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On contractions in Hilbert space

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Dedicated to the memory of Professor Béla Sz.-Nagy

Abstract. In the first part of the paper we study the decompositions of a (bounded linear) operator similar to a normal operator in Hilbert space into the orthogonal sum of a normal (self-adjoint, unitary) part and of a part free of the given property, respectively. In the second part we investigate in a finite dimensional Hilbert space the minimal unitary power dilations (till the exponent k) of a contraction. We determine the general form of such dilations, examine their spectra, and the question of their isomorphy. The first step of the study here is also the decomposition of the contraction into unitary and completely non-unitary parts.

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1. Introduction

The paper consists of two parts. In Section 2 we study the decompositions of a (bounded linear) operator *similar to a normal* operator in Hilbert space into the orthogonal sum of a normal (self-adjoint, unitary) part and of a part completely free of the given property, respectively. The first results of this type for a *general* Hilbert space operator were obtained apparently by Livsic [18] and Brodskii [4] for the self-adjoint, and by Apostol [1] for the normal and unitary parts (see also Durszt [9]), respectively.

For a *contraction* in Hilbert space the basic decomposition result into *unitary and completely non-unitary (c.n.u.)* parts is due to Sz.-Nagy and Foias [22], see also Langer [16]. It is well known that this result has played a very important role in subsequent investigations. It may be interesting to note that further studies of different general decompositions of this type were undertaken, e.g., by the papers [19],[24],[12],[11] and [5].

The case of an operator similar to a normal can and will be studied here with the help of a general result on bounded Boolean algebras of idempotents in Hilbert space, one of the standard proofs of which is based on an elegant result by B. Sz.-Nagy [21] (see also Dixmier [6], Dunford and Schwartz [8] and Dowson [7]).

In Section 3 we study in a *finite dimensional* Hilbert space the *minimal unitary power dilations (till the exponent k)* of a contraction. Dilations of this type were studied first by Egervary [10], see also Riesz and Sz.-Nagy [20]. We determine the general form of such a dilation by using results of Arsene and Gheondea [2], Thompson and Kuo [25] and Benhida, Gorkin and Timotin [3]. Since two such dilations of a contraction are, in general, not isomorphic, we study their spectra, the multiplicity functions, and the question of their isomorphy. The first step of the study in this part is also the decomposition of the contraction into unitary and completely non-unitary parts.

Terminology and notation will be standard or explained in the text. Norm in the Hilbert space or operator norm will be denoted by $|\cdot|$, adjoint by *. When an *equivalent scalar product* is introduced (with the help of an operator B), then the corresponding notation will be $|\cdot|_B$ for the norms, and T_B for the B-adjoint of the operator T. Note that a *resolution of the identity* will be, in general, a spectral measure the values of which are *bounded idempotents.* Normal (equivalently, self-adjoint) idempotents will be called *projections.* $L(H)$ will denote the algebra of bounded linear operators in the Hilbert space H. If H is finite dimensional, we shall often identify an operator T with its matrix in a suitably fixed orthonormal basis. $\mathbf{C}, \mathbf{D}, \mathbf{N}, \mathbf{R}, \mathbf{T}$ denote the sets of all complex, in modulus less than 1, positive integer, real, and in modulus equal to 1 numbers, respectively.

2. Parts of an operator similar to a normal

Let T be a bounded linear operator in the complex Hilbert space H. It is well known that for a number of *properties* the operator is a uniquely determined orthogonal sum of two *parts*, $T = T_0 \oplus T_1$, where T_0 *has* the property, and T_1 is *completely free* of that property.

A systematic study of such decompositions was initiated by Szymanski [24], see also Fujii, Kajiwara, Kato and Kubo [12], Ernest [11], and Brown, Fong and Hadwin [5]. They studied as important examples, e.g., the properties of being a normal, self-adjoint or unitary operator, respectively.

One of the most important and consistently applied classical examples is the situation when T is a *contraction* in the complex Hilbert space H. Sz.-Nagy and Foias [22] (see also Langer [16]) showed that there is then an orthogonal decomposition as above such that the part T_0 is unitary, and the part T_1 is completely nonunitary. This decomposition is unique: the subspace H_0 of T_0 consists of those vectors $h \in H$ for which

$$|T^n h| = |h| = |T^{*n} h| \qquad (n \in \mathbf{N}),$$

and $T = T_0 \oplus T_1$ is called *the canonical decomposition* (with respect to being unitary) of T. It is clear that the subspace $H_0(T)$ is characterized as the maximal T-invariant subspace of H onto which the restriction of T is a unitary operator (see, e.g., Gokhberg and Krein [14, pp.288-290]).

Recall that for any operator $T \in L(H)$ the linear subspace

$$A(T) := \{h \in H : T^n h = T^{*n} h \quad \text{for every} \quad n \in \mathbf{N}\}$$

is the maximal linear subspace on which T and T^* coincide, and which is invariant with respect to both (i.e. T-reducing). Clearly, $A(T) = A(T^*)$, and the following holds (cf. [14, pp.275-276]):

For every operator $T \in L(H)$ there is a unique orthogonal sum decomposition

$$H = K_0(T) \oplus K_1(T)$$

such that the subspaces are orthogonally T-reducing, the operators T, T^* coincide on $K_0 \equiv K_0(T)$, and the restriction $T|K_1$ is *completely non-selfadjoint* or, equivalently, *simple* in the sense that the restrictions of T and T^* to *any* jointly invariant subspace $K \subset K_1$ do *not coincide.* For this (uniquely determined) decomposition $K_0(T) = A(T)$.

Note that there are *two further equivalent descriptions of the subspace* $A(T)$ in the case $T \in L(H)$:

1. it is the maximal subspace $M \equiv M(T)$ orthogonally reducing T on which the restriction $T|M$ is self-adjoint,

2. it is the T-invariant (closed) subspace generated by all the vectors

$$\{h \in H : Th = T^* h\}.$$

We shall here prove only that $A(T) = M(T)$. It is clear that $A(T) \subset M(T)$. Interestingly, the converse direction is proved in ([14, pp.276-277]) only under the assumption that $T \in L(H)$ is dissipative (which we shall not suppose).

Let T_I denote the imaginary part of T, i.e., $T_I := (T - T^*)/2i$. If $h \in M(T)$, then $(T_I h, h) = \Im(Th, h) = 0$. Hence the numerical range of the *bounded* operator $T_I|M(T)$ is the singleton $\{0\}$. The spectrum of a bounded operator is contained in the closure of the numerical range (see, e.g., Kato [15, V.3.3]), hence $\sigma(T_I|M(T)) = \{0\}$. Since $T_I|M(T)$ is self-adjoint, it is the zero operator. Hence $Th = T^* h$ for every $h \in M(T)$. By the equivalent description 2. above, $M(T) \subset A(T)$, as stated.

Assume now that S is a (bounded linear) operator *similar to a normal operator* or, equivalently, *a spectral operator of scalar type in the sense of Dunford* [8], and has resolution of the identity $E(\cdot)$:

$$S = \int_{\sigma(S)} \lambda E(d\lambda).$$

By [21], [6], [8], [7], there is then a (strictly) positive self-adjoint operator B in H such that the operator $N := BSB^{-1}$ is normal and, equivalently, its resolution of the identity

$$P(\cdot) = BE(\cdot)B^{-1}$$

consists of self-adjoint projections. We clearly have then $\sigma(N) = \sigma(S)$. If S is, in addition, a *contraction*, then

$$\sigma(N) = \sigma(S) \subset \overline{\mathbf{D}} = \mathbf{D} \cup \mathbf{T}.$$

Defining (for a general S)

$$N_0 := \int_{\mathbf{T}} \lambda P(d\lambda), \qquad N_1 := \int_{\mathbf{C}\setminus\mathbf{T}} \lambda P(d\lambda),$$

we see that the self-adjoint projections $P(\mathbf{T}), P(\mathbf{C} \setminus \mathbf{T})$ satisfy $P(\mathbf{T})P(\mathbf{C} \setminus \mathbf{T}) = 0$, hence $N = N_0 \oplus N_1$, N_0 is unitary and N_1 is completely nonunitary. Hence: if S is a contraction, then N is also. Further: the decomposition described above is the canonical one *for any normal contraction* N.

The operator B from the preceding paragraph determines on H *a new scalar product* defined by

$$(h, k)_B := (Bh, Bk) \qquad (h, k \in H).$$

This induces the B-norm $|h|_B = (h, h)_B^{1/2}$, which is equivalent to the old one and, together with the original Hilbert space $[H, (,)]$, we can also consider the new $H_B := [H, (,)_B]$. Any linear operator T is everywhere defined and bounded in H, in sign $T \in L(H)$, exactly when $T \in L(H_B)$. Further, the adjoints $T^* \in L(H)$ and $T_B \in L(H_B)$ are connected as follows: for any $x, y \in H$ we have

$$(BT_B B^{-1} Bx, By) = (BT_B x, By) = (T_B x, y)_B = (x, Ty)_B = (Bx, BTy) = (Bx, BTB^{-1}By).$$

Thus we obtain

$$BT_B B^{-1} = [BTB^{-1}]^*, \qquad \text{hence} \qquad T_B = B^{-2} T^* B^2.$$

Since $S = B^{-1}NB$, we have

$$SS_B = B^{-1}NBB^{-2}S^*B^2 = B^{-1}NN^*B = B^{-1}N^*NB = B^{-2}S^*BBSB^{-1}B = S_B S.$$

This means that the operator S is *normal with respect to the B-scalar product.* Further, if S is a *contraction* in $[H, (,)]$, then S is a contraction in H_B, i.e., with respect to the B-scalar product. Indeed, since $BS = NB$, and N is also a contraction, we obtain

$$|S|_B^2 = \sup_{|h|_B=1} |Sh|_B^2 = \sup(Sh, Sh)_B = \sup(BSh, BSh) = \sup_{|Bh|=1} |BSh|^2 = \sup_{|Bh|=1} |NBh|^2 \le |N|^2 \le 1.$$

Since $S \in L(H_B)$ is a normal contraction, the preceding paragraph yields its canonical decomposition (orthogonal *in the B-scalar product*).

Now we want to obtain *the canonical decompositions* for the spectral operator of scalar type $S \in L(H)$ *with respect to the original scalar product.* Apply the preceding notation, and consider the following three "good properties" of a part: 1) normal, 2) self-adjoint, 3) unitary.

Let $Y_k \equiv H_0(S; k)$ denote the subspace of the part of S for the property k) above ($k = 1, 2, 3$). Since Y_k is the maximal subspace of H on which $S|Y_k$ has property k), $\sigma(S|Y_k) \subset \sigma(S)$ is contained in $\mathbf{C}, \mathbf{R}, \mathbf{T}$, respectively.

For any Borel subset $b \subset \mathbf{C}$ consider the subspace $A[E(b)]$ defined above. The idempotent property implies that

$$A[E(b)] = \{h \in H : E(b)h = E(b)^*h\}.$$

Further, define

$$K(E) := \cap_{b \subset \mathbf{C}} A[E(b)].$$

Theorem 1. *Assume that the operator S is spectral of scalar type with resolution of the identity E, and apply the preceding notation. Then*

$$H_0(S; 1) = K(E),$$

(i.e., loosely speaking, the "maximal normal" subspace for S is identical with the maximal subspace on which all the relevant idempotents $E(b)$ are orthogonal).

Proof. If Y_1 is the maximal reducing subspace for S as above, we have $\sigma(S|Y_1) \subset \sigma(S)$. Further, by [7, Theorem 12.2], for every Borel set $b \subset \mathbf{C}$ we have $E(b)Y_1 \subset Y_1$, and the resolution of the identity F for $S|Y_1$ satisfies $F(b) = E(b)|Y_1$. The resolution of the identity for the normal operator $S|Y_1$ (is uniquely determined and) consists of orthogonal projections, hence each $E(b)|Y_1$ is self-adjoint. By the paragraph preceding the Theorem, we obtain $H_0(S;1) \subset K(E)$.

Denote the subspace $K(E)$ by Z. For each Borel set $c \subset \mathbf{C}$ the idempotent $E(c)$ leaves Z invariant. Indeed, if $h \in Z$, then $E(c)h = E(c)^*h$. Hence, for every Borel set $b \subset \mathbf{C}$ we have

$$E(b)E(c)h = E(b \cap c)h = E(b \cap c)^*h = E(b)^*E(c)^*h = E(b)^*E(c)h,$$

thus $E(c)h \in Z$. It follows that the restrictions $\{E(c)|Z : c \subset \mathbf{C}\}$ form a Boolean σ-algebra of idempotents on the subspace Z, and each $E(c)|Z$ is self-adjoint. It is clear that

$$S|Z = \int_{\mathbf{C}} \lambda E(d\lambda)|Z.$$

Hence the restriction $S|Z$ has property 1) with adjoint $(S|Z)^* = \int_{\mathbf{C}} \overline{\lambda} E(d\lambda)|Z$. Thus $K(E) \subset H_0(S;1)$. △

Corollary. *The maximal subspaces* $Y_k \equiv H_0(S;k) \quad (k = 2,3)$ *are obtained with the help of the maximal normal operator* $S|Z$ *in the well-known way:*

$$H_0(S;2) = E(\mathbf{R})Z, \qquad H_0(S;3) = E(\mathbf{T})Z.$$

△

Finally, we want to record the connection between the maximal self-adjoint ($\equiv$ normal) subspaces Z for the spectral measure E in the original topology and W for the spectral measure P in the B-topology.

Proposition. *Apply the preceding notation. The subspace* Z *is the maximal self-adjoint subspace for the spectral measure* E *in the original topology if and only if* $W := B^{-1}Z$ *is the maximal self-adjoint subspace for the spectral measure* P *in the* B*-topology.*

Proof. Fix any Borel set $b \in \mathbf{C}$, and write $E := E(b),\ P := P(b)$, etc. until the end of this proof. Since $E^* = E$ on the subspace Z, and $BE = PB$, we have

$$BPB^{-1}z = BP^*B^{-1}z = E^*z = Ez = B^{-1}PBz \qquad (z \in Z).$$

Let $w := B^{-1}z$, hence $B^2w = Bz$. We obtain then

$$Pw = B^{-2}PB^2w = B^{-2}P^*B^2w = P_Bw \qquad (w \in W).$$

Hence the maximal B-self-adjoint subspace for the spectral measure P contains W. Since the steps above are reversible, it is equal to W. △

3. Minimal unitary power dilations of a contraction till the exponent k in finite dimension

It is well known that for any contraction A in the Hilbert space H and $k \in \mathbf{N}$ there is a unitary operator U in a Hilbert space $K \supset H$ such that A^n is a projection of U^n for $n = 0,1,\ldots,k$. For the space K we may take the orthogonal sum $\oplus_{j=0}^k H$, and such a U is called a *unitary power dilation of* A *till the exponent* k.

Levy and Shalit [17] called a unitary power dilation till the exponent k *minimal*, if $K = span\{U^jh : h \in H, j = 0,1,\ldots,k\}$. They showed ([17, Theorem 1.3]) that *in an* n*-dimensional Hilbert space* (where $n \in \mathbf{N}$) for every $k \in \mathbf{N}$ each contraction A has a *minimal unitary power dilation till the exponent* k on a space K of *dimension* $n + kd_A$. Here, as usual, we apply the notation

$$D_A := (I - A^*A)^{1/2}, \quad d_A := \dim[ran(D_A)] \equiv rank(D_A),$$

where *ran* denotes range. We also apply the similar notation $D_{A^*},\ d_{A^*}$ for the adjoint contraction A^*. As we shall see in the proof of the next theorem, the minimality of the dimension $n + kd_A$ was established already in [25, Theorem 2]. Note that the unitary power dilation till the exponent k constructed by Egervary is, in general, not minimal ([17, p.3]). We shall show that the Egervary dilation is *minimal for any completely non-unitary* contraction.

Recall that two power dilations M and M' acting in the Hilbert spaces $K, K' \supset H$, respectively, of the contraction A in the Hilbert space H are called (see, e.g., [23, I.4]) *isomorphic*, if there is a unitary map U of K' onto K which is the identity on H and satisfies $M' = U^*MU$. Note that [25, p.349] calls such a relation a *unitary similarity preserving* A. It is well known that minimal unitary power dilations (for *all* nonnegative integers) of a given contraction A are isomorphic. In a *finite dimensional* Hilbert space for *minimal unitary power dilations of A till the exponent k* the situation is different (cf. [17, p.3]).

It is well known that in a finite dimensional Hilbert space two normal operators are unitarily equivalent if and only if their *spectral lists* are the same: the latter means the list of the eigenvalues taking into account their multiplicities (in other words: the *multiplicity function*).

Theorem 2. *Assume that A is a contraction in a finite dimensional Hilbert space with unitary part A_u and c.n.u. part A_c, and let d_A denote its defect number. Then*

$$A = A_u \oplus A_c, \qquad d_A = d_{A_c}.$$

Let $k \in \mathbf{N}$. If M is any unitary power dilation of A_c till the exponent k, then $M(A) := A_u \oplus M$ is a unitary power dilation of A till the exponent k, and every unitary power dilation of A till the exponent k has this form with M being a unitary power dilation till the exponent k of A_c. M is minimal (for A_c) if and only if $M(A)$ is minimal (for A).

The general form of (the matrix of) M of a c.n.u. contraction A in a suitable orthonormal basis is given by

$$M = \begin{pmatrix} A & D_{A^*}V_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & U_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & U_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & U_{k-1} \\ V_1 D_A & -V_1 A^* V_2 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where, with the notation $d := d_A$, each block is $d \times d$, and the operators $V_1, V_2, U_1, U_2, \dots, U_{k-1} : \mathbf{C}^d \to \mathbf{C}^d$ are unitaries. Define the operator

$$W \equiv W_k := V_1^*(V_2 U_1 U_2 \cdots U_{k-1})^*.$$

Then the spectral list of the minimal unitary dilation M till the exponent k is determined by W, and is given by the list of the solutions $z \in \mathbf{T}$ (with multiplicities) of the equation

$$\det[\Theta_A(z)^{-1} - z^k W] = 0, \qquad (sp)$$

where

$$\Theta_A(z) := -A + z D_{A^*}(I - zA^*)^{-1} D_A : \mathbf{C}^d \to \mathbf{C}^d$$

is the characteristic function of the operator A. In the converse direction: any possible list of the eigenvalues (with multiplicities) of a minimal unitary dilation M till the exponent k of a completely non-unitary contraction A is determined by a unitary ($d \times d$ matrix) W and the equation (sp) above.

Proof. The first part of the proof is based on a result of Thompson and Kuo [25, Theorem 2]. Let A be an arbitrary contraction in a finite dimensional Hilbert space. Let $n \in \mathbf{N}$ denote the dimension of the space in which the contraction A acts, and let $d \equiv d(A)$ denote its *unitary deficiency* defined as the number (with multiplicity) of the *singular values* of (the matrix of) A (strictly) *less than 1.* It is clear that d is the rank of $I - A^*A$ or, equivalently, the rank of $I - AA^*$. In operator terms, d is the common value of the defect numbers $d_A = d_{A^*}$, and is 0 if and only if the operator A is unitary.

The contraction A is the orthogonal sum of its unitary part and of its completely non-unitary (c.n.u.) part A_c, and for the defect numbers we have $d(A) = d(A_c)$. Invoking [25, Theorem 2], we see that any contraction matrix A has a unitary power dilation M till the exponent k, with M having m rows more than A, if and only if $m \geq kd$ (this is the cited result in [17, Theorem 1.3]).

If $M(A)$ is a unitary power dilation of A till the exponent k, then we have in a suitable orthonormal basis for $j = 0, 1, \ldots, k$

$$M(A)^j = \begin{pmatrix} A^j & * \\ * & * \end{pmatrix} = \begin{pmatrix} A_u^j \oplus A_c^j & * \\ * & * \end{pmatrix} = \begin{pmatrix} A_u^j & 0 & 0 \\ 0 & A_c^j & * \\ 0 & * & * \end{pmatrix}.$$

It follows that

$$M(A) = M(A_u \oplus A_c) = A_u \oplus M(A_c).$$

Hence the general form of the *minimal* dilation of the (general) contraction A will be the orthogonal sum of the *minimal* dilation of the c.n.u. part A_c plus the unitary part A_u of the original contraction.

Denote in this paragraph the dimension of the space of any operator T by $\dim T$. By [25, Theorem 2], for a minimal dilation $M(A)$ we obtain $\dim M(A) - \dim A = kd(A)$. Taking into account $\dim A = \dim H = n$, and $d(A) = d_A = d_{A^*} = d$, this gives $\dim M(A) = n + kd_A$, which is exactly [17, Theorem 1.3]. Further, we clearly have from above $\dim M(A) = \dim A_u + \dim M(A_c)$, where $M(A_c)$ denotes a minimal dilation of A_c. If we *start with a c.n.u.* contraction A_c, then its space has $\dim A_c = \dim H = n$, and $rank(I - A_c^* A_c) \equiv d(A_c)$ is also equal to n. Hence $\dim M(A_c) = d(A_c) + kd(A_c) = (k+1)d$. On the other hand, if we *start with a general* contraction $A \in L(H)$ such that $\dim H > d(A)$, and apply Egervary's original construction [10] of a unitary power dilation of A till the exponent k (see also [20]), then we do *not* obtain a *minimal* dilation of A till the exponent k.

Therefore *we may and will assume that the operator (matrix) A is c.n.u.*, that is $d = d(A) = n$, and apply [25, Lemma 5]. We obtain that there is an orthonormal basis in which the matrix M of the *minimal unitary power dilation till the exponent k* has the following partitioned form. Since the number of the new rows is $m = kd = kn$, we have blocks

$$\{M_{ij} : \ 0 \le i, j \le k\} = M$$

such that

(a) $M_{00} = A$ and the other diagonal blocks are square,

(b) each block is the zero matrix except perhaps for the subscripts

$$\{00, 01, 11, k0, k1, \text{ and } (i, i+1) \text{ for } 1 \le i < k\},$$

(c) the blocks with subscripts $(j-1, j) \quad (3 \le j \le k)$ are each unitary and $d \times d$,

(d) block M_{12} has d columns and at least d rows; block M_{k0} has d rows forming a linearly independent system.

Specifying to our situation, we can establish additionally that: since $n = d$, by (d) we obtain that M_{12} has d rows and d columns. Since the latter are the only nonzero parts of the corresponding columns of the big unitary matrix M, they form an orthonormal basis in $\mathbf{C}^d$, hence the block M_{12} is a unitary matrix. It follows that the matrix M_{11} is $d \times d$ and, since the *rows* of the block M_{12} also form an orthonormal system in $\mathbf{C}^d$, and the (long) rows in the block matrix M_{1*} must form an orthonormal system in $\mathbf{C}^{(k+1)d}$, that $M_{11} = 0$.

So we have obtained the following block matrix decomposition of M:

$$M = \begin{pmatrix} A & M_{01} & 0 & 0 & \ldots & 0 \\ 0 & 0 & U_1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & U_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & U_{k-1} \\ M_{k0} & M_{k1} & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Here all the blocks are $d \times d$, and we have written $U_j := M_{j,j+1}$, referring to the fact that they are unitaries for $j = 1, \ldots, k-1$. Introduce now the notation

$$V := \begin{pmatrix} A & M_{01} \\ M_{k0} & M_{k1} \end{pmatrix}.$$

This block submatrix of M is, in view of the structure of M, itself unitary. Since in our case $rank(D_A) = rank(D_{A^*}) = d$, and V is a unitary dilation (till the exponent 1) of A, we can apply the "folklore" Proposition 3.1 of [3] by Benhida, Gorkin and Timotin proving, with the help of a result of Arsene and Gheondea [2], that there are two $d \times d$ unitary matrices V_1, V_2 such that

$$V = \begin{pmatrix} I & 0 \\ 0 & V_1 \end{pmatrix} \begin{pmatrix} A & D_{A^*} \\ D_A & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V_2 \end{pmatrix}.$$

It follows that we have

$$M = \begin{pmatrix} I & 0 & 0 & 0 & \dots & 0 \\ 0 & U_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & U_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & U_{k-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & V_1 \end{pmatrix} \begin{pmatrix} A & D_{A^*} & 0 & 0 & \dots & 0 \\ 0 & 0 & I & 0 & \dots & 0 \\ 0 & 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I \\ D_A & -A^* & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 & \dots & 0 \\ 0 & V_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & 0 & \dots & I \end{pmatrix}.$$

From this we see that, in general, the dilation M is not isomorphic to the middle factor, (which we can call *the Egervary dilation*).

In order to *study the spectrum* of the dilation M, let $z \in \mathbf{C}$ and consider the matrix

$$M - zI_{(k+1)d} = \begin{pmatrix} A - zI_d & D_{A^*}V_2 & 0 & 0 & \dots & 0 \\ 0 & -zI_d & U_1 & 0 & \dots & 0 \\ 0 & 0 & -zI_d & U_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & U_{k-1} \\ V_1 D_A & -V_1 A^* V_2 & 0 & 0 & \dots & -zI_d \end{pmatrix}.$$

(Here and in what follows I_p denotes the identity operator in $\mathbf{C}^p$.) We shall index the block rows and columns above by $0, 1, \dots, d$. The number $z \in \mathbf{T}$ is in the spectrum of the unitary dilation M if and only if $\det(M - zI_{(k+1)d}) = 0$.

Let $z \in \mathbf{T}$. Since A is completely nonunitary, we have $z \notin \sigma(A)$. In order to establish the value of $\det(M - zI_{(k+1)d})$, as the first step, add the 0th block row of $M - zI_{(k+1)d}$ multiplied from the left by $-V_1 D_A (A - zI_d)^{-1}$ to the last block row. The latter will then be modified to the block row

$$(\,0 \quad V_1[-A^* - D_A(A - zI_d)^{-1} D_{A^*}]V_2 \quad 0 \quad 0 \quad \dots \quad -zI_d\,).$$

Multiply now the first block row of $M - zI_{(k+1)d}$ from the left by $V_1[-A^* - D_A(A - zI_d)^{-1} D_{A^*}]V_2/z$, and add it to the (modified) last block row. The latter will then be modified to

$$(\,0 \quad 0 \quad V_1[-A^* - D_A(A - zI_d)^{-1} D_{A^*}]V_2 U_1/z \quad 0 \quad \dots \quad -zI_d\,).$$

Multiply now the second block row of $M - zI_{(k+1)d}$ from the left by $V_1[-A^* - D_A(A - zI_d)^{-1} D_{A^*}]V_2 U_1/[z^2]$, and add it to the (modified) last block row. The latter will then be modified to

$$(\,0 \quad 0 \quad 0 \quad V_1[-A^* - D_A(A - zI_d)^{-1} D_{A^*}]V_2 U_1 U_2/[z^2] \quad \dots \quad -zI_d\,).$$

Proceeding in this way, in step k multiply the $(k-1)$st block row of $M - zI_{(k+1)d}$ from the left by $V_1[-A^* - D_A(A - zI_d)^{-1} D_{A^*}]V_2 U_1 U_2 \cdots U_{k-2}/[z^{k-1}]$, and add it to the (modified) last block row. Since the indicated transformations do not change the value of the determinant (see, e.g., Gantmakher [13]), we obtain then

$$\det(M - zI_{(k+1)d}) =$$

$$= \det \begin{pmatrix} A - zI_d & D_{A^*}V_2 & 0 & 0 & \dots & 0 \\ 0 & -zI_d & U_1 & 0 & \dots & 0 \\ 0 & 0 & -zI_d & U_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & U_{k-1} \\ 0 & 0 & 0 & 0 & \dots & V_1[-A^* - D_A(A - zI_d)^{-1}D_{A^*}]V_2U_1U_2\cdots U_{k-1}/[z^{k-1}] - zI_d \end{pmatrix}.$$

The value of the right-hand side determinant of the upper triangular block matrix is the product of the determinants of the diagonal blocks. Thus we obtain

$$\det(M - zI_{(k+1)d}) =$$

$$= \det(A - zI_d)[\det(-zI_d)]^{(k-1)} \det\{V_1[-A^* - D_A(A - zI_d)^{-1}D_{A^*}]V_2U_1U_2\cdots U_{k-1}/[z^{k-1}] - zI_d\}.$$

Introduce the notation $U := V_2U_1U_2\cdots U_{k-1}$. Then U and V_1 are unitary operators in $\mathbf{C}^d$. The operator valued function $z \mapsto -A^* - D_A(A - zI_d)^{-1}D_{A^*}$ is clearly connected with the characteristic function Θ of A and A^* (cf. [23, pp. 237-238]), and is identical with

$$\Theta_{A^*}(1/z) \equiv \Theta_A(z)^{-1} \qquad (|z| = 1),$$

(the operators mapping $\mathbf{C}^d$ to $\mathbf{C}^d$). Thus we obtain for $z \in \mathbf{T}$

$$\det(M - zI_{(k+1)d}) = (-z)^{(k-1)d} \det(A - zI_d) \det\{V_1\Theta_A(z)^{-1}U/[z^{k-1}] - zI_d\}.$$

Since the contraction A is completely nonunitary, its spectrum lies in the open disk $\mathbf{D}$. Hence for any $|z| = 1$ we have

$$\det(M - zI_{(k+1)d}) = 0 \Longleftrightarrow \det[\Theta_A(z)^{-1} - z^kW] = 0. \qquad (*)$$

Here we have applied the notation W for the unitary operator $V_1^{-1}U^{-1}$, and used the fact that the modulus of the determinant of a unitary matrix is 1. $\triangle$

Theorem 3. *Let $k \in \mathbf{N}$. The spectral list of any minimal unitary dilation till the exponent k of a completely nonunitary contraction A in $\mathbf{C}^d$ is the list of all solutions $z \in \mathbf{T}$ of the equation*

$$\det[z^k\Theta_A(z) + Y] = 0,$$

where Y is a unitary operator in $\mathbf{C}^d$. Equivalently, it is the list of all solutions $z \in \mathbf{T}$ of the equation

$$\det[z^{k+1}I_d - z^kA + (I_d - zA^*)D_{A^*}^{-1}YD_A] = 0.$$

It follows that if $z_1, z_2, \dots, z_{(k+1)d}$ is the solutions list, then the trace $tr\,(A)$ satisfies

$$tr\,(A) = z_1 + z_2 + \dots + z_{(k+1)d}.$$

In geometrical terms: the point $tr\,(A)/(k+1)d \in \mathbf{D}$ is the centre of gravity of the polygon determined by the points $z_1, z_2, \dots, z_{(k+1)d} \in \mathbf{T}$ of the list. This shows, in particular, that not every list $z_1, z_2, \dots, z_{(k+1)d} \in \mathbf{T}$ is the spectral list of **some** *minimal unitary dilation till the exponent k of the c.n.u. contraction A in $\mathbf{C}^d$. On the other hand, we have*

$$z_1 \cdot z_2 \cdot \dots \cdot z_{(k+1)d} = (-1)^{(k+1)d} \det[Y].$$

This shows how the product of the elements of the spectral list is connected to the number $\det[Y] \in \mathbf{T}$*, which clearly varies with the unitary matrix Y.*

Proof. By formula $(*)$ from the preceding proof, for any $z \in \mathbf{T}$ we have

$$\det(M - zI_{(k+1)d}) = 0 \Longleftrightarrow \det[\Theta_A(z)z^k - W^*] = 0.$$

In view of [23, VI.(1.2)], we have

$$[\Theta_A(z)z^k - W^*]D_A = D_{A^*}(I_d - zA^*)^{-1}(zI_d - A)z^k - W^*D_A.$$

The operators $D_{A^*}(I_d - zA^*)^{-1}$ are (boundedly) invertible for z in a neighbourhood of $\mathbf{T}$, hence in this neighbourhood the function $z \mapsto \det[[D_{A^*}(I_d - zA^*)^{-1}]^{-1}]$ is not 0. Multiplying the equation above from the left by the inverses, we obtain

$$[D_{A^*}(I_d - zA^*)^{-1}]^{-1}[\Theta_A(z)z^k - W^*]D_A = (zI_d - A)z^k - (I_d - zA^*)D_{A^*}^{-1}W^*D_A.$$

By the preceding remarks, for any $z \in \mathbf{T}$ we have

$$\det[\Theta_A(z)z^k - W^*] = 0 \Longleftrightarrow \det[(zI_d - A)z^k - (I_d - zA^*)D_{A^*}^{-1}W^*D_A] = 0.$$

The matrix $Y := -W^*$ is clearly unitary. Consider the matrix polynomial

$$P(z) := z^{k+1}I_d - z^kA + (I_d - zA^*)D_{A^*}^{-1}YD_A,$$

which is *monic* of degree $k+1$, and has $d \times d$ coefficient matrices. Denoting the entries of the matrix of A (in the considered orthonormal basis) by a_{ij}, the leading coefficients in the matrix of $P(z)$ are

$$\begin{pmatrix} z^{k+1} - z^k a_{11} & -z^k a_{12} & \dots & -z^k a_{1d} \\ -z^k a_{21} & z^{k+1} - z^k a_{22} & \dots & -z^k a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ -z^k a_{d1} & -z^k a_{d2} & \dots & z^{k+1} - z^k a_{dd} \end{pmatrix},$$

(plus in each entry we find at most linear terms in the variable z). This shows that the *determinant of* $P(z)$ obtains the highest degree terms (in z) from the main diagonal of the matrix. They are

$$z^{(k+1)d} - z^{(k+1)d-1}[a_{11} + a_{22} + \dots + a_{dd}]$$

(plus lower degree terms in z). Denoting the roots of the scalar polynomial $\det P(z)$ by $z_1, z_2, \dots, z_{(k+1)d}$, the Viéte formulae show that

$$z_1 + z_2 + \dots + z_{(k+1)d} = tr(A).$$

It follows that for an arbitrary minimal unitary dilation till the exponent k of A in $\mathbf{C}^d$, the centre of gravity of the elements of the spectral list of the dilation is $tr(A)/(k+1)d$, as stated.

The constant term in the polynomial $\det P(z)$ obtains if we calculate from the matrix of $P(z)$ the determinant with entries from the constant (matrix) term of $P(z)$. Hence it is equal to

$$\det[D_{A^*}^{-1}YD_A] = \det[D_{A^*}^{-1}]\det[D_A]\det[Y].$$

The Viéte formulae show that the product of the roots of $\det P(z)$ satisfy

$$z_1 \cdot z_2 \cdot \ldots \cdot z_{(k+1)d} = (-1)^{(k+1)d}\det[D_{A^*}^{-1}]\det[D_A]\det[Y].$$

Let $\lambda_1[\cdot], \dots, \lambda_d[\cdot]$ denote the eigenvalues (with multiplicities) of any operator in $\mathbf{C}^d$. Since the eigenvalues of A^*A and AA^* are identical, it is clear that

$$(-1)^d\det[D_A] = \prod_{i=1}^d \lambda_i[D_A] = \prod_{i=1}^d \lambda_i[(I_d - A^*A)^{1/2}] = \prod_{i=1}^d \lambda_i[(I_d - AA^*)^{1/2}] =$$

$$= \prod_{i=1}^d \lambda_i[D_{A^*}] = (-1)^d\det[D_{A^*}].$$

It follows that $\det[D_{A^*}^{-1}]\det[D_A] = 1$, hence

$$z_1 \cdot z_2 \cdot \ldots \cdot z_{(k+1)d} = (-1)^{(k+1)d}\det[Y],$$

as stated. △

For the basic properties of matrix polynomials in the next result see, e.g., [26].

Theorem 4. *Let $k \in \mathbf{N}$. Any minimal unitary dilation M till the exponent k of the c.n.u. contraction A in $\mathbf{C}^d$ is uniquely determined by an operator $B := D_{A^*}^{-1} Y D_A$, where $Y := -W^* = -V_2 U_1 U_2 \dots U_{k-1} V_1$. Two such dilations $M(B)$ and $M(B')$ are unitarily equivalent exactly when the first companion matrices $C(B)$ and $C(B')$ of the corresponding matrix polynomials $P(B, z)$ and $P(B', z)$ (see below) are similar.*

Proof. The preceding proof shows that any minimal unitary dilation M till the exponent k is uniquely determined by the unitary operator

$$Y := -W^* = -V_2 U_1 U_2 \dots U_{k-1} V_1,$$

and its spectral list is determined by the relation

$$\det(M - zI_{(k+1)d}) = 0 \Longleftrightarrow \det[(zI_d - A)z^k + (I_d - zA^*)D_{A^*}^{-1} Y D_A] = 0 \qquad (z \in \mathbf{T}).$$

Introduce the notation $B := D_{A^*}^{-1} Y D_A$, and

$$P(B, z) := z^{k+1} I_d - z^k A - zA^* B + B.$$

Then the map $Y \mapsto B$ is a bijection, and for the corresponding dilation $M \equiv M(Y) \equiv M(B)$ we have

$$\det[M(B) - zI_{(k+1)d}] = 0 \Longleftrightarrow \det[P(B, z)] = 0 \qquad (z \in \mathbf{T}).$$

It is well known that a monic matrix polynomial has the same spectrum as its first companion matrix C, i.e., $\det[P(B, z)] = 0$ if and only if

$$\det[C(B) - zI_{(k+1)d}] = \det \begin{pmatrix} -zI_d & I_d & 0 & 0 & \dots & 0 \\ 0 & -zI_d & I_d & 0 & \dots & 0 \\ 0 & 0 & -zI_d & I_d & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I_d \\ -B & A^*B & 0 & 0 & \dots & A - zI_d \end{pmatrix} = 0.$$

In addition, the invariant polynomials (hence the partial multiplicities of every eigenvalue) are identical. Consider now two unitaries Y and Y' in $\mathbf{C}^d$, the corresponding B and B', and the dilations $M(B)$ and $M(B')$. Since they are unitaries, $M(B)$ and $M(B')$ are unitarily equivalent exactly when they have the same spectral lists. This is the case exactly when $\det[P(B, z)]$ and $\det[P(B', z)]$ have the same zeros, i.e., when the two polynomials are identical:

$$\det[P(B, z)] \equiv \det[P(B', z)].$$

As mentioned above, this holds if and only if (with understandable notation)

$$\det[C(B) - zI_{(k+1)d}] \equiv \det[C(B') - zI_{(k+1)d}].$$

Since both companion matrices are diagonalizable, the equality of the characteristic equations is equivalent to the fact that $C(B)$ and $C(B')$ are similar. △

References

[1] C. Apostol, Sur la partie normale d'un ensemble d'operateurs de l'espace de Hilbert, Acta Math. Acad. Sci. Hungar., 17 (1966), 1-4.

[2] G. Arsene and A. Gheondea, Completing matrix contractions, J. Operator Theory, 7 (1982), 179-189.

[3] C. Benhida, P. Gorkin and D. Timotin, Numerical ranges of $C_0(N)$ contractions, arXiv:1009.2249v2 [math.FA] 2 Dec 2010.

[4] M.S. Brodskii, Triangular and Jordan representations of linear operators, Nauka, Moscow, 1969 (in Russian).

[5] A. Brown, C-K. Fong and D.W. Hadwin, Parts of operators on Hilbert space, Illinois J. Math., 22 (1978), 306-314.

[6] J. Dixmier, Les moyennes invariantes dans les semi-groupes et leurs applications, Acta Sci. Math., 12A (1950), 213-227.

[7] H.R. Dowson, Spectral theory of linear operators, Academic Press, London, 1978.

[8] N. Dunford and J.T. Schwartz, Linear Operators, Part III, Wiley-Interscience, New York, 1971.

[9] E. Durszt, On the unitary part of an operator on Hilbert space, Acta Sci. Math., 31 (1970), 87-89.

[10] E. Egervary, On the contractive linear transformations of n-dimensional vector space, Acta Sci. Math. 15 (1954), 178-182.

[11] J. Ernest, Charting the operator terrain, Mem. Amer. Math. Soc., No. 171, 1976.

[12] M. Fujii, M. Kajiwara, Y. Kato and F. Kubo, Decompositions of operators in Hilbert spaces, Math. Japonicae, 21 (1976), 117-120.

[13] F.R. Gantmakher, Theory of matrices, 4th ed., Nauka, Moscow, 1988 (in Russian).

[14] I.C. Gohberg and M.G. Krejn, Introduction in the theory of linear non-self-adjoint operators in Hilbert space, Nauka, Moscow, 1965 (in Russian).

[15] T. Kato, Perturbation theory for linear operators, Springer, Berlin, 1966.

[16] H. Langer, Ein Zerspaltungssatz für Operatoren im Hilbertraum, Acta Math. Hung., 12 (1961), 441-445.

[17] E. Levy and O.M. Shalit, Dilation theory in finite dimensions: the possible, the impossible and the unknown, arXiv:1012.4514v2 [math.FA] 15 Jul 2011.

[18] M.S. Livsic, On spectral decomposition of linear non-self-adjoint operators, Mat. Sbornik, 34 (1954), 144-199 (in Russian).

[19] B.B. Morrel, A decomposition for some operators, Indiana Univ. Math. J. 23 (1973), 497-511.

[20] F. Riesz and B. Sz.-Nagy, Lecons d'analyse fonctionnelle, 6me ed., Akad. Budapest, 1972.

[21] B. Sz.-Nagy, On uniformly bounded linear transformations in Hilbert space, Acta Sci. Math., 11 (1947), 152-157.

[22] B. Sz.-Nagy and C. Foias, Sur les contractions de l'espace de Hilbert. IV, Acta Sci. Math., 21 (1960), 251-259.

[23] B. Sz.-Nagy and C. Foias, Harmonic analysis of operators on Hilbert space, Akad. Budapest, 1970.

[24] W. Szymanski, Decompositions of operator-valued functions in Hilbert spaces, Studia Math. 50 (1974), 265-280.

[25] R.C. Thompson and C-C.T. Kuo, Doubly stochastic, unitary, unimodular, and complex orthogonal power embeddings, Acta Sci. Math. 44 (1982), 345-357.

[26] I.C. Gohberg, P. Lancaster and L. Rodman, Matrix polynomials, Academic Press, New York, 1982.

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