC *that there is no tower in $\mathcal{I}^*$ for each tall analytic P-ideal $\mathcal{I}$.*

Versions of Hechler's theorem

Theorem. (Hechler's original theorem [5]) *Let $(Q, \leq)$ be a partial ordered set such that each countable subset of Q has a strict upper bound in Q. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ a cofinal subset of $(\omega^\omega, \leq^*)$ is order isomorphic to $(Q, \leq)$.*

Let $\mathcal{N}$ be the ideal of measure zero subsets of the reals and $\mathcal{M}$ be the ideal of meager subsets of the reals.

Theorem. ([1], [2]) *Let $(Q, \leq)$ be as above. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ a cofinal subset of $(\mathcal{M}, \subseteq)$ (resp. $(\mathcal{N}, \subseteq)$) is order isomorphic to $(Q, \leq)$.*

Theorem. (Farkas) *Let $(Q, \leq)$ be as above. Then there is a ccc forcing notion $\mathbb{P}$ such that in $V^{\mathbb{P}}$ a cofinal subset of $(\mathcal{I}, \subseteq^*)$ is order isomorphic to $(Q, \leq)$ for each tall analytic P-ideal $\mathcal{I}$ from V.*

$\mathcal{I}$-bounding and $\mathcal{I}$-dominating

A *supported relation* (or simply *relation*) is a triple $\mathcal{R} = (A, R, B)$ where $R \subseteq A \times B$, $\mathrm{dom}(R) = A$, $\mathrm{ran}(R) = B$, and we always assume that for each $b \in B$ there is an $a \in A$ such that $\langle a, b\rangle \notin R$. $\mathcal{R}$ is Borel if $A, B \subseteq \omega^\omega$ and $R \subseteq (\omega^\omega)^2$ are Borel sets.

Let $\mathcal{R}_1 = (A_1, R_1, B_1)$ and $\mathcal{R}_2 = (A_2, R_2, B_2)$ be (Borel) supported relations. A pair of (Borel) functions $\phi : A_1 \to A_2$, $\psi : B_2 \to B_1$ is a *(Borel) Galois-Tukey connection from $\mathcal{R}_1$ to $\mathcal{R}_2$*, $(\phi, \psi) : \mathcal{R}_1 \preceq^{(\mathrm{B})}_{\mathrm{GT}} \mathcal{R}_2$, if $\langle a_1, \psi(b_2)\rangle \in R_1$ whenever $\langle \phi(a_1), b_2\rangle \in R_2$.

$$\begin{array}{ccc} \psi(b_2) \in B_1 & \xleftarrow{\ \psi\ } & B_2 \ni b_2 \\ R_1 & \Longleftarrow & R_2 \\ a_1 \in A_1 & \xrightarrow{\ \phi\ } & A_2 \ni \phi(a_1) \end{array}$$

Theorem. ([3], [4], [7]) $(\omega^\omega, \leq_{\mathcal{I}}, \omega^\omega) \equiv^{\mathrm{B}}_{\mathrm{GT}} (\omega^\omega, \leq^*, \omega^\omega) \preceq^{\mathrm{B}}_{\mathrm{GT}} (\mathcal{I}, \subseteq^*, \mathcal{I}) \preceq^{\mathrm{B}}_{\mathrm{GT}} (\omega^\omega, \sqsubseteq^*, \mathcal{S}) \equiv^{\mathrm{B}}_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$

where $f \leq_{\mathcal{I}} g$ if $\{n : f(n) > g(n)\} \in \mathcal{I}$; as usual $\leq^*$ stands for $\leq_{\mathrm{fin}}$; $\mathcal{I}$ is a tall analytic P-ideal (tallness is used in the second connection); $\mathcal{S} = \mathsf{X}_{n\in\omega}[\omega]^{\leq n}$ is the set of slaloms; and if $f \in \omega^\omega$ and $S \in \mathcal{S}$ then $f \sqsubseteq^* S$ iff $\forall^\infty\, n$ $f(n) \in S(n)$.

Let $\mathcal{R} = (A, R, B)$ be a Borel relation. A forcing notion $\mathbb{P}$ is *$\mathcal{R}$-bounding* if

$$\Vdash_{\mathbb{P}} \forall\, a \in A \cap V[\dot{G}]\ \exists\, b \in B \cap V\ \langle a, b\rangle \in R,$$

$\mathbb{P}$ is *$\mathcal{R}$-dominating* if

$$\Vdash_{\mathbb{P}} \exists\, b \in B \cap V[\dot{G}]\ \forall\, a \in A \cap V\ \langle a, b\rangle \in R.$$

Almost all classical bounding or dominating-like properties of forcing notions can be written in this form. Examples: A forcing notion $\mathbb{P}$ is ω^ω-bounding (resp. adds dominating real) iff $\mathbb{P}$ is $(\omega^\omega, \leq^*, \omega^\omega)$-bounding (resp. -dominating); $\mathbb{P}$ has the Sacks property iff $\mathbb{P}$ is $(\omega^\omega, \sqsubseteq^*, \mathcal{S})$-bounding; and $\Vdash_{\mathbb{P}}$"$\bigcup(\mathcal{N} \cap V)$ has measure zero" iff $\mathbb{P}$ is $(\mathcal{N}, \subseteq, \mathcal{N})$-dominating etc.

The following observation shows that we can translate questions about implications between properties of forcing notions to combinatorial ones: Assume $\mathcal{R}_1 \preceq^{\mathrm{B}}_{\mathrm{GT}} \mathcal{R}_2$. If $\mathbb{P}$ is $\mathcal{R}_2$-bounding / dominating then it is $\mathcal{R}_1$-bounding / dominating as well.

Using the above mentioned Borel GT connections we obtain for example that the Sacks property (resp. adding a slalom capturing all ground model reals) implies the $\mathcal{I}$-bounding (resp. -dominating) property which implies the ω^ω-bounding property (resp. adding dominating real) for each tall analytic P-ideal $\mathcal{I}$.

The converse of the second implication is false (see the Random and resp. the Hechler forcing). The converse of the first implication is a more difficult question.

Theorem. ([3] using [4]) *$\mathbb{P}$ is $\mathcal{Z}$-bounding iff $\mathbb{P}$ has the Sacks property.*

Problem. Does the $\mathcal{I}$-bounding property imply the Sacks property for each tall analytic P-ideal $\mathcal{I}$? Does the $\mathcal{Z}$-dominating (or $\mathcal{I}$-dominating) property imply adding slalom capturing all ground model reals?

References

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